Steiner systems $S(2, 4, v)$ - a survey

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Abstract

We survey the basic properties and results on Steiner systems $S(2, 4, v)$, as well as open problems.

1 Introduction

A *Steiner system* $S(t, k, v)$ is a pair $(V, \mathcal{B})$ where V is a v-element set and $\mathcal{B}$ is a family of k-element subsets of V called *blocks* such that each t-element subset of V is contained in exactly one block. For basic properties and results on Steiner systems, see [51].

By far the most popular and most studied Steiner systems are those with $t = 2$ and $k = 3$, called *Steiner triple systems* (STS). There exists a very extensive literature on STSs (see, eg., [54] and the bibliography therein). Steiner systems with $t = 3$ and $k = 4$ are known as *Steiner quadruple systems* and are commonly abbreviated as SQS; there exist (at least) two extensive surveys on SQSs (see [122], [95]); see also [51]). When $t = 2$, one often speaks of a *Steiner* 2*-design*.

Quite a deal of attention has also been paid to the case of Steiner systems $S(2, 4, v)$, but to the best of our knowledge, no survey of known results and problems on these Steiner 2-designs with block size 4 has been published. It is the purpose of this article to fill this gap by bringing together known results and problems on this quite interesting class of Steiner systems. No "catchy" name appears to be in circulation about Steiner systems $S(2, 4, v)$, as opposed to the case of STSs or SQSs.

2 Existence of Steiner systems $S(2, 4, v)$

Since every pair of distinct elements of an $S(2, 4, v)$ is contained in a unique block, and each block contains six such pairs, the total number of pairs $\binom{v}{2}$ must be divisible by 6, and thus 12 must divide $v(v-1)$. Also, any element occurs in a block with three further elements, so the total number of other elements, $v-1$, must be divisible by three. These two conditions imply that $v \equiv 1$ or $4 \pmod{12}$ is necessary for an $S(2,4,v)$ to exist. Hanani [92] was the first to prove the sufficiency of this condition.

Theorem 2.1. *A Steiner system $S(2,4,v)$ exists if and only if $v \equiv 1$ or $4 \pmod{12}$.*

The system of order 1 (with one element and no blocks) and system of order 4 (with four elements and one block) are *trivial systems*. All systems of order at least 13 are *nontrivial*.

In 1939, Bose [15] gave a construction of two infinite classes of $S(2,4,v)$s using finite fields (cf. also [91], Theorems 15.3.5 and 15.3.6).

Theorem 2.2. *Let $v = p^n = 12t+1$, p prime, and if α is a primitive root of $GF(p^n)$ such that $\alpha^{4t} - 1 = \alpha^q$ for some odd q, then the blocks*
$\{0, \alpha^{2i}, \alpha^{4t+2i}, \alpha^{8t+2i}\}$, $i = 0, 1, \dots, t-1$
are the base blocks for an $S(2,4,v)$.

Theorem 2.3. *Let $4t+1 = p^n$, p prime, and let α be a primitive root of $GF(p^n)$. Then there exists a pair of integers c, q such that $(\alpha^c+1)/(\alpha^c-1) = \alpha^q$. Let $V = GF(p^n) \times \{1,2,3\} \cup \{\infty\}$. Then the blocks*
$\{{\alpha_1}^{2i}, {\alpha_1}^{2t+2i}, {\alpha_2}^{2i+c}, \alpha_2^{2t+2i+c}\}$
$\{{\alpha_2}^{2i}, {\alpha_2}^{2t+2i}, {\alpha_3}^{2i+c}, \alpha_3^{2t+2i+c}\}$
$\{{\alpha_3}^{2i}, {\alpha_3}^{2t+2i}, {\alpha_1}^{2i+c}, \alpha_1^{2t+2i+c}\}$, $i = 0, 1, \dots, t-1$
$\{\infty, 0_1, 0_2, 0_3\}$
are the base blocks with respect to the additive group of $GF(p^n)$ of an $S(2,4,12t+4)$.

One of the most frequently used recursive constructions is the so-called

"$v \to 3v+1$ **rule**". Let $(V, \mathcal{B})$ be an $S(2,4,v)$, and let X be a set such that $|X| = 2v+1$, $X \cap V = \emptyset$. Let $(X, \mathcal{C})$ be a resolvable STS$(2v+1)$ and let $\mathcal{R} = \{R_1, \dots, R_v\}$ be a resolution of $(X, \mathcal{C})$, that is, let $(X, \mathcal{C}, \mathcal{R})$ be a Kirkman triple system of order $2v+1$; since $v \equiv 1, 4 \pmod{12}$, such a system exists. Form the set of quadruples $D_i = \{\{v_i, x, y, z\} : v_i \in V, \{x,y,z\} \in R_i\}$, and put $\mathcal{D} = \bigcup_i D_i$. Then $(V \cup X, \mathcal{B} \cup \mathcal{D})$ is an $S(2,4,3v+1)$.

Another construction that deserves mention is the

"direct product rule". If there exists an $S(2,4,v)$ and an $S(2,4,w)$ then there exists an $S(2,4,v.w)$.

If $(V, \mathcal{B})$ and $(W, \mathcal{C})$ are two such systems, form an $S(2,4,v.w)$ on the set $V \times W$; if $B = \{p,q,r,s\}$ and $B' = \{x,y,z,u\}$ are two blocks from $\mathcal{B}$ and $\mathcal{C}$ respectively, form an $S(2,4,16)$ on the set $B \times B'$, making sure that $\{(p,j),(q,j),(r,j),(s,j)\}$ and $\{(i,x),(i,y),(i,z),(i,u)\}$ are blocks, for $i,j = 1,2,3,4$.

Hanani's proof of the existence of $S(2,4,v)$s as given in [92] (cf. also [93]) is inductive. As the induction basis, one constructs directly systems of small orders (i.e., of orders $13, 16, 25, 28, 37$). Then one uses a pairwise balanced design of index one, PBD$(u, K, 1)$, where $K = \{4,5,8,9,12\}$; such a PBD exists for all $u \equiv 0, 1 \pmod 4$ (see [51]). If $(X, \mathcal{B}$

is such a PBD, one takes the $(3v+1)$-set $V = X \times \{1,2,3\} \cup \{\infty\}$; for each block $B \in \mathcal{B}$, construct an $S(2,4,3|B|+1)$ on the set $B \times \{1,2,3\} \cup \{\infty\}$, making sure that for $x \in X$, $\{\infty,(x,1),(x,2),(x,3)\}$ is always a block. The result is an $S(2,4,3v+1)$.

Unfortunately, there is a marked increase in the level of difficulty as the block size increases from three to four. This is reflected among other things in the fact that to-date there does not exist a simple direct proof of the existence of $S(2,4,v)$s. By comparison, one has the wonderful Bose's and Skolem's constructions for Steiner triple systems (in their most general form, cf. [54]), and the existence proof via the cyclic STSs. A simple direct proof of the existence of $S(2,4,v)$s appears very desirable.

3 Automorphisms of Steiner systems $S(2,4,v)$

Let $(V,\mathcal{B})$ and $(W,\mathcal{C})$ be two Steiner systems $S(2,4,v)$. An *isomorphism* from $(V,\mathcal{B})$ to $(W,\mathcal{C})$ is a 1-1 mapping $\alpha : V \to W$ such that $\bar{\alpha}(\mathcal{B}) = \mathcal{C}$ (where $\bar{\alpha}$ is the mapping induced on blocks by α). Isomorphism is an equivalence relation, and the systems $(V,\mathcal{B})$ and $(W,\mathcal{C})$ are *isomorphic*. An *automorphism* of $(V,\mathcal{B})$ is an isomorphism of $(V,\mathcal{B})$ to itself.

3.1 Cyclic Steiner systems $S(2,4,v)$

A Steiner system $S(2,4,v)$ is *cyclic* if it contains an automorphism which consists of a single cycle of length v; this automorphism is called a *cyclic automorphism*. If $(V,\mathcal{B})$ is a cyclic $S(2,4,v)$, one may assume $V = Z_v$, and $\alpha : i \to i+1 \ (mod\ v)$ to be its cyclic automorphism. The blocks of an $S(2,4,v)$ are partitioned into orbits under the action of the cyclic group generated by α. Each orbit of blocks is completely determined by any of its blocks, and $\mathcal{B}$ is determined by a collection of blocks called *base* blocks (sometimes also called *starter* blocks or *initial* blocks) containing one block from each orbit.

The possible orbit lengths of blocks in a cyclic $S(2,4,v)$ are v and $\frac{v}{4}$; an orbit of length v is *full*, that of length $\frac{v}{4}$ is *short*. In a cyclic $S(2,4,v)$ with $v \equiv 1 \ (mod\ 12)$ there are $\frac{v-1}{12}$ full orbits of blocks (and no short orbit) while if $v \equiv 4 \ (mod\ 12)$, there are $\frac{v-4}{12}$ full orbits and one short orbit.

Any set of base blocks of a cyclic $S(2,4,v)$ with $v \equiv 1 \ (mod\ 12)$ yields a $(v,4,1)$-*difference family*; the latter is a collection of 4-element sets $D_1, D_2, \ldots, D_n$ such that each nonzero residue modulo v can be written in a unique way as $d_i - d_j$ with $d_i, d_j \in D_m$ for some m. Conversely, any $(v,4,1)$-difference family forms a set of base blocks of a cyclic $S(2,4,v)$ (with $v \equiv 1 \ (mod\ 12)$).

In a similar manner, for v divisible by 4, a *modified* $(v,4,1)$-*difference family* is a collection of 4-element sets $D_1, D_2, \ldots, D_n$ such that each nonzero residue modulo v except for $\pm\frac{v}{4}$ and $\frac{v}{2}$ can be written in a unique way as $d_i - d_j$ with $d_i, d_j \in D_m$ for some m. Clearly, adjoining the base block $\{0, \frac{v}{4}, \frac{v}{2}, \frac{3v}{4}\}$ to a modified $(v,4,1)$-difference family results in a set of base blocks of a cyclic $S(2,4,v)$ with $v \equiv 4 \ (mod\ 12)$, and conversely, the set of base blocks of a cyclic $S(2,4,v)$ with $v \equiv 4 \ (mod\ 12)$ minus the base block corresponding to the short orbit yields a modified $(v,4,1)$-difference family.

Bose's first construction (Theorem 2.2 above) yields cyclic $\mathrm{S}(2,4,v)$s whenever $v = 12t+1$ is a *prime*. The smallest cyclic systems thus obtained are those of orders $13, 37, 61, 73, 97, \ldots$

There exists no cyclic $\mathrm{S}(2,4,v)$ for $v = 16, 25, 28$. This is established by showing the nonexistence of a $(v,4,1)$- or a modified $(v,4,1)$-difference family, as the case may be.

Further progress on the existence of cyclic $\mathrm{S}(2,4,v)$s is more recent. It is shown in [24], [45] that a cyclic $\mathrm{S}(2,4,v)$ and a cyclic $\mathrm{S}(2,4,4v)$ exists whenever v is a product of primes, each congruent to 1 modulo 12. Buratti [23] has shown that a cyclic $\mathrm{S}(2,4,4v)$ exists for any v such that any prime factor p of v is congruent to 1 modulo 6 and $gcd(\frac{p-1}{6}, 20!) \neq 1$. In [29], Buratti provided some explicit constructions of cyclic $\mathrm{S}(2,4,v)$s and also showed that there exists a cyclic $\mathrm{S}(2,4,4p)$ if $p = 6n+1$ is a prime > 7 and the least prime factor of n is not greater than 19. Chang [38] has shown that there exists a cyclic $\mathrm{S}(2,4,4^n u)$ where $n \geqslant 3$ is a positive integer and u is a product of primes, each congruent to 1 modulo 6, or $n = 2$ and u is a product of primes each congruent to 1 modulo 6 such that $gcd(u, 7.13.19) \neq 1$.

Several recursive constructions for cyclic $\mathrm{S}(2,4,v)$s similar to recursive constructions for 1-rotational $\mathrm{S}(2,4,v)$s (cf. Section 3.2 below) were obtained in [56], [103], [104].

For small orders, it has been established [2], [44] that cyclic Steiner systems $\mathrm{S}(2,4,v)$ exist for all $v \equiv 1$ or 4 $(mod\ 12)$, $v \leqslant 613$, except for $v = 16, 25, 28$, for which they do not exist.

Even though it has been conjectured that cyclic $\mathrm{S}(2,4,v)$s exist for all admissible $v \geqslant 37$, the existence question for cyclic $\mathrm{S}(2,4,v)$s remains open.

3.2 1-rotational Steiner systems S$(2,4,v)$

A Steiner system $\mathrm{S}(2,k,v)$ is *k-rotational* if it has an automorphism consisting of a fixed point and k disjoint cycles of length $\frac{v-1}{k}$. Thus a 1-rotational $\mathrm{S}(2,4,v)$, $(V,\mathcal{B})$, may be assumed to have $V = Z_{v-1} \cup \{\infty\}$, with the 1-rotational automorphism given by $i \to i+1 (mod\ v-1)$, $\infty \to \infty$. For the existence of a 1-rotational $\mathrm{S}(2,4,v)$ it is necessary that $v \equiv 4$ $(mod\ 12)$. The designs of points and lines in affine spaces $\mathrm{AG}(d,4)$ provide examples of 1-rotational $\mathrm{S}(2,4,v)$s. It follows from a construction of Moore [139] that a 1-rotational $\mathrm{S}(2,4,v)$, $v \equiv 4$ $(mod\ 12)$, exists whenever $\frac{v-1}{3}$ is a prime. Colbourn and Colbourn [50] were apparently the first to provide a recursive construction for 1-rotational $\mathrm{S}(2,k,v)$s. From a given 1-rotational $\mathrm{S}(2,4,3u+1)$ and a 1-rotational $\mathrm{S}(2,4,3v+1)$, where $gcd(u,6) = 1$, they construct a 1-rotational $\mathrm{S}(2,4,3uv+1)$. As a corollary they obtain that a 1-rotational $\mathrm{S}(2,4,v)$ exists for all $v = 3.5^n + 1$, $n \geqslant 0$. Another recursive construction for 1-rotational $\mathrm{S}(2,4,v)$s using difference matrices was given by Jimbo [101], [103]. Liaw [118], apparently unaware of the results of Colbourn, Colbourn and Jimbo, has shown that if there exists a 1-rotational $\mathrm{S}(2,4,3u+1)$ and a 1-rotational $\mathrm{S}(2,4,3v+1)$ such that 9 does not divide $v-1$ then there exists a 1-rotational $\mathrm{S}(2,4,3uv+1)$. As a corollary he obtains that there exists a 1-rotational $\mathrm{S}(2,4,v)$ whenever $v = 3p_1^{\alpha_1} \ldots p_n^{\alpha_n} + 1$ where p_i is prime, $p_i \equiv 1 (mod\ 4)$, or $p_i = \frac{4^m-1}{3}$, and there is at most one p_i such that $p_i = \frac{4^m-1}{3}$ where $m \equiv 0 (mod\ 3)$, and $\alpha_i = 1$ for this p_i. It is pointed out in [106] that for all such orders v there exists a Z-cyclically resolvable 1-rotational $\mathrm{S}(2,4,v)$ (i.e. one in which the parallel classes are generated cyclically).

It has been conjectured that a 1-rotational $\mathrm{S}(2,4,v)$ exists for all $v \equiv 4$ $(mod\ 12)$, except for $v = 28$ for which it is known not to exist, but this remains unresolved.

4 Enumeration of Steiner systems $\mathrm{S}(2,4,v)$

Let $N(v)$ be the number of nonisomorphic $\mathrm{S}(2,4,v)$s. The smallest nontrivial $\mathrm{S}(2,4,v)$ is the unique $\mathrm{S}(2,4,13)$ which is the design of points and lines of $PG(2,3)$, the projective plane of order 3. The design is cyclic, and may be presented as

$$(V, \mathcal{B}) = (Z_{13}, \{0,1,4,6\}\ mod\ 13).$$

Its full automorphism group is doubly transitive of order 5616.

The Steiner system $\mathrm{S}(2,4,16)$ is also unique; it is the design of points and lines of $AG(2,4)$, the affine plane of order 4. The design is not cyclic but is 1-rotational, and is doubly transitive as well. Its full automorphism group is of order 16.15.12.2.

The early papers towards enumeration of $\mathrm{S}(2,4,v)$ for the next two orders 25 and 28 include [150], [151], [166], [17], [19], [157], [110]. In [109] it was shown that there are exactly 16 Steiner systems $\mathrm{S}(2,4,25)$ with a nontrivial automorphism group. Spence [178] has completed the enumeration of $\mathrm{S}(2,4,25)$ by showing that there are exactly 18 nonisomorphic systems. These are listed in full in [129]; their automorphism groups have orders $504, 150, 63, 21, 9$ (three times), $6, 3$ (eight times), and 1 (twice).

The exact number of systems $\mathrm{S}(2,4,28)$ is not known at present. But Krčadinac [108] has shown recently that there are exactly 4466 nonisomorphic $\mathrm{S}(2,4,28)$s with a nontrivial automorphism group. Together with some $\mathrm{S}(2,4,28)$s with trivial automorphism group found in [12], this implies that there are at least 4653 nonisomorphic $\mathrm{S}(2,4,28)$s.

These results, together with some known bounds for systems of other small orders, are summarized in the table below. The lower bound for $N(37)$ is from [112], the other values are from [129].

v	13	16	25	28	37	40	49
$N(v)$	1	1	18	$\geqslant 4653$	$\geqslant 51402$	$\geqslant 1108800$	$\geqslant 769$

v	52	61	64
$N(v)$	$\geqslant 206$	$\geqslant 18132$	$\geqslant 14.10^{30}$

Concerning $N(v)$ when v is large, Doyen was the first to show that $N(v)$ tends to infinity with v. In [62] he obtained an inequality which, when applied to $\mathrm{S}(2,4,v)$s, yields

$$N(v) \leqslant 2^{v^2 log_2 v}.$$

Some additional enumeration results on restricted classes of $\mathrm{S}(2,4,v)$s deserve mention. Colbourn and Mathon [57] have enumerated small cyclic $\mathrm{S}(2,4,v)$s. As already mentioned, there is a unique S$(2,4,13)$ which is cyclic, and there is no cyclic S$(2,4,v)$ for $v = 16, 25$ or 28. The number of cyclic S$(2,4,v)$s for $v = 37, 40, 49, 52, 61$, and 64 is $2, 10, 224, 206, 18132$, and 12048, respectively.

Let $NR(v)$ be the number of nonisomorphic resolutions of S$(2,4,v)$s (cf. Section 10 below). Very recently, Kaski and Östergård [108] established $NR(28) = 7$, i.e. there are exactly 7 nonisomorphic resolutions of S$(2,4,28)$. Each is coming from a different system except that the so-called Rees unital admits two nonisomorphic resolutions which means that there are exactly 6 nonisomorphic resolvable S$(2,4,28)$s. All these resolutions have been known previously (cf., e.g., [129]), but in [108] it is shown that there are no others.

Krčadinac [112] has enumerated all S$(2,4,37)$s with an automorphism of order 11; there are exactly 284 of these. [In the same paper, he obtained the above mentioned lower bound on $N(37)$.] Gropp [89] has established that there are exactly 226 nonisomorphic abelian S$(2,4,49)$.

The complexity of the isomorphism problem for S$(2,4,v)$s is unknown. It follows from [6] that isomorphism of S$(2,4,v)$s can be tested in subexponential time (cf. also [55]).

5 Steiner systems S$(2,4,v)$ and algebras

The close relationship, indeed an equivalence, between Steiner triple systems and idempotent totally symmetric quasigroups (sometimes called Steiner quasigroups) is well known (cf. [78], [54]). In what follows we describe the relationship of Steiner systems S$(2,4,v)$ to another class of algebras, namely to *Stein quasigroups*.

In the literature one finds described two distinct types of quasigroups, both termed Stein quasigroups. The first kind of these, which appears more common, is defined as a quasigroup $(V, \circ)$ satisfying the law

$$x \circ (x \circ y) = y \circ x. \qquad (1)$$

This is *Stein's law*, sometimes called the *first Stein's law* (cf. [59]), or *law of semisymmetry.*

It is easy to see that Stein quasigroups (S-quasigroups for short) are idempotent, anticommutative (i.e, $x \circ y \neq y \circ x$ for $x \neq y$), and nonassociative (cf. [200]).

An example of an S-quasigroup of order 4 is

$\circ$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

The second kind of Stein quasigroups (cf., e.g., [78], [200]) are those satisfying the laws

$x \circ x = x$

$(x \circ y) \circ y = y \circ x$

$(y \circ x) \circ y = x$.

(The third of these laws is called in [59] the *law of left semisymmetry.*) We will call quasigroups satisfying these laws S^**-quasigroups.*

It can be shown that each S^*-quasigroup is an S-quasigroup, but the converse need not hold. However, the S-quasigroup of order 4 given above is also an S^*-quasigroup (cf. [200]).

Given an S^*-quasigroup $(V,\circ), |V| = v$, define

$\mathcal{B} = \{\{x, y, x \circ y, y \circ x\} : x, y \in V, x \neq y\}$.

Then $(V, \mathcal{B})$ is an $\mathrm{S}(2,4,v)$.

Conversely, given an $\mathrm{S}(2,4,v)$, $(V,\mathcal{B})$, we can obtain from it an S^*-quasigroup as follows. For every block $B \in \mathcal{B}$, choose arbitrarily a bijection $\phi_B : B \to \{1,2,3,4\}$. On V, define a binary operation $*$ by

$x * x = x \circ x \ (= x)$

$x * y = \phi_B^{-1}(\phi_B(x) \circ \phi_B(y))$

for $x \neq y, x, y \in B$ (where $\circ$ is a binary operation on $\{1,2,3,4\}$ as defined by the table above). Then $(V, *)$ is an S^*-quasigroup.

6 Subsystems, partial systems and embeddings

A (partial) Steiner system $\mathrm{S}(2,4,w)$, $(W,\mathcal{C})$, is a *subsystem* of a Steiner system $\mathrm{S}(2,4,v)$, $(V,\mathcal{B})$, if $W \subseteq V$ and $\mathcal{C} \subseteq \mathcal{B}$. We also say in this case that $(W,\mathcal{C})$ is *embedded* in $(V,\mathcal{B})$.

Ganter [74] and Quackenbush [152] have shown that every partial system $\mathrm{S}(2,4,w)$ can be embedded in a finite $\mathrm{S}(2,4,v)$. Their result is commonly referred to as "Ganter's Theorem". However, their proof only provides an embedding which at worst is exponential in the size of the partial system. A proof of the existence of a polynomial size embedding, as it was proved to exist in the case of partial Steiner triple systems (cf. [54]), remains elusive so far.

Theorem 6.1 ((Ganter's theorem)). *Every partial Steiner system S($2,4,w$) can be embedded in some Steiner system S($2,4,v$).*

Brouwer and Lenz [20], [21] were the first to study embeddings of $\mathrm{S}(2,4,w)$s into $\mathrm{S}(2,4,v)$s. It is easy to see that the necessary conditions for the existence of such an embedding are $w = v$ (trivial embedding) or $v \geqslant 3w+1$ (proper embedding). Wei and Zhu [190], [191] have shown the sufficiency of these conditions for all $w > 85$, $w \neq 133$, and subsequently Rees and Stinson [155] have completed the proof of what is now referred to as the Rees-Stinson Theorem.

Theorem 6.2 ((Rees-Stinson Theorem)). *Let w, v be positive integers, $w, v \equiv 1$ or 4 (mod 12). An S($2,4,w$) can be properly embedded in some S($2,4,v$) if and only if $v \geqslant 3w+1$.*

The paper [155] contains many involved constructions and is a fine example of skillful use of design-theoretic techniques. The main result of [155] actually contains a more general theorem than Theorem 6.2 above, as it also deals with embeddings of Steiner systems $\mathrm{S}(2,4,v)$ with a *hole*.

An $S(2,4,v)$ with a hole of size w (also called an *incomplete design* $\mathrm{IPBD}(v,w;\{4\})$, cf. [51]) is a triple $(V,W,\mathcal{B})$ where $W \subset V$, $\mathcal{B}$ is a set of 4-tuples of elements of V such that every 2-subset of V is contained in exactly one 4-tuple of $\mathcal{B}$ except when both elements of the 2-subset belong to W in which case this 2-subset is contained in no 4-tuple of $\mathcal{B}$. For example, if one deletes the 4-tuples of an $S(2,4,w)$ which is embedded in an $S(2,4,v)$, then an $S(2,4,v)$ with a hole of size w (or an $\mathrm{IPBD}(v,w,\{4\})$) results. However, the size of the hole is not necessarily an admissible order congruent to 1, or 4 (mod 12). If, on the other hand, the hole *has* an admissible order then *any* subsystem of that order can be put in the hole. This is often referred to as the *replacement property* of Steiner systems $S(2,4,v)$. The following is the general version of the Rees-Stinson theorem.

Theorem 6.3. *A Steiner system S($2,4,v$) with a hole of size $w < v$ exists if and only if $v \geqslant 3w+1$, and*
(i) $v \equiv w \equiv 1$ *or* 4 (mod 12), *or*
(ii) $v \equiv w \equiv 7$ *or* 10 (mod 12).

Very recently, embeddings of resolvable $S(2,4,v)$s into resolvable $S(2,4,v)$s were considered in [81] (for resolvable $S(2,4,v)$s, cf. Section 10 below). It is shown in [81] that a resolvable $S(2,4,w)$ can be embedded in a resolvable $S(2,4,v)$ if and only if $v \geqslant 4w, v \equiv w \equiv 4$ (mod 12) with 179 possible exceptions, but certainly whenever $w > 1840$ (for details, see [81]).

Another type of an embedding is that of an $S(2,4,v)$ into an $S(3,4,v)$, that is, into a Steiner quadruple system of the same order v. (The term "completion" would be more appropriate here.) For this to be possible, it is necessary that $v \equiv 4$ (mod 12). If an $S(2,4,v)$ is embedded into an $\mathrm{SQS}(v)$, the latter is said to contain a *spanning block design*. Lindner conjectured (cf. [95] or [100]) that an $\mathrm{SQS}(v)$ with a spanning block design exists for all $v \equiv 4$ (mod 12). Historically, the problem which was considered first, and whose solution would imply the existence of an SQS with a spanning block design, was that of the existence of an $\mathrm{SQS}(v)$ partitionable into $\frac{v-2}{2}$ disjoint $S(2,4,v)$s, a much stronger requirement (cf.Section 7). Semakov and Zinoviev [174] and Baker [8] have shown that such a partitionable SQS is provided by (the points and the planes of) $\mathrm{AG}(2m,2)$, the even-dimensional affine geometry over $\mathrm{GF}(2)$. Fu [68] has given a quadruplication construction for SQSs with a spanning block design, provided a solution for $v = 28$ and obtained as a corollary that these exist for all $v = 7.4^n$. Finally, in [101] and [100] it was shown that an $\mathrm{SQS}(v)$ with a spanning block design exists if and only if $v \equiv 4$ (mod 12).

Theorem 6.4. *There exists an S($2,4,v$) which can be embedded in an SQS(v) if and only if $v \equiv 4$ (mod 12).*

We note that the replacement property does not hold here any more.

When $v \equiv 1$ (mod 12), a completion of $S(2,4,v)$ to an SQS of the same order is not possible but one may look for an embedding into an $\mathrm{SQS}(v+1)$. If such an embedding exists, the $\mathrm{SQS}(v+1)$ is said to contain an *almost spanning block design*. In [100], the authors present some constructions for SQSs with almost spanning block designs but not much work appears to be done on these designs, and their existence for general $v \equiv 2$ (mod 12) is still open.

We note that since every $\mathrm{S}(2,4,v)$ may be viewed as a partial SQS, it follows from [74] that *every* $\mathrm{S}(2,4,v)$ can be embedded in an $\mathrm{SQS}(u)$ for some finite u. However, nothing seems to be known about bounds on the size of such an embedding.

Yet another type of embedding involving Steiner systems $\mathrm{S}(2,4,v)$ is that of an embedding of Steiner triple systems, $\mathrm{STS}(w)$, into $\mathrm{S}(2,4,v)$s (see [132]). An $\mathrm{STS}(w)$, $(W,\mathcal{C})$, is said to be embedded into a Steiner system $\mathrm{S}(2,4,v)$, $(V,\mathcal{B})$, if $W \subset V$ and $\mathcal{B}|W = \mathcal{C}$. In other words, $\mathcal{C} = \{C : C \subset B$ for some $B \in \mathcal{B}\}$. A well-known example of embeddings of STSs into $\mathrm{S}(2,4,v)$s is provided by the class of affine spaces $\mathrm{AG}(n,3)$ which can be embedded into projective spaces $\mathrm{PG}(n,3)$ by adding a hyperplane at infinity (cf. [96]). The existence, for example, of the century design (cf. Section 9 below) is equivalent to the existence of an embedding of an $\mathrm{STS}(45)$ [or of an $\mathrm{STS}(55)$] into an $\mathrm{S}(2,4,100)$.

It is shown in [132], by using Ganter's Theorem, that every $\mathrm{STS}(w)$ can be embedded into an $\mathrm{S}(2,4,v)$ for some v. The embedding spectrum $E(W,\mathcal{C})$ for an $\mathrm{STS}(w)$, $(W,\mathcal{C})$, is defined as $E(W,\mathcal{C}) = \{v$: there exists $\mathrm{S}(2,4,v)$ containing $(W,\mathcal{C})$ as a subsystem$\}$, and the embedding spectrum $E(w)$ for an admissible order w is defined as $E(w) = \bigcup E(W,\mathcal{C})$ where the union is taken over all $\mathrm{STS}(w)$.

In [132], the spectra $E(7)$ and $E(9)$ have been determined completely ($E(9)$ with one possible exception which was subsequently eliminated by Krcadinac [111]).

Theorem 6.5. *(i)* $E(7) = \{v : v \equiv 1,4$ (mod 12)*,* $v \geqslant 25\}$.

(ii) $E(9) = \{13,28\} \cup \{v : v \equiv 1,4$ (mod 12)*,* $v \geqslant 40\}$.

Also in [132], bounds are provided on $m(w)$ and $q(w)$ where $m(w) = min\ E(w)$, $q(w) = min\{q$: there exists an $\mathrm{S}(2,4,v)$ containing some $\mathrm{STS}(w)$ as a subsystem for all admissible $v \geqslant q\}$. Improved bounds on these quantities are provided for small v, $13 \leqslant v \leqslant 27$.

Finally, another kind of embedding is provided by the *extension* of an $\mathrm{S}(2,4,v)$ into an $\mathrm{S}(3,5,v+1)$. If there exists an extension of an $\mathrm{S}(2,4,v)$, $(V,\mathcal{B})$ into an $\mathrm{S}(3,5,v+1)$, say, $(V \cup \{\infty\},\mathcal{C})$ then $(V,\mathcal{B})$ is a *derived* design of $(V \cup \{\infty\},\mathcal{C})$ through the point ∞, obtained, obviously, by taking all blocks of $\mathcal{C}$ containing ∞ and deleting ∞ from the latter. Since an affine space $\mathrm{AG}(n,q)$ can be extended to an inversive space $\mathrm{S}(3,q+1,q^n+1)$, it follows that for each positive integer n there exists an $\mathrm{S}(2,4,4^n)$ which can be extended to an $\mathrm{S}(3,5,4^n+1)$. So, for example, the (unique) $\mathrm{S}(2,4,16)$ can be extended to the (unique) $\mathrm{S}(3,5,17)$. On the other hand, since a necessary condition for the existence of an $\mathrm{S}(3,5,u)$ is $u \equiv 2,5,17,26,41$, or 50 (mod 60), it follows that a necessary condition for the existence of an $\mathrm{S}(2,4,v)$ extendable to an $\mathrm{S}(3,5,v+1)$ is that $v \equiv 1,4,16,25,40$, or 49 (mod 60). The $\mathrm{S}(2,4,25)$ No.1 in [129] (in [51]) can be extended to the only known $\mathrm{S}(3,5,26)$; the latter is cyclic but it is not known whether it is unique up to isomorphism. But the necessary and sufficient conditions for the existence of an $\mathrm{S}(3,5,v)$ are not known. In fact, it is not even known whether an $\mathrm{S}(3,5,41)$ or an $\mathrm{S}(3,5,50)$ exists or not.

7 Disjoint systems, intersections and large sets

Two Steiner systems S$(2,4,v)$ $(V,\mathcal{B}),(V,\mathcal{B}')$ are said to *intersect* in s blocks if $|\mathcal{B} \cap mathcalB'| = s$; if $\mathcal{B} \cap \mathcal{B}' = \emptyset$ then they are said to be *disjoint*. Extending a well-known result of Teirlinck for STSs, Ganter, Pelikán and Teirlinck [77] (cf. also [55]) proved the following.

Theorem 7.1. *Let* $(V,\mathcal{B}_1)$*,* $(V,\mathcal{B}_2)$ *be any two Steiner systems S*$(2,4,v)$*. Then there exist two disjoint systems* $(V,{\mathcal{B}_1}')$*,* $(V,{\mathcal{B}_2}')$ *such that* $(V,\mathcal{B}_1)$ *is isomorphic to* $(V,{\mathcal{B}_1}')$ *and* $(V,\mathcal{B}_2)$ *is isomorphic to* $(V,{\mathcal{B}_2}')$*.*

The *intersection problem* for S$(2,4,v)$ is to determine the set $J(v)$ of all integers s for which there exist two Steiner systems S$(2,4,v)$ intersecting in s blocks. The intersection problem for S$(2,4,v)$s was considered by Colbourn, Hoffman and Lindner in [52]. It has been known previously that $J(13) = \{0,1,2,3,4,5,7,13\}$ and $J(16) = \{0,1,2,3,4,5,6,8,12,20\}$. They determined the set $J(v)$ completely for all orders $v \equiv 1,4 \ (mod\ 12)$, with some possible exceptions for orders $25, 28$ and 37. Let $b_v = \frac{v(v-1)}{12}$, and let $I(v) = \{0,1,\ldots,b_v\} \setminus \{b_v-7, b_v-5, b_v-4, b_v-3, b_v-2, b_v-1\}$. It is shown in [52] that $J(v) = I(v)$ for all admissible $v \geqslant 40$.

However, $J(25) \neq I(25)$ since, as is shown in [52], $42 \notin J(25)$, and $44 \notin J(25)$. On the other hand, $\{0,1,\ldots,30\} \cup \{32,35,36,38,50\} \subseteq J(25)$, which leaves each $s \in \{31,33,34,37,39,40,41\}$ in doubt as to its belonging to $J(25)$. Moreover, it is shown in [52] that $I(28) \setminus \{44,46,49,50,52,53,54,57\} \subseteq J(28)$, and

$$I(37) \setminus \{64,66,76,82,84,85,88,90,91,92,93,94,96,97,98,99,100,101\} \subseteq J(37).$$

In particular, the above results imply that for every admissible v, there exists a pair of disjoint S$(2,4,v)$s. Not much is known, however, about larger cardinality sets of mutually disjoint S$(2,4,v)$s. The maximum possible number of mutually disjoint S$(2,4,v)$s equals $\frac{(v-2)(v-3)}{2}$; any set of $\frac{(v-2)(v-3)}{2}$ disjoint S$(2,4,v)$s is called a *large set* of S$(2,4,v)$s and is denoted by $LS(2,4,v)$. However, for only two values is a large set of S$(2,4,v)$s known: an $LS(2,4,13)$ containing 55 mutually disjoint S$(2,4,13)$s [46], and an $LS(2,4,16)$ containing 91 disjoint S$(2,4,16)$s [128].

The difficulty of constructing large sets of S$(2,4,v)$s is partly due to the fact that the number of systems in a large set $LS(2,4,v)$ is quadratic in v. So, for example, an $LS(2,4,25)$ must contain 253 disjoint systems; the existence of such a large set is unknown at present.

Quite recently, the so-called *triangle intersection problem* for S$(2,4,v)$s was considered in [39]. Given two Steiner systems S$(2,4,v)$, say, $(V,\mathcal{B}_1)$, $(V,\mathcal{B}_2)$ on the same set, and two blocks B, B', $B \in \mathcal{B}_1$, $B' \in \mathcal{B}_2$, the number of triangles that these blocks may have in common is $0, 1$, or 4. In the triangle intersection problem, one is asked to determine the possible number of triangles that the two systems may have in common. In [39], the set of these numbers is completely determined for all $v \geqslant 121$ and for $v = 40$, and for all v, $49 \leqslant v \leqslant 112$, it is determined with one possible exception.

8 Independent sets and arcs in Steiner systems $\mathbf{S}(2,4,v)$

A subset $S \subset V$ in a design $\mathcal{D} = (V, \mathcal{B})$ is *independent* if there is no $B \in \mathcal{B}$ contained in S. An independent set is *maximal* if for all $x \in V \setminus S$, the set $S \cup \{x\}$ is not independent. The independence number $\alpha(\mathcal{D})$ of $\mathcal{D}$ is defined as $\alpha(\mathcal{D}) = \{max|S| : S$ is an independent set in $\mathcal{D}\}$.

If S is an independent set in an $\mathrm{S}(2,k,v)$, then $|S| \leqslant \frac{k-2}{k-1}(v-1)+1$. Thus for an $\mathrm{S}(2,4,v)$, $|S| \leqslant \frac{2v+1}{3}$ where equality can be attained for $v \equiv 4, 13 \ (mod\ 36)$. Indeed, taking an $\mathrm{S}(2,4,v)$, $(V,\mathcal{B})$, and embedding it into an $\mathrm{S}(2,4,3v+1)$, $(W,\mathcal{C})$, by applying the $v \to 3v+1$ rule (cf. Section 2), it is readily seen that the set $W \setminus V$ is a (maximum) independent set. This independent set is also an $(0,3)$-set (see below).

For example, the unique $\mathrm{S}(2,4,13)$ contains independent sets of any size up to 9, maximal independent sets of sizes 7 and 9, and its independence number $\alpha = 9$. The unique $\mathrm{S}(2,4,16)$ contains independent sets of sizes up to 9, maximal independent sets of size 9, and its independence number $\alpha = 9$ (it also contains a maximum arc of size 6, see below).

On the other hand, a lower bound for the independence number of $\mathcal{D}$ is obtained in [86] where it is shown that for sufficiently large v, $\alpha(\mathcal{D}) \geqslant cv^{\frac{2}{3}}(log\ v)^{\frac{1}{3}}$ for some constant c. It was shown later in [165] that for sufficiently large v there exists an $\mathrm{S}(2,4,v)$ $\mathcal{D}^*$ whose independence number $\alpha(\mathcal{D}^*)$ satisfies

$$cv^{\frac{2}{3}}(log\ v^{\frac{1}{3}}) \leqslant \alpha(\mathcal{D}^*) \leqslant dv^{\frac{2}{3}}(log\ v^{\frac{1}{3}})$$

for some constants c, d.

A special type of an independent set is an *arc*. An s-arc in a Steiner system $\mathrm{S}(2,k,v)$ is a set of s points no three of which are contained in a block. An arc is *complete* if any point of the $\mathrm{S}(2,k,v)$ is contained in at least one secant block (a block which contains exactly two elements of the arc). In other words, a complete arc cannot be contained in a larger arc. Examples of complete arcs can be found in [159], [161], [162]. It is shown in [159] that the minimum possible size s of a complete arc in an $\mathrm{S}(2,k,v)$ satisfies $s^2(k-2) - s(k-4) - 2v = 0$. It follows that the minimum possible size of a complete arc in an $\mathrm{S}(2,4,v)$ equals $\sqrt{v}$.

In the same paper [159], the question is posed whether it is possible to partition the set of elements of an $\mathrm{S}(2,4,v)$, v a square, into $\sqrt{v}$ complete $\sqrt{v}$-arcs. Examples of such a partition are given for the unique $\mathrm{S}(2,4,16)$ and for one of the $\mathrm{S}(2,4,25)$s (No.6 in [110]). Some further such partitions are presented in [149] where an infinite class of $\mathrm{S}(2,4,v)$s admitting such a partition is given.

If for each element x of a complete arc S there exists a unique block B containing x but no other element of S, then the complete arc is called an *oval* (and such a block is called a *tangent block*). There seems to have been very little work done on ovals in Steiner systems $\mathrm{S}(2,4,v)$, although some observations are made in [196].

An arc with maximum size $\frac{v+2}{3} = r+1$ is a *maximum arc* or a *hyperoval*. The necessary condition for the existence of a hyperoval in an $\mathrm{S}(2,4,v)$ is $v \equiv 4 \ (mod\ 12)$ [61]. While examples of hyperovals in $\mathrm{S}(2,4,v)$s were constructed in [60], [134], the above necessary condition was shown to be sufficient in [88] and independently in [125]. An $\mathrm{S}(2,4,v)$ with a maximum arc admits a 2-colouring of type AC (cf. Section 9). Some further recursive constructions for $\mathrm{S}(2,4,v)$s with maximum arcs are given in [133].

Another type of independent sets are the s-sets of type (m,n) introduced and studied in [156]. In general, an s-set of type (m,n), $0 \leqslant m < n \leqslant k-1$, in a Steiner system $\mathrm{S}(2,k,v)$, $(V,\mathcal{B})$, is a subset $S \subset V$, $|S| = s$ such that $|S \cap B| = m$ or n for all $B \in \mathcal{B}$. In the case of $\mathrm{S}(2,4,v)$s, one is interested in the s-sets of type $(0,2)$, $(0,3)$, and $(1,3)$. Sets of type $(0,2)$ are hyperovals. The necessary and sufficient condition for the existence of a $(0,3)$-set in an $\mathrm{S}(2,4,v)$ is $v \equiv 4, 13$ $(mod\ 36)$ [156], [196]. The most interesting of the above cases is that of $(1,3)$-sets. It is shown in [156] that a necessary condition for an $\mathrm{S}(2,4,v)$ to contain a $(1,3)$-set is $v \equiv 4$ $(mod\ 24)$, and v must be a square; the size s of such an $(1,3)$-set is $s = (v - \sqrt{v})/2$ or $s = (v + \sqrt{v})/2$. The smallest v satisfying this necessary condition is $v = 100$, and it was shown very recently [65], [66] that such an $\mathrm{S}(2,4,100)$ containing $(1,3)$-sets of sizes 45 and 55, respectively, indeed exists (cf. Section 9 below, colouring of type B).

Finally, a special class of independent sets is provided by blocking sets (independent sets whose complements are also independent, cf. Section 9 below). Not every system $\mathrm{S}(2,4,v)$ contains a blocking set since every such $\mathrm{S}(2,4,v)$ must be 2-colourable.

9 Colourings of Steiner systems $\mathrm{S}(2,4,v)$

The colourings of Steiner systems $\mathrm{S}(2,4,v)$ are basically of two kinds: element-colourings and block-colourings (although one may consider colouring both, elements and blocks, simultaneously). In this section we consider the first of these two kinds.

Let $S = (V,\mathcal{B})$ be a Steiner system $\mathrm{S}(2,4,v)$. A (proper element-) *colouring* of S is a mapping $\phi : V \to C$ such that for all $B \in \mathcal{B}$, $|\phi(B)| > 1$ where $\phi(B) = \cup_{x \in B} \phi(x)$. Here C is the set of *colours*. For each $c \in C$, the set $\phi^{-1}(c) = \{x : \phi(x) = c\}$ is a *colour class*. If $|C| = m$, we speak of an m-colouring of S. The *chromatic number* of S, $\chi(S)$, is the smallest m for which there exists an m-colouring of S. If $\chi(S) = m$, the system S is said to be m-chromatic.

In other words, in a colouring of S, no block is monochromatic. This type of colouring has sometimes been called *weak* (and is even occasionally referred to as *classical* colouring), in accordance with Berge's definition of weak and strong colouring of hypergraphs, respectively. However, strong colourings of hypergraphs require all elements of a block (=hyperedge) to be coloured with a different colour, and thus are of no interest with regard to colouring Steiner systems (or other block designs) as this requirement would imply that each element of V must get a different colour (but see the classical strong colourings mentioned below among colourings with specified block-colour patterns).

Unlike Steiner triple systems whose chromatic number must be at least 3 if $v > 3$, Steiner systems $\mathrm{S}(2,4,v)$ may be 2-chromatic. In fact, it is easily seen that for $v \in \{4, 13, 16\}$, the unique $\mathrm{S}(2,4,v)$ is 2-chromatic. Two of the 18 nonisomorphic $\mathrm{S}(2,4,25)$ are also 2-chromatic. It was shown in [97] that for all $v \equiv 1$ or 4 $(mod\ 12)$, except possibly for $v = 37, 40, 73$, there exists a 2-chromatic $\mathrm{S}(2,4,v)$. More recently, the existence of a 2-chromatic $\mathrm{S}(2,4,v)$ with $v \in \{37, 40, 73\}$ was established in [67], thus we have:

Theorem 9.1. *A* 2*-chromatic S(*$2,4,v$*) exists for all admissible orders* $v \equiv 1$ *or* 4 (*mod* 12)*.*

Let us observe that the above result was established in terms of the existence of a blocking set in $\mathrm{S}(2,4,v)$. A *blocking set* in $(V, \mathcal{B})$ is a subset X of V which is independent and whose complement $V \setminus X$ in V is also independent. Thus a blocking set in an $\mathrm{S}(2,4,v)$ and its complement are the two colour classes in a 2-colouring of V.

The smallest 3-chromatic $\mathrm{S}(2,4,v)$ has order $v = 25$. The following was proved in [164].

Theorem 9.2. *A* 3*-chromatic S(*$2,4,v$*) exists if and only if* $v \equiv 1$ *or* 4 (*mod* 12) *and* $v \geqslant 25$*.*

Combining an early result of Erdős with Ganter's theorem on completion of partial $\mathrm{S}(2,4,v)$s (see Section 6 above), one obtains immediately that there exist Steiner systems $\mathrm{S}(2,4,v)$ with arbitrarily high chromatic number. However, in a result paralleling a similar result for STSs, Linek and Wantland [124] established the following much stronger statement.

Theorem 9.3. *For each* $m \geqslant 2$*, there exists a constant* v_m *such that for all admissible orders* $v \geqslant v_m$*, there exists an* m*-chromatic S(*$2,4,v$*).*

Theorems 9.1 and 9.2 show that $v_2 = 4$, $v_3 = 25$ but the value of v_m remains unknown for $m \geqslant 4$.

The complexity of computing the chromatic index of $\mathrm{S}(2,4,v)$s appears to be unknown.

Voloshin's concept of mixed hypergraph colourings (cf. [189]) where each block is considered as a C-edge and a D-edge simultaneously requires the absence of not only monochromatic but also of polychromatic blocks, that is no block in a colouring can have all its elements coloured the same or all its elements coloured with pairwise distinct colours. A refinement of this concept leads to a consideration of several types of colourings with specified block-colour patterns. If the block size is k, there are P_k possible colour patterns on a block where P_k is the number of partitions of k. For Steiner systems $\mathrm{S}(2,4,v)$, the block size is 4, and there are altogether 5 partitions of the number 4: $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$ which we denote, in this order, as of type A, B, C, D, or E. If we take $S \subseteq \{A, B, C, D, E\}$, $S \neq \emptyset$, we can define a colouring of type S as a colouring where each block $B \in \mathcal{B}$ is coloured with a pattern from S only. Since there are 31 distinct nonempty subsets of $\{A, B, C, D, E\}$, there are 31 distinct types of colouring one might consider. For example, our "classical" colouring is of type $BCDE$, the strong colouring mentioned at the beginning of this section is of type E, and the Voloshin-type colouring (cf. [189]) is of type BCD.

A new phenomenon which arises here is that of *uncolourability.* Unlike in the classical case, it may happen that for certain given types S of colourings, a given Steiner system

S(2, 4, v) cannot be coloured with block-colour patterns from S; in this case, it is *S-uncolourable*. Clearly, if an S(2, 4, v) is S-uncolourable then $S \subseteq \{B, C, D\}$. In [165], [86] it is shown that for sufficiently large orders v there exist S(2, 4, v) with maximal independent sets of size $o(v)$. This implies that any such S(2, 4, v) is BCD-uncolourable (i.e., there is no colouring satisfying Voloshin's condition).

The question one asks in connection with colourings of type S is not only about the existence or nonexistence of colourings but also about how many colours are possible in a colouring if one exists. Here one must insist that in an m-colouring, all m colours are used (a requirement unimportant in classical type colourings) since the existence of an m-colouring no longer implies the existence of an $(m+1)$-colouring.

Given an S(2, 4, v), $(V, \mathcal{B})$, define the *colouring spectrum*

$\Omega_S(V, \mathcal{B}) = \{m : (V, \mathcal{B})$ has an m-colouring of type $S\}$, and

$\Omega_S(v) = \cup \Omega_S(V, \mathcal{B})$

where the union is taken over all systems of order v.

Clearly, we have $\Omega_{BCDE}(V, \mathcal{B}) = \{\chi, \chi + 1, \ldots, v\}$, and

$\Omega_{BCDE}(v) = \{2, 3, \ldots, v\}$

where χ is the (classical) chromatic number of $(V, \mathcal{B})$.

Trivially, $\Omega_A(v) = \{1\}$, $\Omega_E(v) = \{v\}$, $\Omega_{ABCDE}(v) = \{1, 2, \ldots, v\}$.

Furthermore, while the trivial S(2, 4, 4) obviously has a colouring of type C, a simple calculation shows that any S(2, 4, v) with $v \geqslant 13$ is C-uncolourable [134].

A quite interesting case of colourings is provided by colourings of type B. In such a colouring, each block must have three elements coloured with the same colour, and the fourth element with a different colour. Starting with the trivial S(2, 4, 4) coloured in such a way, and applying the $v \to 3v + 1$ construction (cf. Section 2) consecutively $k - 2$ times, one obtains an S(2, 4, v) where $v = \frac{3^k - 1}{2}$ with a k-colouring of type B ([134]). The type B colourable Steiner systems S(2, 4, v) obtained in this way include the projective spaces PG(k, 3); although the class of S(2, 4, v)s in question is quite extensive, their order is necessarily of the form $v = \frac{3^k - 1}{2}$.

Are there any other S(2, 4, v)s which admit colouring of type B? The following necessary conditions are taken from [134] (conditions (i), (ii), (iii)) and from [65] (conditions (iv) and (v)).

Lemma 9.4. *If $(V, \mathcal{B})$ is an S(2, 4, v) with a k-colouring of type B and x_i is the size of the ith colour class then*

(i) $x_i \equiv 1, 3 \pmod 6$ for $1 \leqslant i \leqslant k$, with precisely one $x_i \equiv 1 \pmod 6$;

(ii) $\Sigma \binom{x_i}{2} = \Sigma x_i x_j = \frac{v(v-1)}{4}$;

(iii) $x_i \leqslant \frac{2v+1}{3}$ for $1 \leqslant i \leqslant k$;

(iv) $x_i = \frac{2v+1}{3}$ for some i if and only if the S(2, 4, v) can be obtained from an S(2, 4, $v - x_i$) using the $v \to 3v + 1$ rule;

(v) for $1 \leqslant i < j \leqslant k$, $(x_i - x_j)^2 \geqslant x_i + x_j$.

These conditions are quite stringent, since, besides the orders of the projective spaces PG(k, 3), there are only three solutions for $v \leqslant 121$ that satisfy the conditions of Lemma 9.4:

1. $v = 61$, $(x_1, x_2, x_3) = (3, 19, 39)$
2. $v = 100$, $(x_1, x_2) = (45, 55)$
3. $v = 109$, $(x_1, x_2, x_3) = (1, 45, 63)$.

The case (2) is most intriguing as it has been around for quite some time (cf., e.g, [156]). The problem of finding a system with a colouring of type B in this case is equivalent to the problem of finding an $\mathrm{S}(2,4,100)$ having a (classical) 2-colouring without blocks coloured with the pattern $2+2$; it has been dubbed the "century design". Until recently, no solution to either of 1., 2. or 3. was known. In a recent superb paper [65], a solution was found for each of these cases (cf. also [66] for the century design). The $v \to 3v+1$ rule implies the existence of three new infinite series of type B colourable systems $\mathrm{S}(2,4,v)$, namely those of orders $v = 3^n v_0 + \frac{3^n-1}{2}$ where $v_0 \in \{61, 100, 109\}$, and n is a nonnegative integer. For example, there exists an $\mathrm{S}(2,4,184)$ with a 4-colouring of type B. In fact, the number of such systems is enormous, as there exists an enormous number of Kirkman triple systems of order 123, cf. [129].

Colourings of type D are interesting in that systems admitting such colouring with a small number of colours are rather exceptional. In particular, for $k \leqslant 5$ there are only finitely many $\mathrm{S}(2,4,v)$s with a k-colouring of type D. A 3-colouring of type D exists only if $v = 4$; no 4-colouring of type D exists for any $\mathrm{S}(2,4,v)$; and a 5-colouring of type D exists only if $v \in \{13, 16, 25\}$ [134]. Thus if there is an m-colouring of type D with $v > 25$ then $m \geqslant 6$. It is shown in [134] that $\Omega_D(13) = \{5, 6\}$, and $\Omega_D(16) = \{5, 6, 7\}$ but determining $\Omega_D(v)$ for larger v appears more difficult.

One interesting result regarding colourings of type D obtained in [134] is as follows.

Theorem 9.5. *Let k be an arbitrary integer $\geqslant 2$, and assume there exists a k-colouring of S$(2,4,v)$ of type D in which all colour classes have the same cardinality. Then $k = 5$, and $v = 25$.*

The colouring spectrum for colourings of type AC (only monochromatic blocks or blocks with colour pattern $2+2$ are allowed) has also been completely determined. It was shown in [134] that $\Omega_{AC}(v) = \{1\}$ whenever $v \equiv 1 \ (mod\ 12)$; on the other hand, if $v \equiv 4 \ (mod\ 12)$, then $\Omega_{AC}(v) \subseteq \{1, 2\}$, and it was conjectured in [134] that equality always holds. This has been proved in [88] where an $\mathrm{S}(2,4,v)$ with a maximal arc was constructed for all $v \equiv 4 \ (mod\ 12)$; it is not hard to see that an $\mathrm{S}(2,4,v)$ with a maximal arc admits a 2-colouring of type AC.

Further results on certain colourings of type S when $|S| = 2$ are contained in [134], [84], in particular, for colourings of type AD, AE, and BC; colourings of the latter type are called *bicolourings* in [84].

10 Block-colourings and resolvability

A Steiner system $\mathrm{S}(2,4,v)$ $(V, \mathcal{B})$ is *resolvable* if $\mathcal{B}$ can be partitioned into $\frac{v-1}{3}$ parallel classes. Here a *parallel class* $\mathcal{P}$ is a subset of $\mathcal{B}$ such that each $x \in V$ is contained in exactly one block of $\mathcal{P}$, i.e., $\mathcal{P}$ is a partition of V.

The (up to an isomorphism) unique $S(2,4,16)$ which consists of points and lines of $AG(2,4)$ is, of course, resolvable.

Since clearly 4 must divide v, a necessary condition for the existence of a resolvable $S(2,4,v)$ is $v \equiv 4$ (mod 12). In [94] it is shown that this condition is also sufficient.

Theorem 10.1. *A resolvable $S(2,4,v)$ exists if and only if $v \equiv 4$ (mod 12).*

In what follows we sketch the proof (cf. [3]).

(1) If $q = 4t+1$ is a prime power then there exists a resolvable $S(2,4,3q+1)$.

Indeed, let $V = GF(q) \times Z_3 \cup \infty$, and let $\mathcal{B}$ be determined by the base blocks $\{0_0, 0_1, 0_2, \infty\}$, $\{x^i{}_j, x^{i+2t}{}_j, x^{i+t}{}_{j+1}, x^{i+3t}{}_{j+1}\}$, $i = 0, 1, \ldots, t-1, j \in Z_3$ where x is a primitive element of $GF(q)$. Then $(V, \mathcal{B})$ is a resolvable $S(2,4,3q+1)$ [92].

(2) If there exists a resolvable $S(2,4,v)$ and a resolvable $S(2,4,w)$ then there exists a resolvable $S(2,4,vw)$ [92].

(3) For $v \in \{100, 172, 232, 388\}$, there exists a resolvable $S(2,4,v)$ (a direct construction is provided for each of these orders in [92]; see also [3]).

(4) If there exists a resolvable $S(2,4,12t+4)$ and a resolvable $S(2,4,12h+4)$ where $0 \leqslant h \leqslant t$ and there exist at least 4 mutually orthogonal latin squares of order t then there exists a resolvable $S(2,4,60t+12h+4)$.

(5) Main induction step: assume the existence of a resolvable $S(2,4,\ 12u+4)$ for al $u \leqslant 30k$, $k \geqslant 3$. Then write

(i) $12u+4 = 12(30k+h)+4 = 60(6k-1)+12(h+5)+4$ if $30k \leqslant u \leqslant 30k+4$;

(ii) $12u+4 = 60(6k+1)+12h+4$ if $30k+5 \leqslant u \leqslant 30k+24$; and

(iii) $12u+4 = 60(60k+5)+12k+4$ if $30k+25 \leqslant u \leqslant 30k+29$;

then apply (4) to obtain the existence of a resolvable $S(2,4,12u+4)$ for all $u, 30k \leqslant u \leqslant 30(k+1)$.

Given a Steiner system $S(2,4,v)$, say $\mathcal{S} = (V, \mathcal{B})$, a *block-colouring* of $\mathcal{S}$ is a mapping $\psi : \mathcal{B} \to C$ (where C is the set of colours) such that if $\psi(B) = \psi(B')$ for $B, B' \in \mathcal{B}$, $B \neq B'$, then $B \cap B' = \emptyset$. If the set of colours has cardinality k then ψ is a k-block colouring. For each $c \in C$, $\psi^{-1}(c)$ is a *block colour class*. The *chromatic index* of $\mathcal{S}$, denoted $\chi'(\mathcal{S})$, is the smallest k for which $\mathcal{S}$ admits a k-block-colouring.

Since any block colour class is a partial parallel class, we must have $\chi'(\mathcal{S}) \geqslant |\mathcal{B}|/\lfloor v/4 \rfloor$ for any $S(2,4,v)$ $\mathcal{S}$. In other words, $\chi' \geqslant \frac{v-1}{3}$. Theorem 10.1 on the existence of resolvable $S(2,4,v)$s can therefore be restated as a result on the chromatic index.

Theorem 10.2. *For every $v \equiv 4$ (mod 12) there exists a Steiner system $S(2,4,v)$ with minimum chromatic index $\frac{v-1}{3}$.*

When $v \equiv 1$ (mod 12), very little is known about the existence of $S(2,4,v)$s with minimum chromatic index. Clearly, such a system cannot be resolvable, and its chromatic index must be at least $\frac{v+2}{3}$. The chromatic index of the unique $S(2,4,13)$ equals 13 since any two of its blocks intersect. The chromatic index of the unique $S(2,4,16)$ equals 5 as it is resolvable. The chromatic index of each of the 18 nonisomorphic $S(2,4,25)$s was determined by M. Meszka [131]. One system has chromatic index 10, thirteen of the systems have chromatic index 11, and four systems have chromatic index 12. Thus for $v \leqslant 25$ there exists no $S(2,4,v)$ with chromatic index equal to the minimum possible value $\frac{v+2}{3}$. Whether such systems exist for $v = 37$ or for any greater order remains an open problem.

As for upper bounds, it is well known (cf. [49]) that $\chi'(\mathcal{S}) \leqslant \frac{4v-1}{3}$ for any system $\mathcal{S}$. By contrast, the chromatic index of a *cyclic* $S(2,4,v)$ cannot exceed v [56].

The complexity of computing the chromatic index of $S(2,4,v)$s is unknown.

Two resolutions $\mathcal{R}_1, \mathcal{R}_2$ of an $S(2,4,v)$ are *orthogonal* if any parallel class of $\mathcal{R}_1 = \{P_1, \ldots, P_{\frac{v-1}{3}}\}$ has at most one block in common with any parallel class of $\mathcal{R}_2 = \{Q_1, \ldots, Q_{\frac{v-1}{3}}\}$. An $S(2,4,v)$ possessing a pair of orthogonal resolutions is said to be *doubly resolvable*. To each doubly resolvable $S(2,4,v)$, $(V,\mathcal{B})$, corresponds a $\frac{v-1}{3} \times \frac{v-1}{3}$ square array (called a Kirkman square of block size 4) with the property that every element of V is contained in exactly one cell of each row and each column and each cell of the array is either empty or contains a block of $\mathcal{B}$. Such an array is obtained from $(V,\mathcal{B})$ by indexing the rows and columns of the array with the parallel classes of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively; in the cell labelled (P_i, Q_j) one places $P_i \cap Q_j$, for all $i,j = 1, \ldots, \frac{v-1}{3}$. If $P_i \cap Q_j = \emptyset$, the cell is empty. There exists a doubly resolvable $S(2,4,4^n)$ for all $n \geqslant 3$ (see [71], [130]) but the spectrum for doubly resolvable $S(2,4,v)$ remains undetermined.

In an analogous way, a set of d ($\geqslant 3$) pairwise orthogonal resolutions of an $S(2,4,v)$ yields a d-dimensional Kirkman cube of block size 4. Recently, Lamken [114] has shown that for $d \geqslant 2$ and $k \geqslant 3$ there exists a constant $v_0 = v_0(k)$ such that for all $v \geqslant v_0, v \equiv k (mod\ k(k-1))$ there exists an $S(2,k,v)$ with d pairwise orthogonal resolutions. Thus when $k = 4$, there exists a Steiner system $S(2,4,v)$ with d orthogonal resolutions, $d \geqslant 2$ (and thus a d-dimensional Kirkman cube with block size 4) for all $v \equiv 4$ (mod 12) and v sufficiently large.

11 Packings and coverings

Let $v \geqslant k \geqslant t$. A $t-(v,k,\lambda)$ *packing* is a pair $(V,\mathcal{B})$ where V is a v-element set and $\mathcal{B}$ is a collection of k-subsets of V such that each t-subset of V is contained in *at most* λ blocks of $\mathcal{B}$. The *packing number* $D_\lambda(v,k,t)$ is the largest possible number of blocks in a $t-(v,k,\lambda)$ packing. A $t-(v,k,\lambda)$ packing is *optimal* (or *maximum*) if $|\mathcal{B}| = D_\lambda(v,k,t)$. Thus an $S(2,4,v)$ is an optimal packing with $D_\lambda(v,k,t) = \frac{v(v-1)}{12}$.

By iterating the inequality

$$D_\lambda(v,k,t) \leqslant \lfloor \frac{v}{k} D_\lambda(v-1,k-1,t-1) \rfloor,$$

known as the *first Johnson bound*, Schönheim [173] obtained an upper bound on $D_\lambda(v,k,t)$:

$$D_\lambda(v,k,t) \leqslant \lfloor \frac{v}{k} \lfloor \frac{v-1}{k-1} \ldots \lfloor \frac{v-t+1}{k-t+1} \lambda \rfloor \ldots \rfloor \rfloor = U(v,k,t).$$

Thus for the case $\lambda = 1, k = 4, t = 2$ we obtain

$$D(v,4,2) \leqslant U(v,4,2) = \lfloor \frac{v}{4} \lfloor \frac{v-1}{3} \rfloor \rfloor.$$

The existence problem for maximum $2-(v,4,1)$ packings was settled by Brouwer [18] who showed that

$$U(v,4,2) - D(v,4,2) = \begin{cases} 1 & \text{if } v \equiv 7,10 \ (mod\ 12), v \neq 10,19 \text{ or } v = 9,17 \\ 2 & \text{if } v = 8,10,11 \\ 3 & \text{if } v = 19 \\ 0 & \text{otherwise} \end{cases}$$

(The results for $v = 17, 19$ are due to Stinson [182]).

An interesting question that arises from packings concerns the leave graphs. Given a $2-(v,4,1)$ packing $(X, \mathcal{B})$, the *leave graph* of $(X, \mathcal{B})$ is the graph $(X, \mathcal{E})$ where the edges in $\mathcal{E}$ are the 2-subsets of X that are not covered by blocks of $\mathcal{B}$. In the table below, the leave graphs of optimal $2-(v,4,1)$ packings are given for $v \notin \{8,9,10,11,17,19\}$.

$v \equiv$	$D(v,4,2)$	Leave graph
$1,4$ $(mod\ 12)$	$\frac{v^2-v}{12}$	empty
$0,3$ $(mod\ 12)$	$\frac{v^2-3v}{12}$	$\frac{v}{3}K_3$
$2,8$ $(mod\ 12)$	$\frac{v^2-2v}{12}$	$\frac{v}{2}K_2$
$5,11$ $(mod\ 12)$	$\frac{v^2-2v-3}{12}$	$K_{1,4} \cup \frac{v-5}{2}K_2$
$7,10$ $(mod\ 12)$	$\frac{v^2-v-18}{12}$	$K_{3,3}$
$6,9$ $(mod\ 12)$	$\frac{v^2-3v-6}{12}$	$(K_6 \setminus K_4) \cup \frac{v-6}{3}K_3$

A $t-(v,k,\lambda)$ packing $(X, \mathcal{B})$ is *resolvable* if $\mathcal{B}$ can be partitioned into sets of $\frac{v}{k}$ disjoint blocks each, called parallel classes. For $v \equiv 4$ $(mod\ 12)$, a resolvable $2-(v,4,1)$ packing is the same as a resolvable $S(2,4,v)$ which exists for all such v (cf. Section 10). When $v \equiv 0$ $(mod\ 12)$, resolvable packings exist for all such $v \geqslant 24$ [93]. When $v \equiv 8$ $(mod\ 12)$, there exist resolvable packings for all such $v \geqslant 32$, except possibly for $v \in \{68, 92, 104, 140, 164, 188, 200, 236, 260, 284, 356, 368, 404, 428, 476, 500, 668, 692\}$. (The results for $v \equiv 0, 8$ $(mod\ 12)$ can be found in [82].)

A *cyclic* $2-(v,4,1)$ packing is defined analogously to a cyclic $S(2,4,v)$. A block orbit is *full* if there are v distinct blocks in it, and *short* otherwise. A cyclic $2-(v,4,1)$ packing having only full block orbits is denoted by $2-CP(4,1;v)$. In [138] and [30], the asymptotic existence of $2-CP(4,1;v)$s was investigated using the concept of a *strong* *difference family* and the theorem of Weil on multiplicative character sums which is well known in number theory.

The known results for $2-CP(4,1;v)$s can be found in [193], [69], [70], [194], [57], [2], [40]. Note that a cyclic $2-(v,4,1)$ packing is not necessarily optimal.

A $2-(v,4,1)$ packing is *maximal* if it ceases to be a packing whenever a further block is adjoined to it. In [144], Novák considers the size of maximal $2-(v,4,1)$ packings with a minimum possible number of blocks. Let $P_{min}(v,4,2)$ denote such a packing, and let $|P_{min}(v,4,2)| = m(v)$. Novák proves the following results for $m(v)$:
$m(4) = m(5) = m(6) = 1, m(7) = m(8) = m(9) = 2, m(10) = m(11) = m(12) = 3, m(13) = m(14) = 6,$
$m(36t+3) = 36t^2+t, m(36t+6) = 36t^2+9t+1, m(36t+9) = 36t^2+15t+2, m(36t+12) = 36t^2+21t+3.$

Based on these results, Novák conjectures that $m(v) \leqslant m(v+1)$ for all $v \geqslant 4$. While the value of $m(v)$ remains undetermined for other values of v, upper bounds are given in [144] for $m(v)$.

Let $v \geqslant k \geqslant t$. A $t-(v,k,\lambda)$ *covering* is a pair $(V,\mathcal{B})$ where V is a set of v elements and $\mathcal{B}$ is a collection of k-subsets of V such that each t-subset of V is contained in *at least* λ blocks of $\mathcal{B}$. The *covering number* $C_\lambda(v,k,t)$ is the smallest possible number of blocks in a $t-(v,k,\lambda)$ covering. A $t-(v,k,\lambda)$ covering is *optimal* (or *minimum*) if $|\mathcal{B}| = C_\lambda(v,k,t)$. Thus an S$(2,4,v)$ is an optimal covering with $C_\lambda(v,4,1) = \frac{v(v-1)}{12}$.

By iterating the inequality

$$C_\lambda(v,k,t) \leqslant \lfloor \frac{v}{k} C_\lambda(v-1,k-1,t-1) \rfloor,$$

Schönheim [172] obtained a lower bound on $C_\lambda(v,k,t)$:

$$C_\lambda(v,k,t) \geqslant \lceil \frac{v}{k} \lceil \frac{v-1}{k-1} \dots \lceil \frac{v-t+1}{k-t+1} \lambda \rceil \dots \rceil \rceil = L(v,4,2).$$

In particular, for the case $\lambda = 1, k = 4, t = 2$ one obtains

$$C(v,4,2) \geqslant L(v,4,2) = \lceil \frac{v}{4} \lceil \frac{v-1}{3} \rceil \rceil.$$

The covering number $C(v,4,2)$ was determined for general v by Mills in [135], [136] (some earlier results can be found in [99]):

$$C(v,4,2) - L(v,4,2) = \begin{cases} 1 & \text{if } v = 7,9,10 \\ 2 & \text{if } v = 19 \\ 0 & \text{otherwise} \end{cases}$$

An interesting question that arises from coverings concerns the shape of excess graphs. Given a $2-(v,4,1)$ covering $(X,\mathcal{B})$, the *excess graph* (actually, a multigraph) of $(X,\mathcal{B})$ is the graph $(X,\mathcal{E})$ where $\mathcal{E}$ consists of 2-subsets of X with mutliplicity $|\{B \in \mathcal{B} : \{x,y\} \subseteq B\}| - 1$. The table below lists the excess graphs for the optimal $2-(v,4,1)$-coverings, $v \notin \{7,9,10,19\}$.

$v \equiv$	$C(v,4,2)$	Excess Graph
$1,4\ (mod\ 12)$	$\frac{v^2-v}{12}$	empty
$0,6\ (mod\ 12)$	$\frac{v^2}{12}$	$\frac{v}{2}K_2$
$3,9\ (mod\ 12)$	$\frac{v^2+3}{12}$	$K_{1,4} \cup \frac{v-5}{2}K_2$
$7,10\ (mod\ 12)$	$\frac{v^2-v+6}{12}$	triple edge
$8,11\ (mod\ 12)$	$\frac{v^2+v}{12}$	G^*
$2,5\ (mod\ 12)$	$\frac{v^2+v+6}{12}$	G^{**}

where G^* is a 2-regular multigraph with v vertices, and G^{**} is a multigraph with v vertices in which two vertices have degree 5 and the remaining $v-2$ vertices have degree 2, or a multigraph in which one vertex has degree 8 and the remaining $v-1$ vertices have degree 2.

A $t-(v,k,\lambda)$ covering $(X,\mathcal{B})$ is *resolvable* if $\mathcal{B}$ can be partitioned into sets of $\frac{v}{k}$ disjoint blocks each (parallel classes). For $v \equiv 4\ (mod\ 12)$, an optimal resolvable $2-(v,4,1)$ covering is the same as a resolvable $\mathrm{S}(2,4,v)$ (which exists for all such v, [94], cf. Section 10). It was shown in [1], [115] that there exists an optimal resolvable $2-(v,4,1)$ covering for all $v \equiv 0\ (mod\ 4)$, except possibly when $v \in \{108,116,132,156,204,212\}$.

12 Systems with higher λ and directed systems

There are several ways to extend and generalize the concept of Steiner systems $\mathrm{S}(2,4,v)$. We discuss very briefly two of these; there exists a vast literature which we do not attempt to survey here.

Related to Steiner systems $\mathrm{S}(2,4,v)$ are balanced incomplete designs with the same block size 4 but with higher index λ, $\mathrm{BIBD}(v,4,\lambda)$. For the existence of a $\mathrm{BIBD}(v,4,\lambda)$, $v \geqslant 4$, it is necessary that

$$\lambda(v-1) \equiv 0\ (mod\ 3), \lambda v(v-1) \equiv 0\ (mod\ 2).$$

Hanani [92] (see also [93]) was the first to determine that these necessary conditions are also sufficient. The necessary conditions for λ and $\gcd(\lambda,6)$ are the same, thus it suffices to consider $\lambda \in \{1,2,3,6\}$. If $\lambda = 1$, we have the case of Steiner systems $\mathrm{S}(2,4,v)$; a $\mathrm{BIBD}(v,4,2)$ exists if and only if $v \equiv 1\ (mod\ 3)$; a $\mathrm{BIBD}(v,4,3)$ exists if and only if $v \equiv 0,1\ (mod\ 4)$; and a $\mathrm{BIBD}(v,4,6)$ exists for all $v \geqslant 4$.

Another possible extension of the notion of Steiner systems is to directed systems. A *directed balanced incomplete design*, DBIBD, with parameters (v,b,r,k,λ) is a

BIBD$(v, b, r, k, 2\lambda)$ where the blocks are ordered k-tuples and where each ordered pair occurs in exactly λ blocks. For example, the block $(0, 1, 4, 6)$ contains the ordered pairs $(0, 1), (1, 4), (0, 6), (1, 4), (1, 6)$, and $(4, 6)$; when developed modulo 7, this base block gives a DBIBD$(7, 4, 1)$. It is easy to see that the existence of a BIBD(v, b, r, k, λ) implies the existence of a DBIBD(v, b, r, k, λ): just write the blocks of the BIBD again in reverse order.

In [183], the following is proved for $k = 4$.

Theorem 12.1. *Let λ, and $v \geqslant 4$ be positive integers. Then a DBIBD$(v, b, r, 4, \lambda)$ exists if and only if*

$$\lambda(v-1) \equiv 0 \ (mod\ 3), \lambda v(v-1) \equiv 0 \ (mod\ 6).$$

For proof, the authors of [183] used directed group divisible designs which are group divisible designs with directed blocks in which each ordered pair of elements from different groups is contained in the same number of blocks.

13 Miscellanea

13.1 Halving S$(2, 4, v)$s

An S$(2, 4, v)$ $(V, \mathcal{B})$ is said to have the *halving property* (or simply can be *halved)* if there is a partition of the set of blocks $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are isomorphic as hypergraphs. Any permutation α such that $\alpha\mathcal{B}_1 = \mathcal{B}_2$ is called a halving permutation. An obvious necessary condition for the existence of an S$(2, 4, v)$ with the halving property is $v \equiv 1$ or 16 $(mod\ 24)$, as the number of blocks must be even. In [61] it was shown that for $v = 16, 25, 40, 64$ and for infinitely many other orders there exist S$(2, 4, v)$ which can be halved. This follows form the fact that if all block orbits under an automorphism α of the S$(2, 4, v)$ are of even length then the system can be halved. Thus, in particular, any cyclic S$(2, 4, v)$ with $v \equiv 16$ $(mod\ 24)$ and any 2-rotational S$(2, 4, v)$ with $v \equiv 1$ $(mod\ 24)$ can be halved. Phelps [148] has established the existence of an S$(2, 4, v)$ that can be halved for all orders $v \equiv 1$ or 16 $(mod\ 24)$, except possibly for six orders: $136, 184, 222, 328, 424, 616$. The existence of a cyclic S$(2, 4, v)$ has since been proved for the first five of these orders (cf. Section 3.1), and so we have:

Theorem 13.1. *There exists an S$(2, 4, v)$ with the halving property if and only if $v \equiv 1$ or 16 $(mod\ 24)$ except possibly when $v = 616$.*

13.2 Block-intersection graphs of S$(2, 4, v)$s

Given an S$(2, 4, v)$, $(V, \mathcal{B})$, its *block intersection graph* (BIG) has as its vertices the blocks of $\mathcal{B}$; two vertices are adjacent precisely when the corresponding blocks intersect. The BIGs of S$(2, 4, v)$s are strongly regular with parameters $(v, \frac{4(v-4)}{3}, \frac{v+20}{3}, 16)$. The BIGs of the unique S$(2, 4, v)$s of orders 13 and 16 are imprimitive (being the complete graph

K_{13}, and the complete 5-partite graph $K_{4,4,4,4,4}$, respectively). The BIGs of S$(2,4,v)$ with $v \geqslant 25$ are primitive, and smallest among these have parameters $(50,28,15,16)$. Every BIG of an S$(2,4,v)$ is Hamiltonian [98].

13.3 Decompositions of S$(2,4,v)$s into configurations

A *configuration* in an S$(2,4,v)$ is a "small" partial system. When dealing with configurations in S$(2,4,v)$s, one usually speaks of points and lines rather then of elements and blocks. There are two distinct (4-point) two-line configurations: two disjoint (parallel) lines (B_1), and two intersecting lines (B_2). There are five three-line configurations: the star, the hut, the 3-path, the triangle and the 3-PPC (three mutually parallel lines)); cf. [98] where these configuration are displayed when the lines have three points each. Since the block-intersection graph of any S$(2,4,v)$ is Hamiltonian (cf. Section 13.2), it follows, similarly as in [98] for Steiner triple systems that *every* S$(2,4,v)$ with an even number of blocks can be decomposed into copies of B_1 or into copies of B_2.

A necessary condition for the existence of a decomposition of an S$(2,4,v)$ into copies of any 3-line configuration is $v \equiv 1$ $(mod\ 36)$. It is shown in [14] that there exists an S$(2,4,v)$ whose lines can be decomposed into triangles if and only if $v \equiv 1$ $(mod\ 36)$.

A necessary condition for the existence of a decomposition of an S$(2,4,v)$ into copies of any 4-line configuration is $v \equiv 1$ or 16 $(mod\ 48)$. It is shown in the same paper [14] that if L is a 4-gon or a complete quadrilateral then this necessary condition is also sufficient for the existence of an S$(2,4,v)$ decomposable into copies of L. Here the 4-gon (l_1,l_2,l_3,l_4) is the four-line configuration in which l_i and l_{i+1} are intersecting lines but l_i and l_{i+2} are parallel (subscripts reduced modulo 4); in the complete quadrilateral, the four lines pairwise intersect but no three intersect in the same point; the complete quadrilateral may be viewed as obtained from the Pasch configuration (with three points per line) by "inserting" an extra point on each line.

Any configuration with no more than three lines cannot be avoided in an S$(2,4,v)$. It appears that unlike for Steiner triple systems, the questions of avoidance of individual configurations with four or more lines in S$(2,4,v)$ has not been systematically addressed so far. The importance of this topic is highlighted in [43] where an analysis is performed of the configurations that need to be avoided when constructing good erasure-resilient codes.

13.4 Metamorphosis

The concept of metamorphosis of one kind of design into another kind of design is relatively recent and is due to Lindner [123].

Start with a Steiner system S$(2,4,v)$, $(V,\mathcal{B})$. For each block $B \in \mathcal{B}$, viewed as a complete graph K_4, and a vertex $x \in B$, the block B decomposes into a triangle and a 3-star centered at x. Choosing a vertex in each block of $\mathcal{B}$ results in a collection of triples, say T, and a collection of 3-stars, say S. If the edges of S can be re-assembled into a collection of triangles (triples) S^* then the Steiner triple system $(V, T \cup S^*)$ is said to be a *metamorphosis* of $(V, \mathcal{B})$. For example, one may take the cyclic $S(2,4,13)$ with blocks $\{i, i+1, i+4, i+6\}, i = 0, 1, \ldots, 12$ $(mod\ 13)$, delete from each block the last element, and re-assemble the edges of the resulting 3-stars into triples $\{i, i+2, i+7\}, i = 0, 1, \ldots, 12 (mod\ 13)$. Clearly, for a metamorphosis of an $S(2,4,v)$ into an $S(2,3,v)$ to exist, it is necessary that $v \equiv 1$ $(mod\ 12)$, as v must be an admissible order for both types of designs. Using skew Room squares with holes of size 2, it is shown in [123] that there exists a Steiner system $S(2,4,v)$ having a metamorphosis into an $S(2,3,v)$ if and only if $v \equiv 1$ $(mod\ 12)$. Similar results are shown also for higher indices λ.

There exist many further results on the metamorphosis of Steiner systems $S(2,4,v)$ into other types of designs, such as K_4-e designs, 4-cycle systems, kite systems etc.; for a survey of these, see [13].

14 Open problems

2.1 Find a direct proof of the existence of $S(2,4,v)$s.

3.1 Show that a cyclic $S(2,4,v)$ exists for all admissible $v \geqslant 37$.

3.2 Show that a 1-rotational $S(2,4,v)$ exists for all $v \equiv 4$ $(mod\ 12)$, $v \neq 28$.

4.1 Determine the number of automorphism-free $S(2,4,28)$s.

4.2 Show that an automorphism-free $S(2,4,v)$ exists for all admissible $v \geqslant 25$.

4.3 What is the complexity of the isomorphism problem for $S(2,4,v)$s?

6.1 Find a polynomial size embedding for partial systems $S(2,4,v)$.

7.1 Determine the sets $J(25)$, $J(28)$ and $J(37)$ (cf. Section 7).

7.2 Find good lower bounds for the maximum number of mutually disjoint $S(2,4,v)$s.

7.3 Do there exist large sets of disjoint $S(2,4,v)$s for $v \geqslant 25$?

9.1 Determine the complexity of computing the chromatic number of $S(2,4,v)$s.

9.2 Find new infinite classes of type B colourable $S(2,4,v)$s.

10.1 Find bounds on the chromatic index of $S(2,4,v)$s.

10.2 Do there exist, for $v \equiv 1$ $(mod\ 12)$, $v \geqslant 37$, Steiner systems $S(2,4,v)$ with the (minimum) chromatic index equal to $\frac{v+2}{3}$?

10.3 Determine the complexity of computing the chromatic index of $S(2,4,v)$s.

10.4 Determine the spectrum for doubly resolvable $S(2,4,v)$s.

11.1 Complete the determination of $m(v)$, the size of maximal $2-(v,4,1)$ packing with minimum number of blocks.

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