Sequenceable Groups and Related Topics

M. A. Ollis
Marlboro Institute for Liberal Arts and Interdisciplinary Studies,
Emerson College,
120 Boylston Street, Boston, MA 02116, USA.
matt_ollis@emerson.edu

Submitted Nov 15, 2001; Accepted: Aug 12, 2002.
Version 1 published: Aug 12, 2002
Version 2 published: May 31, 2013
Version 3 published: Feb 14, 2025

Abstract

In 1980, about 20 years after sequenceable groups were introduced by Gordon to construct row-complete latin squares, Keedwell published a survey of all the available results concerning sequencings. This was updated (jointly with Dénes) in 1991 and a short overview, including results about complete mappings and R-sequencings, was given in the CRC Handbook of Combinatorial Designs in 1995. In Sections 1 and 2 we give a survey of the current situation concerning sequencings, including details of the most important constructions. In Section 3 we consider some concepts closely related to sequenceable groups: R-sequencings, harmonious groups, supersequenceable groups (also known as super P-groups), terraces, the Gordon game and subset sequencings. We also look at constructions for row-complete latin squares that do not use sequencings.

Mathematics Subject Classifications: 20D60, 05B15

Author comment on Version 3 (2025). For compatability with existing references to the survey, I have organised the new material so that the sections and results from Version 2 remain the same, with one exception: the summary of results that was previously Section 2.5 is now 2.6. I have also maintained the more detailed proofs for these older results as some are in difficult to access papers. For the many new results, especially in the expanding landscape of related problems, I mostly state the results in consistent notation and give references for the papers that contain their proofs.

There has been notable progress in various areas since the previous version of the survey. In particular:

- the existence questions for R-sequencings and terraces in abelian groups have been fully resolved (Sections 3.1 and 3.4),

- it is now known that sequencings exist for all sufficiently large non-abelian groups (Section 2.5), as do R-sequencings and harmonious sequences when the group meets a long-known necessary condition (Sections 3.1 and 3.2),
- there has been renewed interest in, and consolidation of, questions concerning the sequencing of subsets of groups, leading to significant progress (Section 3.7).

Contents

1 Introduction

The problem of finding sequencings for groups was introduced by Gordon in 1961 [79], although similar ideas for cyclic groups go back at least as far as 1892 [105]. Our focus here is on the sequenceable group problem of Gordon; we give the necessary definitions and motivation in this section and look at the progress made in the next one. We consider some related problems: R-sequencings, harmonious groups, supersequenceable groups (also known as super P-groups), terraces, subset sequencings and the Gordon game in Section 3 and also give the conclusion to the row-complete latin square question that was one of the motivators for the original problem.

Among topics not covered here are Fibonacci sequences in groups, the study of which appears to have initiatied in [146] and sometimes includes the phrase "sequenceable group" [139], and the sequenceability of Steiner triple systems, for which see [5, 47, 69, 89, 101, 102]. Also not included are problems that naturally arise in looking for paths or cycles in

Cayley graphs, where using a small number of generating elements for the Cayley graph is usually desired.

Unless explicitly stated, group theoretic terms may be found in [138]. Sequencings have been previously surveyed by Keedwell [96, 99], Dénes and Keedwell [66, chapter 3], Evans [70], and, with a slightly different emphasis, Alspach [3].

A non-trivial finite group G of order n is said to be *sequenceable* if its elements can be arranged in a sequence $(b_1, b_2, \ldots, b_n)$ in such a way that the partial products $(a_1, a_2, \ldots, a_n)$, where $a_i = b_1 b_2 \cdots b_i$, are distinct. The sequence $(b_1, b_2, \ldots, b_n)$ is called a *sequencing* for G. If $(b_1, b_2, \ldots, b_n)$ is a sequencing for G then $b_1 = e$, where e is the identity of G (if $b_i = e$ for some $i \neq 1$ then $a_{i-1} = a_i$). The sequence $(a_1, a_2, \ldots, a_n)$ is called a *basic directed terrace* (for any element $g \in G$ the sequence $(ga_1, ga_2, \ldots, ga_n)$ is called a *directed terrace*—observe that Theorem 1 still holds for this more general definition).

Note that a sequencing uniquely determines a basic directed terrace and that a basic directed terrace uniquely determines a sequencing. It is common practice in more recent papers to omit the initial identity element of the sequencing.

A *latin square* of order n is an $n \times n$ array defined on a set X with n elements such that every element of X appears once in each row and once in each column. The notation $L = (l_{ij})$ represents a latin square L with l_{ij} in the ith row and jth column. A latin square is said to be *based* on a group G if the latin square can be bordered with the elements of G to form the Cayley table of G.

An $n \times n$ latin square is said to be *row complete* if every pair $\{x, y\}$ of distinct elements of X occurs exactly once in each order in adjacent horizontal cells. A latin square is said to be *column complete* if every pair $\{x, y\}$ of distinct elements of X occurs exactly once in each order in adjacent vertical cells. If a latin square is both row complete and column complete then it is said to be *complete*.

Another application is to graph theory. If there is a row-complete latin square of order n then the complete directed graph on n vertices can be decomposed into n disjoint Hamiltonian paths (a *Hamiltonian path* is a path which passes through each vertex exactly once; paths are *disjoint* if they have no edges in common). This is done by associating each symbol in the latin square with a vertex in the graph and taking a path to traverse the vertices in the order a row lists the symbols. As we are using a latin square the paths are Hamiltonian (since each symbol occurs exactly once in each row). As the latin square is row complete each ordered pair of symbols (x, y) occurs exactly once in adjacent horizontal cells, thus no edge is repeated and the paths are disjoint. Example 2 demonstrates this for $n = 4$. Observe that we have not used the property that each symbol occurs once in each column. If this property is removed from the definition of a row-complete latin square then we have a *Tuscan square*. A Tuscan square of order n is equivalent to a decomposition of the complete graph on n vertices into n Hamiltonian paths. See [78] for more details about Tuscan squares.

Theorem 1. [79] *Let G be a sequenceable group and $(b_1, b_2, \ldots, b_n)$ be a sequencing with associated basic directed terrace $(a_1, a_2, \ldots, a_n)$. Then $L = (l_{ij})$, where $l_{ij} = {a_i}^{-1} a_j$ for $1 \leqslant i, j \leqslant n$, is a complete latin square.*

Proof. Suppose $l_{ij} = l_{ik}$ for some $1 \leqslant i, j, k \leqslant n$. Then ${a_i}^{-1}a_j = {a_i}^{-1}a_k$, giving $a_j = a_k$. Therefore $j = k$ and L has no repeated entries in any row. Similarly, L has no repeated entries in any column. Therefore L is a latin square.

To show that L is row complete we need ${a_i}^{-1}a_j = x$ and ${a_i}^{-1}a_{j+1} = y$ to have a unique solution for i and j given any ordered pair (x, y) of distinct elements of G.

Inverting both sides of the first equation and post-multiplying by the second gives ${a_j}^{-1}a_{j+1} = x^{-1}y$, that is $b_{j+1} = x^{-1}y$, uniquely determining j. Now ${a_i}^{-1}a_j = x$ uniquely determines i, and L is row complete.

An analogous argument shows that L is also column complete. Therefore L is a complete latin square. $\square$

Example 2. Let $G = \mathbb{Z}_4$, the additively written cyclic group of order 4. Then $(0, 3, 2, 1)$ is a sequencing of G with basic directed terrace $(0, 3, 1, 2)$. The corresponding complete latin square L is given in Figure 2. Figure 2 shows how this leads to a decomposition of the complete directed graph on 4 vertices into disjoint hamiltonian paths.

0	3	1	2
1	0	2	3
3	2	0	1
2	1	3	0

Figure 1: L

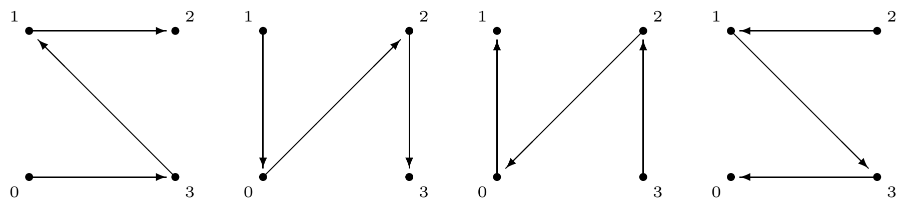

Figure 2: Decomposition of the complete directed graph with 4 vertices

Sequencings with special properties have been used to solve problems concerning bipartite tournaments balanced for carry-over effects [21, 45, 129], to give additional balance properties to latin squares [13, 15], to study 1-rotational Hamiltonian cycle systems of the complete graph [38, 49, 51], to solve some cases of the Oberwolfach problem [50, 125, 127], to construct rainbow-difference paths [103], and to construct Hamiltonian double latin squares [87, 119, 141]. Bate and Jones [41] give a survey of the use of sequencings and similar ideas in the field of experimental design.

If a group is sequenceable then the Cayley table of the group has an almost transversal. This result follows because a sequencing gives rise to a near-complete mapping; see [99, 71] for an explanation of this and other results concerning complete and near-complete mappings.

Vanden Eynden [145] extends the idea of a sequencing to groups of countably infinite order, showing that all such groups are sequenceable. Caulfield [54] shows that this notion corresponds to the ability to construct a quarter-plane complete infinite latin square and uses similar ideas to show that full-plane complete infinite latin squares also exist. These ideas are developed further by Evans, Martin, Minde and Ollis in [72].

2 Classifying Sequenceable Groups

In his paper [79], which introduced the concept of a sequencing, Gordon also completely classified the sequenceable abelian groups: see Section 2.1. He also noted that the quaternion group, Q_8, of order 8 and D_6 and D_8, the dihedral groups of order 6 and 8 respectively, are not sequenceable. He did find, however, that D_{10} is sequenceable. In 1968 Dénes and Tőrők [67] confirmed these results and added D_{12}, D_{14}, D_{16} and the non-abelian group of order 21 (the smallest non-abelian group of odd order) to the list of known sequenceable groups. Also in 1968 Mendelsohn [108] published an independently obtained sequencing for the non-abelian group of order 21. In 1973 Keedwell [95] sequenced the non-abelian group of order 27 with exponent 9 and Wang [152] sequenced the non-abelian groups of orders 39, 55 and 57.

In 1976 more significant headway started to be made with the question of which dihedral groups are sequenceable: see Section 2.2. Also in 1976 the concept of a symmetric sequencing was introduced. This set in motion the now nearly complete classification of sequenceable binary groups: see Section 2.3. We define a *binary group* to be a group with a single element of order 2. This does not contradict Dickson's [68] use of the term and fits well as a generalisation of binary polyhedral groups: see [63]. In the literature on sequencings, binary groups are usually called Λ-*groups*. In 1981 Keedwell was the first to give sequencings of infinitely many non-abelian groups of odd order: see Section 2.4.

Throughout this section we give the constructions for sequencings of the groups in question but usually refer the reader to the relevant papers for proofs of their correctness.

2.1 Abelian Groups

In this section we shall write abelian groups additively. In [79] Gordon proved the following theorem, which shows exactly which abelian groups are sequenceable. Recall that a binary group is defined to be a group with a single element of order 2.

Theorem 3. [79] *A finite abelian group G is sequenceable if and only if G is a binary group.*

Proof. ($\Rightarrow$): Suppose $(b_1, \ldots, b_n)$ is a sequencing for G with associated basic directed terrace $(a_1, \ldots, a_n)$. Since G is abelian we have that a_n is the sum of the elements of G written in any order.

We first suppose that G has no elements of order 2. For each $g \in G \setminus \{0\}$ we have $g \neq -g$, so the non-identity elements of the sequencing will cancel in pairs. This gives $a_n = 0$, contradicting $a_1 = 0$.

Now suppose that G has k elements, h_i, of order 2, where $k > 1$. These elements, along with 0, form a subgroup H of G of order $k + 1 = 2^l$ for some $l > 1$. Then H has a basis $\{u_1, \ldots, u_l\}$ for some $u_1, \ldots, u_l \in H$, thus each h_i is expressible in the form $\epsilon_1 u_1 + \cdots + \epsilon_l u_l$ with each $\epsilon_i \in \{0, 1\}$. Each expression of this form represents one of the elements of order 2. Therefore 2^{l-1} elements of H involve the generator u_i for each i. Since each element u_i occurs an even number of times in the expression for a_n, and $2u_i = 0$ for all i, we again reach the contradiction $a_n = 0$.

Note that if G has exactly one element, h, of order 2 then $a_n = h$.

($\Leftarrow$): Gordon gave a direct construction of sequencings in abelian binary groups. However, later results have made possible a simpler proof; we give this simpler proof in Section 3.4. $\square$

Example 4. Consider the cyclic group of even order, $\mathbb{Z}_{2n}$. The following $\mathbf{b}$ is a sequencing and has corresponding basic directed terrace $\mathbf{a}$.

$$\mathbf{b} = (0, 1, 2n-2, 3, 2n-4, 5, \ldots, 4, 2n-3, 2, 2n-1)$$

$$\mathbf{a} = (0, 1, 2n-1, 2, 2n-2, 3, \ldots, n+2, n-1, n+1, n).$$

This sequencing was first given (implicitly) by Lucas (who gave credit to Walecki) in [105], where it was used to solve a problem concerning schoolchildren performing round-dances. It is often referred to as the *Lucas-Walecki-Williams (directed) terrace/sequencing*, with the addition of "Williams" due to his similar construction for a non-directed terrace, see Section 3.4. Further solutions to this problem which use sequencings and related ideas are given in [38].

Combining Example 4 and Theorem 1 gives a method for constructing a complete latin square of any even order.

2.2 Dihedral Groups

Let $n \geqslant 3$. We describe the dihedral group D_{2n}, of order $2n$, as the set of ordered pairs (x, ϵ) with $x \in \mathbb{Z}_n$ and $\epsilon \in \mathbb{Z}_2$ and multiplication defined by:

$$\begin{aligned}(x, 0)(y, \delta) &= (x + y, \delta) \\ (x, 1)(y, \delta) &= (x - y, 1 + \delta).\end{aligned}$$

In 1976 Anderson [7] showed that D_{2p} is sequenceable if p is a prime with a primitive root r such that $3r \equiv -1 \pmod{p}$. Also in 1976 Friedlander [76] showed that D_{2p} is sequenceable if p is prime and $p \equiv 1 \pmod{4}$. In 1981 Hoghton and Keedwell [88] added the groups D_{2p}, where p is a prime such that $p \equiv 7 \pmod{8}$ and p has a primitive root r such that $2r \equiv -1 \pmod{p}$.

All of these results were obtained using quotient sequencings (see Section 2.4) and number theoretic arguments of varying intricacy.

In 1987 Anderson [8] used a computer search to show that all dihedral groups D_{2n} for $5 \leqslant n \leqslant 50$ are sequenceable. In 1990 Isbell [91] produced a general argument which, when allied to Anderson's computer search, covered all of the infinite classes mentioned above and more:

Theorem 5. [91] *The dihedral groups D_{2n}, of order $2n$, are sequenceable for all n, where $n \neq 3$ (D_6 is not sequenceable) and $n \neq 4k$.*

Proof construction. We split the construction into five cases and then some anomalous small examples. For the first three cases we exhibit a sequencing of the form $\mathbf{b} = (e, \alpha, \beta, \gamma)$ where e is the identity, α and γ partition the remaining elements of the form $(x, 0)$ and β consists of the elements of the form $(x, 1)$.

Case 1; $n = 4k + 1$: We define α, β and γ as follows:

$$\begin{aligned}
\alpha &= (2k-1,0),(2-2k,0),(2k-3,0),(4-2k,0),\ldots,(3,0),(-2,0),\\
&\quad\ (1,0),(2k,0)\\
\beta &= (0,1),(1,1),(2,1),\ldots,(2k-1,1),(4k,1),(2k,1),(2k+1,1),\ldots,\\
&\quad\ (4k-1,1)\\
\gamma &= (-2k,0),(-1,0),(2,0),(-3,0),(4,0),\ldots,(3-2k,0),(2k-2,0),\\
&\quad\ (1-2k,0).
\end{aligned}$$

Case 2; $n = 8k + 7$, ($k \geqslant 1$): We produce β in the same manner as before:

$$\beta = (0,1),(1,1),\ldots,(4k+2,1),(8k+6,1),(4k+3,1),\ldots,(8k+5,1).$$

Now, working in $\mathbb{Z}_{8k+7}$, consider the following sequence:

$$\begin{aligned}
\sigma &= -(2k+1),(4k+2),-(4k+1),4k,\ldots,-(2k+3),(2k+2),\\
&\quad\ -1,-2k,(2k-1),-(2k-2),\ldots,3,-2.
\end{aligned}$$

Define α to be the sequence in D_{2n} with σ in the first co-ordinates and 0's in the second, followed by $(-(4k+3),0)$. Define γ to be $(4k+3,0)$ followed by the sequence with $-\sigma$ in the first co-ordinates and 0's in the second. Now the sequence $(e, \alpha, \beta, \gamma)$ lists all elements of D_{2n} and is the required sequencing.

Case 3; $n = 8k+3$, ($k \neq 1, 2, 4$): Again we use the list $(e, \alpha, \beta, \gamma)$ but here β is slightly more complicated:

$$\begin{aligned}
\beta &= (0,1),(1,1),\ldots,(4k-1,1),(8k,1),(8k+1,1),(8k+2,1),(4k,1),\\
&\quad\ (4k+1,1),\ldots,(8k-1,1).
\end{aligned}$$

Similarly to the $n = 8k + 7$ case we look at sequences in $\mathbb{Z}_{8k+3}$ first. Define

$$X_{(k)} = \{x : -k \leqslant x \leqslant k-1, x \neq 1, -1\}.$$

We construct orderings X_k of $X_{(k)}$ beginning with 2 and ending with $-k$ such that the $2k-3$ differences between consecutive elements contain exactly one of i and $-i$ for $2 \leqslant i \leqslant 2k-2$. This condition is satisfied by the following three orderings:

$$\begin{aligned} X_3 &= (2, -2, 0, -3) \\ X_5 &= (2, 0, -4, 4, -3, 3, -2, -5) \\ X_7 &= (2, -5, 4, 0, -2, 3, -3, 5, -6, 6, -4, -7). \end{aligned}$$

We now extend inductively from k to $k+3$. Note that the penultimate element in each case is $-(k-3)$; this condition will also be preserved by the induction.

To order $X_{(k+3)}$ list X_k as far as the penultimate element $-(k-3)$, then continue $k+2, -(k+2), k+1, -(k+1), k, -k, -(k+3)$. This satisfies the conditions.

Consider the sets $Y_{(k)}$ $(\supseteq X_{(k)})$ of integers defined as follows:

$$Y_{(k)} = \{x : -(2k-1) \leqslant x \leqslant 2k, x \neq -1\}.$$

We define an ordering Y_k of $Y_{(k)}$ beginning with 2, ending with 1 and having differences of consecutive elements exactly one of i and $-i$ for $1 \leqslant i \leqslant 4k-2$:

$$(\underbrace{2, \ldots, -k}_{X_k}, k, -(k+1), k+1, \ldots, -(2k-1), (2k-1), 2k, 1).$$

Let τ_k be the sequence of differences of consecutive elements in this ordering Y_k: then the partial sums of τ_k list the translate $-2 + Y_{(k)}$ without repetition. Define α to be the sequence with τ_k in the first co-ordinates and 0's in the second, followed by $(-(4k+1), 0), (4k-1, 0), (-4k, 0)$. Define γ to be $(4k, 0)$ followed by the sequence with $-\tau_k$ in the first co-ordinates and 0's in the second, finishing with $(4k+1, 0), (-(4k-1), 0)$. We now have $(e, \alpha, \beta, \gamma)$ listing D_{2n} without repetition and this is the required sequencing.

Case 4; $n = 4k+2$, k even $(k \geqslant 2)$: For this we use the sequence $(e, \beta, \alpha, \delta, \gamma)$ where e is the identity, α and γ partition the remaining elements $(x, 0)$ (here α and γ are not of equal length), δ is $(4k+1, 1)$ and β covers the other elements $(x, 1)$. We construct β in the same manner as in case $n = 4k+1$, that is

$$\begin{aligned} \beta \;=\;& (0,1), (1,1), (2,1), \ldots, (2k-1, 1), (4k, 1), (2k, 1), (2k+1, 1), \ldots, \\ & (4k-1, 1). \end{aligned}$$

Consider the following two sequences in $\mathbb{Z}_{4k+2}$:

$$\begin{aligned} \sigma_1 &= -3, 5, -7, 9, \ldots, 2k-3, \underbrace{1-2k, 2k-2}, -(2k-3), 2k-5, \ldots, -5, 3 \\ \sigma_2 &= -2, 4, -6, 8, \ldots, 2k-4, \underbrace{-(2k-2), 1, 2k-1, -1}, -(2k-4), \\ & \quad 2k-6, \ldots, -4, 2. \end{aligned}$$

Define α to be $(2k+2, 0)$ followed by the sequence with σ_1 in the first co-ordinates and 0's in the second, followed by $(2k, 0), (2k+1, 0)$. Define γ to be the sequence with

σ_2 as the first co-ordinates and 0's as the second. Now α and γ cover all elements $(x, 0)$ such that $x \neq 0$, and $(e, \beta, \alpha, \delta, \gamma)$ is a sequencing of D_{2n}.

Case 5; $n = 4k + 2$, k odd ($k \geqslant 3$): We define the sequencing $(e, \beta, \alpha, \delta, \gamma)$ as in the previous case, but we need to modify σ_1 and σ_2 slightly as the length of the list each side of the braces is now odd, meaning that the sign alternation causes a problem. This problem is rectified by reversing the order of the terms in the braces, that is

$$\begin{aligned}
\sigma_1 &= -3, 5, -7, 9, \ldots, -(2k-3), \underbrace{2k-2, 1-2k}, 2k-3, -(2k-5), \ldots, \\
&\quad -5, 3 \\
\sigma_2 &= -2, 4, -6, 8, \ldots, -(2k-4), \underbrace{-1, 2k-1, 1, -(2k-2)}, 2k-4, \\
&\quad -(2k-6), \ldots, -4, 2.
\end{aligned}$$

The construction now goes through as before.

The anomalous cases: The sequencings given here are those due to Anderson [8], though Isbell did produce sequencings for D_{14}, D_{22}, D_{38} and D_{70} similar in style to his sequencings of the infinite classes. For brevity, we identify $(i, 0)$ with $i + 1$ and $(i, 1)$ with $n + i + 1$ (exactly as in [144]). Recall that D_6 is not sequenceable.

Below, $\mathcal{S}_{2n}$ is a sequencing for D_{2n}.

$$\begin{aligned}
\mathcal{S}_{12} &: (1, 11, 2, 7, 3, 9, 12, 10, 6, 5, 8, 4) \\
\mathcal{S}_{14} &: (1, 8, 2, 10, 7, 6, 9, 5, 11, 4, 14, 13, 12, 3) \\
\mathcal{S}_{22} &: (1, 8, 18, 15, 16, 5, 10, 4, 6, 13, 11, 3, 19, 17, 7, 9, 22, 2, 14, 12, 20, 21) \\
\mathcal{S}_{38} &: (1, 32, 15, 24, 23, 8, 38, 14, 22, 19, 37, 34, 5, 33, 36, 26, 12, 25, 13, 6, 28, \\
&\quad\; 21, 7, 29, 10, 4, 20, 11, 31, 18, 31, 35, 32, 16, 17, 27, 9) \\
\mathcal{S}_{70} &: (1, 3, 45, 10, 22, 33, 11, 16, 32, 54, 47, 61, 43, 62, 31, 12, 53, 20, 67, 35, 8, \\
&\quad\; 46, 29, 21, 7, 60, 25, 39, 34, 57, 64, 59, 6, 55, 66, 4, 38, 63, 65, 51, 70, 2, 13, \\
&\quad\; 68, 28, 37, 26, 50, 30, 24, 23, 58, 5, 40, 27, 69, 15, 48, 19, 42, 56, 9, 18, 36, \\
&\quad\; 17, 41, 44, 49, 14, 52)
\end{aligned}$$

□

In 1997 Li [104] completed the classification of sequenceable dihedral groups by sequencing D_{2n} where $n \equiv 0 \pmod 4$, $n \neq 4$. Recourse to Anderson's computer search [8] was again needed for some small cases.

Theorem 6. [104] *The dihedral groups D_{2n} are sequenceable when $n = 4k$, except when $n = 4$.*

Construction: The construction varies slightly as k varies modulo 4. For each case the sequencing is ((a), (b), …, (s)) from the appropriate table amongst Tables 1, 2, 3 and 4. Note that for some small values of k some of the components may be empty.

Table 1: $k \equiv 0 \pmod 4$, $k \geqslant 4$

Sequencing		No. of terms
(a)	$(0,0)$	1
(b)	$(0,1),(1,1),(2,1),\ldots,(2k-2,1)$	$2k-1$
(c)	$(4k-2,1)$	1
(d)	$(2k-1),(2k,1),(2k+1,1),\ldots,(4k-3,1)$	$2k-1$
(e)	$(2k,0)$	1
(f)	$(4k-3,0),(5,0),(4k-7,0),(9,0),\ldots,(2k-3,0)$	$k-2$
(g)	$(2k+2,0)$	1
(h)	$(2k-1,0),(2k+3,0),(2k-5,0),(2k+7,0),\ldots,(3,0)$	$k-1$
(i)	$(4k-2,0)$	1
(j)	$(4k-1,1)$	1
(k)	$(2,0),(4k-4,0),(6,0),(4k-8,0),\ldots,(k-2,0)$	$k/2-1$
(l)	$(1,0)$	1
(m)	$(3k-4,0),(k+6,0),(3k-8,0),(k+10,0),\ldots,(2k-2,0)$	$k/2-2$
(n)	$(3k,0)$	1
(o)	$(2k+1,0)$	1
(p)	$(k+2,0)$	1
(q)	$(2k-4,0),(2k+6,0),(2k-8,0),(2k+10,0),\ldots,(3k-2,0)$	$k/2-2$
(r)	$(4k-1,0)$	1
(s)	$(k,0),(3k+2,0),(k-4,0),(3k+6,0),\ldots,(4,0)$	$k/2-1$

Table 2: $k \equiv 1 \pmod 4$, $k \geqslant 5$

Sequencing		No. of terms
(a)	$(0,0)$	1
(b)	$(0,1),(1,1),(2,1),\ldots,(2k-2,1)$	$2k-1$
(c)	$(4k-2,1)$	1
(d)	$(2k-1),(2k,1),(2k+1,1),\ldots,(4k-3,1)$	$2k-1$
(e)	$(2k,0)$	1
(f)	$(4k-3,0),(5,0),(4k-7,0),(9,0),\ldots,(2k-1,0)$	$k-1$
(g)	$(2k+2,0)$	1
(h)	$(2k-3,0),(2k+5,0),(2k-7,0),(2k+9,0),\ldots,(3,0)$	$k-2$
(i)	$(4k-2,0)$	1
(j)	$(4k-1,1)$	1
(k)	$(2,0),(4k-4,0),(6,0),(4k-8,0),\ldots,(k,0)$	$(k-3)/2$
(l)	$(1,0)$	1
(m)	$(3k-3,0),(k+5,0),(3k-7,0),(k+9,0),\ldots,(2k-4,0)$	$(k-5)/2$
(n)	$(3k+1,0)$	1
(o)	$(2k+1,0)$	1
(p)	$(k+1,0)$	1
(q)	$(2k-2,0),(2k+4,0),(2k-6,0),(2k+8,0),\ldots,(3k-1,0)$	$(k-1)/2$
(r)	$(4k-1,0)$	1
(s)	$(k-1,0),(3k+3,0),(k-5,0),(3k+7,0),\ldots,(4,0)$	$(k-3)/2$

Table 3: $k \equiv 2 \pmod 4$, $k \geqslant 6$

Sequencing		No. of terms
(a)	$(0,0)$	1
(b)	$(0,1),(1,1),(2,1),\dots,(2k-2,1)$	$2k-1$
(c)	$(4k-2,1)$	1
(d)	$(2k-1),(2k,1),(2k+1,1),\dots,(4k-3,1)$	$2k-1$
(e)	$(2k,0)$	1
(f)	$(4k-3,0),(5,0),(4k-7,0),(9,0),\dots,(2k-3,0)$	$k-2$
(g)	$(2k+2,0)$	1
(h)	$(2k-1,0),(2k+3,0),(2k-5,0),(2k+7,0),\dots,(3,0)$	$k-1$
(i)	$(4k-2,0)$	1
(j)	$(4k-1,1)$	1
(k)	$(2,0),(4k-4,0),(6,0),(4k-8,0),\dots,(k,0)$	$k/2$
(l)	$(4k-1,0)$	1
(m)	$(3k-2,0),(k+4,0),(3k-6,0),(k+8,0),\dots,(2k-2,0)$	$k/2-1$
(n)	$(3k,0)$	1
(o)	$(2k+1,0)$	1
(p)	$(k+2,0)$	1
(q)	$(2k-4,0),(2k+6,0),(2k-8,0),(2k+10,0),\dots,(3k-4,0)$	$k/2-3$
(r)	$(1,0)$	1
(s)	$(k-2,0),(3k+4,0),(k-6,0),(3k+8,0),\dots,(4,0)$	$k/2-2$

Table 4: $k \equiv 3 \pmod 4$, $k \geqslant 7$

Sequencing		No. of terms
(a)	$(0,0)$	1
(b)	$(0,1),(1,1),(2,1),\dots,(2k-2,1)$	$2k-1$
(c)	$(4k-2,1)$	1
(d)	$(2k-1),(2k,1),(2k+1,1),\dots,(4k-3,1)$	$2k-1$
(e)	$(2k,0)$	1
(f)	$(4k-3,0),(5,0),(4k-7,0),(9,0),\dots,(2k-1,0)$	$k-1$
(g)	$(2k+2,0)$	1
(h)	$(2k-3,0),(2k+5,0),(2k-7,0),(2k+9,0),\dots,(3,0)$	$k-2$
(i)	$(4k-2,0)$	1
(j)	$(4k-1,1)$	1
(k)	$(2,0),(4k-4,0),(6,0),(4k-8,0),\dots,(k-1,0)$	$(k-1)/2$
(l)	$(4k-1,0)$	1
(m)	$(3k-1,0),(k+3,0),(3k-5,0),(k+7,0),\dots,(2k-4,0)$	$(k-3)/2$
(n)	$(3k+1,0)$	1
(o)	$(2k+1,0)$	1
(p)	$(k+1,0)$	1
(q)	$(2k-2,0),(2k+4,0),(2k-6,0),(2k+8,0),\dots,(3k-3,0)$	$(k-3)/2$
(r)	$(1,0)$	1
(s)	$(k-3,0),(3k+5,0),(k-7,0),(3k+9,0),\dots,(4,0)$	$(k-5)/2$

The anomalous cases: As in the proof of Theorem 5 we identify $(i, 0)$ with $i + 1$ and $(i, 1)$ with $n + i + 1$. Recall that D_8 is not sequenceable. Here we give Anderson's sequencing $\mathcal{S}_{2n}$ for D_{2n} [8]:

$$\begin{aligned} \mathcal{S}_{16} &: (1, 13, 11, 16, 4, 14, 3, 5, 6, 15, 8, 7, 9, 12, 2, 10) \\ \mathcal{S}_{24} &: (1, 17, 4, 11, 2, 8, 9, 13, 10, 3, 23, 24, 22, 15, 14, 6, 20, 18, 16, 7, 21, 19, 12, 5) \end{aligned}$$

We have now covered all required values of n. □

2.3 Binary Groups

Recall that a binary group is defined to be a group with a unique element of order 2. If G is a binary group then we denote the unique subgroup of order 2 by $\Lambda(G)$. The subgroup $\Lambda(G)$ is necessarily normal. Let G be a binary group of order $2n$ with z as its unique element of order 2. A sequencing $\mathbf{b}$ of G is said to be *symmetric* if it is of the form

$$\mathbf{b} = (e, b_2, b_3, \ldots, b_n, z, b_n^{-1}, \ldots, b_3^{-1}, b_2^{-1}).$$

Note that as z is the only element of order 2 we have immediately that $b_i \neq b_i^{-1}$ for $2 \leqslant i \leqslant n$. Gordon's construction (Theorem 3) for sequencing abelian groups gives symmetric sequencings, as does our new proof of that theorem in Section 3.4.

The aim of this section is to find symmetric sequencings for binary groups. We begin by considering the structure of binary groups.

The class of binary groups has arisen in several different contexts. For example, a Frobenius complement of even order (in particular, the multiplicative group of a nearfield) is a binary group [134, chapter 3.18], as is the automorphism group of a switching class of tournaments [36]. We have already noted that the binary polyhedral groups are binary groups. Coxeter[63, p. 82] posed the problem of classifying the binary groups, which is now solved (as we outline below). It is unclear who first solved this problem. Babai and Cameron [36] give a classification due to Glauberman but report that "[t]his result is known to some group theorists, but we are not aware of a proof in the literature".

If G is a binary group, then so is any subgroup of even order; in particular, each Sylow 2-subgroup. Now, 2-groups with a unique involution are known [52, p. 132]: they are cyclic or generalised quaternion groups. Here, the *generalised quaternion group* Q_{2^n} is defined by

$$Q_{2^n} = \langle u, v \colon u^{2^{n-1}} = e, v^2 = u^{2^{n-2}}, vuv^{-1} = u^{-1} \rangle.$$

The Sylow 2-subgroups of $G/\Lambda(G)$ have the form $S/\Lambda(S)$ for Sylow 2-subgroups S of G. The quotient $S/\Lambda(S)$ is cyclic or dihedral according as S is cyclic or generalised quaternion.

Conversely, a cohomological argument due to Glauberman, reported in [36], shows that, if H is a finite group with cyclic or dihedral Sylow 2-subgroups, then there is a unique binary group G with $G/\Lambda(G) \cong H$.

So the classification of binary groups reduces to that of groups with cyclic or dihedral Sylow 2-subgroups.

This classification is provided by Burnside's Transfer Theorem [52, p. 155] and the Gorenstein–Walter Theorem [80, 46]. The result is as follows. Recall that $O(G)$ is the largest normal subgroup of G of odd order. Let H be a finite group with Sylow 2-subgroup T. Then

- if T is cyclic, then $H/O(H) \cong T$;
- if T is dihedral, then $H/O(H)$ is isomorphic to the alternating group A_7, or to a subgroup of $\mathrm{P\Gamma L}(2, q)$ containing $\mathrm{PSL}(2, q)$ (where q is an odd prime power), or to T.

In particular, if G is a soluble binary group, then $G/O(G)\Lambda(G)$ is isomorphic to A_4, S_4, V, or a cyclic or dihedral 2-group. (V denotes the elementary abelian 2-group of order 4).

It is not completely straightforward to describe the corresponding binary groups. Glauberman's argument gives a description in cohomological terms.

The search for symmetric sequencings was effectively initiated by Lucas [105], although he did not use this terminology. Following on from Gordon's construction, further results have been obtained by Bailey and Praeger [40], Nilrat and Praeger [115] and Anderson, working alone and with Ihrig and Leonard. Symmetric sequencings with special properties have been used to solve some cases of the Oberwolfach problem [125]. Theorem 5 is an early result which the rest of the work can be seen as generalising:

Theorem 7. [7] *If G is a sequenceable group of odd order n then $G \times C_2$ has a symmetric sequencing.*

Proof. Let z be the non-identity element of C_2. Observe first that $G \times C_2$ is a binary group with (e, z) as its unique element of order two. Let $(e, d_2, \ldots, d_n)$ be a sequencing of G. Since G is of odd order, every non-identity element is distinct from its inverse. Partition $G \setminus \{e\}$ into $(n-1)/2$ two-element subsets of the form $\{g, g^{-1}\}$ and choose an element from each subset.

We now define a symmetric sequencing $\mathbf{b}$, where $\mathbf{b} = (b_1, b_2, \ldots, b_{2n})$, for $G \times C_2$:

$$(b_1, b_2, \ldots, b_n) = ((e, e), (d_2, \epsilon_2), \ldots, (d_n, \epsilon_n))$$

where

$$\epsilon_i = \begin{cases} z & \text{if } d_i \text{ is a chosen element} \\ e & \text{otherwise,} \end{cases}$$

$$b_{n+1} = (e, z)$$

and

$$(b_{n+2}, b_{n+3}, \ldots, b_{2n}) = (({d_n}^{-1}, \epsilon_n), ({d_{n-1}}^{-1}, \epsilon_{n-1}) \ldots, ({d_2}^{-1}, \epsilon_2)).$$

Now $\mathbf{b}$ lists $G \times C_2$ without repetition and does so symmetrically (since $(b_i, \epsilon_i)^{-1} = ({b_i}^{-1}, \epsilon_i)$).

Also, all of the elements in $G \times C_2$ are in the sequence of partial products. The partial products move through the basic directed terrace for G, with associated e's and z's in the second co-ordinate. Then, from the $(n+1)$th position, they move back through G's basic directed terrace with the e's and z's switched, finishing on (e, z). ☐

A key concept on which the work relies is that of a 2-sequencing (or equivalently a basic terrace), introduced by Bailey [37]. A 2-*sequencing* of H, a group of order n, is a sequence of elements $(e, d_2, d_3, \ldots, d_n)$, not necessarily distinct, such that:

- the associated partial products $e, ed_2, ed_2d_3, \ldots, ed_2 \cdots d_n$ are all distinct; this sequence is called a *basic terrace*,
- if $h \in H$ and $h \neq h^{-1}$ then
$$|\{i : 2 \leqslant i \leqslant n,\ d_i \in \{h, h^{-1}\}\}| = 2,$$
- if $h \in H$ and $h = h^{-1}$ then
$$|\{i : 1 \leqslant i \leqslant n,\ d_i = h\}| = 1.$$

Note that a sequencing for H is also a 2-sequencing for H. We look at the theory of terraces and 2-sequencings more in Section 3.4.

The following generalisation of Theorem 7 is pivotal:

Theorem 8. [9] *Let G be a binary group of order $2n$. Then G has a symmetric sequencing if and only if $G/\Lambda(G)$ has a 2-sequencing.*

Proof. Let $\pi : G \to G/\Lambda(G)$ be the natural projection. Then it is straightforward to check that if $(e, b_2, \ldots, b_n, z, {b_n}^{-1}, \ldots, {b_2}^{-1})$ is a symmetric sequencing of G then $(\pi(e), \pi(b_2), \ldots, \pi(b_n))$ is a 2-sequencing of $G/\Lambda(G)$.

Let z be the element of order 2 in G. If $y \in G/\Lambda(G)$ then $y = \{x, xz\}$ for some $x \in G$. Suppose we are given a 2-sequencing of $G/\Lambda(G)$. Lift this back to a sequence in G as follows:

(i) If $y \in G/\Lambda(G)$, $y \neq y^{-1}$ and y (equivalently y^{-1}) occurs twice in our 2-sequencing, the two occurrences of $y = \{x, xz\}$ can be lifted back to x and xz (in either order).

(ii) If $y \in G/\Lambda(G)$, $y \neq y^{-1}$ and both y and y^{-1} occur once in our 2-sequencing, say $y = \{x, xz\}$ and $y^{-1} = \{x^{-1}, x^{-1}z\}$, we can either lift y to x and y^{-1} to $x^{-1}z$ or y to xz and y^{-1} to x^{-1}.

(iii) If $y \in G/\Lambda(G)$, $y = y^{-1}$ and $y \neq \{e, z\}$ then y must occur once in our 2-sequencing. Now $y = \{x, x^{-1}\}$ and y may be lifted back to either x or x^{-1}.

(iv) If $y = \{e, z\} \in G/\Lambda(G)$ then y must be lifted to e.

This process clearly gives a sequence of the form $(e, b_2, \ldots, b_n)$ in G where $b_i \neq b_j$ and ${b_i}^{-1} \neq b_j$ for $i, j \leqslant n$, where $i \neq j$. Extend this to a sequence of all the elements of G: $(e, b_2, \ldots, b_n, z, {b_n}^{-1}, \ldots, {b_2}^{-1})$.

We claim that this is a symmetric sequencing of G.

The partial products are

$$(e, a_2, \ldots, a_n, a_n z, a_{n-1} z, \ldots, a_2 z).$$

We therefore need that $\{e, a_2, \ldots, a_n\}$ is a transversal of $\{\{w, wz\} : w \in G\}$. This follows because each a_i is either w_i or $w_i z$ for some $w_i \in G$ and the elements $\{e, z\}, \{w_2, w_2 z\}, \ldots,$ $\{w_n, w_n z\}$ of $G/\Lambda(G)$ are distinct as we started from a 2-sequencing. The sequence is clearly symmetric, so the claim is verified and the proof is complete. □

Note that the construction of Theorem 8 gives $2^{(n+k-1)/2}$ different symmetric sequencings of G for each 2-sequencing of $G/\Lambda(G)$, where k is the number of elements of order 2 in $G/\Lambda(G)$.

Anderson and Ihrig extend this further to show:

Theorem 9. [18] *If G is a binary group and $G/O(G)\Lambda(G)$ has a 2-sequencing then G has a symmetric sequencing.*

The groups A_4 and S_4 have sequencings [8] and hence have 2-sequencings. We have also seen that cyclic and dihedral 2-groups have sequencings (Example 4 and Theorem 6). Prior to the proof of Theorem 6, Anderson had shown that all dihedral groups have 2-sequencings [9, 11, 12].

The case $G/O(G)\Lambda(G) \cong V$ only arises when we consider binary groups with Q_8 as their Sylow 2-subgroup. The group Q_8 itself is not sequenceable, but Anderson and Leonard [20] show that the groups $Q_8 \times B$, where B is a non-trivial abelian group of odd order, have symmetric sequencings. These groups are Hamiltonian groups; that is, each is a non-abelian group every subgroup of which is normal. In fact, they are the only Hamiltonian binary groups. Anderson and Ihrig [18] show that all soluble binary groups G with $G/O(G)\Lambda(G) \cong V$, where $G \neq Q_8$, have symmetric sequencings. Theorem 9 now gives:

Theorem 10. [18] *All finite soluble binary groups, except Q_8, have symmetric sequencings.*

In [19] Anderson and Ihrig consider the structure of insoluble binary groups. They show that to find sequencings of all insoluble binary groups it is sufficient to find 2-sequencings of A_7, $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ for q an odd prime power greater than 3. They also show that there is no redundancy here; finding a 2-sequencing of each of these groups gives symmetric sequencings for an infinite set of insoluble binary groups and these sets are disjoint.

An early result in this direction is the sequencing of $\mathrm{PSL}(2, 5) \cong A_5$ [10], showing that such infinite sets of sequenceable insoluble binary groups do exist. More recently, 2-sequencings have been constructed for $\mathrm{PSL}(2, q)$, for $q \in \{7, 9, 11\}$, and for $\mathrm{PGL}(2, q)$, for $q \in \{5, 7\}$ [123].

2.4 Groups of Odd Order

Early in the development of the theory of sequenceable groups, Keedwell [97] and Wang [149] sequenced some non-abelian semi-direct groups of odd order which have a cyclic normal subgroup with prime index. More recently, Ollis and Tripp have shown that all such groups are sequenceable [128]. We do not give the full constructions here, merely introduce the two crucial concepts.

The first concept is the quotient sequencing (this concept was introduced by Friedlander [76]). Let G be a group of order pq with normal subgroup H of order q. A sequence $\mathcal{Q}$ of length pq containing elements of G/H is said to be a *quotient sequencing* of G/H if each element of G/H occurs q times in both $\mathcal{Q}$ and the partial product sequence (the *basic quotient directed terrace*) of $\mathcal{Q}$. Note that the natural map $G \to G/H$ maps a sequencing of G to a quotient sequencing of G/H; however, most quotient sequencings cannot be lifted to a sequencing of the parent group.

Suppose, with the above notation, that $G/H \cong C_p$ for some odd prime p, where $C_p = \langle u : u^p = e \rangle$. Let β be a primitive root of p such that $\beta/(\beta-1)$ is also a primitive root of p (Wang [149] reports that such a β exists, using the results of [55]). Wang [149] gives a quotient sequencing for G/H. Here we just give the associated basic quotient directed terrace as that may be expressed more simply:

$$e, e, \ldots, e, x \quad (q \text{ elements})$$

followed by $q-2$ copies of the sequence

$$x^{\beta^{p-2}}, x^{\beta^{p-3}}, \ldots, x \quad (p-1 \text{ elements})$$

and finishing with

$$x^{\beta^{p-2}}, x^{\beta^{p-3}}, \ldots, x^{\beta}, x^{\beta}, x^{\beta^2/(\beta-1)}, x^{\beta^3/(\beta-1)^2}, \ldots, x^{\beta-1}, e \quad (2(p-1) \text{ elements}).$$

Wang observes that this is a generalisation of the quotient directed terrace used by Keedwell [97].

The second important concept is the R-sequencing. An *R-sequencing* (also known as a *near-sequencing* or a *rotational sequencing*) of a group G is a sequence $(e, b_2, b_3, \ldots, b_n)$ of all the elements of G such that the partial products $(e, eb_2, eb_2b_3, \ldots, eb_2b_3 \cdots b_{n-1})$ are distinct and $eb_2b_3 \cdots b_n = e$. Keedwell and Wang both consider groups G with a normal cyclic subgroup of order q and index a prime p. The method they use is to find an R-sequencing of C_q and use the first $q-1$ elements of this for the first $q-1$ elements of the sequencing, filling the rest of the sequencing in a way that is also compatible with the above quotient directed terrace.

Suppose that p and q are odd primes, with $p < q$. Then there is a non-abelian group of order pq if and only if $q = 2ph + 1$ for some positive integer h. This group has a cyclic normal subgroup of order q. Keedwell [97] found sequencings of groups of this type whenever 2 is a primitive root of p. Wang showed that it is sufficient to to find an R-sequencing of C_q in which $x^{r-r^{1-\beta}}$ and $x^{r-r^{1-\beta}-1}$ are adjacent for some r with $r^p \equiv 1$

(mod q) and $r \not\equiv 1 \pmod q$. In [148] Wang gives some examples of such R-sequencings where 2 is not a primitive root of p.

Wang [149] also finds a sequencing which is compatible with the above quotient directed terrace for the unique non-abelian group of order p^m that has a cyclic normal subgroup of index p, where p is an odd prime and $m > 3$.

Let G be a group of odd order n. A sequencing $(e, b_2, \ldots, b_n)$, is said to be a *starter-translate* sequencing (Anderson [14] abbreviates this to *st-sequencing*) if both of the sets $\{b_2, b_4, \ldots, b_{n-1}\}$ and $\{b_3, b_5, \ldots, b_n\}$ contain precisely one of g and g^{-1} for each $g \in G \backslash \{e\}$. Anderson [14] shows that if G and H are groups with st-sequencings then $G \times H$ also has an st-sequencing. He also shows that Keedwell's sequencing of the non-abelian group of order pq is starter-translate whenever both p and q are congruent to 3 modulo 4, considerably extending the set of odd integers n for which a sequenceable group of order n is known to exist.

Two papers by Ollis and Tripp [121, 128] build on the ideas above to find sequencings for many more non-abelian semi-direct groups with an abelian normal subgroup of prime index. The most general construction family is collected in Theorem 11.

Theorem 11. [128] *Let q be an odd prime and let m be an odd integer with either $q^2 \mid m$ or $p \mid m$ for some prime $p \equiv 1 \pmod q$. Let α be an automorphism of $\mathbb{Z}_m$ of order q. The group $\mathbb{Z}_q \ltimes_\alpha \mathbb{Z}_m$ is sequenceable.*

Further constructions are also given for some particular families of groups that have a non-cyclic group in place of $\mathbb{Z}_m$ in Theorem 11, chosen to prove the following result.

Theorem 12. [128] *If there is a non-abelian group of odd order n, then there is at least one sequenceable group of order n.*

2.5 Small and Large Groups

Anderson [8, 10] introduced a hill-climbing algorithm to find sequencings for all non-abelian groups of order n in the range $10 \leqslant n \leqslant 32$ and the groups A_5 and S_5, the alternating and symmetric groups on 5 symbols. This was developed further in [123] giving the following result for small groups:

Theorem 13. [8, 10, 123] *If G is a group of order n with n in the range $10 \leqslant n \leqslant 255$, then G is sequenceable.*

In [123], the algorithm was also used to find sequencings for the groups A_6, S_6, $\mathrm{PSL}(2, q_1)$ and $\mathrm{PGL}(2, q_2)$, where q_1 and q_2 are prime powers with $3 \leqslant q_1 \leqslant 11$ and $3 \leqslant q_2 \leqslant 8$.

At the other end of the scale, Müyesser and Pokrovskiy have shown the following result.

Theorem 14. [114] *Every sufficiently large non-abelian group is sequenceable.*

Theorem 14 is proved using a generalization of the Hall-Paige Conjecture, giving a new method that seems to be a powerful tool for problems of this type. We see it again in Section 3.1.

How large is sufficiently large? Müyesser reports that "considering groups of order at least 10^{50} or so should be comfortably enough for everything to go through in the paper as written, but I'd guess somebody with sufficient motivation could bring this down" [113].

2.6 Summary

Collecting together the results of the section, we find the following list of main families of groups that are known to be sequenceable.

- Dihedral groups of order at least 10
- Soluble (including abelian) binary groups, except Q_8
- Insoluble binary groups G with A_5 as their only non-abelian composition factor
- At least one of the non-abelian groups of each odd order at which a non-abelian group exists.
- Non-abelian groups of order n, where $10 \leqslant n \leqslant 255$,
- A_5, S_5, A_6, S_6, $\mathrm{PSL}(2, q_1)$ and $\mathrm{PGL}(2, q_2)$, where q_1 and q_2 are prime powers with $3 \leqslant q_1 \leqslant 11$ and $3 \leqslant q_2 \leqslant 8$,
- Sufficiently large non-abelian groups.

The only groups known to be non-sequenceable are abelian groups which do not have a unique element of order 2 and the non-abelian groups D_6, D_8 and Q_8.

Conjecture 15. (Keedwell) D_6, D_8 and Q_8 are the only non-abelian non-sequenceable groups.

A milder conjecture is

Conjecture 16. (Anderson) Q_8 is the only binary group which does not have a symmetric sequencing.

3 Related Concepts

In this section we look at some concepts related to sequencings: R-sequencings, harmonious groups, supersequenceable groups (also known as super P-groups), terraces, the Gordon Game, row-complete latin squares and subset sequencings. Related concepts that are not included in this version of the survey are directed terraces without repeated differences at greater distances (see, for example, [123]), weak sequenceability (see, for example, [57]), Γ-sequenceability in graphs (see, for example, [56]) and double sequenceability (see, for example, [93]).

3.1 R-sequencings

A pair of latin squares (l_{ij}) and (l'_{ij}) are said to be *orthogonal* if every ordered pair of symbols occurs exactly once among the n^2 pairs (l_{ij}, l'_{ij}). It is shown in [133] that the existence of a group of order n having an R-sequencing (see page 16 for the definition) is a sufficient condition for there to exist a pair of orthogonal latin squares of order n. (More specifically, it is shown that having an R-sequencing is a sufficient condition for a group to have a complete mapping and having a complete mapping in a group of order n is sufficient to produce a pair of orthogonal latin squares of order n. See [99] for a summary of these and related topics.)

The study of orthogonal latin squares was originally motivated by a problem of Euler, set in 1779: "Thirty-six officers of six different ranks and taken from six different regiments, one of each rank in each regiment, are to be arranged, if possible, in a solid square formation of six by six, so that each row and each column contains one and only one officer of each rank and one and only one officer from each regiment". The solution of this problem is equivalent to the construction of a pair of orthogonal latin squares of order 6. In 1782 Euler conjectured that no such pairs of latin squares exist for orders $n = 4k + 2$; this was proved true for $n = 6$ by Tarry in 1900 and false for all $n > 6$ by Bose, Shrikhande and Parker in 1960. It is easily seen to be true for $n = 2$. See [65, chapter 5] for a thorough account of orthogonal latin squares and the history of Euler's conjecture.

For abelian groups the properties of being sequenceable and R-sequenceable are mutually exclusive as the final element of the partial product sequence is invariant.

Theorem 17 records the main points of progress towards the classification of abelian R-sequenceable groups achieved by different authors since the problem was introduced and Theorem 18 states the full resolution achieved by Alspach, Kreher and Pastine in 2017.

Theorem 17. *The following types of abelian group are R-sequenceable:*

(i) C_n*, where* n *is odd* [77],

(ii) abelian groups of odd order with (possibly trivial) cyclic Sylow 3*-subgroups* [77],

(iii) C_3^n*, where* $n \geqslant 2$ [77],

(iv) $C_3 \times C_{3n}$*, where* $n \geqslant 2$ [150],

(v) abelian groups with Sylow 3*-subgroups of the form* $C_3^\rho \times C_9^\rho \times C_{27}^\sigma$ *or* $C_3^\rho \times C_9^{\rho+1} \times C_{27}^\sigma$ [132],

(vi) abelian groups with Sylow 3*-subgroups of the form* $C_3^\rho \times C_9^\rho \times C_{27}^\sigma \times C_{81}^\tau$ *or* $C_3^\rho \times C_9^\rho \times C_{27}^\sigma \times C_{81}^\tau \times C_{3^k}$*, where* k *has the same parity as* $\rho + \sigma$ [106],

(vii) direct products of cyclic groups with each cyclic factor having order congruent to 1 (mod 4) [106],

(viii) $C_2 \times C_{4n}$*, where* $n \geqslant 1$ [77],

(ix) abelian groups with Sylow 2*-subgroups* C_2^n*, where* $n = 2$ *or* $n \geqslant 4$ [77],

(x) abelian groups with Sylow 2*-subgroups* $C_2 \times C_{2^n}$*, where* n *is odd* [77],

(xi) abelian groups of even order whose Sylow 2*-subgroups are neither* $C_2 \times C_4$ *nor* $C_2 \times C_2 \times C_2$ [82],

(xii) infinitely many abelian groups whose Sylow 2*-subgroups are non-cyclic of order* 8 [132].

Alternative constructions for R-sequencings of C_n are given in [1] and it is reported in [66, Chapter 3] that Ringel claims to have shown that $C_2 \times C_{6n+2}$ is R-sequenceable for $n \geqslant 1$.

Theorem 18. [4] *An abelian group is R-sequenceable if and only if its Sylow* 2*-subgroups are trivial or non-cyclic.*

McGill and Ollis [107] have shown that the number of R-sequencings for C_{2n+1} grows exponentially with n.

The following theorem gives early results for non-abelian groups. Recall that D_{2n} is the dihedral group of order $2n$ and the dicyclic group Q_{4n} is

$$Q_{4n} = \langle a, b : a^{2n} = e, b^2 = a^n, ab = ba^{-1} \rangle.$$

If n is a power of 2 then Q_{4n} is a generalised quaternion group.

Theorem 19.

(i) The dihedral group D_{2n} *of order* $2n$ *is R-sequenceable if and only if* n *is even* [98].

(ii) The dicyclic group Q_{4n} *is R-sequenceable if and only if* n *is an even integer greater than* 2 [151].

(iii) The non-abelian groups of order pq*, where* p *and* q *are odd primes, are R-sequenceable* [98, 151].

(iv) The two non-abelian groups of order 27 *are R-sequenceable* [44].

Using methods similar to those employed in the abelian cases of Theorem 17, a variety of non-abelian groups were shown to be R-sequenceable in [122]; a selection of these are described in Theorem 20.

Theorem 20. [122] *Let* $s \geqslant 1$*. For* i *in the range* $1 \leqslant i \leqslant s$*, take*

$$H_i \in \{C_2^2, C_2^3, D_8, C_6 \times C_2, D_{12}, A_4\}.$$

Let K *be a (possibly trivial) group of one of the following types:*

- *abelian groups of even order with a non-cyclic Sylow* 2*-group of order not equal to* 8*,*

- *abelian groups of odd order coprime to* 6*,*
- *non-abelian groups of order* pq*, for odd primes* p *and* q*,*
- D_{4k} *with* $k > 3$ *and* $k \not\equiv 0, 1 \pmod 6$*,*
- Q_{4k} *for* $k \equiv 4$ *or* $8 \pmod{12}$*.*

Let N *be a (possibly trivial) nilpotent group of odd order. Then*

$$H_1 \times \cdots \times H_s \times K \times N$$

is R-sequenceable.

Finally, similarly to the case for the sequenceability of large groups, Müyesser and Pokrovskiy have shown the following result.

Theorem 21. [114] *Every sufficiently large group whose Sylow* 2*-subgroups are either trivial or non-cyclic is R-sequenceable.*

3.2 Harmonious Groups

Similarly to R-sequencings, a harmonious or #-harmonious sequence for a group of order n gives rise to a complete mapping of the group and hence a pair of orthogonal latin squares. They were introduced in [42].

Let G be a non-trivial group of order n and let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an arrangement of the elements of G. Let $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be defined by $b_i = a_i a_{i+1}$ for $i < n$ and $b_n = a_n a_1$. If the elements of $\mathbf{b}$ also include all of the elements of G then $\mathbf{a}$ is a *harmonious sequence* and G is *harmonious*.

The situation for abelian groups is completely settled:

Theorem 22. [42] *An abelian group is harmonious if and only if it has non-cyclic or trivial Sylow* 2*-subgroups and is not an elementary abelian* 2*-group.*

For a non-abelian group to be harmonious it must satisfy the same necessary condition as an abelian group: its Sylow 2-subgroups must be trivial or non-cyclic. Known results for non-abelian groups:

Theorem 23. *The following non-abelian groups are harmonious:*

(i) non-abelian groups of odd order [42]*,*

(ii) D_{8t} *and* D_{24t-12}*, for* $t \geqslant 1$ [42]*,*

(iii) Q_{8t}*, for* $t \geqslant 2$ [151]*,*

(iv) sufficiently large groups that have trivial or non-cyclic Sylow 2*-subgroups* [114]*.*

The notion of #-harmoniousness is similar, but with the identity of G removed. Let G be a non-trivial group of order n with identity e and let $\mathbf{a} = (a_1, a_2, \dots, a_{n-1})$ be an arrangement of the elements of $G \setminus \{e\}$. Let $\mathbf{b} = (b_1, b_2, \dots, b_{n-1})$ be defined by $b_i = a_i a_{i+1}$ for $i < n-1$ and $b_{n-1} = a_{n-1}a_1$. If the elements of $\mathbf{b}$ also include all of the elements of $G \setminus \{e\}$ then $\mathbf{a}$ is a *#-harmonious sequence* and G is *#-harmonious*.

Theorem 24. [42] *An abelian group is #-harmonious if and only if it has non-cyclic or trivial Sylow 2-subgroups and is not $\mathbb{Z}_3$.*

Javaheri and de Wolf extend the notion of harmoniousness to arbitrary subsets in the natural way and show that if G is an abelian binary group with involution z, then $G \setminus \{z\}$ is harmonious [94].

3.3 Supersequenceable Groups

Let G be a group of order n with derived group G'. As the elements of G commute modulo G', the products of all the elements of G lie in the same coset hG' of G', regardless of the order of multiplication. It is known [64, 137] that each element of this special coset may be expressed as the product of all of the group elements in some order (groups with this property were originally known as *P-groups*, but this name is now redundant).

In 1983 Keedwell [98] defined super P-groups, which we now call supersequenceable groups. A *supersequenceable group* is a finite group G in which each element g of the special coset is either

- the last element of some basic directed terrace, or
- the last element of the partial product sequence associated with some R-sequencing.

The second condition is used only when $g = e$ a situation that occurs only when $hG' = G'$.

The following two theorems, due mostly to Keedwell [98], give non-abelian supersequenceable groups.

Theorem 25. [98] *Let G be an abelian group. Then G is a supersequenceable group if and only if G is sequenceable or R-sequenceable.*

Proof. Observe that $G' = \{0\}$, so the relevant coset of G' has just one element. This element is the identity if G is not a binary group, and is the unique element of order 2 if G is a binary group (see Theorem 3). Thus, if G is not a binary group then G is a supersequenceable group if and only if G is R-sequenceable. If G is a binary group then G is a supersequenceable group if and only if G is sequenceable. ☐

Theorem 26. *The following groups are supersequenceable groups:*

(i) Dihedral groups D_{2n} where $n \geqslant 5$ is odd [98]*,*

(ii) Dihedral groups D_{2n} where n is twice an odd prime [98]*,*

(iii) Groups of order pq where p and q are primes, $p < q$ and 2 is a primitive root of p [98]*,*

(iv) The two non-abelian groups of order 27 [44].

3.4 Terraces

As we noted in Section 2.3, terraces and 2-sequencings are equivalent. We say that a group that has a terrace is *terraced*. Terraces were introduced by Bailey [37] to prove Theorem 27—an analogue of Theorem 1 for quasi-complete latin squares. An $n \times n$ latin square is said to be *row quasi-complete* if each distinct pair of symbols $\{x, y\}$ occurs in adjacent horizontal cells twice (in either order). It is said to be *column quasi-complete* if each pair of distinct symbols $\{x, y\}$ occurs in adjacent vertical cells twice (in either order). A latin square that is both row quasi-complete and column quasi-complete is said to be *quasi-complete*.

Row-quasi-complete latin squares were used by Williams [154] for designing experiments where carry-over effects are thought to be present. He uses them in pairs, one containing the reverses of the other's rows, giving a design in which each pair of treatments occurs twice in each order as row-neighbours. He gives an example of such a design being used in practice to study the effect of diet on the milk yield of cows. An application where quasi-completeness is the natural requirement, rather than being used when completeness is unavailable, is given in [39]. The experiment described concerns five methods of controlling insects on spring beans. A quasi-complete latin square of order 5 is advocated because it was felt that there may be neighbour effects between adjacent plots from insects overspilling from a plot containing spring beans with a treatment that does little (or nothing) to repel them. It is pointed out that row neighbours should be kept distinct from column neighbours as plots in this type of experiment are rarely square—in this instance they measured 1.2m $\times$ 1m. A similar experiment is described in [142]. However, this experiment has six treatments and their quasi-complete latin square is also complete.

Quasi-complete latin squares have also been considered by Freeman [73, 74] and Campbell and Geller [53].

Theorem 27. [37] *Let G be a terraced group with terrace $(a_1, a_2, \ldots, a_n)$. Then the square $(l_{ij}) = ({a_i}^{-1}a_j)$, where $1 \leqslant i, j \leqslant n$, is a quasi-complete latin square.*

Proof. Similar to the proof of Theorem 1. □

We would therefore like to know which groups are terraced. The following result is originally due to Williams [154] and has been rediscovered, in various guises, by many authors and used in many ways, see [2].

Theorem 28. [154] *For all positive integers n, the cyclic group $\mathbb{Z}_n$ is terraced.*

Proof construction. The sequence

$$(0, 1, n-1, 2, n-2, 3, \ldots)$$

is a terrace for $\mathbb{Z}_n$. Its 2-sequencing is

$$(0, 1, n-2, 3, n-4, 5, \ldots). \qquad \square$$

When n is even the terrace given in Theorem 28 is directed. This is the one given in Example 4, explaining the Lucas-Walecki-Williams name given there.

Much work has been done on finding terraces for cyclic groups that have special properties. Sometimes the interest is in the latin squares or other desgins that can be constructed from them [13, 15, 24, 116, 118, 125, 127, 143]; sometimes it is in properties of the terraces themselves [22, 23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 126, 130, 136].

Returning to the question of which groups are terraced, the following two results are due to Bailey.

Theorem 29. [37] *$G = \mathbb{Z}_2{}^n$ is not 2-sequenceable for $n > 1$.*

Proof. For each $g \in G$, we have $g = -g$. Thus G is 2-sequenceable if and only if G is sequenceable. However, G is not sequenceable, by Theorem 3. □

Theorem 30. [37] *Abelian groups of odd order are terraced.*

Proof. Let $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$ be an abelian group of odd order. We use induction on the number of summands. By Theorem 28, $\mathbb{Z}_{n_1}$ is terraced.

Suppose that $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{k-1}}$ is terraced with terrace $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ and let $\mathbf{c} = (c_1, c_2, \ldots, c_{n_k})$ be the Williams terrace for $\mathbb{Z}_{n_k}$. For $i > 0$ let

$$\begin{array}{rcll} \alpha^{(0)} & = & (a_1, 0), (a_2, 0), \ldots, (a_m, 0) & \\ \alpha^{(i)} & = & (a_m, c_i), (a_{m-1}, c_{i+1}), (a_{m-2}, c_i), (a_{m-3}, c_{i+1}), \ldots, (a_1, c_i) & \text{if } i \text{ is odd} \\ \alpha^{(i)} & = & (a_1, c_i), (a_2, c_{i-1}), (a_3, c_i), (a_4, c_{i-1}), \ldots, (a_m, c_i) & \text{if } i \text{ is even} \end{array}$$

We claim that $(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(n_k)})$ is a terrace for $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$:

An allowable set of differences with zeros in the second co-ordinate occurs within the differences produced by $\alpha^{(0)}$ as $\mathbf{a}$ is a terrace. An allowable set of differences with zeros in the first co-ordinate occurs within the differences produced by the juxtapositions of $\alpha^{(i)}$ and $\alpha^{(i+1)}$ as $\mathbf{c}$ is a terrace. These are the only occurrences of differences with zero in either co-ordinate.

Let x be a non-zero element of $\mathbb{Z}_{n_k}$ such that $c_{i+1} - c_i = x$ for some odd i (this covers exactly one of x and $-x$ for each $x \in \mathbb{Z}_{n_k}$). Then x or $-x$ occurs in the second co-ordinate of the differences in $\alpha^{(i)}$ and $\alpha^{(i+1)}$ (and nowhere else). As $\mathbf{a}$ is a terrace, for each non-zero y, where $y \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{k-1}}$, we get one of the following combinations of differences among the differences produced by $\alpha^{(i)}$ and $\alpha^{(i+1)}$:

- two occurrences each of (y, x) and $(-y, x)$ (and no occurrences of $(y, -x)$ or $(-y, -x)$)
- one occurrence each of (y, x), $(-y, x)$, $(y, -x)$ and $(-y, -x)$
- two occurrences each of $(y, -x)$ and $(-y, -x)$ (and no occurrences of (y, x) or $(-y, x)$).

None of these combinations contravene the definition of a terrace, so $(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(n_k)})$ is a terrace as claimed. The result now follows by induction on k. □

We can now complete the proof of Theorem 3; that is, abelian binary groups have directed terraces:

Proof of Theorem 3. Let A be an abelian binary group, say $A = \mathbb{Z}_{2^m} \times B$ where $m > 1$ and B is an abelian group of odd order. Then $A/\mathbb{Z}_2 \cong \mathbb{Z}_{2^{m-1}} \times B$ and Theorem 8 says that if $A/\mathbb{Z}_2$ is terraced then A has a directed terrace. The preceding theorem gives a terrace for B, so the result now follows by induction on m. $\square$

Building on earlier work constructing terraces for different families of abelian groups [20, 37, 117, 120, 130], the question of which abelian groups are terraced is now settled:

Theorem 31. [132] *An abelian group is terraced if and only if it is not a non-cyclic elementary abelian* 2*-group.*

The situation for non-abelian groups is less clear-cut. The semidihedral group of order $8t$, denoted SD_{8t} and another similarly structured group of the same order that we denote M_{8t}, are given by:

$$\begin{aligned} SD_{8t} &= \langle u, v : u^{4t} = e = v^2,\ vu = u^{2t-1}v \rangle \\ M_{8t} &= \langle u, v : u^{4t} = e = v^2,\ vu = u^{2t+1}v \rangle \end{aligned}$$

Call a non-trivial abelian group a *generalised Klein group* if it as a composition series with all factors isomorphic to the Klein 4-group C_2^2.

Theorem 32. *The following non-abelian groups are terraced:*

(i) all non-abelian sequenceable groups [37],

(ii) all non-abelian groups of odd order [17],

(iii) all non-abelian groups with a terraced normal subgroup of odd index and non-abelian groups with a odd-order normal subgroup with a terraced quotient group [17, 19],

(iv) SD_{8t} *and* M_{8t}*, for* $t \geqslant 2$ [131],

(v) various groups with a central subgroup isomorphic to C_2^2*, including those of the form* $A \times G$ *where* A *is a generalised Klein group and* G *is* D_{8t}*,* SD_{8t}*,* M_{8t}*, for* $t \geqslant 2$*, or a non-abelian group of order* 12*,* 16 *or* 20 [131],

(vi) all non-abelian groups of order up to 511 *(except possibly* 256 *and* 384*)* [16, 123].

Conjecture 33. (Bailey) All finite groups, except the elementary abelian 2-groups of order at least 4, are terraced.

In 1988 Morgan [109] generalised the concept of a terrace as follows. Let G be a group of order n and let $\mathbf{a}$ be a list $(a_1, a_2, \ldots, a_p)$ of elements of G (repeats and omissions of elements permitted) where $p = 1 + m(n-1)/2$ for some integer m. Note that if n is even then m must also be even. For such an $\mathbf{a}$ let $\mathbf{b} = ({a_1}^{-1}a_2, {a_2}^{-1}a_3, \ldots, {a_{p-1}}^{-1}a_p)$. We say that $\mathbf{a}$ is an *m-terrace* of G if each element of G occurs in $\mathbf{a}$ either $\lfloor p/n \rfloor$ or $\lfloor p/n \rfloor + 1$ times ($\lfloor k \rfloor$ denotes the least integer greater than $k-1$) and if $\mathbf{b}$ consists of

- $m/2$ occurrences of each non-identity element g which satisfies $g = g^{-1}$
- m total occurrences from the pair $\{g, g^{-1}\}$ for each element g which does not satisfy $g = g^{-1}$.

Observe that a 2-terrace is a terrace as previously defined. The cyclic groups $\mathbb{Z}_n$ have 2- and 4-terraces when n is even and 1-, 2-, 3- and 4-terraces when n is odd [109]. That all abelian groups of odd order have a 1-terrace follows immediately from the existence of a half-and-half terrace for such groups, proved in [126].

The purpose of this generalisation is for use in the construction of "polycross designs". We refer the reader to the papers [109, 110, 111, 112, 155] for more on this topic.

3.5 The Gordon Game

In 1992 Isbell [92] introduced the idea of competitive sequencing: the Gordon game $\Gamma(G)$ for a given finite group G is played as follows.

A counter is placed on the identity, e, of G. White and Black then take turns (White first) to move the counter around the group subject to the condition that the $(n+1)$st move (to x_{n+1}) must satisfy

$$x_{n+1} \notin \{e, x_1, \ldots, x_n\}$$

and

$$x_n{}^{-1}x_{n+1} \notin \{x_1, x_1{}^{-1}x_2, \ldots, x_{n-1}{}^{-1}x_n\}.$$

That is, if a game contained as many moves as the group had non-identity elements then the sequence $(e, x_1, \ldots, x_{|G|-1})$ would be a directed terrace for G. The first player unable to make a move loses.

Isbell investigated the Gordon game for groups of small order, finding the following results. Here W and B denote forced wins for White and Black respectively.

C_2 :	W	C_3 :	W	C_4 :	W
$C_2 \times C_2$:	B	C_5 :	W	C_6 :	B
D_6 :	W	C_7 :	B	C_8 :	B
$C_2 \times C_4$:	W	$C_2 \times C_2 \times C_2$:	B	D_8 :	B
Q_8 :	W	C_9 :	B	$C_3 \times C_3$:	B
C_{10} :	W	C_{11} :	B	C_{13} :	B

Isbell tentatively suggests

Conjecture 34. Black wins $\Gamma(C_p)$ for primes $p > 5$.

The reasoning behind this conjecture is (as Isbell freely admits) shaky. Fix $h \in C_p \backslash \{e\}$. White's first move is irrelevant as for each $g \in C_p \setminus \{e\}$ there is an automorphism which maps g to h. However, this automorphism is unique and the argument for Black winning is that "in the unique game which Black faces after White's first move in $\Gamma(C_p)$ the $p-3$ possible opening moves are all different (i.e. inequivalent by automorphisms). For large p, it is very unlikely that all are losing moves".

In 2013, Frenk analyzed more instances of the game [75]. He confirmed Isbell's results above and added the following cases:

$$\begin{array}{rlrlrl}
C_{12}: & W & C_2 \times C_6: & B & Q_{12}: & B \\
D_{12}: & B & A_4: & W & C_{14}: & W \\
C_{15}: & B & C_{16}: & W & C_{17}: & B \\
C_{18}: & W & C_2 \times C_3 \times C_3: & B & C_{19}: & B \\
C_{20}: & W & C_2 \times C_{10}: & W & &
\end{array}$$

Note that the wins for Black in C_{17} and C_{19} are consistent with Isbell's conjecture. Frenk observes that White wins the game in C_{2m} for $5 \leqslant m \leqslant 10$, but with only one possible opening move: choose the involution. He gives a heuristic argument for why this might be the case and might continue to be the case for larger m.

3.6 Row-Complete Latin Squares

We have already noted that sequencings were primarily investigated because they can be used to construct row-complete latin squares. However, with Theorems 3 and 12 giving the complete spectrum of orders at which a sequenceable group exists, that method for constructing row-complete latin squares of new orders has run its course. In this section we outline another construction method.

Let q be an odd prime power. Let A be a $q \times mq$ array of symbols from $\mathbb{F}_q \times \mathbb{Z}_m$, where $\mathbb{F}_q$ is the field with q elements. Write $A_{ij} = (x_{ij}, y_{ij})$ for $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant mq$. Then A is a *generating array* if the following conditions hold:

- each symbol appears once in each row of A;
- if $x_{ij} = x_{i'j}$ then $i = i'$;
- if $y_{i,j+1} - y_{ij} = y_{i',j'+1} - y_{i'j'}$ and $(x_{ij}, x_{i,j+1}) = (x_{i'j'}, x_{i',j'+1})$ then $(i,j) = (i',j')$.

Given a $q \times mq$ generating array, A, define L to be the $mq \times mq$ array (with symbols from $\mathbb{F}_q \times \mathbb{Z}_m$) with

$$L_{kq+i,j} = (x_{ij}, y_{ij} + k)$$

where $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant mq$ and $0 \leqslant k \leqslant m-1$.

Theorem 35. [35] *L, as defined above, is an $mq \times mq$ row-complete latin square.*

Thus to construct an $mq \times mq$ row-complete latin square we need only to construct a $q \times mq$ generating array.

Example 36. [35] Let $n = 9$, $\mathbb{F}_3 = \{0,1,2\}$, $\mathbb{Z}_3 = \{0,1,2\}$. Then the array A given in Figure 3 is a generating array. The corresponding row-complete latin square L is given in Figure 4. It is obtained by using the map $\phi : \mathbb{F}_3 \times \mathbb{Z}_3 \to \{1,2,\ldots,9\}$, $(x,y) \mapsto 3x+y+1$ as integers.

Figure 3: A 3×9 generating array, A.

$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,2)$	$(2,1)$	$(1,1)$	$(2,2)$	$(0,2)$
$(1,0)$	$(0,1)$	$(0,0)$	$(2,1)$	$(2,0)$	$(0,2)$	$(2,2)$	$(1,1)$	$(1,2)$
$(2,0)$	$(2,1)$	$(1,2)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,2)$	$(0,0)$	$(2,2)$

Figure 4: A 9×9 row-complete latin square, L.

1	4	7	2	6	8	5	9	3
4	2	1	8	7	3	9	5	6
7	8	6	5	2	4	3	1	9
2	5	8	3	4	9	6	7	1
5	3	2	9	8	1	7	6	4
8	9	4	6	3	5	1	2	7
3	6	9	1	5	7	4	8	2
6	1	3	7	9	2	8	4	5
9	7	5	4	1	6	2	3	8

It seems that this method was used by Mertz and Sonneman to construct row-complete latin squares of orders 9 and 15 respectively, reported in [83] (they are given as column-complete latin squares there; transposing gives row-complete squares). Archdeacon, Dinitz, Stinson and Tillson [35] first formally defined generating arrays; they also constructed row-complete latin squares of orders 9 and 15 and found examples of orders 21 and 27 not based on groups. In [66, chapter 3] it is reported that Owens, Dinitz and Stinson have found examples of orders 25 and 33. Until 1997 squares constructed from sequencings were the only other known row-complete latin squares of odd order. Higham [85] combined these squares to make larger ones, showing that if there is a sequenceable group of odd order m and a row-complete latin square of odd order n then there is a row-complete latin square of order mn. In 1998 he returned to the concept of a generating array to show the following result.

Theorem 37. [86] *Row-complete latin squares of all odd composite orders exist.*

Proof. We will show how to construct the appropriate generating arrays. Detailed proofs of their correctness may be found in [86].

Let q be an odd prime power, $q > 3$. We construct a $q \times mq$ generating array A, where $A_{ij} = (x_{ij}, y_{ij})$. Note that every odd composite number except 9 has a proper prime power divisor greater than 3 and we have already seen a 9×9 row complete latin square.

Case 1, $m = 4r + 1$: We set the y_{ij}'s first. Choose y_{ij} to be constant within each column, $y_{ij} = s_j$ say.

Now,

$$(0, 4r, 1, 4r-1, \ldots, r-2, 3r+2, r-1, 3r+1, r,$$
$$3r-1, r+1, 3r-2, \ldots, 2r-2, 2r+1, 2r-1, 2r, 0)$$

is the sequence of partial sums of an R-sequencing of $\mathbb{Z}_m$ (see Section 3.1).

Let w be the sequence obtained from this by removing the final 0, adding $r+1$ to each term and cyclically shifting the new sequence forward $2r$ places. That is,

$$\begin{aligned} w =& (2r+1, 4r, 2r+2, 4r-1, \ldots, 3r-1, 3r+2, 3r, 3r+1, \\ & r+1, r, r+2, r-1, \ldots, 2r-1, 2, 2r, 1). \end{aligned}$$

Let s_j be the jth element of the sequence that begins with $q-1$ 0's, then has $q-1$ copies of w, then the reverse of w and finishes with a 0.

To allocate the x_{ij}'s we produce m $q \times q$ *component latin squares*, $C^{(k)}$, $0 \leqslant k \leqslant m-1$. The kth component square is then matched with the symbol k where it occurs in the second co-ordinate of the generating array. More specifically, put the lth column of $C^{(k)}$ in the first co-ordinates of the lth column of A which consists of the symbol k in the second co-ordinate $(1 \leqslant l \leqslant q)$.

We now define these component squares. Let σ be a primitive element of $\mathbb{F}_q$ such that $\sigma \neq 2$. The field $\mathbb{F}_q$ has such a σ for q an odd prime power $\geqslant 5$. Let $\mathbb{F}_q = \{f_1, f_2, \ldots f_q\}$. Then

$$C^{(k)}{}_{ij} = \begin{cases} a_k f_i + b_k \sigma^j + c_k & \text{if } 1 \leqslant j < q \\ a_k f_i + c_k & \text{if } j = q \end{cases}$$

where $a_0 = \cdots = a_{2r} = 1$, $a_{2r+1} = \cdots = a_{4r} = -1$, $b_0 = 1$, $b_1 = 1 - \sigma$, $b_2 = \cdots = b_r = 1$, $b_{r+1} = \cdots = b_{2r} = -1$, $b_{2r+1} = \cdots = b_{3r} = 1/2 - 1/\sigma$, $b_{3r+1} = \cdots = b_{4r} = 1/2$, $c_0 = -\sigma/2$ and $c_1 = \cdots = c_{4r} = 0$.

We now have a generating array for $m = 4r+1$.

Case 2, $m = 4r+3$: This differs only slightly from the previous case. Again choose $y_{ij} = s_j$

Now,

$$\begin{aligned} &(0, 4r+2, 1, 4r+1, \ldots, r-2, 3r+4, r-1, 3r+3, r, \\ &\quad 3r+1, r+1, 3r, r+2, 3r-1, \ldots, 2r-1, 2r+2, 2r, 2r+1, 0) \end{aligned}$$

is the sequence of partial sums of an R-sequencing in $\mathbb{Z}_m$.

Let w be the sequence obtained from this by removing the final 0, adding $r+1$ to each term, reversing the sequence and cyclically shifting the new sequence forward $2r+1$ places. That is,

$$\begin{aligned} w =& (2r+1, 1, 2r, 2, \ldots, r+3, r-1, r+2, r, r+1, \\ & 3r+2, 3r+1, 3r+3, 3r, 3r+4, \ldots, 4r, 2r+3, 4r+1, 2r+2, 4r+2). \end{aligned}$$

Again, let s_j be the jth element of the sequence that begins with $q-1$ 0's then has $q-1$ copies of w, then the reverse of w and finishes with a 0.

We use component squares as before to allocate the x_{ij}'s. Let σ be a primitive element of $\mathbb{F}_q$ such that $\sigma \neq 2$ and $3\sigma \neq 2$. The field $\mathbb{F}_q$ has such a σ for q an odd prime power power $\geqslant 5$. Let $\mathbb{F}_q = \{f_1, f_2, \ldots, f_q\}$. Then

$$C^{(k)}{}_{ij} = \begin{cases} a_k f_i + b_k \sigma^j + c_k & \text{if } 1 \leqslant j < q \\ a_k f_i + c_k & \text{if } j = q \end{cases}$$

where $a_0 = \cdots = a_{2r} = 1$, $a_{2r+1} = \cdots = a_{4r+1} = -1$, $a_{4r+2} = 1$, $b_0 = \cdots = b_r = 1$, $b_{r+1} = \cdots = b_{2r} = -1$, $b_{2r+1} = 1/2 - 1/\sigma$, $b_{2r+2} = \cdots = b_{3r+1} = 1$, $b_{3r+2} = \cdots = b_{4r+1} = -1$, $c_0 = -\sigma/2$ and $c_1 = \cdots = c_{4r} = 0$.

We now have a generating array for $m = 4r + 3$ and hence we have a row complete latin square of every odd composite order. $\square$

It has long been known that there are no $n \times n$ row-complete latin squares for $n = 3, 5$ or 7. More recently, Darcy Best and Ian Wanless have shown that no row-complete latin square of order 11 exists [153]. The question for other odd primes remains open.

3.7 Sequencings of Subsets

Given a sequence $(b_1, b_2, \ldots, b_k)$ of elements of a finite group G, let $(a_0, a_1, \ldots, a_k)$, where $a_0 = e$ and $a_i = b_1 b_2 \cdots b_i$ for $i > 0$, be its sequence of partial products.

Let $S \subseteq G \setminus \{e\}$ with $|S| = k$. If there is an ordering of the elements of S whose sequence of partial products has no repeated elements, then the call the ordering a *linear sequencing* and the subset *linearly sequenceable.* If the only repeat in the partial products is $a_0 = a_k = e$, then call the ordering a *rotational sequencing* and the subset *rotationally sequenceable.* If a subset is at least one of linearly and rotationally seuqenceable, then call it *sequenceable.* Note that in the literature, the word sequenceable is not always used in exactly this way.

If every subset of $G \setminus \{e\}$ is sequenceable then G is *strongly sequenceable* [6]. If every subset of $G \setminus \{e\}$ that may be ordered to have the product not be the identity is linearly sequenecable and every subset of $G \setminus \{e\}$ that cannot be ordered to have the product not be the identity is rotationally sequenecable, then G is *very strongly sequenceable.*

In an abelian group, every subset has a uniquely defined product and so the notions of strong seuqenceability and very strong sequenceability coincide. Abelian groups have received the most consideration and we address the work on this first. We write groups abelian groups additively while summarising the work done and denote the uniquely defined sum of all of the elements of the subset S by ΣS.

The earliest instance of a question regarding sequenceability of subsets seems to be from Graham in 1971 [81], who asked whether subsets S of the cyclic group $\mathbb{Z}_p$, where p is prime are sequencable. Motivated by an application to Heffter arrays, in 2016, Archdeacon, Dinitz, Mattern and Stinson conjectured that the answer to Graham's question is yes when $\Sigma S = 0$, even without the restriction that p be prime (although their work was independent of Graham) [34]. See [135] for more on Heffter arrays.

When $\Sigma S \neq 0$, the first appearance in the literature seems to be Bode and Harborth's reference to a conjecture of Alspach that every subset S of $\mathbb{Z}_n$ with $\Sigma S \neq 0$ is linearly sequenceable [48].

In 2018, Costa, Morini, Pasotti and Pellegrini observed that these various questions and conjectures may be generalized to arbitrary abelian groups [61].

Tying this all together we have the following conjecture for abelian groups. Setting the task of resolving this was credited to Alspach and Kalinowski in [6] (without the conjecture that the answer is positive).

Conjecture 38. Every finite abelian group is strongly sequenceable.

Theorem 39 collects some of the initial progress towards Conjecture 38.

Theorem 39. *Let G be an abelian group of order $n = pt$, where p is the largest prime dividing n, and $S \subseteq G \setminus \{0\}$ with $|S| = k$. Then S is sequenceable in the following cases:*

1. $k \leqslant 9$ [6],
2. $k \leqslant 11$ *and* $t \leqslant 5$ [84, 60],
3. $k = 12$ *and* $t \leqslant 4$ [84, 60, 62].
4. $k = 13$ *and* $t \in \{2, 3\}$, *provided S contains at least one element not in the subgroup of order p* [60],
5. $k = 14$ *and* $t = 2$, *provided S contains at least one element not in the subgroup of order p* [60],
6. $k = 15$ *and* $t = 2$, *provided S does not contain exactly* 0, 1, 2 *or* 15 *elements of the subgroup of order p* [60],
7. $k = n - 3$ *when* $t = 1$ *and* $\Sigma S \neq 0$ [84],
8. $k = n - 2$ *when G is cyclic and* $\Sigma S \neq 0$ [48],
9. $k = n - 1$ [4, 79],
10. $n \leqslant 21$ *and* $n \leqslant 23$ *when* $\Sigma S = 0$ [61],
11. $n \leqslant 25$ *when G is cyclic and* $\Sigma S = 0$ [34].

Parallel to some of the cases of Theorem 39, we have the following result for when most of the prime divisors of the order of the group are large.

Theorem 40. [60] *Let $n = mt$ where all the prime factors of m are bigger than $k!/2$. Then subsets S of size k of $\mathbb{Z}_n \setminus \{0\}$ are sequenceable in the following cases:*

1. $k \leqslant 11$ *and* $t \leqslant 5$,
2. $k = 12$ *and* $t \leqslant 4$,
3. $k = 13$ *and* $t \in \{2, 3\}$, *provided S contains at least one element not in the subgroup of order m,*
4. $k = 14$ *and* $t = 2$, *provided S contains at least one element not in the subgroup of order m,*
5. $k = 15$ *and* $t = 2$, *provided S does not contain exactly* 0, 1, 2 *or* 15 *elements of the subgroup of order m.*

Independently, Kravitz [100] and Sawin [140] have shown sequenceability of subsets in $\mathbb{Z}_p$, for p a large prime, for subsets S without a fixed upper bound on the size of S, specifically all S with $|S| \leqslant \log p / \log\log p$. This is improved by Bedert and Kravitz to give the following result.

Theorem 41. [43] *Let p be a large prime and let G be an abelian group with no non-zero elements of order less than p. For any positive constant c, subsets $S \subseteq G \setminus \{0\}$ satisfying*

$$|S| \leqslant e^{c(\log p)^{1/4}}$$

are sequenceable.

The question of subset sequenceability for non-abelian groups was introduced in [61]. Upon observing that the dihedral group of order 6 is neither sequenceable nor R-sequenceable, we see that Conjecture 38 cannot be immediately applied to non-abelian groups. The little work done in this situation so far has mostly focused on dihedral groups, with Theorem 42 containing the main progress.

Theorem 42. *Let $S \subseteq D_{2m} \setminus \{0\}$. Then S has a sequencing when:*

1. *$|S| \leqslant 9$, when m is prime with $m > 3$* [124]*,*
2. *$|S| = k$, provided that the prime factors of m are each larger than $k!$* [59]*,*
3. *$|S| = 2m - 2$, when m is even or m is an odd prime* [124].

Further, the group D_{10} is strongly sequenceable [124] and subsets of size k of the dicyclic group Q_{4m} of order $4m$ are sequenceable, provided that the prime factors of m are each larger than k^k [59].

In addition to the work described in this section, questions of subset sequenceability have been asked regarding infinite groups, see [61, 62]. There is also a closely related question of "weak sequenceability" of subgroups, which allows repeated elements among the partial products provided they are sufficiently far apart, see [57, 58].

4 Index Of Notation

$\mathbb{Z}_n$	:	The integers modulo n (considered as the additively written cyclic group of order n)
C_n	:	The (multiplicatively written) cyclic group of order n
D_{2n}	:	The dihedral group of order $2n$
SD_{2n}	:	The semidihedral group of order $2n$
Q_{4n}	:	The dicyclic group of order $4n$; this is a generalised quaternion group if n is a power of 2
A_n	:	The alternating group on n symbols
S_n	:	The symmetric group on n symbols
$\mathbb{F}_q$	:	The field with q elements (q must be a prime power)
$\mathrm{PSL}(2,q)$	:	The projective special linear group of 2×2 matrices over $\mathbb{F}_q$
$\mathrm{PGL}(2,q)$	:	The projective general linear group of 2×2 matrices over $\mathbb{F}_q$
$\mathrm{P\Gamma L}(2,q)$	:	The automorphism group of $\mathrm{PSL}(2,q)$
G'	:	The derived subgroup of a group G
$O(G)$	:	The largest normal subgroup of odd order of a group G
$\Lambda(G)$	:	The normal subgroup of order 2 of a binary group G
$\lfloor k \rfloor$	:	The smallest integer greater than $k-1$

Acknowledgements

I am grateful to Prof. R. A. Bailey (Queen Mary, University of London) for reading preliminary versions of this paper and for many useful suggestions. I am also grateful for the useful suggestions of two referees, including the material on the structure of binary groups in Section 2.3

References

[1] A. Ahmed, M. I. Azimli, I. Anderson and D. A. Preece, Rotational terraces from rectangular arrays, *Bulletin Inst. Combinatorics Appl.* **63** (2011) 4–12.

[2] B. Alspach, The wonderful Walecki construction. *Bulletin of the I.C.A.* **52** (2008) 7–20.

[3] B. Alspach, Variations on the sequenceable theme, in 50 *Years of Combinatorics, Graph Theory, and Computing.* Chapman and Hall/CRC, (2019) 37–53.

[4] B. Alspach, D. L. Kreher and A. Pastine, The Friedlander-Gordon-Miller conjecture is true, *Australas. J. Combin.* **67** (2017) 11–24.

[5] B. Alspach, D. L. Kreher and A. Pastine, Sequencing partial Steiner triple systems. *J. Combin. Des.* **28** (2020) 327–343.

[6] B. Alspach and G. Liversidge, On strongly sequenceable abelian groups, *Art Discrete Appl. Math.* **3** (2020) 19pp.

[7] B. A. Anderson, Sequencings and starters, *Pacific J. Math.* **64** (1976) 17–24.

[8] B. A. Anderson, A fast method for sequencing low order non-abelian groups, *Ann. Discrete Math.* **34** (1987) 27–42.

[9] B. A. Anderson, Sequencings of dicyclic groups, *Ars Combin.* **23** (1987) 131–142.

[10] B. A. Anderson, S_5, A_5 and all non-abelian groups of order 32 are sequenceable, *Congr. Numer.* **58** (1987) 53–68.

[11] B. A. Anderson, Sequencings of dicyclic groups II, *J. Combin. Math. Combin. Comp.* **3** (1988) 5–27.

[12] B. A. Anderson, All dicyclic groups of order at least 12 have symmetric sequencings, *Contemp. Math.* **111** (1990) 5–21.

[13] B. A. Anderson, Some quasi-2-complete latin squares, *Congr. Numer.* **70** (1990) 65–79.

[14] B. A. Anderson, A product theorem for 2-sequencings, *Discrete Math.* **87** (1991) 221–236.

[15] B. A. Anderson, A family of $N \times N$ Tuscan-2-squares with $N + 1$ composite, *Ars Combin.* **32** (1991) 33–55.

[16] B. A. Anderson, Bailey's conjecture holds through 87 except possibly for 64, *J. Combin. Math. Combin. Comp.* **12** (1992) 187–195.

[17] B. A. Anderson and E. C. Ihrig, All groups of odd order have starter-translate 2-sequencings, *Australas. J. Combin.* **6** (1992) 135–146.

[18] B. A. Anderson and E. C. Ihrig, Every finite solvable group with a unique element of order two, except the quaternion group, has a symmetric sequencing, *J. Combin. Des.* **1** (1993) 3–14.

[19] B. A. Anderson and E. C. Ihrig, Symmetric sequencings of non-solvable groups, *Congr. Numer.* **93** (1993) 73–82.

[20] B. A. Anderson and P. A. Leonard, Symmetric sequencings of finite Hamiltonian groups with a unique element of order 2, *Congr. Numer.* **65** (1988) 147–158.

[21] I. Anderson and R. A. Bailey, Completeness properties of conjugates of latin squares based on groups, and an application to bipartite tournaments, *Bull. Inst. Combin. Appl.* **21** (1997), 95–99.

[22] I. Anderson, L. H. M. Ellison, Further results on logarithmic terraces, *Discrete Math.* **308** (2008) 684–695.

[23] I. Anderson, M. A. Ollis and D. A. Preece, On power-sequence and matryoshka terraces for $\mathbb{Z}_n$ *Bulletin Inst. Combin. Appl.* **81** (2017) 98–117.

[24] I. Anderson and D. A. Preece, Locally balanced change-over designs, *Utilitas Math.* **62** (2002) 33–59.

[25] I. Anderson, D. A. Preece, Power-sequence terraces for $\mathbb{Z}_n$ where n is an odd prime power, *Discrete Math.* **261** (2003) 31–58.

[26] I. Anderson, D. A. Preece, Some narcissistic half-and-half power-sequence $\mathbb{Z}_p$ terraces with segments of different lengths, *Congr. Numer.* **163** (2003) 5–26.

[27] I. Anderson, D. A. Preece, Narcissistic half-and-half power-sequence terraces for $\mathbb{Z}_n$ with $n = pq^t$, *Discrete Math.* **279** (2004) 33–60.

[28] I. Anderson, D. A. Preece, Some power-sequence terraces for Z_{pq} with as few segments as possible, *Discrete Math.* **293** (2005) 29–59.

[29] I. Anderson, D. A. Preece, Logarithmic terraces, *Bull. I.C.A.* **46** (2006) 49–60.

[30] I. Anderson, D. A. Preece, Some *da capo* directed power-sequence Z_{n+1} terraces with n an odd prime power, *Discrete Math.* **308** (2008) 192–206.

[31] I. Anderson, D. A. Preece, Some $\mathbb{Z}_{n+2}$ terraces from $\mathbb{Z}_n$ power-sequences, n being an odd prime, *Discrete Math.* **308** (2008) 4086–4107.

[32] I. Anderson, D. A. Preece, Combinatorially fruitful properties of $3 \cdot 2^{-1}$ and $3 \cdot 2^{-2}$ modulo p, *Discrete Math.* **310** (2010) 312–324.

[33] I. Anderson, D. A. Preece, Some narcissistic power-sequence Z_{n+1} terraces with n an odd prime power, *Ars Combin.* **97** (2010) 33–57.

[34] D. S. Archdeacon, J. H. Dinitz, A. Mattern and D. R. Stinson, On partial sums in cyclic groups, *J. Combin. Math. Combin. Comput.* **98** (2016) 327–342.

[35] D. S. Archdeacon, J. H. Dinitz, D. R. Stinson and T. W. Tillson, Some new row-complete latin squares, *J. Combin. Theory Ser. A* **29** (1980) 395–398.

[36] L. Babai and P. J. Cameron, Automorphisms and enumeration of switching classes of tournaments, *Electronic J. Combinatorics* **7** (2000), #R38.

[37] R. A. Bailey, Quasi-complete latin squares: construction and randomization, *J. R. Stat. Soc. Ser. B Stat. Methodol.* **46** (1984) 323–334.

[38] R. A. Bailey, M. A. Ollis and D. A. Preece, Round-dance neighbour designs from terraces, *Discrete Math.* **266** (2003) 69–86.

[39] R. A. Bailey and R. W. Payne, Experimental design: statistical research and its application. In *Institute of Arable Crops Research Report for 1989*, J. Abbott (ed.) (1989) 107–112. Harpenden: Agriculture and Food Research Council, Institute of Arable Crops Research.

[40] R. A. Bailey and C. E. Praeger, Directed terraces for direct product groups, *Ars Combin.* **25A** (1988) 73–76.

[41] S. T. Bate and B. Jones, A review of uniform cross-over designs, *J. Statist. Plann. Inference* **138** (2008) 336–351.

[42] R. Beals, J. A. Gallian P. Headley and D. S. Jungreis, Harmonious groups, *J. Comb. Theory, Ser. A* (1991) 223–238.

[43] B. Bedert and N. Kravitz, Graham's rearrangement conjecture beyond the rectification barrier, arXiv:2409.07403 (2024).

[44] D. Bedford, On groups of orders p, p^2, pq and p^3, p, q prime: their classification and a discussion as to whether they are super P-groups, Undergraduate Special Study, University of Surrey (1987).

[45] D. Bedford, M. A. Ollis and R. M. Whitaker, On bipartite tournaments balanced with respect to carry-over effects for both teams, *Discrete Math* **231** (2001) 81–87.

[46] H. Bender, Finite groups with dihedral 2-Sylow subgroups, *J. Algebra* **70** (1981) 216–228.

[47] S. R. Blackburn, and T. Etzion. Block-avoiding point sequencings. *J. Combin. Des.* **29** (2021) 339–366.

[48] J.-P. Bode and H. Harborth, Directed paths of diagonals within polygons. *Discrete Math.* **299** (2005) 3–10.

[49] M. Buratti, Existence of 1-rotational k-cycle systems of the complete graph, *Graphs Combin.* **20** (2004), 41–46.

[50] M. Buratti and G. Rinaldi, 1-rotational k-factorizations of the complete graph and new solutions to the Oberwolfach Problem, *J. Combin. Des.* **16** (2008) 87–100.

[51] M. Buratti, G. Rinaldi and T. Traetta, Some results on 1-rotational Hamiltonian cycle systems, *J. Combin. Des.* **22** (2014) 231–251.

[52] W. Burnside, *Theory of Groups of Finite Order*, Dover Publications (reprint), New York, 1955.

[53] G. Campbell and S. Geller, Balanced latin squares, *Purdue University Department of Statistics Mimeoseries* **80-26** (1980).

[54] M. J. Caulfield, Full and quarter plane complete infinite latin squares, *Disc. Math.* **159** (1996) 251–253.

[55] S. D. Cohen and G. L. Mullen, Primitive elements in finite fields and Costas arrays, *Appl. Algebra Engr. Comm. Comput.* **2** (1991) 45–53.

[56] S. Costa, A Littlewood-Offord kind of problem in $\mathbb{Z}_p$ and Γ-sequenceability, arXiv:2308.04284 (2023).

[57] and S. Della Fiore, Weak sequenceability in cyclic groups, *J. Combin. Des.* **30** (2022) 735–751.

[58] S. Costa and S. Della Fiore, Alternating parity weak sequencing, *J. Combin. Des.* **32** (2024) 308–327.

[59] S. Costa and S. Della Fiore, Weak Freiman isomorphisms and sequencings of small sets, arXiv:2407.15785 (2024).

[60] S. Costa, S. Della Fiore, M. A. Ollis and S. Z. Rovner-Frydman On sequences in cyclic groups with distinct partial sums, *Electron. J. Combin.* **29** #P3.33 (2022) 21pp.

[61] S. Costa, F. Morini, A. Pasotti and M. A. Pellegrini, A problem on partial sums in abelian groups, *Discrete Math.* **341** (2018) 705–712.

[62] S. Costa and M. A. Pellegrini, Some new results about a conjecture by Brian Alspach, *Archiv der Mathematik* **115** (2020) 479–488.

[63] H. S. M. Coxeter, *Regular Complex Polytopes*, Cambridge University Press, Cambridge, 1974.

[64] J. Dénes and P. Hermann, On the product of all the elements of a finite group, *Ann. Discrete Math.* **15** (1982) 105–109.

[65] J. Dénes and A. D. Keedwell, *Latin Squares and their Applications,* Akadémiai Kiadó, Budapest/English Universities Press, London/Academic Press (1974).

[66] J. Dénes and A. D. Keedwell, *Latin Squares: New Developments in the Theory and Applications*, North-Holland (1991).

[67] J. Dénes and E. Tőrők, Groups and Graphs, *Combinatorial Theory and its Applications*, North Holland (1970) 257–289.

[68] L. E. Dickson, On finite algebras, *Nachrichten der Königlichen Gesellschaft der Wissenschaften in Göttingen (Math.-Phys. Klasse)* (1905) 358–393.

[69] G. Erskine and T. S. Griggs Good point sequencings of Steiner triple systems, *Australas. J. Combin.* **84** (2022) 431–439.

[70] A. B. Evans, Complete mappings and sequencings of finite groups, in *Handbook of Combinatorial Designs*, Chapman and Hall/CRC (2006) 371–378.

[71] A. B. Evans, *Orthogonal Latin squares based on groups*, Springer (2018).

[72] A. B. Evans, G. N. Martin, K. Minden and M. A. Ollis, Infinite latin squares: neighbor balance and orthogonality. *Electron. J. Combin.* (2020) #P3.52.

[73] G. H. Freeman, Some two-dimensional designs balanced for nearest neighbours, *J. R. Statist. Soc. B* **41** (1979) 88–95.

[74] G. H. Freeman, Further results on quasi-complete latin squares, *J. R. Statist. Soc. B* **43** (1981) 314–320.

[75] A. Frenk, The Gordon Game, *EWU Masters Thesis Collection* (2013) **235**.

[76] R. Friedlander, Sequences in groups with distinct partial products, *Aequationes Math.* **14** (1976) 59–66.

[77] R. J. Friedlander, B. Gordon and M. D. Miller, On a group sequencing problem of Ringel, *Congr. Numer.* **21** (1978) 307–321.

[78] S. W. Golomb and H. Taylor, Tuscan squares—a new family of combinatorial designs, *Ars Combin.* **20B** (1985) 115–132.

[79] B. Gordon, Sequences in groups with distinct partial products, *Pacific J. Math.* **11** (1961) 1309–1313.

[80] D. Gorenstein and J. H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups, I, II and III, *J. Algebra* **2** (1965) 85–151, 218–270, 334–393.

[81] R. L. Graham, On sums of integers taken from a fixed sequence, *Proceedings, Washington State University Conference on Number Theory* (1971) 22–40.

[82] P. Headley, R-sequenceability and R*-sequenceability of abelian 2-groups, *Discrete Math.* **131** (1994) 345–350.

[83] A. Hedayat and K. Afsarinejad, Repeated measurements designs II, *Ann. Statist.* **6** (1978) 619–628.

[84] J. Hicks, M. A. Ollis and J. R. Schmitt, Distinct partial sums in cyclic groups: polynomial method and constructive approaches, *J. Combin. Des.* **27** (2019) 369–385.

[85] J. Higham, A product theorem for row-complete latin squares, *J. Combin. Des.* **5** (1997) 311–318.

[86] J. Higham, Row-complete latin squares of every composite order exist, *J. Combin. Des.* **6** (1998) 63–77.

[87] A. J. W. Hilton, M. Mays, C. A. Rodger and C. St.J. A. Nash-Williams, Hamiltonian double latin squares, *J. Combin. Thy. Ser. B* **87** (2003) 81–129.

[88] G. B. Hoghton and A. D. Keedwell, On the sequenceability of dihedral groups, *Ann. Discrete Math.* **15** (1982) 253–258.

[89] D. Horsley and P. Ó Catháin, Block avoiding point sequencings of partial Steiner systems, *Des. Codes and Crypto.* **90** (2022) 2375–2383.

[90] P. D. Isaac, A. M. Dean and T. Ostrom, Sequentially counterbalanced latin squares, *Statistics and Applications* **3** (2001) 25–46.

[91] J. Isbell, Sequencing certain dihedral groups, *Discrete Math.* **85** (1990) 323–328.

[92] J. Isbell, The Gordon game of a finite group, *Amer. Math. Monthly* **99** (1992) 567–569.

[93] M. Javaheri, Doubly sequenceable groups. *J. Combin. Des.* **32** (2024) 371–387.

[94] M. Javaheri and L. de Wolf, Harmonious sequences in groups with a unique involution, `arXiv:2408.16207` (2024).

[95] A. D. Keedwell, Some problems concerning complete latin squares, *L.M.S. Lecture Notes* **13**: *Combinatorics: Proc. of the British Comb. Conf. Aberystwyth 1973 (Eds: T. P. McDonough and V. C. Mavron)* (1974) 89–96.

[96] A. D. Keedwell, Sequenceable groups: a survey, *L.M.S. Lecture Notes* **49**: *Finite Geometries and Designs, Proc. of the 2nd Isle of Thorns Conference. (Eds: P. J. Cameron, J. W. P. Hirschfield and D. R. Hughes)* (1980) 205–215.

[97] A. D. Keedwell, On the sequenceability of non-abelian groups of order pq, *Discrete Math.* **37** (1981) 203–216.

[98] A. D. Keedwell, On the R-sequenceability and R_h-sequenceability of groups, *Ann. Discrete Math.* **18** (1983) 535–548.

[99] A. D. Keedwell, Complete mappings and sequencings of finite groups, *The CRC Handbook of Combinatorial Designs. (Eds. C. J. Colbourn and J. H. Dinitz)* CRC press (1996) 246–253.

[100] N. Kravitz, Rearranging small sets for distinct partial sums, `arXiv:2407.01835v2` (2024).

[101] D. L. Kreher and D. R.Stinson, Block-avoiding sequencings of points in Steiner triple systems. *Australas. J. Combin.* **74** (2019) 498–509.

[102] D. L. Kreher and D. R.Stinson, Nonsequenceable Steiner triple systems, `arXiv:1903.08719`.

[103] V. F. Lev, Sums and differences along Hamiltonian cycles, *Discrete Math.* **310** (2010) 575–584.

[104] P. Li, Sequencing the dihedral groups D_{4k}, *Discrete Math.* **175** (1997) 271–276.

[105] E. Lucas, *Récréations Mathématiques, Tôme II.* Albert Blanchard, Paris, 1892, reprinted 1975.

[106] G. N. Martin and M. A. Ollis, R-sequencings and strong half-cycles from narcissistic terraces, *Australas. J. Combin.* **63** (2015) 346–367.

[107] J. McGill and M. A. Ollis, On the asymptotic growth of bipartite graceful permutations, *Discrete Math.* **342** (2019) 793–799.

[108] N. S. Mendelsohn, Hamiltonian decomposition of the complete directed n-graph, *Proc. Colloq. Tihany: 1966*, Academic Press (1968) 237–241.

[109] J. P. Morgan, Balanced polycross designs, *J. R. Stat. Soc. Ser. B Stat. Methodol.* **50** (1988) 93–104.

[110] J. P. Morgan, Polycross designs with complete neighbour balance, *Euphytica* **39** (1988) 59–63.

[111] J. P. Morgan, Terrace constructions for bordered, two-dimensional neighbour designs, *Ars Combin.* **26** (1988) 123–140.

[112] J. P. Morgan, Some series constructions for two-dimensional neighbor designs, *J. Statist. Plann. Inference* **24** (1990) 37–54.

[113] A. Müyesser, personal communication (2023).

[114] A. Müyesser and A. Pokrovskiy, A random Hall-Paige conjecture, arXiv:2204.09666.

[115] C. K. Nilrat and C. E. Praeger, Complete latin squares: terraces for groups, *Ars Combin.* **25** (1988) 17–29.

[116] M. A. Ollis, Protection against premature termination of experiments based on Williams squares with circular structure, *Utilitas Math.* **63** (2003) 143–149.

[117] M. A. Ollis, On terraces for abelian groups, *Discrete Math.* **305** (2005) 250–263.

[118] M. A. Ollis, Some cyclic solutions to the three table Oberwolfach problem, *Electron. J. Combin.* (2005) #R58.

[119] M. A. Ollis, A method for constructing symmetric Hamiltonian double latin squares, *Australas. J. Combin.* **36** (2006) 265–278.

[120] M. A. Ollis, A note on terraces for abelian groups, *Australas. J. Combin.* **52** (2012) 229–234.

[121] M. A. Ollis, New complete latin squares of odd order, *Europ. J. Combin.* **41** (2014) 35–46.

[122] M. A. Ollis, Constructing R-sequencings and terraces for groups of even order, *Alg. Discrete Math.* **20** (2015) 297–316.

[123] M. A. Ollis, Terraces for small groups, *J. Combin. Math. Comput.* **108** (2019) 231-244.

[124] M. A. Ollis, Sequences in dihedral groups with distinct partial products, *Australas. J. Combin.* **78** (2020) 35–60.

[125] M. A. Ollis and D. A. Preece, Sectionable terraces and the (generalised) Oberwolfach problem, *Discrete Math.* **266** (2003) 399–416.

[126] M. A. Ollis and P. Spiga, Every abelian group of odd order has a narcissistic terrace, *Ars Combin.* **76** (2005) 161–168.

[127] M. A. Ollis and A. D. Sterr, From graceful labellings of paths to cyclic solutions of the Oberwolfach problem, *Discrete Math.* **309** (2009) 4877–4882.

[128] M. A. Ollis and C. R. Tripp, The spectrum of group-based complete latin squares, *Electron. J. Combin.* **26** (2019) #P3.15 13pp.

[129] M. A. Ollis and R. M. Whitaker, On invertible terraces for non-abelian groups, *J. Combin. Des.* **15** (2007) 437–447.

[130] M. A. Ollis and D. T. Willmott, On twizzler, zigzag and graceful terraces, *Australas. J. Combin.* **51** (2011) 243–257.

[131] M. A. Ollis and D. T. Willmott, An extension theorem for terraces, *Electron. J. Combin.* **20** (2013) #P34 10pp.

[132] M. A. Ollis and D. T. Willmott, Constructions for terraces and R-sequencings, including a proof that Bailey's Conjecture holds for abelian groups, *J. Combin. Des.* **23** (2015) 1–17.

[133] L. J. Paige, Complete mappings of finite groups, *Pacific J. Math.* **1** (1951) 111–116.

[134] D. S. Passman, *Permutation Groups*, Benjamin, New York, 1968.

[135] A. Pasotti and J. H. Dinitz. A survey of Heffter arrays, *International Conference on New Advances in Designs, Codes and Cryptography*, Springer Nature, Switzerland (2022).

[136] D. A. Preece, Zigzag and foxtrot terraces for $\mathbb{Z}_n$, *Australas. J. Combin.* **42** (2008) 261–278.

[137] A. R. Rhemtulla, On a problem of L. Fuchs, *Studia Sci. Math. Hungar.* **4** (1969) 195–200.

[138] J. J. Rotman, *An Introduction to the Theory of Groups (4th Edition),* Springer-Verlag (1994).

[139] A. Sadeghieh and H. Doostie, Non-abelian sequenceable groups involving α-covers, *Journal of Sciences, Islamic Republic of Iran* **20** (2009) 277–282.

[140] W. Sawin, comment on the post "Ordering subsets of the cyclic group to give distinct partial sums", MathOverflow (2015) `https://mathoverflow.net/q/202857`.

[141] J. Z. Schroeder, A tripling construction for mutually orthogonal symmetric Hamiltonian double latin squares, J. Combin. Des. **27** (2019) 42–52.

[142] L. E. Smart, M. M. Blight, J. A. Pickett and B. J. Pye, Development of field strategies incorporating semiochemicals for the control of the pea and bean weevil, *Sitona lineatus* L., *Crop Protection* **13** (1994) 127–136.

[143] D. J. Street, Some repeated measurements designs, *Comm. Statist. Theory Methods* **17(1)**, (1988) 87–104.

[144] A. D. Thomas and G. V. Wood, *Group Tables,* Shiva Publishing (1980).

[145] C. Vanden Eynden, Countable sequenceable groups, *Disc. Math.* **23** (1978) 317–318.

[146] D. D. Wall, Fibonacci series modulo m, *American Math. Monthly* **67** (1960) 525–532.

[147] C.-D. Wang, On the harmoniousness of dicyclic groups, *Discrete Math.* **120** (1993), 221-225.

[148] C.-D. Wang, Sequenceability, R-sequenceability, and harmoniousness of finite groups, Dissertation, Arizona State University, August 2000.

[149] C.-D. Wang, Complete latin squares of order p^n exist for odd primes p and $n > 2$, *Discrete Math.*, **252** (2002) 189–201.

[150] C.-D. Wang, More R-sequenceable groups, *Australas. J. Combin.* **37** (2007) 215–224.

[151] C.-D. Wang and P. A. Leonard, More on sequences in groups, *Australas. J. Combin.* **21** (2000) 187–196.

[152] L. L. Wang, A test for the sequencing of a class of finite groups with two generators, *Amer. Math. Soc. Notices* **20** (1973) 73T–A275.

[153] I. M. Wanless, personal communication (2025).

[154] E. J. Williams, Experimental designs balanced for the estimation of residual effects of treatments, *Aust. J. Sci. Res. A* **2** (1949) 149–168.

[155] C. E. Wright, A systematic polycross design, *Res. Exp. Rec. Min. Agric. N.I.* **11:1** (1961) 7–8.