Macaulay Posets

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Abstract

Macaulay posets are posets for which there is an analogue of the classical Kruskal-Katona theorem for finite sets. These posets are of great importance in many branches of combinatorics and have numerous applications. We survey mostly new and also some old results on Macaulay posets, where the intention is to present them as pieces of a general theory. In particular, the classical examples of Macaulay posets are included as well as new ones. Emphasis is also put on the construction of Macaulay posets, and their relations to other discrete optimization problems.

1 Introduction

Macaulay posets are, informally speaking, posets for which an analogue of the classical Kruskal-Katona theorem for finite sets holds. They are related to many other combinatorial problems like isoperimetric problems on graphs [15, 34] (see also Section 3) and problems arising in polyhedral combinatorics [28, 30, 94, 100]. Several optimization problems can be solved within the class of Macaulay posets, or at least for Macaulay posets with additional properties (cf. Section 5). Therefore, Macaulay posets are very useful and interesting objects.

A few years ago, the classical Macaulay posets listed in Section 2 were the only known essential examples, and, consequently, the theory of Macaulay posets was more or less the theory of these examples. In his book [52, chapter 8], Engel made a first attempt to unify the theory of Macaulay posets. Although the book appeared quite recently, a number of new examples, relations and applications have been found in the meantime. In this paper, our objective is to give a survey on Macaulay posets that includes these new results and updates [52]. We also present some older results and applications which are not mentioned in [52].

We start with some basic facts and definitions in Section 1 and the classical examples in Section 2. For all definitions not included here we refer to Engel's book [52]. In

Section 3 we proceed with constructions for Macaulay posets and relations to isoperimetric problems. New examples of Macaulay posets are presented in Section 4. Section 5 is devoted to optimization problems on Macaulay posets and some other applications of this theory. We also present some open questions and propose directions for further research throughout the paper.

1.1 Some basic definitions

Let P be a partially ordered set (briefly, poset) with the associated partial order $\leq$. For $x, y \in P$, we say that y *covers* x, denoted by $x \lessdot y$, if $x \leq y$ and there is no $z \in P$ such that $z \neq x, y$ and $x \leq z \leq y$. A subset $X \subseteq P$ is called a *chain* if $x, y \in X$ implies that $x \leq y$ or $y \leq x$. An *antichain* is defined as a subset $X \subseteq P$ such that the conditions $x, y \in X$ and $x \leq y$ imply $x = y$. The *width* of P is the largest size of an antichain in P and is denoted by $d(P)$.

A subset $X \subseteq P$ is an *ideal* (or *downset*) if the conditions $x \in X$ and $y \leq x$ imply $y \in X$. If X is an antichain, then the set $I(X) := \{y \in P \mid y \leq x \text{ for some } x \in X\}$ is an ideal, which is called *ideal generated by* X. Conversely, if I is an ideal, then the set $\max(I) := \{x \in I \mid x \not\leq y \text{ for any } y \in I,\ y \neq x\}$ is an antichain, which is called the *set of maximal elements* of I.

A *rank function* on P is a function $r : P \mapsto I\!N$ such that $r(x) = 0$ for some minimal element x of P and $r(y) = r(z) - 1$ whenever $y \lessdot z$. The poset P is called *ranked*, if a rank function on P exists. The *rank* of P is defined by $r(P) := \max\{r(x) \mid x \in P\}$, where $r(P) = \infty$ is allowed. A ranked poset P is called *graded* if all minimal elements have rank 0, and all maximal elements have rank $r(P)$.

The *dual* P^* of P is the poset on the same set of elements with the partial order defined by: $x \leq^* y$ iff $y \leq x$. If P is ranked with $r(P) < \infty$, then P^* is ranked. If P is ranked with $r(P) = \infty$, then P^* is not ranked in the usual sense. In this case $r^*(x) := -r(x)$ will considered to be the rank function for P^*.

If P is ranked, then the set $\{x \in P \mid r(x) = i\}$ is called the i-th *level* of P and is denoted by $N_i(P)$ or P_i. The *(lower) shadow* of an element $x \in P_i$ is the set $\Delta(x) := \{y \in P \mid y \lessdot x\}$, and its *upper shadow* is $\nabla(x) := \{y \in P \mid x \lessdot y\}$. The lower shadow $\Delta(X)$ (resp. upper shadow $\nabla(X)$) of a subset $X \subseteq P_i$ is defined as the union of the lower (resp. upper) shadows of its elements. For given integers i and m with $1 \leq i \leq r(P)$ and $1 \leq m \leq |P_i|$, the *shadow minimization problem* (SMP) consists in finding an m-element subset $X \subseteq P_i$ such that $|\Delta(X)| \leq |\Delta(Y)|$ for all $Y \subseteq P_i$ with $|Y| = m$. We say that a subset $X \subseteq P_i$ is *optimal* if it has minimum shadow among all subsets of P_i of the same size. Obviously, the SMP is at least NP-hard, since it implies a solution to the Minimum Cover Problem.

The *(cartesian) product* $P \times Q$ of two posets P and Q is the set of all pairs (x, y) with $x \in P$, $y \in Q$, where the partial order is given by: $(x, y) \leq_{P \times Q} (x', y')$ iff $x \leq_P x'$, $y \leq_Q y'$. If P and Q are ranked, then the poset $P \times Q$ is ranked, too, and the rank function for $P \times Q$ is given by: $r(x, y) := r_P(x) + r_Q(y)$. The n-th *(cartesian) power* of a poset P is the poset $P^n := P \times P \times \cdots \times P$ (n times).

1.2 Macaulay posets

Let P be a ranked poset and consider some total order $\preceq$ of its elements. Note that we do not claim the order $\preceq$ to be a *linear extension* of P. For a subset $X \subseteq P$ and a natural number $m \leq |X|$ we will use the notation $C(m, X)$ (resp. $L(m, X)$) for the set of the first (resp. last) m elements of X w.r.t. $\preceq$. In particular, for $X \subseteq P_i$ we abbreviate $C(|X|, P_i)$ and $L(|X|, P_i)$ by $C(X)$ and $L(X)$, respectively. The operation of replacing $X \subseteq P_i$ with $C(X)$ is called *compression*, and we say that X is *compressed* if $X = C(X)$. Compressed subsets will also be called *initial segments* (IS), whereas a *final segment* of P_i is a subset $X \subseteq P_i$ with $X = L(X)$. A *segment* of P_i simply is a set of elements of P_i which are consecutive w.r.t. $\preceq$ (restricted to P_i). For an element $x \in P_i$, the initial segment of P_i whose last element w.r.t. $\preceq$ is x is denoted by $\mathcal{F}_i(x)$.

The poset P is said to be a *Macaulay poset* if there exists a total order $\preceq$ of its elements (called *Macaulay order*) such that

$$\Delta(C(X)) \subseteq C(\Delta(X)) \text{ for all } X \subseteq P_i \text{ and for all } i = 1, \ldots, r(P). \tag{1}$$

If (1) is satisfied for a ranked poset P with a partial order $\leq$ and for a total order $\preceq$ of the elements of P, then the triple $(P, \leq, \preceq)$ is called *Macaulay structure*.

It is easy to verify (cf, [52] for details) that (1) holds iff the conditions $\mathbf{N}_1$ and $\mathbf{N}_2$ given below are satisfied for all $X \subseteq P_i$ and for all $i = 1, \ldots, r(P)$:

$\mathbf{N}_1$: $|\Delta(C(X))| \leq |\Delta(X)|$,

$\mathbf{N}_2$: $C(\Delta(C(X))) = \Delta(C(X))$.

According to $\mathbf{N}_1$, compressed subsets are optimal for the Macaulay poset P. Therefore, $\mathbf{N}_1$ is called the condition of *nestedness* (of the optimal subsets). By $\mathbf{N}_2$, the shadow of a compressed set is a compressed set again. That is why $\mathbf{N}_2$ is said to be the condition of *continuity*.

If $\preceq$ is a total order of the elements of P, then the dual of $\preceq$ is the total order given by: $x \preceq^* y$ iff $y \preceq x$. By definition, the upper shadow of an element x in P is equal to the lower shadow of x in P^*. Moreover, we have the following relation.

Proposition 1 (Bezrukov [14]). *$(P, \leq, \preceq)$ is a Macaulay structure iff so is $(P^*, \leq^*, \preceq^*)$.*

Obviously, if the conditions $\mathbf{N}_1$ and $\mathbf{N}_2$ are satisfied for some total order $\preceq$, then they are satisfied for any total order $\preceq'$ with $x \preceq y$ iff $x \preceq' y$ for all $x, y \in P_i$ and for all i. In other words, for $\mathbf{N}_1$ and $\mathbf{N}_2$ to be satisfied, only the restrictions of the order $\preceq$ to the levels P_i, $i = 0, 1, \ldots, r(P)$, are relevant, rather than how elements of distinct levels are compared by $\preceq$.

For many applications it turns out to be natural and useful to choose a Macaulay order rank greedily. We say that a total order $\preceq$ is *rank greedy* (on P), if it is a linear extension of the partial order $\leq$ (i.e. if $x \leq y$ implies $x \preceq y$), and if, in addition, $r(x) = r(y) + 1$ implies $x \preceq y$ whenever the last element of $\Delta(x)$ w.r.t. $\preceq$ precedes y in the order $\preceq$. The notion rank greedy refers to the following fact: If we are already given some initial part of the rank greedy order $\preceq$, then the next element w.r.t. $\prec$ is one of those whose shadow is already completely contained in this initial part and whose rank is maximum under this condition.

It can be easily shown (see e.g. [52]) that for every Macaulay poset there exists a rank greedy Macaulay order of its elements. The proof for this and the next assertion can be found in [52].

Proposition 2 *If a total order $\preceq$ is rank greedy for a Macaulay poset P, then $\preceq^*$ is rank greedy for P^*.*

If we associate a rank greedy total order with some Macaulay poset P, then we also say that P is rank greedy. Note that all Macaulay orders presented in Sections 2 and 4 are rank greedy.

1.3 The shadow function

Let P be a Macaulay poset. The *shadow function* sf_i assigns with each subset $X \subseteq P_i$ the number $sf_i(X) = |\Delta(C(X))|$. We briefly discuss some properties of the shadow function which are important for many applications.

The lower and upper *new shadows* of an element $x \in P$ are defined by:

$$\begin{aligned} \Delta_{new}(x) &:= \{y \in P \mid y \lessdot x \text{ and there is no } z \in P \text{ with } z \preceq x,\ z \neq x,\ y \lessdot z\}, \\ \nabla_{new}(x) &:= \{y \in P \mid x \lessdot y \text{ and there is no } z \in P \text{ with } x \preceq z,\ z \neq x,\ z \lessdot y\}, \end{aligned}$$

respectively. Note that the upper new shadow of x in P is exactly the lower new shadow of x in P^*. The lower new shadow $\Delta_{new}(X)$ (resp. upper new shadow $\nabla_{new}(X)$) of a subset $X \subseteq P$ is the union of the lower (resp. upper) new shadows of its elements. The shadow function sf_i is called *additive* if the inequality

$$|\Delta_{new}(X)| \geq |\Delta_{new}(Y)| \geq |\Delta_{new}(Z)|$$

is satisfied for all segments $X, Y, Z \subseteq P_i$ with X being initial, Z being final, and $|X| = |Y| = |Z|$. We say that P is additive if sf_i is additive for all $i = 0, \ldots, r(P)$.

Proposition 3 (Engel [52]). *Let P be a Macaulay poset. P is graded and additive iff its dual P^* is graded and additive.*

The shadow function sf_i is called *little-submodular* if

$$sf_i(X) + sf_i(Y) \geq sf_i(X \cup Y) + sf_i(X \cap Y)$$

holds for all $X, Y \subseteq P_i$ with $X \cap Y = \emptyset$ or $X \cup Y = P_i$.

Proposition 4 (Engel [52]). *The shadow function sf_i is additive iff sf_i is little-submodular.*

The Macaulay poset P is called *shadow increasing* if for all $i = 0, \ldots, r(P)-1$ and for any initial segments $X \subseteq P_i$ and $Y \subseteq P_{i+1}$ with $|X| = |Y|$ the inequality $|\Delta(X)| \leq |\Delta(Y)|$ holds.

We say that P is *final shadow increasing* if we have $|\Delta_{new}(X)| \leq |\Delta_{new}(Y)|$ for all $i = 0, \ldots, r(P)-1$ and for any final segments $X \subseteq P_i$ and $Y \subseteq P_{i+1}$ with $|X| = |Y|$.

Finally, P is said to be *weakly shadow increasing* if $|\Delta_{new}(X)| \leq |\Delta_{new}(Y)|$ holds for any segments $X \subseteq P_i$ and initial segments $Y \subseteq P_j$ such that $i \leq j$, $|X| = |Y|$ and $X \cup Y$ is an antichain.

Proposition 5 (Engel, Leck [54]). *Let P be a Macaulay poset.*

a. If P is final shadow increasing, then P^* is shadow increasing.

b. Let P be graded, additive, and shadow increasing. If P^* is shadow increasing, then P is final shadow increasing.

c. If P is a graded, additive and shadow increasing, then P is weakly shadow increasing.

2 Some known Macaulay posets

2.1 Boolean lattices

Boolean lattices are certainly the most popular examples of Macaulay posets. For a natural number n the Boolean lattice B^n is defined as the collection of all subsets of $[n] := \{1, 2, \ldots, n\}$ partially ordered by inclusion, i.e. $X \leq Y$ for $X, Y \subseteq [n]$ iff $X \subseteq Y$. The unique rank-function on B^n maps a set $X \subseteq [n]$ to $|X|$. Representing the subsets of $[n]$ by their characteristic vectors, it is obvious that B^n is isomorphic to the n-th cartesian power of the chain $0 \lessdot 1$ of length one. As an example, the Hasse diagram of B^4 is shown in Fig. 1a (parenthesis and commas are omitted).

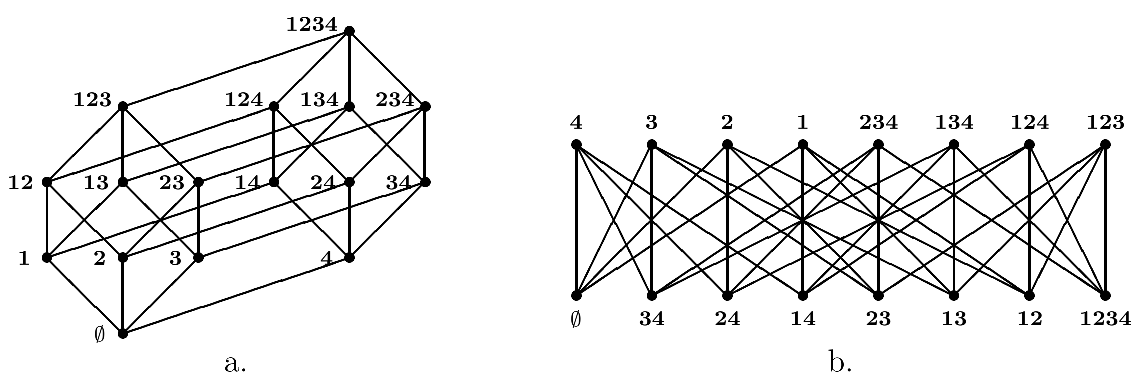


Figure 1: The Boolean lattice B^4 (a) and the poset Q^4 (b).

The *lexicographic order* of the elements of B^n is defined by $X \preceq_{lex} Y$ iff $\max(X \setminus Y) \leq \max(Y \setminus X)$, where $\max(\emptyset) := 0$. The following theorem, which meanwhile became a classical one, was found by Schützenberger [95] (proof incomplete), Kruskal [66] and Katona [63].

Theorem 1 (Kruskal-Katona theorem). *$(B^n, \subseteq, \preceq_{lex})$ is a Macaulay structure.*

It should be mentioned that the proof given by Kruskal in [66] is quite complicated and consists of 30 pages. Katona's proof [63] is done by purely discrete methods, is about 10 pages long, and based on isoperimetric type inequalities and manipulations with binomial coefficients. Many short proofs consist just of 1.5 - 2 pages [48, 56, 97]. This theorem is an immediate corollary of a vertex-isoperimetric problem (VIP) solved by Harper [60] (see also Section 3.3). For short proofs of Harper's theorem and more details we refer to the survey [14].

Let $A = C(m, B^n_k)$ with respect to the lexicographic order. Then m is uniquely represented in the form

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t} \tag{2}$$

for some $a_k, \ldots, a_t$ with $a_k > a_{k-1} > \cdots > a_t \geq t$. Following Kruskal [66] define the (i,k)*th pseudopower* of m to be

$$m^{(i/k)} = \binom{a_k}{i} + \cdots + \binom{a_t}{i-k+t}. \tag{3}$$

Then (cf. [63]), $|\Delta(C(m, B^n_k))| = m^{(k-1/k)}$ for $k \geq 1$.

It is interesting that the formula (3) does not involve n. A useful lower bound is due to Lovász: if $m = \binom{x}{k}$ for some real $x \geq k$, then $|\Delta(C(m, B^n_k))| \geq \binom{x}{k-1}$. In [82] Maire showed that $|\Delta(C(m, B^n_k))| \sim \frac{k}{\sqrt[k]{k!}} m^{1-1/k}$ as k is a constant and $m \to \infty$.

The solution to the SMP provided by the Kruskal-Katona theorem is not unique, in general. However, for at least 2^{n-1} cardinalities m the IS of the lexicographic order of size m is essentially a unique optimal subset, as it is shown in the next theorem.

Theorem 2 (Füredi, Griggs [55], Mörs [84]). *If $(m+1)^{(k-1/k)} > m^{(k-1/k)}$ for some $k \geq 1$, then the set $C(m, B^n_k)$ is a unique optimal subset of size m (up to isomorphism).*

This result, however, is a corollary of more general results [13, 14] which concern the VIP. In particular, [13] implies a complete specification of all optimal subsets of the size m such that $\left((m+1)^{(k-1/k)}\right)^{(k-2/k-1)} > \left(m^{(k-1/k)}\right)^{(k-2/k-1)}$ for some $k \geq 2$. In this case, however, there are nonisomorphic optimal subsets, in general. It turns out that the number of cardinalities m, $0 \leq m \leq 2^n$, which satisfy the above condition and the one in Theorem 2 is asymptotically equal to $\frac{3}{4}2^n$. Without going into details, for which the readers are referred to the survey [14], we mention another corollary of results on the VIP.

Theorem 3 (Bezrukov [13]). *If $A \subseteq B^n_k$ is optimal for some $k \geq 0$, then so is $\Delta(A)$.*

Presently it is not known if this property is valid for other Macaulay posets.

The following poset, denoted here by Q^n, is closely related to B^n. This poset is formed on the element set of B^n and has just two levels. The level Q^n_0 (resp. Q^n_1) consists of all vertices of B^n with an even (resp. odd) number of ones. For $x \in Q^n_0$ and $y \in Q^n_1$ we write $x \leq y$ if the Hamming distance between x and y is 1 (cf. Fig. 1b). The SMP for Q^n was solved independently by three groups of authors.

Theorem 4 (Bezrukov [8, 10], Körner, Wei [69, 70], Tiersma [96]).
The poset Q^n is Macaulay for any $n \geq 1$.

The Macaulay order on Q^n is a restriction of the order introduced by Harper [60] for the VIP on B^n. It turns out that the Macaulayness of Q^n implies a solution for the VIP for B^n and vice versa (cf. [14] for more details).

2.2 Chain products

Cartesian products of chains, called also *lattices of multichains*, are well-studied generalizations of Boolean lattices. For positive integers n and $k_1 \leq k_2 \leq \cdots \leq k_n$ the chain product $S(k_1, k_2, \ldots, k_n)$ consists of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $x_i \in \{0, 1, \ldots, k_i\}$ for $i = 1, 2, \ldots, n$. The partial order is a coordinatewise one: $\mathbf{x} \leq \mathbf{y}$ iff $x_i \leq y_i$ for $i = 1, 2, \ldots, n$. Again we have a uniquely determined rank-function, namely $r(\mathbf{x}) = \sum_{i=1}^n x_i$. Obviously, $S(k_1, k_2, \ldots, k_n)$ is the cartesian product of the chains $0 \lessdot 1 \lessdot \cdots \lessdot k_i$, $i = 1, 2, \ldots, n$.

A natural extension of the lexicographic order to chain products is established by: $\mathbf{x} \preceq_{lex} \mathbf{y}$ iff $\mathbf{x} = \mathbf{y}$ or $x_j < y_j$, where j is the smallest index with $x_j \neq y_j$. As an example, $S(1, 2, 3)$ is shown in Fig. 2a, with the elements of each level placed in increasing lexicographic order from left to right.

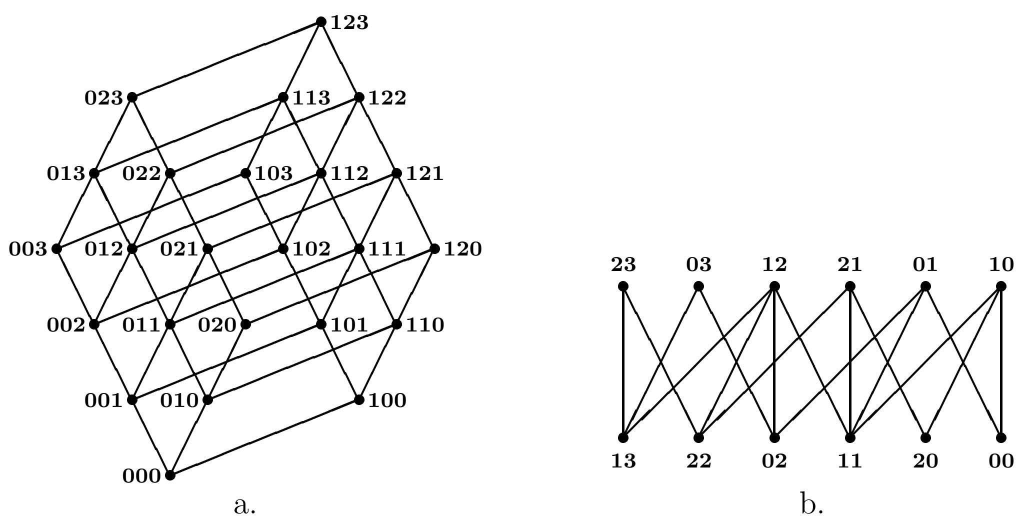


Figure 2: The chain product $S(1, 2, 3)$ (a) and the poset $R(2, 3)$ (b).

The following theorem was found by Clements and Lindström, appeared in [43], and is qualified in [29] as a famous and influential result. In the special case $k_1 = \cdots = k_n = \infty$ the theorem can be derived from a paper of Macaulay [81] (presented in algebraic terms there, cf. section 5.7), who actually provided the chain products as a first example for a Macaulay poset in 1927.

Theorem 5 (Clements-Lindström theorem). *$(S(k_1, \dots, k_n), \leq, \preceq_{lex})$ is a Macaulay structure.*

A short proof of this theorem can be obtained by using the approach of [97] for the Boolean lattice, which is based on compression. Another short proof is based on the shifting technique and is published in [72]. A principally different approach used in [27] for the MWI problem (cf. Section 5.2) implies a short proof, too. The properties of chain products given in the following theorem are important for many applications (see Section 5.1 for instance).

Theorem 6 (Clements [35]). *Chain products are additive and shadow increasing.*

The original proofs of these properties are rather complicated. Shorter proofs can be found in the book of Engel [52] and, for the shadow increase property, in [42]. These elegant proofs are based on an idea of Kleitman, consisting in embedding a chain product S into a chain product S' of higher dimension, and to obtain the above properties of S as corollaries of the Macaulayness of S'. A similar technique works well for establishing these properties for the Macaulay posets considered in the next two sections.

We conjecture that an analogue of Theorem 3 is valid for the chain products. Moreover, we guess that it is possible to construct a Macaulay poset $R(k_1, \dots, k_n)$ with two levels, consisting of the vertices of the "odd" and "even" levels of $S(k_1, \dots, k_n)$, respectively, as it is done for the poset Q^n (as an example, see $R(2,3)$ in Fig. 2b). Our guess is based on a similarity between the VIP on the Boolean poset and on chain products.

2.3 The star posets

Another natural way to generalize Boolean lattices is to consider the chain $0 \lessdot 1$ as a star with just two vertices. This leads to cartesian products of stars which, in contradistinction to chain products, are not self-dual if at least one of the stars in the product has more than two vertices.

For positive integers n and $k_1 \leq k_2 \leq \cdots \leq k_n$ the star poset $T(k_1, k_2, \dots, k_n)$ consists of all vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ such that $x_i \in \{k_n - k_i, k_n - k_i + 1, \dots, k_n\}$ for $i = 1, 2, \dots, n$, where the partial order is given by: $\mathbf{x} \leq \mathbf{y}$ iff $x_i = y_i$ or $y_i = k_n$ for $i = 1, 2, \dots, n$. The unique rank-function on $T(k_1, k_2, \dots, k_n)$ is given by $r(\mathbf{x}) = |\{i \mid x_i = k_n\}|$. According to the above definition, $T(k_1, k_2, \dots, k_n)$ is the cartesian product of the stars $T(k_i)$, $(i = 1, 2, \dots, n)$, shown in Fig. 3a.

To introduce a Macaulay order $\preceq$ on $T(k_1, k_2, \dots, k_n)$, define $\mathbf{x}(j) := \{i \in [n] \mid x_i = j\}$ for $\mathbf{x} \in T(k_1, k_2, \dots, k_n)$ and $j = 0, 1, \dots, k_n$. Now $\preceq$ is defined as follows: $\mathbf{x} \preceq \mathbf{y}$ iff

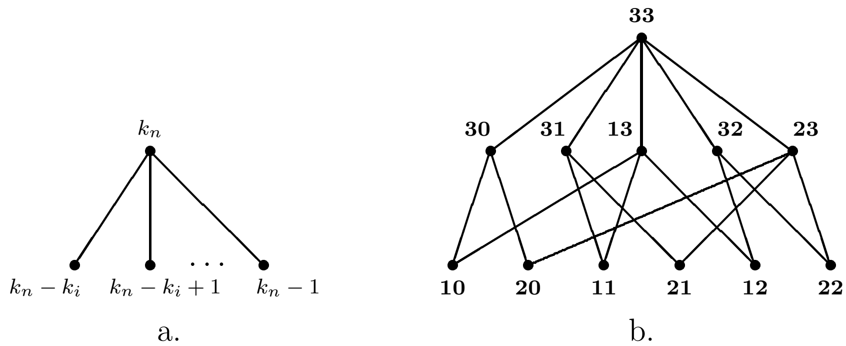


Figure 3: The star $T(k_i)$ (a) and the star poset $T(2,3)$ (b).

$\mathbf{x} = \mathbf{y}$ or $\mathbf{y}(h) \prec_{lex} \mathbf{x}(h)$, where h is the smallest number with $\mathbf{x}(h) \neq \mathbf{y}(h)$. In Fig. 3b, the elements of each level of $T(2,3)$ are placed in increasing order $\preceq$ from left to right.

Theorem 7 $(T(k_1, k_2, \ldots, k_n), \leq, \preceq)$ *is a Macaulay structure.*

This theorem is found by Lindström [79] for the case $k_1 = \cdots = k_n = 2$ (his proof, however, contains a gap), and is proved by Leeb [78] and Bezrukov [12] in the case $k_1 = \cdots = k_n$. Actually, both mentioned proofs can be extended to the case $k_1 \neq k_n$. Explicit proofs for this general case are given in [52, 73].

Theorem 8 *Star products are additive and shadow increasing.*

These two properties of the star poset are very important for applications. The additivity part is again due to Clements [37] (see [52] for simplification), the shadow increase property was shown by Leck [74] using Kleitman's technique mentioned above.

2.4 Colored complexes

Obviously, for $k_n \geq 2$ the star product $T(k_1, k_2, \ldots, k_n)$ is not isomorphic to its dual. Engel [52] observed that the duals of star products are isomorphic to colored complexes which were introduced by Frankl, Füredi and Kalai [59] in the case $k_n - k_1 \leq 1$.

To define colored complexes in general, for positive integers n and $k_1 \leq k_2 \leq \cdots \leq k_n$, and for $i = 1, 2, \ldots, n$, let the i-th *color class* be the set

$$A_i := \{i, n+i, 2n+i, \ldots, (k_i - 1)n + i\}.$$

Now the colored complex $Col(k_1, k_2, \ldots, k_n)$ consists of all subsets $X \subseteq A := \bigcup_{i=1}^{n} A_i$ such that $|X \cap A_i| \leq 1$ for $i = 1, 2, \ldots, n$, i.e. of all subsets of A which meet every color class at most once. The corresponding partial order is the usual set inclusion. According to this definition, $Col(k_1, k_2, \ldots, k_n)$ is the cartesian product of the stars $Col(k_i)$ shown in Fig. 4a.

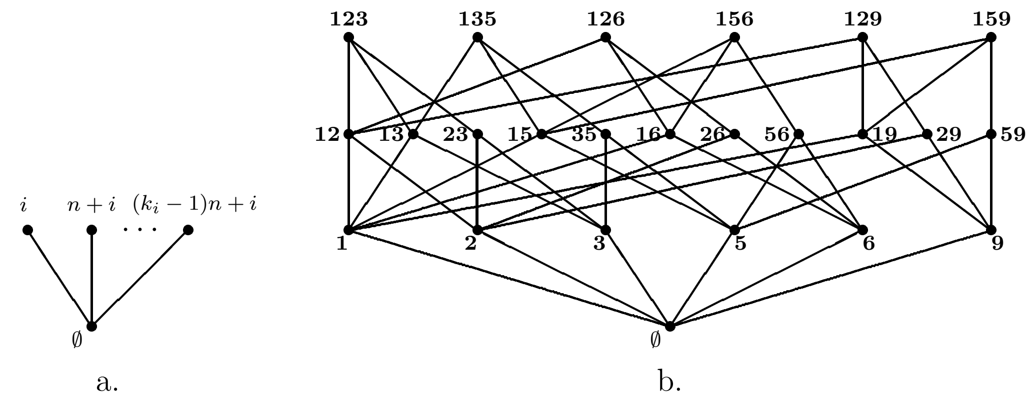


Figure 4: The star $Col(k_i)$ (a) and $Col(1, 2, 3)$ (b).

Engel [52] established the following isomorphism φ from $Col(k_1, k_2, \ldots, k_n)$ to the dual of $T(k_1, k_2, \ldots, k_n)$:

$$\varphi(X) = (z_1, \ldots, z_n), \qquad \text{where } z_i = \begin{cases} k_n - 1 - \frac{x-i}{n} & \text{if } x \equiv i \pmod{n} \\ & \text{for some } x \in X, \\ k_n & \text{otherwise.} \end{cases}$$

It is easily seen that for $X, Y \in Col(k_1, k_2, \ldots, k_n)$ we have $\varphi(X) \preceq \varphi(Y)$ in $T(k_1, k_2, \ldots, k_n)$ iff $Y \preceq_{lex} X$. Consequently, Proposition 1 and Theorem 7 yield the following corollary.

Corollary 1 (Colored Kruskal-Katona theorem).
$(Col(k_1, k_2, \ldots, k_n), \subseteq, \preceq_{lex})$ *is a Macaulay structure.*

The original paper of Frankl, Füredi, Kalai [59] and a simpler proof given by London [80] establish Corollary 1 for $k_n - k_1 \leq 1$ without using Theorem 7 and the above isomorphism.

Since $Col(k_1, k_2, \ldots, k_n)$ is isomorphic to the dual of $T(k_1, k_2, \ldots, k_n)$, Proposition 3 implies that colored complexes are additive.

Theorem 9 (Leck [74]). *Colored complexes are shadow increasing.*

This theorem is the result of another application of the Kleitman's idea mentioned above.

3 Construction of Macaulay posets

In this section we present some constructions of Macaulay posets. We start with a criterion for Macaulayness and some simple observations, and then proceed with deep relations to extremal problems on graphs. In the last part of this section we discuss product theorems for Macaulay posets.

3.1 A criterion for Macaulayness

Let P be a ranked poset with the associated partial order $\leq$ and some total order $\preceq$ of its elements. Denote by P' and P'' the posets obtained from P by deleting its top and bottom levels respectively. Thus, $r(P') = r(P'') = r(P) - 1$. Furthermore, denote by $\leq'$, $\leq''$, $\preceq'$, and $\preceq''$ the restrictions of $\leq$ and $\preceq$ to P' and P'', respectively.

Proposition 6 *$(P, \leq, \preceq)$ is a Macaulay structure iff so are $(P', \leq', \preceq')$ and $(P'', \leq'', \preceq'')$.*

Proof. The "only if" part of the assertion is an immediate consequence of the definitions. On the other hand, if both P' and P'' are Macaulay, then we construct a new total order $\preceq^*$ on P by ordering first the elements of P' within the level P'_0 according to the order $\preceq'$, and then proceeding similarly with the levels $P'_1, P'_2, \ldots, P'_{r(P')}$. Finally we order the elements of $P_{r(P)}$ according to the order $\preceq''$. It is easily seen that the order $\preceq^*$ and its restrictions to P' and P'' are Macaulay orders for P, P', and P''. ■

Unfortunately, this statement does not provide direct constructions for Macaulay posets, since to construct nontrivial Macaulay posets even with just two levels is a difficult problem. However, we believe that this proposition can be useful for answering some open questions mentioned in Section 5.

3.2 Posets with a given shadow function

Here we show that for any shadow function sf_i there exists a Macaulay poset with this shadow function. Obviously, it suffices to construct Macaulay posets with two levels only.

Let P be a ranked poset with $r(P) = 1$ and consider the SMP on its top level P_1. Denote by $\Delta(m)$ the minimal size of the shadow of a set consisting of m elements of P_1. Obviously, the sequence $\{\Delta(m)\}$ is nondecreasing.

Proposition 7 *For any nondecreasing sequence $\{\Delta(1), ..., \Delta(p)\}$ there exists a corresponding Macaulay poset P with $r(P) = 1$.*

Proof. Let $P_1 = \{a_1, \ldots, a_p\}$ and $P_0 = \{b_1, \ldots, b_{\Delta(p)}\}$. We define a partial order $\leq$ on $P = P_0 \cup P_1$ as follows. For any $i = 1, \ldots, p$ set $a_i > b_j$ for $j = 1, \ldots, \Delta(i)$ (cf. Fig. 5).

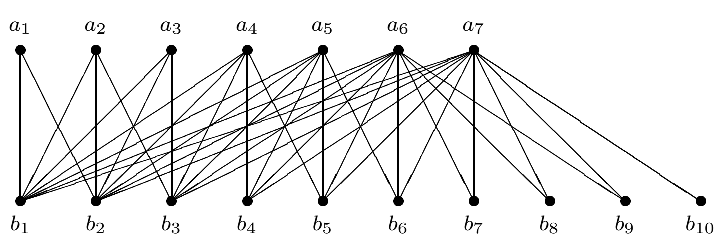


Figure 5: A poset corresponding to the sequence $(2, 3, 3, 5, 6, 9, 10)$.

Obviously, the constructed poset is Macaulay and the labelings of the a_i's and b_i's provide Macaulay orders on P_1 and P_0, respectively. ∎

Similarly, Macaulay posets with more levels can be constructed. This construction is, in a sense, invertible. Given a Macaulay poset $(P, \leq, \preceq)$, construct another poset $Q = (P, \sqsubseteq)$ as follows. Take an element $a \in P_i$ for some $i > 1$ and consider $\mathcal{F}_i(a)$. Then $\Delta(\mathcal{F}_i(a)) = \mathcal{F}_{i-1}(b)$ for some $b \in P_{i-1}$. Let $c \in \mathcal{F}_{i-1}(b)$ and assume $c \not\leq a$. Now we extend the partial order $\leq$ by setting $c \leq a$ (cf. Fig. 6).

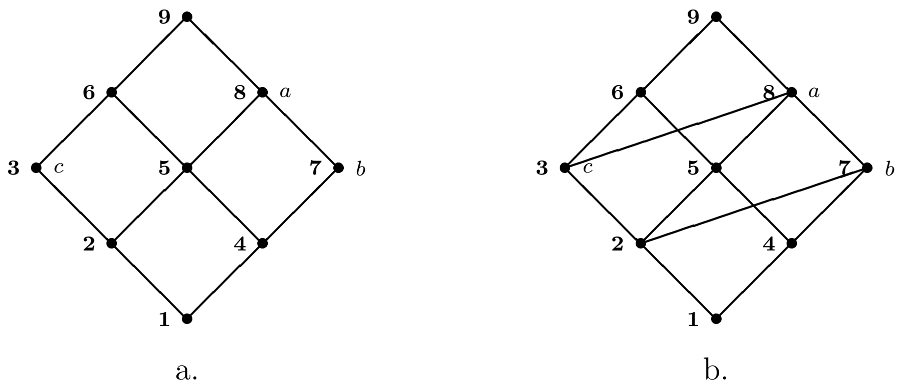


Figure 6: Posets P (a) and Q (b)

Proposition 8 (Bezrukov, Portas, Serra [25]). *The poset Q is Macaulay.*

Proof. Denote $\Delta_P(m, i) = \min |\Delta(A)|$, where the minimum runs over all $A \subseteq P_i$ with $|A| = m$. Since the partial order $\leq$ is a suborder of $\sqsubseteq$, then $\Delta_P(A) \subseteq \Delta_Q(A)$ for any $A \subseteq P_i$, Thus,

$$\Delta_P(m, i) \leq \Delta_Q(m, i) \quad \text{for all} \;\; m = 1, \ldots, |P_i| \;\; \text{and} \;\; i = 1, \ldots, r(P). \tag{4}$$

However, $\Delta_P(\mathcal{F}_i(a)) = \Delta_Q(\mathcal{F}_i(a))$ for any $a \in P_i$. Therefore, since P is Macaulay, then the lower bound (4) for Q is tight. This implies that $(P, \sqsubseteq, \preceq)$ is Macaulay, too. ∎

Hence, applying any number of the above extensions leads to a Macaulay poset. If we add to P all possible relations provided by the above construction, then one will come to Macaulay poset in the *normal form*.

3.3 Posets related to isoperimetric problems on graphs

Let $G = (V_G, E_G)$ be a graph. For $A \subseteq V_G$ denote

$$\begin{aligned} E(A) &= \{(u, v) \in E_G \mid u \in A, \; v \notin A\}, \\ E(m) &= \max_{|A|=m} |E(A)|. \end{aligned}$$

Consider an edge-isoperimetric problem (EIP): for any $m \leq |V_G|$ find $A \subseteq V_G$ such that $|A| = m$ and $|E(A)| = E(m)$. We say that the edge-isoperimetric problem has *nested solutions* if there exists a numbering of V such that each IS is an optimal set. For more information on edge-isoperimetric problems on graphs we refer to the survey [18].

Assume that the EIP has nested solutions for the graph G. We construct a Macaulay poset $(P, \leq)$ with $|P| = |V_G|$ by induction on $|V_G|$ (cf. [17]). If $|V_G| = 1$, then the poset is trivial. For $|V_G| > 1$ let $V_G = \{1, \ldots, |V_G|\}$ and assume that for each $m = 1, \ldots, |V_G|$ the subset $\{v_1, \ldots, v_m\} \subseteq V_G$ is optimal. Note that for $m < |V_G|$ this subset is also optimal for the subgraph G' which is induced by the vertex set $\{1, \ldots, |V_G| - 1\}$. Construct the representing poset $(P', \leq')$ for G' by induction. Now extend P' by adding a new element v at level $i = E(|V_G|) - E(|V_G| - 1)$ and extend the partial order $\leq'$ by setting v to be greater than any element of P' at level $i-1$. This procedure results in the poset $(P, \leq)$.

Applying this algorithm to the Petersen graph (see Fig. 7a) results in a poset P shown in Fig. 7b. (with dotted lines). The elements of P are numbered in Fig. 7b in the same order as they appear in the Petersen graph.

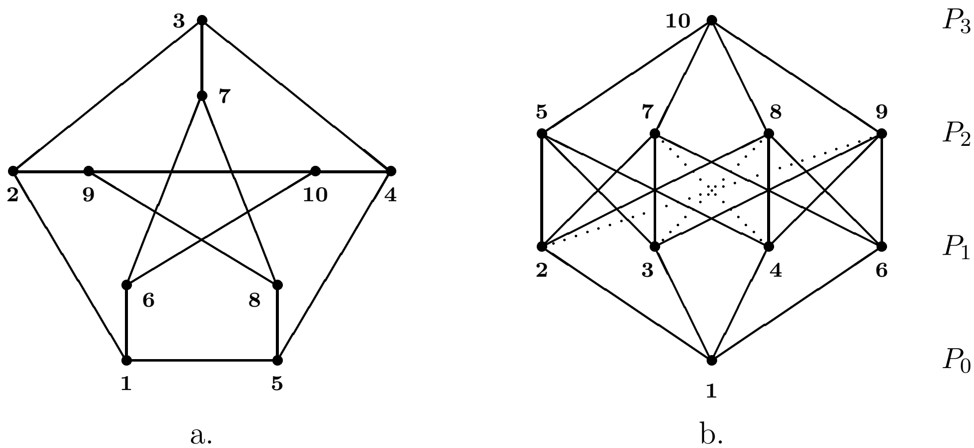


Figure 7: The EIP-construction

Due to the observations in the last subsection the resulting poset is a Macaulay poset in the normal form. Thus, we have the following observation.

Proposition 9 (cf. [17]). *If G has nested solutions in the EIP, then the poset obtained according to the EIP-construction is Macaulay.*

It is easily seen that the relations represented by the dotted lines in Fig. 7b can be deleted from the partial order, and the remaining poset is Macaulay, too [17]. It is interesting that if a poset P represents a graph G, and if P^n is Macaulay, then the EIP on G^n has nested solutions [15, 16]. The inverse proposition is, however, not correct, in general (see an example in Fig. 19). However, the posets P^n are good candidates for being Macaulay (cf. the discussion in Section 5.3).

Note that not every Macaulay poset of the form above represents a graph. Consider, for example, the poset shown in Fig. 8 together with an order of the elements in the construction above. If the corresponding graph G exists, then $E(m)$ for $m = 1, \ldots, 5$ has to be $0, 1, 2, 4, 7$ respectively. Hence, the subgraph of G induced by the first four vertices is a 4-cycle and the fifth vertex has degree 3. However, such a graph necessarily contains a 3-cycle. Thus, the three first values of $E(m)$ should be $0, 1, 3$.

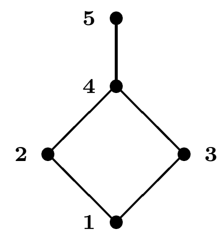


Figure 8: A poset that represents no graph

Now we turn to a *vertex-isoperimetric problem* on $G = (V_G, E_G)$. For $A \subseteq V_G$ denote

$$\begin{aligned}\Gamma(A) &= \{v \in V_G \setminus A \mid (v, u) \in E_G,\ u \in A\},\\ \Gamma(m) &= \min_{|A|=m} |\Gamma(A)|.\end{aligned}$$

The vertex-isoperimetric problem (VIP) consists in finding for a given $m \le |V_G|$ a set $A \subseteq V_G$ such that $|A| = m$ and $|\Gamma(A)| = \Gamma(m)$. Such problems often arise in combinatorics. For a survey we refer to [14].

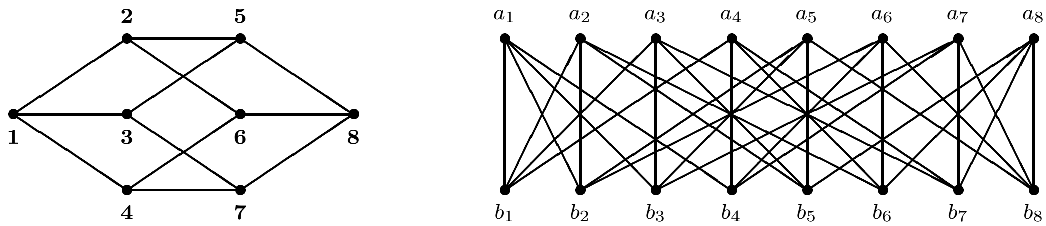
We claim more than nestedness from the VIP. Namely, we additionally assume that for any IS $A \subseteq V_G$ the set $A \cup \Gamma(A)$ is an IS, too. This property corresponds to the continuity in the definition of Macaulay posets and holds for many graph families.

Let $V_G = \{1, \ldots, |V_G|\}$, where any IS represents an optimal set. We construct a poset $(P, \le)$ with $r(P) = 1$ and $|P| = 2|V_G|$ as follows. Let $P_0 = \{b_1, \ldots, b_{|V_G|}\}$ and $P_1 = \{a_1, \ldots, a_{|V_G|}\}$. We set $b_i < a_i$ for $i = 1, \ldots, |V_G|$. Furthermore, if $(i, j) \in E_G$, then set $b_i < a_j$ and $b_j < a_i$.

For example, consider the 3-cube in Fig. 9a. A solution to the VIP for an n-cube is due to Harper [60]. Any IS of the numbering shown in Fig. 9a provides an optimal set. The corresponding poset is shown in Fig. 9b. Note that this poset is isomorphic to the poset Q^4 (cf. Fig. 1b). It can be easily proved that the VIP-construction being applied to B^n results in a poset that is isomorphic to Q^{n+1}.

Proposition 10 *The poset obtained according to the VIP-construction from a graph G is Macaulay iff G satisfies the nestedness and continuity properties with respect to the VIP.*

Proof. The assertion follows from the fact that for $A \subseteq V_G$ one has $\Delta(\{a_i \mid i \in A\}) = \{b_j \mid j \in A \cup \Gamma(A)\}$. ∎



a. b.

Figure 9: The VIP-construction

3.4 Product theorems

Counterexamples show that if P and Q are Macaulay posets, then $P \times Q$ is not necessarily Macaulay. For example, if P is a poset whose Hasse diagram is isomorphic to $K_{p,p}$ for $p \geq 2$ (i.e. we have a special case of a so-called *complete* poset [50]) then $P \times P$ is not Macaulay in contradistinction to a conjecture in [50]. Indeed, if $m \leq p$, then a set of m elements of P_1^2 has minimal shadow iff these elements agree in some entry whose rank in P is 0. However, the shadow of any element of P_2^2 consists of $2p$ elements of P_1^2, which do not contain p elements of the form above.

Thus, a condition on P and Q is needed for a product theorem. The situation is, however, simple if Q is a trivial poset with $r(Q) = 0$. In this case a necessary and sufficient condition for P is found by Clements:

Theorem 10 (Clements [38]). *If $r(Q) = 0$, then $P \times Q$ is additive and Macaulay iff so is P.*

Probably, the next case in this hierarchy are posets of the form $P \times C_q$ with C_q being a chain with q elements. Here a condition for P is required, too, as the following example of a poset Q of the form $P \times C_2$ shows (cf. Fig. 10). For simplicity, some dotted edges are not shown in this figure.

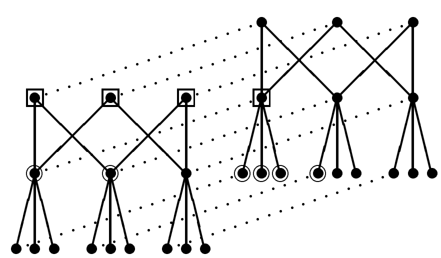


Figure 10: A non Macaulay poset of the form $P \times C_2$

Consider an optimal set of size 4 in Q_2. It is easy to show that any such set contains the 3 leftmost elements of Q_2 in Fig. 10. One of the optimal sets is displayed by squares.

Its shadow consists of 6 elements of Q_1, whose shadow, in turn, consists of 9 elements of Q_0. However, the shadow of an optimal subset of Q_1 of size 6 has only 7 elements (one of such sets is represented by larger circles in Fig. 10).

Theorem 11 *Let P be a poset with $r(P) = 1$ and let $q \geq 1$. Then $P \times C_q$ is a Macaulay poset iff P is Macaulay.*

Proof. Let $\mathcal{P}$ be a rank greedy Macaulay order on P and consider the poset $Q = P \times C_q$. Denote by $\mathcal{L}_Q$ the lexicographic order on the set $Q = P \times \{0, \ldots, q-1\}$. We show that this order is Macaulay. It is easily shown that the order $\mathcal{L}_Q$ satisfies the property $\mathbf{N}_2$.

Furthermore, for $i = 0, \ldots, q-1$ denote by $P(i)$ the subposet of Q with the element set $\{(x, i) \mid x \in P\}$ and the induced partial order (cf. Fig. 11). Obviously, $P(i)$ is isomorphic to P for any i. Without loss of generality we assume $q \geq 2$.

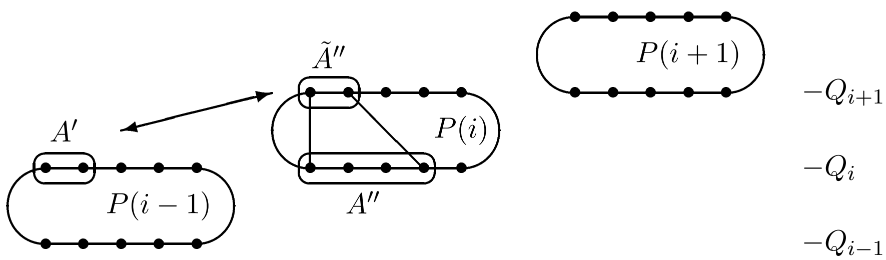


Figure 11: Construction of $\mathcal{Q}_i(m)$ in Theorem 11

We show that for any m and i the subset $C(m, Q_i)$ is optimal. For this, consider a set $A \subseteq Q_i$ with $|A| = m$. If $i = q$, then

$$|\Delta(A)| \geq |A| + \Delta(m). \tag{5}$$

This equality is strict for the set $C(m, Q_q)$. Now for $i \leq q-1$ denote $A' = A \cap P_1(i-1)$ and $A'' = A \cap P_0(i)$ (cf. Fig. 11), and let $m' = |A'|$ and $m'' = |A''|$. If $2 \leq i \leq q-1$, then

$$|\Delta(A)| \geq |\Delta(A') \cap P_1(i-2)| + |\Delta(A'') \cap P_0(i-1)| = m' + m'' = m.$$

This inequality is also strict for $C(m, Q_i)$, which provides its optimality for $2 \leq i \leq q-1$.

It remains to consider the case $i = 1$. Let A be an optimal set. One has

$$|\Delta(A)| \geq \max\{\Delta(m'),\ m''\} \geq m''. \tag{6}$$

Since P is Macaulay, then we can assume that A' and A'' are isomorphic to $C(m', P_1)$ and $C(m'', P_0)$ (in the order $\mathcal{P}$), respectively. In this case the inequalities (6) are strict. Now if $m'' > \Delta(m'+1)$, then replace A'' with $C(m''-1, P_0)$ and replace A' with $C(m'+1, P_1)$ (in the order $\mathcal{P}$). Similarly, if $m'' < \Delta(m')$, then replace A'' with $C(m''+1, P_0)$ and replace

A' with $C(m'-1, P_1)$. In both cases the shadow of the resulting set does not increase. Similar transformations result in an optimal set B such that

$$\Delta(|B'|) \leq |B''| \leq \Delta(|B'|+1),$$

and both B' and B'' are isomorphic to some initial segments of the order $\mathcal{P}$. Hence, B is isomorphic to $C(m, Q_i)$ for the order $\mathcal{L}_Q$.

Now assume that Q is Macaulay. Thus, there exists a total order $\mathcal{Q}$ that satisfies the properties $\mathbf{N}_1$ and $\mathbf{N}_2$. Since for the set $A = C(m, Q_{q+1})$ (in the order $\mathcal{Q}$) the inequality (5) is strict, then the subset $B \subseteq P_1$, which is isomorphic to A, has minimal shadow. This implies that P is Macaulay. ■

3.5 A local-global principle

Consider the SMP on a cartesian power P^n of a Macaulay poset P. There exists a powerful technique for establishing the Macaulayness of such posets, which, in particular, involves induction on the number n of posets in the product. However, the general arguments within this technique work for $n \geq 3$ only. The case $n = 2$ is a special one and must be considered separately.

A similar situation also occurs in the *edge isoperimetric* problem on graphs (see section 3.3) and in the more general problem of the minimization of *submodular functions* on graphs. For a finite set S a function $f : 2^S \mapsto I\!R$ is called submodular if for any $A, B \subseteq S$

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$

If S is the vertex set of a graph $G = (V_G, E_G)$, then the size of the edge cut separating a set $A \subseteq V_G$ from $V_G \setminus A$ is an example of a submodular function. Based on a submodular function f defined on 2^{V_G}, some special functions $f^{(n)}$ on the n^{th} cartesian power of G are considered in [1]. These functions $f^{(n)}$ are, in a sense, decomposable, i.e. they can be represented as certain *sums* of functions $f^{(n-1)}$. Without going into details, consider, for example, the function $f^{(n)}$ defined as the size of an edge cut that separates a subset of vertices of the n-cube from its complement. Then $f^{(n)}$ can be represented as the sum over $i = 1, \ldots, n$ of the number of cut edges which are parallel to the i^{th} dimension.

Ahlswede and Cai proved in [2] that if the *lexicographic order* (see Section 2) provides nestedness (cf. $\mathbf{N}_1$) in minimizing $f^{(n)}$ for $n = 2$, then it is so for any $n \geq 3$. It turns out that the last result, which is called the *local-global principle* in [2], is valid for the edge-isoperimetric problem also with respect to some other total orders [18].

In what concerns the SMP, the above approach can not be directly applied because of the necessity to maintain the level structure of a poset. Another difficulty in applying the results of [1, 2] is that the function $|\Delta(\cdot)|$ is not decomposable in the sense above.

However, similar general principles in the proof techniques for establishing the Macaulayness of posets and solving edge-isoperimetric problems provide a local-global principle with respect to the SMP. It turns out [25] that for the validity of such a principle with

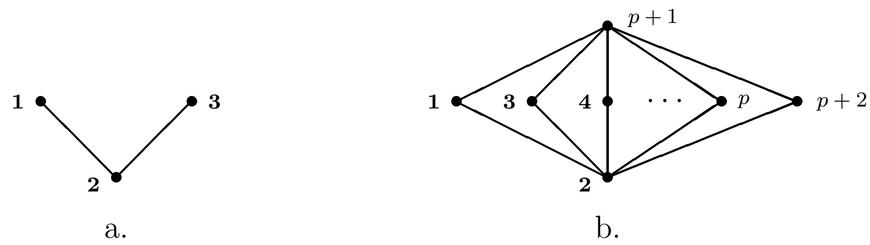


Figure 12: Counterexamples to Theorem 13 for $n = 2$

respect to the lexicographic order it is important that the poset satisfies some additional conditions, which have no analogues for graphs yet.

We call a Macaulay poset P *strongly Macaulay* if it is additive, shadow increasing and final shadow increasing. Note that Theorems 10 and 11 are valid with respect to strongly Macaulay posets, too.

Denote by $\mathcal{M}$ the class of ranked posets having only one maximal and only one minimal element.

Proposition 11 *A poset $P \in \mathcal{M}$ is strongly Macaulay iff so is its dual P^*.*

Theorem 12 (Bezrukov, Portas, Serra [25]). *Let $(P, \leq, \preceq) \in \mathcal{M}$ be strongly Macaulay and rank-greedy. Let the lexicographic order $\preceq^2$ be Macaulay for P^2. Then for any $n \geq 2$ the lexicographic order $\preceq^n$ is a Macaulay order for P^n.*

The assumptions concerning the poset P in Theorem 12 are essential, as the following result shows.

Theorem 13 (Bezrukov, Portas, Serra [25]). *Let $(P, \leq, \preceq)$ be a Macaulay poset. Furthermore, let $r(P) \geq 3$ and assume that the orders $\preceq^2$ and $\preceq^3$ are Macaulay for P^2 and P^3, respectively. Then for any $n \geq 1$*

a. $P^n \in \mathcal{M}$;

b. P^n is rank greedy;

c. P^n is strongly Macaulay.

The last theorem, however, is not true without the conditions concerning $\preceq^3$ and $r(P)$. Indeed, for the poset shown in Fig. 12 the lexicographic order $\preceq^n$ is Macaulay for $n = 2$, however, not for $n = 3$.

Moreover, the Macaulay order for the poset P in Fig. 12 is not rank greedy, while for $p = 1, 2$ the poset P^2 is Macaulay with the lexicographic order as a Macaulay order. Furthermore, for the chain poset C_q with the partial order $0 < 1 < \cdots < q$ and a non rank greedy Macaulay order $0 \succ 1 \succ \cdots \prec q$ the lexicographic order $\preceq^n$ is Macaulay for $n = 1, 2$, but not for $n \geq 3$ if $q > 1$. For $q = 1$ (i.e. for the hypercube) the order $\preceq^n$ is still Macaulay for $n = 3$, but not for $n \geq 4$.

Theorem 12 being applied to chains implies the Kruskal-Katona theorem [63, 66] and the particular case of the Clements-Lindström theorem [43] when all the chains in the product are of the same length.

As another application of the local-global principle consider the following poset $(T(k), \leq) \in \mathcal{M}$ of rank k. For $1 \leq i \leq k-1$ the i^{th} level of $T(k)$ consists of two elements a_i and b_i. Denote by b_0 and a_k the elements of T_0 and T_k, respectively. The partial order is defined as follows: $x < y$ iff $r(x) < r(y)$. The Hasse diagram of $(T(3), \leq)$ is shown in Fig. 13a.

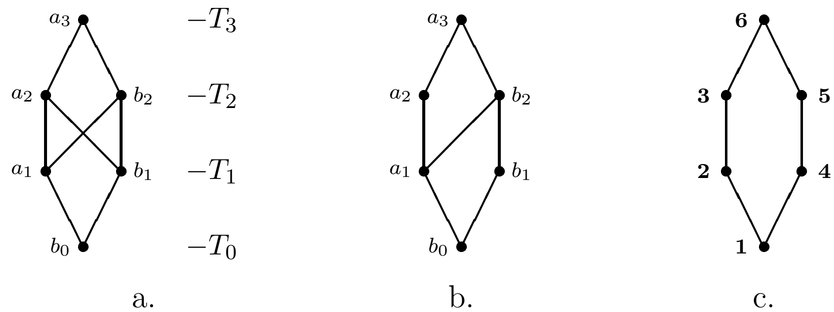


Figure 13: Posets $(T(3), \leq)$ (a.), 3×2 grid (b.), and torus T_3 (c.)

We define the total order $\preceq$ on $T(k)$ by setting $b_{i-1} \prec a_i$ for $i = 1, \ldots, k$ and $a_i \prec b_i$ for $i = 1, \ldots, k-1$. Obviously, the order $\preceq$ is Macaulay on $(T(k), \leq)$.

Theorem 14 (Bezrukov, Portas, Serra [25]). *For any $k \geq 1$ and any $n \geq 1$ the poset $(T^n(k), \leq_\times, \preceq^n)$ is Macaulay.*

It is interesting that if we modify the partial order in $T(k)$ by making the elements a_i and b_{i+1} incomparable for $i = 1, \ldots, k-1$, then the resulting poset Q is isomorphic to the $k \times 2$ grid (cf. Fig. 13b). Being a product of chains, the obtained poset is Macaulay due to the Clements-Lindström theorem [43]. However, no lexicographic order is Macaulay (cf. [25]).

If we further modify the poset $T(k)$ by making the elements b_i and a_{i+1} incomparable for $i = 1, \ldots, k-1$, then this results in a cycle T_k (cf. Fig. 13c). Any cartesian power of T_k (which is isomorphic to a torus) is Macaulay. The Macaulay order can be derived from [62, 86], where a vertex-isoperimetric problem for tori was studied, and is not lexicographic.

Further posets for which the local-global principle is applicable can be constructed using Proposition 8. Let P satisfy the assumptions of Theorem 12, and construct the poset $Q = (P, \sqsubseteq)$ as in Section 3.3. Then Theorem 12 is applicable to Q.

Indeed, the poset Q is Macaulay by Proposition 8. Now consider P^2. Since

$$\Delta_{P^2}(\mathcal{F}_i((x, y))) = \{(x, \xi) \mid \xi \in \Delta_P(\mathcal{F}_{i-r_P(x)}(y))\} \cup \{(\xi, y) \mid \xi \in \Delta_P(\mathcal{F}_{i-r_P(y)}(x))\},$$

then $\Delta_{P^2}(\mathcal{F}_i((x, y)) = \Delta_{Q^2}(\mathcal{F}_i((x, y)))$. Therefore, if P satisfies the assumptions of Theorem 12, then so does Q. On the other hand, since the lexicographic order is Macaulay for P^2, then so it is for P^4, for example. Extending P^2 as shown in Section 3.1 results in a new poset, for which Theorem 12 is applicable. In particular, the Macaulayness of the torus poset T_k^n (cf. Section 4.3) implies a similar result for the powers of $k \times 2$ grids (cf. Fig. 13b,c).

In [26] a similar technique is applied for establishing a local-global principle with respect to a problem related to the theory of Macaulay posets, namely the vertex-isoperimetric problem on graphs (cf. Section 3.3).

4 New Macaulay posets

In this section we present some new families of Macaulay posets. We start with posets which are factorable by using the cartesian product operation in subsections 1 - 3 and proceed with two posets which do not appear to be cartesian products.

4.1 The products of trees and spider posets

Evidently, the classical Macaulay posets mentioned in Section 2 (we mean the Boolean lattice, the chain products, and the star poset) have something in common. Namely, the Hasse diagrams of the underlying posets in the product are trees.

These posets are also *upper semilattices*. For $a, b \in P$ denote by $\sup_P(a, b)$ an element $c \in P$ (if it exists) such that $a \prec c$, $b \prec c$ and $c \prec d$ if $a \prec d$ and $b \prec d$. The poset P is an *upper semilattice* if for any $a, b \in P$, $\sup_P(a, b)$ exists and is unique.

Denote by $\mathcal{P}$ the class of upper semilattices P whose Hasse diagrams are trees. Which posets $P \in \mathcal{P}$ have the property that any cartesian power P^n is Macaulay ? Denote by $Q(k, l) \in \mathcal{P}$ the poset with the element set $\{0, 1, \ldots, (k+1)l\}$, and the partial order $\leq$ being defined as follows: $\alpha \leq \beta$ iff (i) $\alpha = \beta \pmod{k+1}$ and $\alpha \leq \beta$, or (ii) $\beta = (k+1)l$. The Hasse diagram of $Q(k, l)$ is a regular *spider* with $k+1$ legs consisting of l vertices each. As an example, $Q(1, 2)$ is shown in Fig. 14a.

Theorem 15 (Bezrukov [16]). *Suppose for some poset $P \in \mathcal{P}$ that P^n is Macaulay for some integer $n \geq r(P) + 3$. Then P is isomorphic to $Q(k, l)$ for some $k \geq 1$ and $l \geq 1$.*

In the proof it is shown first that if x is a leaf of the Hasse graph of P, then $r_P(x) \in \{0, r(P)\}$. In this part of the proof the condition $n \geq r(P) + 3$ arises. The rest of the proof is devoted to the case when the Hasse graph of P contains a vertex z of degree at least 3. It is shown that the assumption $r_P(z) < r(P)$ leads to a contradiction. It turns out that the inverse theorem is also valid.

Theorem 16 (Bezrukov, Elsässer [22]). *The poset $Q^n(k, l)$ is Macaulay for all integers n, k and l.*

The Macaulay order for $Q^n(k, l)$ is quite complicated and involves, in particular, the star poset order. We refer to [22] for exact definitions. The poset $Q^2(1, 3)$ is shown in Fig. 14b. Although the proof is based on compression techniques, the compression itself

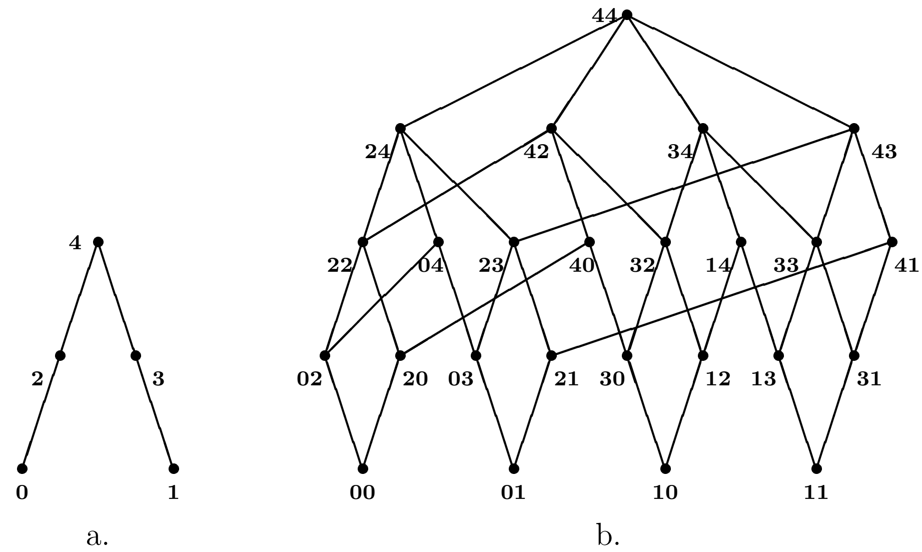


Figure 14: The spider poset $Q(1,2)$ (a.) and $Q^2(1,2)$ (b.)

is a pretty small part of the proof. As a poset becomes more and more complex, the main problem is how to transform a compressed set into an initial segment of the corresponding order. For this, a new technique is proposed in [22], which, hopefully, can be well applied to further posets.

Looking back at Theorem 7 for star posets it is natural to ask if all cartesian products of the form $Q(k_1,l) \times Q(k_2,l) \times \cdots \times Q(k_n,l)$ are Macaulay. We conjecture an affirmative answer. On the other hand, it is easily seen that products of the form $Q(k,l_1) \times Q(k,l_2) \times \cdots \times Q(k,l_n)$ are not Macaulay in general.

A natural question remains open: what if we omit the condition of being a semilattice in the definition of the class $\mathcal{P}$? Products of which trees (considered as posets) are Macaulay ? We conjecture that the condition $r_P(x) \in \{0, r(P)\}$ still should be satisfied for any leaf x, and $r_P(z) \in \{0, r(P)\}$ holds for any vertex of degree at least 3.

4.2 Generalized submatrix orders

Our next example is closely related to colored complexes (and, consequently, to the star posets as well).

Let n and $k_1 \le k_2 \le \dots k_m$ be positive integers such that $k_0 := n - \sum_{i=1}^{m} k_i \ge 0$. Furthermore, let $A_0, A_1, \dots, A_m$ be the sets defined by

$$\begin{aligned} A_0 &:= \{1, 2, \dots, k_0\}, \\ A_i &:= \left\{ \sum_{j=0}^{i-1} k_j + 1, \sum_{j=0}^{i-1} k_j + 2, \dots, \sum_{j=0}^{i} k_j \right\} \text{ for } i = 1, 2, \dots, m. \end{aligned}$$

Clearly, the sets A_i $(i = 0, 1, \dots, m)$ form a partition of $[n] = \{1, 2, \dots, n\}$.

The *generalized submatrix order* $\mathcal{S} := SM(n; k_1, \ldots, k_m)$ consists of all subsets X of $[n]$ such that $A_i \not\subseteq X$ for all $i = 1, 2, \ldots, m$. The corresponding partial order is given by: $X \leq Y$ iff $X \subseteq Y$. According to this definition, $\mathcal{S}$ is isomorphic to the cartesian product $B^{k_0} \times \tilde{B}^{k_1} \times \cdots \times \tilde{B}^{k_m}$, where $\tilde{B}^s$ denotes the Boolean lattice B^s without its maximal element (see Fig. 15a). In particular, in the case $k_0 = 0$, $k_1 = \cdots = k_m = 2$ it is isomorphic to the colored complex $Col(2, 2, \ldots, 2)$.

The name generalized submatrix order refers to the work of Sali [88, 90] who actually considered the dual of $\mathcal{S}$ in the case $m = 2$, $k_0 = 0$. For $m = 2$, $k_0 = 0$ he considered $\mathcal{S}^*$ as the poset of all non-empty submatrices of an $(k_1 \times k_2)$-matrix M with A_1, A_2 being the set of rows and columns, respectively, of M. Now in $\mathcal{S}^*$ every submatrix M' is represented by the union X of the sets of rows and columns that have to be deleted from M to obtain M'. Clearly, if we delete all rows or all columns, then we will not obtain a non-empty submatrix. Therefore, we have the conditions $A_i \not\subseteq X$, $i = 1, 2$. Sali proved for this poset several analogues to classical theorems on finite sets (Sperner, Erdös-Ko-Rado). For this poset, he also solved the problem of minimizing the number of atoms which are covered by an m-element subset of the i-th level for given i, m and conjectured Theorem 17 below in an equivalent form.

Harper introduced the name *orthogonal product of simplices* for the dual of $\mathcal{S}$ with $k_0 = 0$ (i.e. the forbidden subsets form a partition of $[n]$). For $k_0 = 0$, a statement equivalent to Theorem 17 below was conjectured by Moghadam [85].

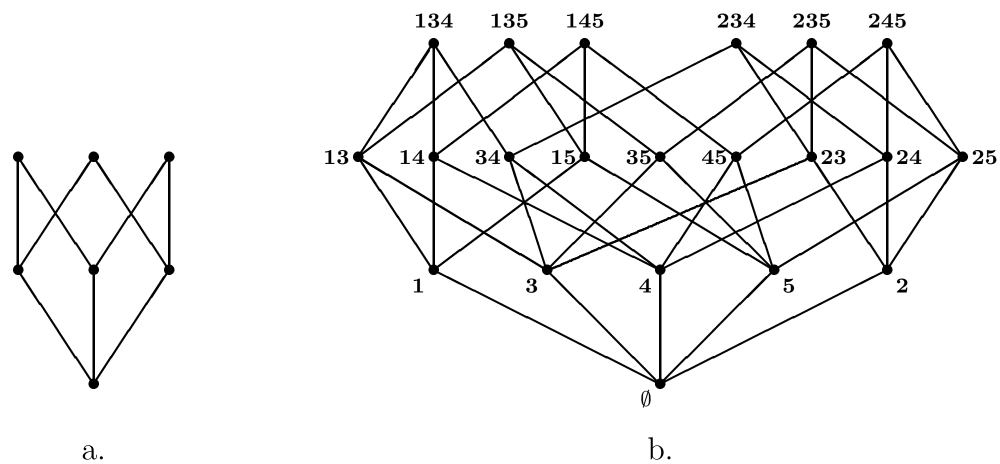


Figure 15: The poset $\tilde{B}^3$ (a) and the submatrix order $SM(5; 2, 3)$ (b).

To introduce a total order of the elements of $\mathcal{S}$, for $X \in \mathcal{S}$ and $i = 1, 2, \ldots, m$ let $p_i(X)$ denote the greatest element of A_i which is not contained in X. Furthermore, define $P(X) := \{p_1(X), p_2(X), \ldots, p_m(X)\}$. Now the total order $\preceq$ on $\mathcal{S}$ is established by the following two conditions:

(1) $X \preceq Y$ if $P(X) \neq P(Y)$ and $\min[(P(X) \cup P(Y)) \setminus (P(X) \cap P(Y))] \in P(Y)$,

(2) $X \preceq Y$ if $P(X) = P(Y)$ and $X \preceq_{lex} Y$.

Note that condition (1) partitions the poset $\mathcal{S}$ into blocks, where each of these blocks is isomorphic to a Boolean lattice. By condition (2), the order $\preceq$ acts lexicographically on the blocks. The poset $SM(5;2,3)$ is shown in Fig. 15b with the elements ordered according to $\preceq$ from left to right on each level.

Theorem 17 (Leck [76, 77]). *$(\mathcal{S}, \subseteq, \preceq)$ is a Macaulay structure.*

Note that Theorem 17 coincides with the colored Kruskal-Katona theorem in the case $k_0 = 0$, $k_1 = \cdots = k_m = 2$. It is natural to ask for the following generalization: Let $1 \leq k < s$ be integers, and let P be the poset obtained from the Boolean lattice B^s by deleting all elements of rank greater than $s-k$, i.e. the subposet $B_0^s \cup B_1^s \cup \cdots \cup B_{s-k}^s$. Is the poset P^n Macaulay for all $n \geq 2$? It is not difficult to show that, in general, the answer is negative. In fact, one can even show that the answer is positive iff $k = 1$ (i.e. P^n is a generalized submatrix order) or $k = s-1$ (i.e. P^n is a colored complex).

Before the above theorem was established, the closely related problem of finding ideals of maximum rank (cf. section 5.3) was solved by Vasta [98] for $\mathcal{S}^*$ with $k_0 = 0$. Using Theorem 17, a more general statement is now implied by Theorem 30.

In the proof of Theorem 17, again the case $m = 2$ required some special treatment, a modification of the well-known shifting operator for finite sets was used to settle this case. The following theorem is used in the proof for $m > 2$, which is done by induction.

Theorem 18 (Leck [77]). *Generalized submatrix orders are additive.*

Another interesting poset which is related to the generalized submatrix orders is the poset M^n of square submatrices of a square matrix of order n ordered by inclusion. This poset also was studied by Sali [87, 89] with respect to Sperner and intersection properties.

In other words the poset M^n can be also represented with the help of the *rankwise direct product* operation introduced by Sali in [87]. Given posets P and Q with $r(P) = r(Q)$, the poset $R = P \times_{\mathrm{r}} Q$ is a poset with $r(R) = r(P)$ and such that $R_i = P_i \times Q_i$, $i = 0, \ldots, r(P)$. The partial relation on R is defined as follows: $(x, y) \leq_R (x', y')$ iff $x \leq_P x'$ and $y \leq_Q y'$. Applying this operation with $P = Q = B^n$ (the Boolean lattice) results in the poset M^n.

For $n \leq 3$ the poset M^n is Macaulay, but not for $n \geq 4$ in contradistinction to a conjecture in [50]. In order to see it consider for $n \geq 4$ an optimal set $A \subseteq M_2^n$ with $|A| = 6$. Brute force methods provide $|\Delta(A)| = 8$, and, moreover, the elements of A must agree in some entry. Adding any new element to such a set results in an increase of the shadow size by at least 2. However, $|\Delta(B)| = 9$ holds for an optimal set $B \subseteq M^n$ with $|B| = 7$.

4.3 The torus poset

Denote by T_k the poset whose Hasse diagram can be obtained from two disjoint chains of length k each by identifying their top and bottom vertices. Obviously, the Hasse diagram of T_k is a cycle of length $2k$ (cf. Fig. 16a).

Let $T^n_{k_1,\ldots,k_n} = T_{k_1} \times \cdots \times T_{k_n}$. The solution to the SMP for this poset follows from a solution to a more general problem: the VIP (cf. Section 3.2). In order to show the relation, let us consider a bipartite graph G. Fix a vertex $v_0 \in V_G$ and denote by G_i the set of all vertices of G at distance i from v_0. This leads to a ranked poset P with $P_i = G_i$ whose Hasse diagram is isomorphic to G. Assume that a solution to the VIP on G satisfies the nestedness and continuity properties. Moreover, we assume that the total order $\mathcal{O}$ which provides a solution to the VIP orders the vertices of G_i in sequence. In other words, if A is an IS of $\mathcal{O}$ and $\sum_{i=0}^{r} |G_i| \leq |A| \leq \sum_{i=0}^{r+1} |G_i|$, then A contains a ball of radius r centered in v_0 and is contained in the ball of radius $r+1$ with the same center.

Obviously, a solution to the SMP with respect to the minimization of $\nabla(\cdot)$ for the subsets of P_r follows. Moreover, each IS of the order $\mathcal{O}$ restricted to P_r provides an optimal set. This problem is equivalent to the SMP with respect to the minimization of $\Delta(\cdot)$ for the dual of P. Thus, both P^* and P are Macaulay.

The Macaulay order for $T^n_{k_1,\ldots,k_n}$, thus, can be obtained from the VIP-order for the torus. This order is first established in [62], mentioned in the survey [14] and recently rediscovered in [86]. We follow [62] to present a solution. Assuming $k_1 \leq k_2 \cdots \leq k_n \leq \infty$ we represent the elements of T_{k_i} as $-k_i+1, \ldots, -1, 0, 1, \ldots, k_i$ in cyclic order with 0 at the bottom level (cf. Fig. 16a). For $\mathbf{x} = (x_1, \ldots, x_n) \in T^n_{k_1,\ldots,k_n}$ denote $|\mathbf{x}| = (|x_1|, \ldots, |x_n|)$, $\sigma(\mathbf{x}) = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i = 1$ if $x_{n-i+1} > 0$ and 0 otherwise. Finally, denote $N(\mathbf{x}) = \sum_{i=1}^{n} |x_i|$. Obviously, $N(\mathbf{x})$ is the level of $T^n_{k_1,\ldots,k_n}$ containing $\mathbf{x}$.

Now we are ready to define the optimal VIP-order $\mathcal{T}$ for the torus. We say $\mathbf{x}$ precedes $\mathbf{y}$ iff

(1) $N(\mathbf{x}) < N(\mathbf{y})$, or

(2) $N(\mathbf{x}) = N(\mathbf{y})$ and $\sigma(\mathbf{y})$ precedes $\sigma(\mathbf{x})$ lexicographically, or

(3) $N(\mathbf{x}) = N(\mathbf{y})$, $\sigma(\mathbf{x}) = \sigma(\mathbf{y})$ and $|\mathbf{y}|$ precedes $|\mathbf{x}|$ lexicographically.

This order is schematically shown in Fig. 16b for $T^2_{3,3}$. To simplify the figure the wrap-around edges of the torus are not shown.

Theorem 19 (Karachanjan [62], Riordan [86]). *Any IS of the $\mathcal{T}$-oder provides a solution to the VIP. Moreover, the $\mathcal{T}$-oder satisfies the continuity property.*

It should be mentioned that if some basic cycle of a torus has an odd length then no nestedness in the VIP exists in general [62]. However, this case is not related to posets.

4.4 Subword orders

Let us now turn to a first example of a Macaulay poset which is not representable as a cartesian product of nontrivial factors.

Let $n \geq 2$ be an integer, and let Ω denote the set $\{0, 1, \ldots, n-1\}$. In the sequel, we call Ω the *alphabet*. The *subword order* $SO(n)$ consists of all strings (called *words*) that contain symbols (called *letters*) from Ω only. The partial order on $SO(n)$ is the subword

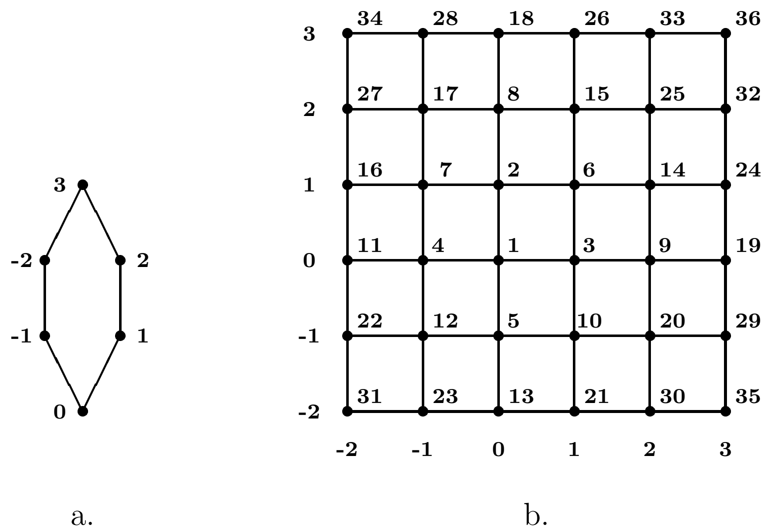


Figure 16: The torus T_3 (a.) and the VIP-order for $T^2_{3,3}$ (b.)

relation, i.e. we have $x_1x_2\dots x_k \le y_1y_2\dots y_l$ iff there is a set $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, l\}$ of indices such that $i_1 < i_2 < \dots < i_k$ and $x_j = y_{i_j}$ for $j = 1, 2, \dots, k$. In other words, $\mathbf{x} \le \mathbf{y}$ holds iff the word $\mathbf{x}$ can be obtained from the word $\mathbf{y}$ by successively deleting letters. By this definition, the rank of an element of $SO(n)$ equals its length, that means $r(x_1x_2\dots x_i) = i$. The only element of $N_0(SO(n))$ is the *empty word* ε.

Consider the case $n = 2$. Clearly, the level $N_i(SO(2))$ consists of all 0-1-words of length i and, therefore, in an obvious way its elements can be considered as the elements of the Boolean lattice B^i. It was shown by Harper [60] that, among all subsets $X \subseteq B^i$ of fixed cardinality, the IS in the *VIP-order* minimizes $|\Gamma_B(X)|$ (the size of the vertex-boundary of X in the Boolean lattice B^i). This order induces a total order of the elements for each level of $SO(2)$. For convenience, we define $w(x_1x_2\dots x_i) := |\{j \mid x_j = 1, 1 \le j \le i\}|$. Now the rank greedy extension of the VIP-order to the whole poset $SO(2)$ is given by the following conditions:

(1) $\mathbf{x} \preceq_{vip} \mathbf{y}$ if $w(\mathbf{x}) < w(\mathbf{y})$,

(2) $\mathbf{x} \preceq_{vip} \mathbf{y}$ if $w(\mathbf{x}) = w(\mathbf{y})$ and there is some $j \le \min\{r(\mathbf{x}), r(\mathbf{y})\}$ such that $x_j > y_j$ and $x_h = y_h$ for $h = 1, 2, \dots, j-1$,

(3) $\mathbf{x} \preceq_{vip} \mathbf{y}$ if $w(\mathbf{x}) = w(\mathbf{y})$, $r(\mathbf{x}) \le r(\mathbf{y})$ and $x_j = y_j$ for $j = 1, 2, \dots, r(\mathbf{x})$.

Fig. 17 shows the first levels of $SO(2)$, where in each level the elements are ordered w.r.t. $\preceq_{vip}$.

The next theorem reflects the importance of the VIP-order. It was proved independently in three different papers almost at the same time.

Theorem 20 (Ahlswede, Cai [3], Daykin, Danh [45, 46], Bezrukov [15]). $(SO(2), \le, \preceq_{vip})$ *is a Macaulay structure.*

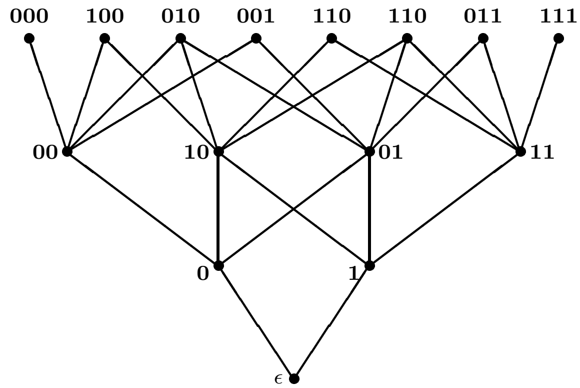


Figure 17: The subword order $SO(2)$.

Let us remark that there are also several other Macaulay orders for $SO(2)$ which were determined by Daykin [51].

Based on the numerical approach of Ahlswede and Cai in [3], Engel and Leck [54] provided a relatively simple proof of Theorem 20. One of the main observations relates the SMP for $SO(2)$ to the VIP for Boolean lattices: If $X \subseteq N_i(SO(2))$ is a final segment, then $|\nabla(X)| = |\Gamma_B(X)| + 2|X|$ holds.

Ahlswede and Cai [3, 4] proved for $SO(2)$ a number of isoperimetric inequalities related to Theorem 20. The following perhaps is the central one.

Theorem 21 (Ahlswede, Cai [4]). *Let $P = SO(2)$, and let m, k, a be natural numbers such that $1 \le m = 2^k - 1 + a$ and $a < 2^k$. Then the set $X = P_0 \cup P_1 \cup \ldots \cup P_{k-1} \cup C(a, P_k)$ has minimum-sized vertex-boundary $\Gamma(X)$ among all m-element subsets of P.*

For proving relations like the one in Theorem 21, the following simple observation turns out to be crucial: If $X \subseteq N_i(SO(2))$, then $C(X)$ and $L(X)$ are isomorphic. The corresponding isomorphism maps every letter x_j of $\mathbf{x} \in C(X)$ to the letter $1 - x_j$, i.e. we interchange the roles of the 0's and 1's. Clearly, this implies $|\Delta(C(X))| = |\Delta(L(X))|$ for all $X \subseteq N_i(SO(2))$ and all i. Macaulay posets satisfying this equality are called *shadow symmetric*.

Theorem 22 (Engel, Leck [54]). *Let P be a Macaulay poset. If P is shadow symmetric, then P additive.*

According to the above theorem, $SO(2)$ and its dual are additive. The next properties are also important for applications.

Theorem 23 (Engel, Leck [54]).

a. The subword order $SO(2)$ is shadow increasing.

b. The dual of $SO(2)$ is weakly shadow increasing.

Unfortunately, the dual of $SO(2)$ is obviously not shadow increasing (just consider the final element of each level). In fact, this poset is even *shadow decreasing*, that means $|\nabla(X)| \geq |\nabla(Y)|$ holds for all final segments $X \subseteq N_i(SO(2))$, $Y \subseteq N_{i+1}(SO(2))$ with $|X| = |Y|$ and for all i (see [54] for a proof). Fortunately, for some applications (see Section 5.1) the weak shadow increase property can serve as a substitute.

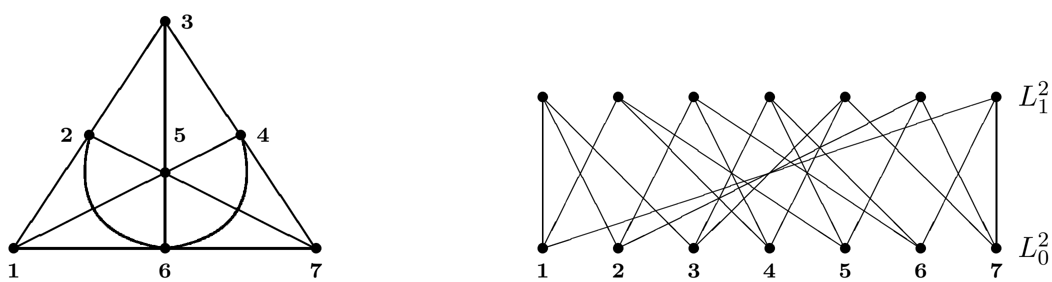
Let us now briefly discuss the case of larger alphabets. In [20] a Kruskal-Katona type theorem for $SO(n)$ with $n \geq 2$ was presented but there is a mistake in the proof, as pointed out by Danh and Daykin [47]. They also provided an example showing that the statement itself is not true at all for $n > 2$.

Daykin [50] introduced the *V-order*, an extension of the VIP-order for $SO(n)$ with $n \geq 2$. He conjectured that this order is a Macaulay order for $SO(n)$. For $n \geq 3$, a counterexample to this conjecture is given in [75]. Even worse, this example and a tedious case study yield the following result.

Theorem 24 (Leck [75]). *If $n > 2$, then the subword order $SO(n)$ is not a Macaulay poset.*

4.5 The linear lattice

The linear lattice L^n is another example of a poset which is not representable as a cartesian product of other posets. This poset is defined to be the collection of all proper nonempty subspaces of PG$(n, 2)$ ordered by inclusion (cf. Fig. 18 for $n = 2$).



a. b.

Figure 18: PG$(2, 2)$ and the poset L^2.

Although it is well known that this poset and the Boolean lattice have some similar features, for $n \geq 3$ the linear lattice is not Macaulay as shown by Bezrukov and Blokhuis in [19]. However, they found some partial analogue of the Kruskal-Katona theorem for the poset L^n.

Note that the $2^{n+1} - 1$ points of PG$(n, 2)$ are just $(n+1)$-dimensional non-zero binary vectors $(\beta_1, \ldots, \beta_{n+1})$. Using the lexicographic ordering of the points, let us represent each subspace $a \in L^n$ by its characteristic vector, i.e. by the $(2^{n+1} - 1)$-dimensional binary vector $(\alpha_{2^{n+1}-1}, \ldots, \alpha_1)$, where α_i corresponds to the i^{th} point of PG$(n, 2)$.

For two subspaces $a, b \in L^n$, we say that a is greater than b in the order $\mathcal{O}$ if the characteristic vector of a is greater than the one of b in the lexicographic order. Such an ordering is shown in Fig. 18a for PG$(2, 2)$, where the points are represented by their lexicographic numbers.

Now for $t > 0$ and $A \subseteq L_t^n$ denote

$$\hat{\Delta}(A) = \{x \in L_0^n \mid x \le y,\ y \in A\}$$

and consider the SMP for the levels L_t^n and L_0^n.

Theorem 25 (Bezrukov, Blokhuis [19]). *Let $n \ge 1$ and $t > 0$. Then any IS of the order $\mathcal{O}_t$ has minimal shadow $\hat{\Delta}(\cdot)$. The shadow $\hat{\Delta}(\cdot)$ of any IS is an IS itself.*

It is known that if one considers the set of hyperplanes of PG$(n, 2)$ as a collection of points, then it is possible to construct a new geometry PG$'(n, 2)$ on this set. It is also known that the two geometries PG$(n, 2)$ and PG$'(n, 2)$ are isomorphic. This implies that L^n is isomorphic to its dual. Thus, it is possible to extend Theorem 25 for the SMP with respect to the levels $n - 1$ (the level of hyperplanes) and $t < n - 1$. The general result concerning the dual of a Macaulay poset (cf. Proposition 1) provides an order $\mathcal{O}'$ of hyperplanes, each initial segment of which has minimal shadow $\hat{\Delta}(\cdot)$ in L_t^n. It is worth to mention that the orders $\mathcal{O}_{n-1}$ and $\mathcal{O}'$ are different in general, while a similar construction in the Boolean lattice leads to two isomorphic (namely lexicographic) orders.

Counterexamples show that none of the orders $\mathcal{O}$ and $\mathcal{O}'$ is Macaulay. Nevertheless, let us mention an interesting phenomenon in the case $n = 3$. In this case the order $\mathcal{O}'$ works for the minimization of $\Delta(\cdot)$ for $t = 2$ and the order $\mathcal{O}$ works the for minimization of $\Delta(\cdot)$ for $t = 1$. Moreover, both orders work for the minimization of $\hat{\Delta}(\cdot)$ for $t = 2$. However, as it is shown in [19], there is no universal order, which would provide the Macaulayness of L^3.

5 Related problems and applications

In this section we will be concerned with some optimization problems for which solutions are known for a rich class of Macaulay posets.

Let P be a poset, and let $I\!R^+$ denote the set of nonnegative real numbers. Furthermore, let there be a *weight function* $w : P \mapsto I\!R^+$ on P. If $w(x) = w(y)$ whenever $r(x) = r(y)$, the function $w(\cdot)$ is called *rank-symmetric*. If $w(\cdot)$ is a rank-symmetric weight function and $w(x) \le w(y)$ whenever $r(x) < r(y)$, then $w(\cdot)$ is called *monotone*. Now define the weight of a subset $X \subseteq P$ as $w(X) = \sum_{x \in X} w(x)$.

5.1 Generated ideals of minimum weight

Consider the problem of constructing an antichain $X \subseteq P$ of given cardinality $m \le d(P)$ such that the ideal generated by X has minimum weight for some monotone weight function.

This problem was considered by Frankl [57] for the Boolean lattice. For chain products, the problem was solved by Clements [36] who generalized preliminary results of Kleitman [64] and Daykin [49]. A further generalization is due to Engel [52] who provided a solution for the class of Macaulay posets P such that P and P^* are graded, additive, and shadow increasing. Unfortunately, the subword order $SO(2)$ is not included in this class since its dual is not shadow increasing (see Section 4.4). Therefore, Engel and Leck [54] gave the following strengthening which applies to the classical Macaulay posets as well as to $SO(2)$.

Theorem 26 (Engel, Leck [54]). *Let P be a Macaulay poset such that P and P^* are weakly shadow increasing. Furthermore, let $m \leq d(P)$ be a positive integer, and put $i := \min\{j \mid m \leq |P_j|\}$ and $a := \min\{b \mid b + |P_{i-1}| - |\Delta(C(b, P_i))| = m\}$. Then the set*

$$X := C(a, P_i) \cup (P_{i-1} \setminus \Delta(C(a, P_i)))$$

is an antichain of size m. Moreover, $w(I(X)) \leq w(I(Y))$ holds for all antichains $Y \subseteq P$ with $|Y| = m$ with respect to any monotone weight function.

This theorem provides a sufficient condition for a poset to be Sperner (cf. [54] for details).

Corollary 2 *Let P be a Macaulay poset such that P is not an antichain. If P and P^* are weakly shadow increasing, then P is graded and has the Sperner property, i.e. the width of P is equal to $\max_i |P_i|$.*

What if the set X is not necessarily an antichain? If $X \subseteq P_i$ for some i then, obviously, the best choice for X is $X = C(|X|, P_i)$. It turns out that a similar approach is well applied if X can be chosen from a couple of consecutive levels of P.

Theorem 27 (Bezrukov, Heijnen [24]). *Let P be a rank-greedy poset with a monotone weight function and let $P_{i,j} = P_i \cup P_{i+1} \cup \cdots \cup P_j$ for some fixed i, j with $0 \leq i \leq j \leq r(P)$. Then the minimum weight ideal generated by a subset $X \subseteq P_{i,j}$ of a fixed size is obtained for $X = C(|X|, P_{i,j})$.*

An interesting question arises if we combine the claims for X in Theorems 26 and 27. Namely, which antichains of $P_{i,j}$ generate ideals of minimum weight? The question is nontrivial even for $j = i + 1$.

5.2 Ideals with the maximum number of maximal elements

Now consider a problem dual to the last one. Namely, we are looking for an ideal of a given size, which has the maximum number of maximal elements. In order to present a solution to this problem, we first introduce quasispheres. A *quasisphere* of size m in a ranked poset P is a set of the form

$$P_0 \cup P_1 \cup \cdots \cup P_i \cup C(a, P_{i+1}),$$

where the numbers a and i are (uniquely) defined by $m = \sum_{j=0}^{i} |P_j| + a$, $0 \leq a < |P_{i+1}|$. Obviously, any quasisphere is an ideal.

Theorem 28 (Engel, Leck [54]). *Let P be a Macaulay poset such that P and P^* are weakly shadow increasing. Then a quasisphere of size m has the maximum number of maximal elements in the class of all ideals of size m in P.*

Clearly, the set of maximal elements of some ideal is an antichain. For Boolean lattices, a related problem was considered by Labahn [71]. He determined the maximum size of an antichains X such that the ideal generated by X contains exactly m elements of P_i.

5.3 Maximum weight ideals

Now consider the problem of finding an ideal $I^* \subseteq P$ such that $w(I^*) \geq w(I)$ for any other ideal $I \subseteq P$ with $|I| = |I^*|$. We call this problem the Maximum Weight Ideal problem (MWI for brevity). Denote $w_i = w(x)$ for any $x \in P_i$.

The MWI problem is closely related to the edge-isoperimetric problems (cf. Section 3.2 and [14, 17] for more details) and was first considered by Bernstein and Steiglitz in [7] for the Boolean lattice and applied to a problem in coding theory.

Theorem 29 (Bernstein, Steiglitz [7]). *If $\preceq$ is a lexicographic order, then for any $m = 0, \ldots, 2^n$ the set $C(m, B^n)$ is a solution to the MWI problem for B^n with respect to any monotone weight function.*

This result can be well applied, for example, to the Boolean lattice for constructing ideals which contain the maximum number of subcubes of a fixed dimension t [31, 32]. To see this, note that the number of subcubes in question contained in an ideal I equals $\sum_{v \in I} \binom{r(v)}{t}$, and, thus, depends just on the cardinalities of the subsets $I \cap P_i$, $i = 0, \ldots, r(P)$, rather than on the subsets themselves. Therefore, the sets $C(m, B^n)$ contain also the maximum number of subcubes of all dimensions.

Clements and Lindström in [43] extended Theorem 29 to the chain products in the case $w_i = i$ for all i, where a similar solution with respect to the lexicographic order was obtained by using Theorem 5. It turns out that the MWI problem is a direct consequence of the shadow minimization problem, as presented in the following theorem (see [9, 52]).

Theorem 30 *Let $(P, \leq, \preceq)$ be a rank-greedy Macaulay structure with a monotone weight function. Then the set $C(m, P)$ is a solution to the MWI problem for P.*

What if the weight function is not monotone? It is easily seen that if $w_0 \geq w_1 \geq \cdots \geq w_n$ then a solution to the MWI problem is attained on a quasisphere for any ranked poset P. For some less trivial nonmonotone weight functions a solution to the MWI is known for the Boolean lattice.

Theorem 31 (Ahlswede, Katona [5]). *Consider the Boolean lattice and let $\preceq$ be the lexicographic order.*

a. If $w_0 \leq w_1 \leq \cdots \leq w_{i-1} \geq w_i \geq \cdots \geq w_n$, then a solution to the MWI problem is attained on an intersection of $C(m', B^n)$ with a quasisphere for some $m' \leq m$.

b. If $w_0 \geq w_1 \geq \cdots \geq w_{i-1} \leq w_i \leq \cdots \leq w_n$, then a solution to the MWI problem is attained on a union of $C(m', B^n)$ with a quasisphere for some $m' \leq m$.

However, the proof technique of [5] is based on manipulations with binomial coefficients and is hardly extendable to other posets. Bezrukov and Voronin in [27] proposed a new approach to this problem which significantly explores the Macaulay property. They showed that a similar result holds for the chain products. We conjecture that the arguments of [27] can be well applied to any rank-greedy Macaulay poset to show that the sets described in Theorem 31 give a solution to the MWI problem.

Note that the methods of neither [5] nor [27] provide exact values of m'. The corresponding results describe the situation just qualitatively and only ensure that such m' does exist. We guess that the approach of [27] can be extended to qualitatively describe maximum weight ideals for any rank-symmetric weight function, at least for the Boolean lattice and the products of chains. We leave the development of this question to interested readers.

Let us go back to Theorem 30. Evidently, the MWI and the SMP are closely related. The principal question is what we should suppose about the solutions to the MWI problem in order to deduce the Macaulayness of the corresponding poset? In other words, assume there exists a total order $\mathcal{L}$ on the poset P, each IS of which provides a solution to the MWI problem (i.e. we have nestedness in the MWI), is it true that the order $\mathcal{L}$ is Macaulay on P ? The interest in this question is explained by constructions for Macaulay posets from the EIP approach (cf. Section 3.3). If the EIP has nested solutions for some graph G, then the MWI problem for the corresponding poset P has nested solutions with respect to the rank function taken as the weight function (i.e. if $w_i = i$ for all i). In this case we refer to such MWI problem as to the *Maximum Rank Ideals* (MRI) problem. It is shown in [17, 18] that the EIP on G^n for $n \geq 2$ has nested solutions, iff so has the MRI on P^n.

Surprisingly enough the nestedness in the MRI problem on a poset P does not imply the Macaulayness of P in general. It is proved in [23] that the lexicographic order provides a solution to the MRI problem on any cartesian power of the poset P shown in Fig. 19. However, it can be shown that for any $n \geq 2$ the poset P^n is not Macaulay. This provides another counterexample to a conjecture in [50] concerning the products of complete posets.

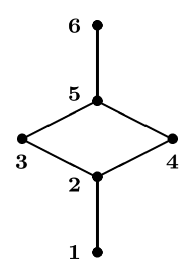


Figure 19: A Macaulay poset any cartesian power of which is not Macaulay

It follows from the above discussion that the SM problem is, in a sense, a more difficult problem than the MWI. Let us try to find a weaker problem that would imply the MWI, too.

Consider the problem of finding a subset $A \subseteq P_k$ such that $|I(A) \cap P_0|$ is minimal among all subsets of P_k of the same size. This problem is solved for the linear lattice (cf. Section 4.5), although this poset is not Macaulay. On the other hand, if P is Macaulay, then, obviously, the IS of the Macaulay order restricted to P_k provides a solution to the new problem. Therefore, this problem is, in a sense, strictly weaker, than the SMP. It would be interesting to investigate under which conditions the nestedness in the new problem implies the nestedness in the MWI.

Another interesting problem, which is closely related to the MWI is to construct a maximum weight ideal among the set of all ideals I with $|I \cap P_0| = m$ for a fixed m. If P is Macaulay then any rank-greedy Macaulay order does the job and provides nestedness in this problem. What if we do not know if P is Macaulay? What are the relations between this problem and the one we considered in the last paragraph?

5.4 Some computational problems

The Kruskal-Katona theorem was intensively used by Korshunov [67] (see also [93]) to compute the asymptotics for the number of Boolean functions on n variables. This problem was posed by Dedekind in 1897. Remember that a Boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ is called *monotone* if $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, f_n)$ whenever $x_i \leq y_i$ for $i = 1, \ldots, n$.

To demonstrate the connection to the SMP, assume that the values of f are determined on levels $B_0^n, \ldots, B_k^n$ for some k. How many ways are there to extend this function to the level B_{k+1}^n preserving its monotonicity? Let $A \subseteq B_k^n$ be the set of vertices where f takes the value 1. Then, due to the monotonicity, f takes the value 1 at any vertex of $\nabla(A)$. The values of f at the vertices of $B_k^n \setminus \nabla(A)$ can be defined arbitrarily keeping f monotone. Thus, there are $2^{\binom{n}{k+1} - |\nabla(A)|}$ possibilities to extent f for B_{k+1}^n. This provides an approach for computing an upper bound for the number of monotone functions used in [67], the most difficult part of the Dedekind's problem.

A similar approach can be well used for a number of related problems, even if they look quite differently. Let us consider the problem of computing the number of binary codes with code distance 2. This problem is among the central problems in coding theory. Korshunov and Sapozhenko in [68] used the poset Q^n (cf. Section 2.1) to derive an asymptotic formula for this number. Let $C \subset B^n$ be a code with distance 2, and let $C_i = C \cap Q_i^n$, $i = 0, 1$. Then $C_1 \cap \nabla(C_0) = \emptyset$. Therefore, for any choice of a set C_0 (note that C_0 is a code with distance 2) there are exactly $2^{|Q_1^n| - |\nabla(C_0)|} = 2^{2^{n-1} - |\nabla(C_0)|}$ possibilities to extend C_0 on B^n preserving the code distance. Thus, we come to the problem of estimating $\min_{|C_0|=m} |\nabla(C_0)|$, which is equivalent to the estimation of $\min_{|C_1|=m'} |\nabla(C_1)|$ and is done in Theorem 4. Therefore, the number of codes in question is

$$2^{2^{n-1}} \sum_{C_0 \subseteq Q_0^n} 2^{-|\nabla(C_0)|}. \tag{7}$$

In order to estimate the sums of the type (7) in arbitrary posets with two levels (satisfying certain conditions), Sapozhenko develops the so-called *method of boundary functionals*, which he applied to a number of computational problems [91, 92, 93]. The SMP plays an important role for this method. The method is well applicable for the Dedekind problem, for computing the number of antichains in ranked posets, for computing the number of pairs of subsets of B^n at distance 2, and to many other problems.

Alon in [6] used a similar approach to derive an upper bound for the number of independent sets in bipartite graphs. Suppose X and Y are two vertex-classes of a bipartite graph G. For a set $A \subseteq X$ let $N(A)$ denote the set of its neighbors in Y. The set A is called an *s-set* if $|N(A)| = s$. Denote by $I(s,t)$ the number of s-sets of size t. Then then total number of independent sets in G is

$$\sum_{t=0}^{|X|} \sum_{s=0}^{|Y|} I(s,t) \cdot 2^{|Y|-s}.$$

Thus, we have the problem of estimating a sum of the type (7). The bipartite graph G can be represented as a poset with two levels corresponding to the sets X and Y. Then $N(A)$ corresponds to the shadow function. The Kruskal-Katona theorem was used in [6] to estimate the sum above.

5.5 Separation in graphs

Let G and H be graphs with $|V_G| \leq |V_H|$. A *separator* of G in H is any bijective mapping $f : V_G \mapsto V_H$. The *minimum distance* of a separator f is defined as $|f| = \min\{\mathrm{dist}_H(f(x), f(y)) \mid (x,y) \in E_G\}$. Now define $\mathrm{sep}(G,H)$, the *separation of G in H*, as the maximum of $|f|$ over all separators f.

The number $\mathrm{sep}(G,H)$ is considered for a number of graphs in the literature. In particular, the case $G = K_p$ and $H = B^n$ is interesting for the theory of binary error-correcting codes. A lot of results on the separation in graphs can be found in the survey [83].

Miller and Pritikin used the Kruskal-Katona theorem in [83] to compute the separation of complete bipartite graphs in the n-cube, i.e., the number $\mathrm{sep}(K_{r,s}, B^n)$. They used a result of Frankl and Füredi [58] to conclude that among all separators $f : K_{r,s} \mapsto B^n$, the maximum value of $|f|$ is achieved when one of the independent sets of $K_{r,s}$ is mapped to a Hamming ball about some vertex α of B^n and the other one is mapped to a Hamming ball about the complement of α.

Let X and Y be the independent sets of $K_{r,s}$, and let $\alpha = (0, \ldots, 0)$. Represent the numbers r and s in the form $r = \sum_{i=0}^{k'} \binom{n}{i} + \delta'$ and $s = \sum_{i=0}^{k''} \binom{n}{i} + \delta''$ with $0 \leq \delta' < \binom{n}{k'+1}$ and $0 \leq \delta'' < \binom{n}{k''+1}$. Then $f(X)$ consists of all vertices of B^n of rank at most k' together with δ' vertices of rank $k'+1$ (denote this set by X'), and $f(Y)$ consists of all vertices of B^n of rank at least $n - k''$ together with some vertices of rank $n - k'' - 1$ (the set Y'). A nontrivial case occurs if $X \neq \emptyset$ and $Y \neq \emptyset$. In this case by using the Kruskal-Katona theorem it is shown in [83] that to maximize $|f|$ one can choose $X' = L(\delta', B^n_{k'+1})$ (with respect to the lexicographic order) and $Y' = C(\delta'', B^n_{n-k''-1})$. This leads to an exact formula for $\mathrm{sep}(K_{r,s}, B^n)$. A similar approach was used in [11] to compute the maximum size of a subset of B^n with a fixed diameter.

Now consider a dual problem: find two subsets A, B in a chain product such that $\max_{a\in A,\, b\in B} \mathrm{dist}(a,b)$ is minimum among all subsets of the same size. Note that we allow the sets A and B to intersect. Interesting enough that this problem has principally different solutions for the Boolean lattice and the chain products with all the chains of even length. Kleitman and Schulman showed in [65] that for the Boolean lattice the best choices for A and B are initial segments of the lexicographic order of vectors $(r(\mathbf{x}), -x_1, \ldots, -x_n)$, where $\mathbf{x} = (x_1, \ldots, x_n) \in B^n$. In other terms, this order is nothing else than the VIP order introduced by Harper in [60].

What concerns the chain product, the same authors [65] used the Clements-Lindström method to show that the best choices for A and B are initial segments of the following order $\mathcal{O}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in S(k_1, \ldots, k_n)$ with $-c_i \le x_i \le c_i$ denote by $(\sigma_1, \ldots, \sigma_n)$ a binary vector with $\sigma_i = 1$ iff $x_i > 0$, $i = 1, \ldots, n$. The order $\mathcal{O}$ is the lexicographic order of the vectors $(r(\mathbf{x}), -\sigma_1, \ldots, -\sigma_n)$

5.6 Network reliability

Suppose we have a network which we represent by a graph $G = (V_G, E_G)$, and let s and t be two distinct vertices of G. Assume that each edge of G is operational with a probability p, and failed with probability $q = 1 - p$. Define the *two-terminal connection probability* for the nodes s, t to be the probability that operational edges include an s,t-path, or, equivalently, that the failed edges do not include an s,t-cut. Denote by F_i the number of sets of i edges which do not contain an s,t-cut, and let $e = |E_G|$. Then $F_i q^i p^{e-i}$ is the probability that exactly i edges fail and the network remains operational. Therefore, the two-terminal connection probability $\mathrm{CP}(p)$ satisfies

$$\mathrm{CP}(p) = \sum_{i=0}^{m} F_i q^i p^{e-i}.$$

Let l be the length of the shortest s,t-path, and denote by c the size of the minimum s,t-cut. Then $F_i = 0$ for $i > e - l$ and $F_i = \binom{c}{i}$ for $i < c$.

The Kruskal-Katona theorem implies [66] that if $F_k = m$, then $F_i \ge m^{(i/k)}$ for $i \le k$, and $F_i \le m^{(i/k)}$ for $i \ge k$ (cf. Section 2.1 for the definition of $m^{(i/k)}$). Using these inequalities, we obtain the *Kruskal-Katona bounds*

$$\begin{aligned}
\mathrm{CP}(p) &\ge \sum_{i=0}^{c-1} \binom{e}{i} p^{e-i} q^i + \sum_{i=c}^{d} F_d^{(i/d)} p^{e-i} q^i, \\
\mathrm{CP}(p) &\le \sum_{i=0}^{c-1} \binom{e}{i} p^{e-i} q^i + \sum_{i=c}^{d-1} \left(\binom{e}{c} - 1 \right)^{(i/c)} p^{e-i} q^i + F_d p^{e-d} q^d,
\end{aligned}$$

where $d = e - l$. For further bounds and more information we refer to [33] and to the excellent book of Colbourn [44].

5.7 Algebra and combinatorial topology

Let M be a collection of monomials in the variables $x_1, x_2, \ldots$. The set M is called an *order ideal of monomials* if whenever $v \in M$ and a monomial u divides v then $u \in M$. Let $m_k(M)$ be the number of monomials in M or degree k, and assume that m_k is represented in the form (2). The theorem of Macaulay [81] says that $(m_0, m_1, \ldots)$ is the m-vector of some order ideal of monomials iff $m_0 = 1$ and $m_{k-1} \geq \binom{a_k-1}{k-1} + \cdots + \binom{a_t-1}{t-1}$ for $k \geq 1$.

Now let $\mathcal{C}$ be a *simplicial complex* on the vertex set $V = \{x_1, \ldots, x_p\}$, i.e., $\mathcal{C}$ is a collection of subsets of V such that $\{x_i\} \in \mathcal{C}$, $i = 1, \ldots, p$, and if $\sigma \in \mathcal{C}$ and $\tau \subseteq \sigma$ then $\tau \in \mathcal{C}$. The elements of $\mathcal{C}$ are called *faces*. A *facet* of $\mathcal{C}$ is a face σ such that $\tau \in \mathcal{C}$ and $\sigma \subseteq \tau$ imply $\sigma = \tau$. Furthermore, let $f_i = f_i(\mathcal{C})$ be the number of faces $\sigma \in \mathcal{C}$ with $|\sigma| = i + 1$. Similarly, let $n_i = n_i(\mathcal{C})$ be the number of facets $\tau \in \mathcal{C}$ with $|\tau| = i + 1$.

In other terms, a simplicial complex $\mathcal{C}$ corresponds to an ideal I in the Boolean lattice, and the set of the facets of $\mathcal{C}$ corresponds to the generating antichain of I. Therefore, for a simplicial complex $\mathcal{C}$ with f-vector $(f_0, f_1, \ldots)$, the condition $f_i^{(i-1/i)} \leq f_{i-1}$ for all $i \geq 1$ is a necessary and sufficient condition for $\mathcal{C}$ to exist (here $f_i^{(i-1/i)}$ is the pseudopower defined in Section 2.1). The complex for which this bound is attained is called *Kruskal-Katona complex*.

Herzog and Hibi in [61] used the Kruskal-Katona theorem to derive an upper bound for the numbers n_i of the form $n_{i-1} \leq f_{i-1} - f_i^{(i-1/i)}$. They also introduced the concept of *j-facets* of $\mathcal{C}$. A face $\sigma \in \mathcal{C}$ is called j-facet if j is equal to the largest integer k for which there exists a face $\tau \in \mathcal{C}$ such that $\sigma \cap \tau = \emptyset$ and $\sigma \cup \tau \in \mathcal{C}$. Let n_i^j denote the number of j-facets of $\mathcal{C}$ of size $i - 1$. It is shown in [61] that if $\mathcal{C}$ is a simplicial complex and $\tilde{\mathcal{C}}$ is the Kruskal-Katona complex with the same f-vector, and if $n_i^0 + \cdots + n_i^{j-1} = \tilde{n}_i^0 + \cdots + \tilde{n}_i^{j-1}$ for all $i \geq 0$, then $n_i^j \leq \tilde{n}_i^j$ for all $i \geq 0$.

Wegner in [99] extended the Kruskal-Katona theorem to c-semilattices. A *c-semilattice* is a ranked poset $(P, \leq)$ such that for any two elements $x, y \in P$ there exists a minimum $x \wedge y \in P$, and for any two compatible elements $x, y \in P$ with $r(y) - r(x) \geq 2$ there exist at least two elements $z_1, z_2 \in P$ with $x < z_1, z_2 < y$. Any simplicial complex is a c-semilattice, but not vice versa. Denote $f_i = |P_i|$. Wegner proved that if $(f_0, f_1, \ldots)$ is the f-vector of a c-semilattice, then $f_i^{(i-1/i)} \leq f_{i-1}$ for all $i \geq 1$, thus, extending the Kruskal-Katona theorem.

For more information on the topic, and for further applications we refer to [28, 30], and to the books of Stanley [94] and Ziegler [100].

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