A Survey of Minimum Saturated Graphs

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Bryan L. Currie, Jill R. Faudree and John R. Schmitt dedicate this paper to the memory of Ralph J. Faudree and Paul S. Wenger.

Abstract

Given a family of (hyper)graphs $\mathcal{F}$ a (hyper)graph G is said to be *$\mathcal{F}$-saturated* if G is F-free for every $F \in \mathcal{F}$ but for any edge e in the complement of G the (hyper)graph $G + e$ contains some $F \in \mathcal{F}$. We survey the problem of determining the minimum size of an $\mathcal{F}$-saturated (hyper)graph and collect many open problems and conjectures.

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1 Introduction

Given a (hyper)graph F, we say that the (hyper)graph G is *F-free* if G has no sub-(hyper)graph isomorphic to F. We say a (hyper)graph G is *F-saturated* if G is F-free but $G + e$ does contain a copy of F for every (hyper)edge $e \in E(\overline{G})$ where $\overline{G}$ denotes the complement of G. For example, any complete bipartite graph is a K_3-saturated graph. Additionally, we have:

$ex(n, F) = \max\{|E(G)| \,:\, |V(G)| = n \text{ and } G \text{ is } F\text{-saturated}\}$,
$Ex(n, F) = \{G \,:\, |V(G)| = n,\ |E(G)| = ex(n, F), \text{ and } G \text{ is } F\text{-saturated}\}$,
$sat(n, F) = \min\{|E(G)| \,:\, |V(G)| = n \text{ and } G \text{ is } F\text{-saturated}\}$,
$Sat(n, F) = \{G \,:\, |V(G)| = n,\ |E(G)| = sat(n, F), \text{ and } G \text{ is } F\text{-saturated}\}$.

Note that the word saturated could be replaced with the word free in the definitions for $ex(n, F)$ and $Ex(n, F)$ but not so in the other two. We will refer to $ex(n, F)$ as the *extremal number* of F and $sat(n, F)$ as the *saturation number*.

We can generalize all the definitions above by replacing the graph F with a family of (hyper)graphs $\mathcal{F}$. So, a (hyper)graph G is $\mathcal{F}$-saturated if G contains no member of $\mathcal{F}$ as a sub(hyper)graph but for every edge $e \in \overline{G}$, there exists $F \in \mathcal{F}$ such that $G + e$ contains F as a sub(hyper)graph. When $\mathcal{F} = \{F\}$ we write F-saturated, $sat(n, F)$, etc. in place of $\mathcal{F}$-saturated, $sat(n, \mathcal{F})$, etc.

The focus of this survey are the functions $sat(n, \mathcal{F})$ and $Sat(n, \mathcal{F})$, along with related notions. For terms and notation used throughout the survey, we refer the reader to the table of notation appearing at the end of the survey.

In 1941, P. Turán [Tur41] introduced the idea of an extremal number and determined $ex(n, K_p)$ and $Ex(n, K_p)$. In particular, he proved that $Ex(n, K_p)$ consists of a single graph (up to isomorphism): the complete $(p-1)$-partite graph, where the n vertices are distributed among the partite sets as evenly as possible.

In 1964, motivated by a conjecture of P. Erdős and T. Gallai [EG61], P. Erdős, A. Hajnal, and J.W. Moon [EHM64] introduced the idea of a saturation number (though not using that terminology) and proved the following.

Theorem 1. *[EHM64] If $2 \leqslant p \leqslant n$, then $sat(n, K_p) = (p-2)(n-p+2) + \binom{p-2}{2} = \binom{n}{2} - \binom{n-p+2}{2}$ and $Sat(n, K_p)$ contains only one graph, $K_{p-2} + \overline{K}_{n-p+2}$.*

Note that $K_{p-2} + \overline{K}_{n-p+2}$ can be thought of as the complete $(p-1)$-partite graph on n vertices such that all but one of the partite sets contains exactly one vertex, i.e. the vertices are distributed as 'unevenly' as possible amongst the $p-1$ parts.

In 1986 L. Kászonyi and Zs. Tuza [KT86] found the best known general upper bound for $sat(n, \mathcal{F})$. Note that $\alpha(F)$ is the independence number of F (i.e. the order of the largest clique in $\overline{F}$) and a *star on n vertices* refers to the complete bipartite graph $K_{1,n-1}$. To state their result, we first define

$$u = u(\mathcal{F}) = \min\{|V(F)| - \alpha(F) - 1 : F \in \mathcal{F}\}$$

and

$$d = d(\mathcal{F}) = \min\{|E(F')| : F' \subseteq F \in \mathcal{F} \text{ is induced by } S \cup x\},$$

where S is an independent set in $V(F)$ and $|S| = \{|V(F)| - u - 1, x \in V(F) \setminus S\}$.

Theorem 2. *[KT86] $sat(n, \mathcal{F}) \leqslant un + (d-1)(n-u)/2 - \binom{u+1}{2}$.*

This theorem is interesting for several reasons. First the proof hinges largely on two simple observations and exploits the power of considering the saturation number of families of graphs. Second, the bound is exact for a great many graphs. Finally the proof is implicitly constructive. That is, for many graphs, the proof describes how to construct an F-saturated graph. In fact, for all graphs F, the proof constructs a graph that must contain an F-saturated graph as a subgraph. An outline of the proof and its consequences now follows.

Given a family of graphs $\mathcal{F}$, Kászonyi and Tuza define the family of deleted subgraphs of $\mathcal{F}$ as $\mathcal{F}' = \{F \backslash x \mid F \in \mathcal{F}, x \in V(F)\}$ and recursively, $\mathcal{F}''$, $\mathcal{F}'''$, and so forth. A first observation is that a graph G on n vertices with a vertex x of degree $n-1$ is $\mathcal{F}$-saturated if and only if $G \backslash x$ is $\mathcal{F}'$-saturated. Now, by the choice of parameter u in the hypothesis of the theorem, the family $\mathcal{F}^u$ must contain a star on $d+1$ vertices from which it immediately follows that any $\mathcal{F}^u$-saturated graph has maximum degree less than d. The upper bound, then, is simply a count of the number of edges in a graph on n vertices such that u of the vertices have degree $n-1$ and the subgraph containing the remaining $n-u$ vertices is $(d-1)$-regular. For reference, we will call this graph $K_u + G'$, where G' is a $(d-1)$-regular graph on $n-u$ vertices.

R. Faudree and R. Gould observed in [FG13] that the bound in [KT86] can be improved slightly by replacing G' by a graph $G^* \in sat(n-u, K_{1,d})$, since the critical fact is that the addition of any edge in G^* will result in a vertex of degree d. This does not change the bound asymptotically, but gives the inequality

$$sat(n, \mathcal{F}) \leqslant un + (d-1)(n-u)/2 - \binom{u+1}{2} - \frac{1}{2}\lfloor d^2/4 \rfloor.$$

This upper bound is sharp in many cases. In particular, in the case that $\mathcal{F}$ contains only the complete graph — the construction gives the unique extremal graph in this case (see Theorem 1 of Erdős, Hajnal, and Moon). Furthermore, in [FG13], the authors establish the existence of infinite families of graphs such that for every member F, $K_u + G^* \in Sat(n, F)$. In [CFG08] this upper bound was also shown to give a sharp bound for the saturation numbers for similar graphs, such as books and generalized books. And in [FG13] it is shown that the saturation numbers for various families of nearly complete graphs are either precisely the Kászonyi-Tuza bound or the bound is asymptotically correct. The bound is also sharp in the case of the very sparse graph $\mathcal{F} = \{K_{1,k-1} + e\} = \{K_1 + (K_2 \cup (k-3)K_1)\}$. In this case, $u = 1$ and $d = 1$, and the construction given by Theorem 2 gives the star graph $K_{1,n-1}$. In some cases the bound is known to be asymptotically correct. (See, for example, Theorem 14.)

Finally, for any graph F, the F-saturated subgraph contained in $K_u + G^*$ can be constructed by beginning with the graph $K_u + \overline{K}_{n-u}$ and adding edges one by one from the graph G^* if and only if their addition does not produce a copy of F. This procedure must end in the desired subgraph.

In many instances the bound in Theorem 2 is neither sharp nor asymptotically correct. (See, for example, Theorem 17 and Theorem 18.)

Note that Theorem 2 implies that $sat(n, \mathcal{F}) = O(n)$, while for the extremal number we have $ex(n, \mathcal{F}) = O(n^2)$ (see [ES66]).

A nontrivial general lower bound has yet to be determined though lower bounds do exist for certain classes of graphs as will be seen later in the survey.

One of the most interesting tools to arise as a result of the study of the saturation function is due to B. Bollobás [Bol65]. We refer to this tool as *Bollobás' inequality* or the *Two Families Theorem*. It allows for simple proofs of many results, including the quantitative part of Theorem 1, which we give after the statement. It was developed however to establish a corresponding result for k-uniform hypergraphs (see Theorem 3), but it also easily adapts to allow for proofs for bipartite graphs in a bipartite setting (see Section 6, in particular Theorem 50). Bollobás' inequality has also found use outside the study of this function; most of these uses lie in Extremal Set Theory where the method of proof is sometimes referred to as the *set-pair method*. For instances of such see Section 10 of the survey by P. Frankl in [GGL95] and the excellent two-part survey on the set-pair method by Zs. Tuza [Tuz94, Tuz96].

Theorem 3. *[Bol65] Let $\{(A_i, B_i) : i \in I\}$ be a finite collection of finite sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. For $i \in I$ set $a_i = |A_i|$ and $b_i = |B_i|$. Then*

$$\sum_{i \in I} \binom{a_i + b_i}{a_i}^{-1} \leqslant 1 \tag{1}$$

with equality if and only if there is a set Y and non-negative integers a and b, such that $|Y| = a + b$ and $\{(A_i, B_i) : i \in I\}$ is the collection of all ordered pairs of disjoint subsets of Y with $|A_i| = a$ and $|B_i| = b$ (and so $B_i = Y \setminus A_i$).
In particular, if $a_i = a$ and $b_i = b$ for all $i \in I$, then $|I| \leqslant \binom{a+b}{a}$. If $a_i = 2$ and $b_i = n - p$ for all $i \in I$, then $|I| \leqslant \binom{n-p+2}{2}$.

We can now easily give a proof of the quantitative part of Theorem 1.

Proof of Theorem 1 (as given in [GGL95], page 1269) Let G be an n-vertex K_p-saturated graph. We show that the number of non-edges l is at most $\binom{n-p+2}{2}$. Let $A_1, \ldots, A_l$ be the pairs of vertices 'belonging' to a non-edge of G. For each such set there is a corresponding $p-$set C_i of vertices in $V(G)$ containing A_i such that $V(C_i)$ induces a $K_p - e$. Set B_i to be the complement of C_i in $V(G)$. Now note that the hypotheses of Theorem 3 are met and so $l \leqslant \binom{n-p+2}{2}$, or rather $sat(n, K_p) \geqslant \binom{n}{2} - \binom{n-p+2}{2}$. □

In this paper, we will summarize known results for $sat(n, \mathcal{F})$ and $Sat(n, \mathcal{F})$. Earlier such surveys may be found in [Tuz88], [GGL95] (see the chapter by B. Bollobás), and the Ph.D. thesis of O. Pikhurko [Pik99b]. In an effort to stimulate further research, we include many open conjectures, questions, and problems. We regard these items with respect to importance and/or interest in the same order.

We now give a brief overview of the contents of this dynamic survey (a first version of which appeared in 2011), though the reader is invited to peruse the the Table of Contents given above. In Section 2 we consider results pertaining to complete graphs, including degree restrictions, unions of cliques, complete partite graphs, and edge coloring problems. These problems and results are among the first and most natural considerations after the introduction of the function in the early 1960s. Some results are arrived at in a straightforward manner, e.g. unions of cliques, others thwarted attack for a long time and required a novel approach, e.g. the results on complete partite graphs. In Section 3 and Section 4 we present results on cycles and trees, respectively. In these sections we begin to get a sense of the challenges of studying this function, whether it be the technical proof involved in determining the value of the function for the five-cycle or the strange behavior of the function exhibited for two trees of a given order with 'similar' structure. In Section 14 we grapple with some of the inherent difficulties of the *sat*-function. One of the main current challenges in the study of the saturation function is that it fails to have the monotonic properties for which one might hope. We discuss these issues in depth and believe that Question 15 is most important to settle. Other open questions and conjectures are sprinkled throughout the survey. Indeed, it is one of our main aims in writing this survey: we wish to propel the research activities of our community in the direction of saturated graphs of minimum size. Section 5 considers the problem for hypergraphs and Section 6 considers the problem for when the 'host' graph is something other than the complete graph. Section 17 shows some relationships that the *sat*-function has with other extremal functions, including the *ex*-function. In particular, it seems that certain aspects of the saturation function are as difficult as some of the most challenging outstanding problems in the whole of extremal graph theory, see Subsection 17.1. In Section 10 we consider the related notion of weak saturation. Though later in our presentation, the topic should not be considered lesser in terms of interest or challenges present. Indeed, this topic has attracted the attention of some of the top combinatorists of the past few decades and as a consequence some beautiful results and techniques have been found. Furthermore, since the first version of our survey appeared in 2011, many new topics have appeared or grown considerably. These include, for instance, induced saturation (Section 13), edge-coloring problems (Section 7), and counting the minimum number of copies of a graph H in an F-saturated graph (Section 12). One topic that is older but was not included in the first version of the survey is that of bootstrap percolation (see Section 11).

It should be noted that while much of this survey is devoted to compiling known results and open problems, we do give some proofs that we feel are particularly novel, striking or beautiful, one such is given above and another is to follow immediately.

Note that in the proof of Theorem 1 we only made use of the "in particular" statement found in Theorem 3. We give a proof of just this part of the theorem (as found in L. Babai and P. Frankl [BF92]) as it brings to light how L. Lovász [Lov77] brought the *linear algebra method* into play for theorems of this type. Generalizations of Bollobás' theorem often allow extensions of this method.

Proof of the "In particular" statement of Theorem 3

Let $Y = (\cup_I A_i) \cup (\cup_I B_i)$. For each $y \in Y$ we associate a vector

$$v(y) = (v_0(y), v_1(y), \ldots, v_a(y)) \in \mathbb{R}^{a+1},$$

such that the set of vectors is in general position; that is, any $a+1$ vectors are linearly independent. Now for each set $Y' \subseteq Y$ we associate a polynomial $f_{Y'}(x)$ in the $a+1$

variables $x = (x_0, x_1, \ldots, x_a)$ as follows:

$$f_{Y'}(x) = \prod_{y \in Y'} (v_0(y)x_0 + v_1(y)x_1 + \cdots + v_a(y)x_a).$$

The above polynomial is homogeneous and has degree equal to the size of the set Y'. It follows from the definition of orthogonal that the polynomial is non-zero only when x is orthogonal to none of the $v(y), y \in Y'$.

We now consider such a polynomial associated with a set B_i and let a_j be a non-zero vector orthogonal to the subspace generated by the a elements of A_j. Note that a_j is orthogonal to $v(y)$ only if $y \in A_j$ (this follows from the fact that the vectors were chosen to be in general position). We are now able to claim that $f_{B_i}(a_j) = 0$ if and only if A_j and B_i intersect; that is, if and only if $i \neq j$.

It can then be shown that the polynomials $f_{B_1}, \ldots f_{B_{|I|}}$ form a linearly independent set. Thus, (by the so-call linear algebra bound) the size of this set is not greater than the dimension of the space of homogeneous polynomials of degree b in $a+1$ variables; that is, $|I| \leqslant \binom{(a+1)+b-1}{b} = \binom{a+b}{a}$. $\square$

2 Complete graphs

Recall that in the original paper by Erdős, Hajnal and Moon [EHM64], their main result was to establish $sat(n, K_p)$ and the uniqueness of the graph in $Sat(n, K_p)$. This section describes results concerning graphs that are 'related' to minimum K_p-saturated graphs, such as the saturation number of K_p with restrictions on the minimum or maximum degree of the host graph, and the saturation number of complete multi-partite graphs, unions of cliques and subdivisions of K_p. (One exception is the generalization to hypergraphs which is discussed in Section 5.) The reader will find that even the set of results close to the original [EHM64] result include a great variety of approaches all of which have natural open problems in their respective directions.

2.1 Degree restrictions

One of the first generalizations considered was to place additional restrictions on the graph. Recall that all the vertices in the unique extremal graph in $Sat(n, K_p)$ either have degree equal to $\Delta = n-1$ or $\delta = p-2$. And, in fact, any K_p-saturated graph has to have minimum degree at least $p-2$. While confirming a conjecture of T. Gallai about the minimum degree of a K_p-saturated without conical (degree $n-1$) vertices, A. Hajnal in [Haj65] asked, what is the minimum number of edges in a K_p-saturated graph if $\Delta \leqslant n-2$? With this question in mind, we define $sat^{\Delta}(n, F)$ to be the minimum number of edges in a F-saturated graph on n vertices with maximum degree no more than Δ. In Z. Füredi and Á. Seress [FS94], the value of $sat^{\Delta}(n, K_3)$ was found precisely for $\Delta \geqslant (n-2)/2$ and n sufficiently large.

Theorem 4. *[FS94] Let $n > 2^{2^{28}}$. Then*

$$sat^{\Delta}(n, K_3) = \begin{cases} 2n-5, & \textit{for } \Delta = n-2, \\ 2n-5+(n-3-\Delta)^2, & \textit{for } n-3-\sqrt{n-10} \leqslant \Delta \leqslant n-3, \\ 3n-15, & \textit{for } (n-2)/2 \leqslant \Delta < n-3-\sqrt{n-10}. \end{cases}$$

Upper and lower bounds are established for other values of Δ. Continuing in this direction, P. Erdős and R. Holzman [EH94] gave the following result.

Theorem 5. *[EH94]*

$$\lim_{n\to\infty} \frac{sat^{cn}(n, K_3)}{n} = \begin{cases} (11-7c)/2, & \textit{for } 3/7 \leqslant c < 1/2, \\ 4 & \textit{for } 2/5 \leqslant c \leqslant 3/7. \end{cases}$$

N. Alon, P. Erdős, R. Holzman, and M. Krivelevich [AEHK96] proved similar results for K_4. Additionally, they construct a K_p-saturated graph with $\Delta = 2p\sqrt{n}$ for all p and sufficiently large n.. K. Amin, J. Faudree, R. Gould, and E. Sidorowicz [AFGS13] studied $sat^{\Delta}(n, K_p)$ for $\Delta \leqslant n-2$.

Theorem 6. *[AFGS13] Let $p, n \in \mathbb{N}$ such that $p \geqslant 3$ and $n \geqslant 3p+12$. Then*

$$sat^{n-2}(n, K_p) = (p-1)n - \frac{p(p-1)}{2} - 2.$$

Problem 1. *Investigate $sat^{n-c}(n, K_p)$, where c is a small positive constant.*

From a slightly different perspective, D. Duffus and D. Hanson [DH86] considered minimally K_p-saturated graphs with minimum degree at least δ, for $\delta \geqslant p-2$. Thus, define $sat_{\delta}(n, F)$ to be the minimum number of edges in an n-vertex F-saturated graph G with minimum degree δ; obviously, one only considers when $\delta(G) \geqslant \delta(F) - 1$ since both endpoints of the added edge e must be contained in the copy of F created. Upper and lower bounds for this function are found in some instances.

Theorem 7. *[DH86]*

$$sat_2(n, K_3) = 2n-5, n \geqslant 5,$$
$$sat_3(n, K_3) = 3n-15, n \geqslant 10.$$

Note that the upper bound for each of the above statements in Theorem 7 can be realized by duplicating a vertex in the 5-cycle and Petersen graph, respectively. This process of duplicating a vertex occurs frequently, but certainly not always, in the extremal graphs for the *sat*-function. In addition, Theorem 7 plays a role in the previously mentioned results found in [FS94] and [AEHK96].

Theorem 7 led B. Bollobás [GGL95] (see page 1271) to ask the following: for $\delta \geqslant 4$, does $sat_{\delta}(n, K_3) = \delta n - O(1)$? Certainly we have $sat_{\delta}(n, K_3) \leqslant \delta(n-\delta)$ as the bipartite graph $K_{\delta, n-\delta}$ is K_3-saturated with minimum degree at least δ. (In [DH86] a different construction is given yielding a slightly better upper bound, and better yet in [FS94].) The more general problem of determining $sat_\delta(n, K_p)$ can also be considered.

As a means of estimating $sat_\delta(n, K_p)$, Duffus and Hanson introduce the idea of a minimally color-critical graph. If we look again at the graph $K_{p-2} + \overline{K}_{n-p+2}$, we see that its chromatic number is $p-1$ and the addition of any edge increases the chromatic number to p. Suppose G is a graph on n vertices with chromatic number $p-1$ and minimum degree at least δ. They define $\chi_\delta(n, p)$ to be the minimum number of edges that G can have such that the addition of any edge to G increases the chromatic number. Such graphs are called *minimal* (χ, δ)*-saturated graphs.* Duffus and Hanson find the value of $\chi_\delta(n, p)$ precisely and show that the extremal graph corresponding to it is unique, consisting of a complete $(p-1)$-partite graph with suitably sized partite sets. More precisely, they give the following.

Theorem 8. *[DH86] For integers n, p, δ, such that $2 \leqslant p \leqslant n$, $\delta \geqslant p-2$, the complete $(p-1)$-partite graph with $(p-2-\lfloor \frac{n-p+1}{n-\delta-1} \rfloor)$ parts of cardinality one, $\lfloor \frac{n-p+1}{n-\delta-1} \rfloor$ parts having cardinality $(n-\delta)$, and one part having cardinality $n-(p-2-\lfloor \frac{n-p+1}{n-\delta-1} \rfloor) - \lfloor \frac{n-p+1}{n-\delta-1} \rfloor (n-\delta)$ is the only n-vertex minimal (χ, δ)-saturated graph.*

It is, in fact, $\chi_\delta(n, p)$ that provides an upper bound for the number of edges in a K_p-saturated graph with prescribed minimum degree.

Duffus and Hanson showed that $sat_\delta(n, K_p) \geqslant n(\delta+p-2)/2 - O(1)$. Subsequently, one of the results of [AEHK96] (see Theorem 2) implies that $sat_\delta(n, K_p) = \delta n + O(\frac{n}{\log \log n})$. Another advance was made by O. Pikhurko [Pik04], who improved the error term and showed that for any fixed $\delta \geqslant p-1$, $sat_\delta(n, K_p) = \delta n + O(\frac{n \log \log n}{\log n})$. Finally, A.N. Day [Day17] offered a dramatic improvement on these results and confirmed a generalization of Bollobás's conjecture.

Theorem 9. *[Day17] Let $t \in \mathbb{N}$. Then there exists a constant $c = c(t)$ such that for all $p \in \mathbb{N}$ with $p \geqslant 3$, $sat_t(n, K_p) \geqslant tn - c$. Furthermore, when n is sufficiently large, $G \in Sat_t(n, K_p)$ implies $\delta(G) = t$.*

For a graph G, let G^* be the graph obtained by adding a new vertex v to G and making v adjacent to all vertices in G, i.e. v is a conical vertex. For a graph G with minimum degree at least t that is K_p-saturated, the graph G^* is K_{p+1}-saturated and has minimum degree at least $t+1$.

Problem 2. *[Day17] For which $n, t, p \in \mathbb{N}$ are all graphs in $Sat_t(n, K_p)$ of the form G^* for some $G \in Sat_{t-1}(n-1, K_{p-1})$?*

Finally, we mention the problem of determining the minimum size of a non-$(p-1)$-partite K_p-saturated graph. For $p = 3$, this was solved by C. Barefoot, K. Casey, D. Fisher, K. Fraughnaugh, and F. Harary [BCF$^+$95b]. For $p = 3$, such a graph has $2n-5$ edges and can be obtained by duplicating two non-adjacent vertices of a C_5. K. Amin, J. Faudree, R. Gould, and E. Sidorowicz [AFGS13] generalize the first part of this result, showing that for $p \geqslant 3$ and $n \geqslant 3(p+4)$, the minimum number of edges in a non-$(p-1)$-partite K_p-saturated graph is $(p-1)n - \binom{p}{2} - 2$. They show that one can construct graphs that achieve this bound in a recursive fashion and the lower bound follows by induction on p.

2.2 Unions of cliques and complete multi-partite graphs

Another extension is to consider graphs which generalize the complete graph. One approach is to consider a union of cliques. For a graph F, let tF denote the disjoint union of t copies of F.

In [FFGJ09b], R. Faudree, M. Ferrara, R. Gould, and M. Jacobson determined $sat(n, tK_p)$ and $sat(n, K_p \cup K_q)$ precisely with extremal graphs given by $K_{p-2} + \{(t-1)K_{p+1} \cup \overline{K}_{n-pt-t+3}\}$ and $K_{p-2} + \{K_{q+1} \cup \overline{K}_{n-q-p+1}\}$ (for $p \leqslant q$), respectively.

Theorem 10. *[FFGJ09b] Let $t \geqslant 1, p \geqslant 3$ and $n \geqslant p(p+1)t - p^2 + 2p - 6$ be integers. Then*

$$sat(n, tK_p) = (t-1)\binom{p+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2).$$

Furthermore, if $t = 2$ or 3, the extremal graph, respectively, is unique.

This was built on previous work of W. Mader [Mad73] who considered the case $p = 2$.

Using similar techniques, Faudree, Ferrara, Gould, and Jacobson [FFGJ09b] were able to establish the saturation number for generalized friendship graphs. That is, for integers t, p, and l, define $F_{t,p,l}$ to be the graph composed of t copies of K_p intersecting in a common K_l; $F_{t,p,l}$ is called the generalized friendship graph.

Theorem 11. *[FFGJ09b] Let $p \geqslant 3, t \geqslant 2$ and $p - 2 \geqslant l \geqslant 1$ be integers. Then, for sufficiently large n,*

$$sat(n, F_{t,p,l}) = (p-2)(n-p+2) + \binom{p-2}{2} + (t-1)\binom{p-l+1}{2}.$$

The value of $sat(n, K_p \cup K_q \cup K_r)$ is still open and, as the authors observe, the construction they use to establish an upper bound for $sat(n, K_p \cup K_q)$ (which they determine exactly) does not apply in this case. Also, it is not known in general if $Sat(n, tK_p)$ is unique for $t \geqslant 4$.

Problem 3. *[FFGJ09b] Investigate $sat(n, K_p \cup K_q \cup K_r)$ and $sat(n, 2K_p \cup K_q)$.*

Another generalization is to complete partite graphs. Let $K_{s_1,\ldots,s_p}$ denote the complete p-partite graph with partite sets of size $s_1, \ldots, s_p$ and $1 \leqslant s_1 \leqslant cldots \leqslant s_p$ and $p \geqslant 2$. Note that the *star* $K_{1,k-1}$ with k vertices will be fully considered in Section 4, and $K_{2,2}$, which is isomorphic to C_4 and has saturation number $\lfloor \frac{3n-5}{2} \rfloor$, in Section 3. O. Pikhurko and J. Schmitt [PS08] considered the graph $K_{2,3}$ and proved that there is a constant c such that for all $n \geqslant 5$ we have $2n - cn^{3/4} \leqslant sat(n, K_{2,3}) \leqslant 2n - 3$.. Later Y.-C. Chen [Che14] was able to accurately describe the properties and structures of the graphs in $Sat(n, K_{2,3})$, which enabled a proof of a conjecture of T. Bohman, M. Fonoberova, and O. Pikhurko [BFP10] as given in the following theorem.

Theorem 12. *[Che14] For $n \geqslant 5$, $sat(n, K_{2,3}) = 2n - 3$.*

More recently, S. Huang, H. Lei, Y. Shi, and J. Zhang [HLSZ19] confirmed a conjecture of Pikhurko and Schmitt [PS08].

Theorem 13. *[HLSZ19]*

$$sat(n, K_{3,3}) = \begin{cases} 2n & \text{if } 6 \leqslant n \leqslant 8, \\ 3n - 9 & \text{if } n \geqslant 9. \end{cases}$$

Each of the proofs for these 'small' bipartite graphs relies on a case-by-case consideration with respect to the minimum degree of the saturated graph G. Obviously, as one considers larger bipartite graphs such an approach becomes more tedious.

O. Pikhurko [Pik04] computed $sat(n, K_{1,\dots,1,s})$ exactly for n sufficiently large, as subsequently did G. Chen, R. Faudree and R. Gould [CFG08] while simultaneously giving better estimates on n. R. Gould and J. Schmitt [GS07] considered the graph $K_{2,\dots,2}$ and determined the extremal graph under the assumption that the graph has a vertex of smallest possible minimum degree. A result of Bohman et al. [BFP10] confirmed that *sat*-function for a complete multipartite graph behaves asymptotically like the upper bound provided for this graph by Theorem 2.

Theorem 14. *[BFP10] Let $p \geqslant 2$, $s_p \geqslant \cdots \geqslant s_1 \geqslant 1$. Then for all large n,*

$$sat(n, K_{s_1,\dots,s_p}) = (s_1 + \cdots + s_{p-1} + \frac{s_p - 3}{2})n + O(n^{3/4}).$$

Additionally, Bohman et al. are able to provide a stability type result — the first such result in the study of this function! That is, $K_{s_1,\dots,s_p}$-saturated graphs with at most $sat(n, K_{s_1,\dots,s_p}) + o(n)$ edges can be changed into the construction provided by Theorem 2 by adding and removing at most $o(n)$ edges. The authors note that the *exact* determination of the saturation number for complete multipartite graphs is an interesting open problem (a conjecture for the exact value of $sat(n, K_{2,\dots,2})$ is given in [GS07]).

Problem 4. *Determine precisely the value of $sat(n, K_{s_1,\dots,s_p})$.*

2.3 Subdivisions of K_p

A *subdivision* of a graph F is a graph obtained from F by replacing the edges of F with internally disjoint paths of arbitrary length. Let $\mathcal{S}(F)$ denote the family of subdivisions of F, which includes F itself.

M. Ferrara, M. Jacobson, K. Milans, C. Tennenhouse, and P. Wenger [FJM$^+$12] introduced the study of $sat(n, \mathcal{S}(F))$. In particular, they give results when F is the complete graph or a cycle. Here we present their results for the complete graph and discuss the results for cycles in Section 3.

Theorem 15. *[FJM$^+$12] Let $p \geqslant 5$. If $n = d(p-1) + r$ for $d \geqslant 2$ and $0 \leqslant r \leqslant p-2$, then*

$$sat(n, \mathcal{S}(K_p)) \leqslant (\frac{p-2}{2} + o(1))n.$$

Further, Ferrara et al. [FJM$^+$12] point out that $\mathcal{S}(K_3)-$saturated graphs are trees, and so $sat(n,\mathcal{S}(K_3)) = sat(n,K_3) = n-1$. They also point out that $sat(n,\mathcal{S}(K_4)) = sat(n,K_4) = 2n-3$ follows from the fact that a graph G is $\mathcal{S}(K_4)$-saturated graph if and only if G is a 2-tree, where a p-tree is defined to be any graph that can be obtained from a K_p by iteratively joining vertices to cliques of size p. Ferrara et al. [FJM$^+$12] also show the following.

Theorem 16. *[FJM$^+$12] For $n \geqslant 10, sat(n,\mathcal{S}(K_5)) = \lceil \frac{3n+4}{2} \rceil$.*

Problem 5. *[FJM$^+$12] Determine if there exists some absolute constant c such that $sat(n,\mathcal{S}(K_p)) < cn$ for all p and large enough n.*

3 Cycles

We now consider C_l-saturated graphs where C_l denotes the cycle on l vertices; we also consider $\mathcal{C}_{\geqslant l}$-saturated graphs, where $\mathcal{C}_{\geqslant l}$ denotes the family of all cycles of length at least l. We begin by discussing the known results for small values of l, after which we focus on the case when $l = n$ (i.e. the Hamilton case). The reader will find that for small values of l exact results are known only for l at most 5. Finding precise values appears to be quite difficult. For $l = n$, the saturation number is established through the collective work of many people. There are several interesting questions regarding the behavior of $sat(n,C_l)$.

3.1 Cycles of small length

In his text on extremal graph theory (p. 167, Problem 39), B. Bollobás [Bol04] gave the problem of estimating $sat(n,C_l)$ for $3 \leqslant l \leqslant n$. When $l = 3$, as $C_3 \cong K_3$, the value of $sat(n,C_3) = n-1$ is given by the result of [EHM64]. In 1972 L.T. Ollmann [Oll72] determined that $sat(n,C_4) = \lfloor \frac{3n-5}{2} \rfloor$ for $n \geqslant 5$ (this differs from the erroneous value for the function for this case given in [Bol78] p.167, Problem 40) and gave the set of extremal graphs. Later, Zs. Tuza [Tuz89] gave a shorter proof. Tuza's proof is a rare instance in which an inductive argument (for a particular case) is used in proving a lower bound on $sat(n,F)$. A slight extension was given by D. Fisher, K. Fraughnaugh, L. Langley [FFL97]. A graph is P_l*-connected* if every pair of nonadjacent nodes is connected by a path with l vertices. (It should be noted that this concept has sometimes been defined as a path with l edges, as opposed to l vertices.) Observe that a C_l-saturated graph is necessarily P_l-connected, though a P_l-connected graph need not be C_l-saturated. Fisher et al. determined the minimum size of a P_4-connected graph, thus generalizing Ollmann's result. This class of extremal graphs properly contains those of Ollmann.

Theorem 17. *[Oll72],[Tuz89],[FFL97] For $n \geqslant 5, sat(n,C_4) = \lfloor \frac{3n-5}{2} \rfloor$.*

Work done by J. Cooper, J. Lenz, T.D. LeSaulnier, P. Wenger, and D. West on uniquely C_4-saturated graphs will be discussed in Section 17.

D. Fisher, K. Fraughnaugh, and L. Langley [FFL95] gave an upper bound for $sat(n,C_5)$ of $\lceil \frac{10}{7}(n-1) \rceil$. Subsequently, a very technical proof given by Y.-C. Chen [Che09] has shown that this upper bound also serves as a lower bound for $n \geqslant 21$. In further work [Che11], Y.-C. Chen has determined $Sat(n, C_5)$ - an impressive feat considering the number and structure of the extremal graphs involved.

Theorem 18. *[Che09][Che11] For $n \geqslant 21, sat(n, C_5) = \lceil \frac{10}{7}(n-1) \rceil$.*

C. Barefoot, L. Clark, R. Entringer, T. Porter, L. Székely, and Zs. Tuza [BCE+96] gave constructions that showed that for $l \neq 8$ or 10 and n sufficiently large there exists a positive constant c such that $sat(n, C_l)$ is bounded above by $n + c\frac{n}{l}$. For small values of l their constructions rely upon, what they call, *C_l-builders*. C_l-builders are C_l-saturated graphs (of generally small order) which are used to "build" C_l-saturated graphs of large order by identifying many copies of the C_l-builder at a particular vertex. The main result in [Che11] implies that most graphs in $Sat(n, C_5)$ have this structure. Note that the particular vertex at which the copies are identified is a cut-vertex. Their construction for $l = 6$ gives that $sat(n, C_6) \leqslant \frac{3n}{2}$ for $n \geqslant 11$.

When $l = 6$, a construction given by R. Gould, T. Łuczak, and J. Schmitt [GŁS06] (see Section 3 of [GŁS06]) yields $sat(n, C_6) \leqslant \lfloor \frac{3n-3}{2} \rfloor$. A different construction method given by M. Zhang, S. Luo, and M. Shigeno [LSZ15] provides the same upper bound. These authors also provide the best known lower bound.

Theorem 19. *1. ([GŁS06], [LSZ15]) For $n \geqslant 9, sat(n, C_6) \leqslant \lfloor \frac{3n-3}{2} \rfloor$;*

2. ([LSZ15]) $sat(n, C_6) \geqslant \lceil \frac{7n}{6} \rceil - 2$.

Further, Zhang et al. [LSZ15] believe the upper bound is tight.

Conjecture 1. *[LSZ15] For $n \geqslant 9, sat(n, C_6) = \lfloor \frac{3n-3}{2} \rfloor$.*

Gould et al. [GŁS06] did improve the constant c of the upper bound given in [BCE+96] for all $l \geqslant 8$. For certain values of l their constructions resemble a bicycle wheel and do not contain cut-vertices. These wheel constructions showed that $sat(n, C_l) \leqslant (1 + \frac{2}{l-\epsilon(l)})n + O(l^2)$, where $\epsilon(l) = 2$ for l even $\geqslant 10$, $\epsilon(l) = 3$ for l odd $\geqslant 17$. Z. Füredi and Y. Kim [FK13] improved upon these bounds with a much simpler construction.

Barefoot et al. also gave the first non-trivial lower bound on $sat(n, C_l)$ for $n \geqslant l \geqslant 5$. Füredi and Kim improved upon their argument to obtain a better lower bound.

The main result of [FK13] is the following.

Theorem 20. *[FK13] For all $l \geqslant 7$ and $n \geqslant 2l - 5$,*

$$(1 + \frac{1}{l+2})n - 1 < sat(n, C_l) < (1 + \frac{1}{l-4})n + \binom{l-4}{2}.$$

The reader will notice that a gap still exists between upper and lower bounds. However, Füredi and Kim believe that the constructions that yield the upper bound are essentially optimal and they pose the following.

Conjecture 2. *[FK13] There exists an l_0 such that $sat(n, C_l) = (1 + \frac{1}{l-4})n + O(l^2)$ holds for each $l > l_0$.*

We now turn our attention to $sat(n, \mathcal{C}_{\geqslant l})$, where $\mathcal{C}_{\geqslant l}$ denotes the family of all cycles of length at least l. In other language, $\mathcal{C}_{\geqslant l}$ is the set of all graphs that include C_l and any subdivision of it. As all $\mathcal{C}_{\geqslant 3}$-saturated graphs are trees, it is trivial to see that $sat(n, \mathcal{C}_{\geqslant 3}) = n-1$, which is the same value as $sat(n, C_3)$. M. Ferrara, M. Jacobson, K.G. Milans, C. Tennenhouse, and P.S. Wenger [FJM+12] initiated the study of $sat(n, \mathcal{C}_{\geqslant l})$, giving the following results.

Theorem 21. *[FJM+12]*

1. *If* $n \geqslant 1$*,* $sat(n, \mathcal{C}_{\geqslant 3}) = n - 1$*;*
2. *If* $n \geqslant 1$*,* $sat(n, \mathcal{C}_{\geqslant 4}) = n + \lfloor \frac{n-3}{4} \rfloor$*;*
3. *If* $n \geqslant 5$*,* $sat(n, \mathcal{C}_{\geqslant 5}) = \lceil \frac{10(n-1)}{7} \rceil$*.*

Subsequently, Y. Ma, X. Hou, D. Hei, J. Gao [MHHG21] determined the value of $sat(n, \mathcal{C}_{\geqslant 6})$ which relied on a detailed analysis of the structure of $\mathcal{C}_{\geqslant 6}$-saturated graphs.

Theorem 22. *[MHHG21] If* $n \geqslant 10$*,* $sat(n, \mathcal{C}_{\geqslant 6}) = \lceil \frac{3n}{2} \rceil$*.*

The observant reader will notice that $sat(n, \mathcal{C}_{\geqslant l}) = sat(n, C_l)$ for $l = 3$ and $l = 5$. Equality does not hold for any other value of l ([MHHG21]).

Among other results, Ma et al. [MHHG21] also give the following.

Theorem 23.

1. *[MHHG21] If* $2l \geqslant n \geqslant l \geqslant 6$*,* $sat(n, \mathcal{C}_{\geqslant l}) \geqslant n + \frac{l}{2}$*.*
2. *[MHHG21] If* $n \geqslant l \geqslant \frac{n}{2} \geqslant 28$*,* $sat(n, \mathcal{C}_{\geqslant l}) = n + \lceil \frac{l}{2} \rceil$*.*

Problem 6. *Determine* $sat(n, \mathcal{C}_{\geqslant l})$ *for* $7 \leqslant l < \frac{n}{2}$*.*

We end this subsection with a list of open problems and questions.

Question 1. *[BCE+96] Is* $sat(n, C_l)$ *a convex function of* l*,* $l > 3$*, for fixed* n*? Or is it convex at least when the parity of* l *is fixed?*

If the answer to this question is in the affirmative, then one ought to be able to find a better upper bound for, say, $l = 9$.

Problem 7. *[BCE+96] Determine the value of* l *which minimizes* $sat(n, C_l)$ *for fixed* n*.*

Question 2. *[BCE+96] Is* $\limsup_n sat(n, C_l)/n$ *a decreasing function of* l*, at least for odd* l *and even* l*, respectively?*

Question 3. *[Łuc]For every* $x \in [0, 1]$ *define a function* $f(x)$ *in the following way:*

$$f(x) = \limsup_{n \to \infty} (sat(n, C_{\lceil xn \rceil}))/n - 1.$$

As $f(1) = \frac{1}{2}$*, and, most probably,* $f(x) = O(1/x)$ *for small* x*, does* $f(x) \to 0$ *as* $x \to 0$*? Is* $f(x)$ *continuous in* $[0, 1]$*? Is it strictly increasing? For instance, is it true that, say,* $f(0.99) = \frac{1}{2}$*?*

3.2 Hamilton cycles

We now turn our attention to the case when $l = n$.

In an effort to understand the structure of Hamiltonian graphs or conditions which imply when a graph is Hamiltonian, authors have often focused on when a graph *just* fails to be Hamiltonian. One such focus is C_n-saturated graphs, often referred to as maximally non-Hamiltonian (MNH) graphs. Thus the question of determining $sat(n, C_n)$ is rather 'natural.'

The first result on C_n-saturated graphs of minimal size is due to A. Bondy [Bon72]. He showed that if such a graph G of order at least 7 has m vertices of degree two then it has size at least $\frac{1}{2}(3n + m)$. As an MNH graph with a vertex of degree one must be a clique with a pendant edge (which in fact implies that the graph is edge maximum), this result implies that $sat(n, C_n) \geqslant \lceil \frac{3n}{2} \rceil$. As a result, it is logical to consider 3-regular graphs in the search for graphs in $Sat(n, C_n)$. Bondy also pointed out that the Petersen graph, which has girth five, is in $Sat(10, C_{10})$.

Another famous 3-regular graph, the Coxeter graph, which has girth seven, was shown to be in $Sat(28, C_{28})$ by L. Clark and R. Entringer [CE83]. Previously, however, W. Tutte [Tut60] had shown it to be non-Hamiltonian and H.S.M. Coxeter himself [Cox81] knew that his graph was an MNH graph.

If a graph is 3-regular and Hamiltonian, then it is 3-edge colorable. This makes 4-edge-chromatic 3-regular graphs suitable candidates for $Sat(n, C_n)$. Over the course of several papers [CE83], [CCES86], [CES92], where each paper included some subset of the following authors - L. Clark, R. P. Crane, R. Entringer, and H.D. Shapiro, it was shown that $sat(n, C_n)$ does indeed equal $\lceil \frac{3n}{2} \rceil$ for even $n \geqslant 36$ and odd $n \geqslant 53$. They showed that graphs which help establish equality include the Isaacs' flower snarks (which R. Isaacs [Isa75] showed were 4-edge-chromatic 3-regular graphs), most of which have girth six, and variations of them. These variations are obtained through "blowing up" a degree three vertex into a triangle. Through the aid of a computer search, X. Lin, W. Jiang, C. Zhang, and Y. Yang [LJZY97] analyzed the remaining small cases and were able to determine that the value of $sat(n, C_n)$ matched the lower bound provided by Bondy except in a few small cases. Together, these results imply the following.

Theorem 24. *For all even $n \geqslant 20$ and odd $n \geqslant 17$, we have $sat(n, C_n) = \lceil \frac{3n}{2} \rceil$.*

P. Horák and J. Širáň [HŠ86] constructed triangle-free MNH graphs of near minimal size using a construction technique of C. Thomassen [Tho74]. Thomassen's technique involves "pasting" together two graphs at two vertices of degree three. Thomassen was interested in constructing families of hypo-Hamiltonian graphs (non-Hamiltonian graphs which become Hamiltonian upon the removal of any vertex) and his technique builds a new hypo-Hamiltonian graph from two smaller ones. Horák and Širáň show that the technique also works for MNH graphs when the smaller graphs are copies of either the Petersen graph or an Isaacs' flower snark. The technique does not decrease the length of the shortest cycle, thus the graphs constructed are triangle-free. L. Stacho [Sta96] also used this technique on copies of the Coxeter graph, yielding MNH graphs of girth seven.

Problem 8. *[HŠ86] Does there exist an MNH graph of girth greater than seven?*

Problem 9. *Furthermore, if there is such a graph, is there one of (near) minimal size?*

L. Stacho [Sta98] also proved that $|Sat(n, C_n)| \geqslant \lceil \frac{3n}{2} \rceil$ for all $n \geqslant 88$ and showed that $\lim_{n\to\infty} |Sat(n, C_n)| = \infty$, answering a question of L. Clark and R. Entringer [CE83].

4 Trees and Forests

Early results concerning stars, paths, and sets of independent edges – results that both identified the saturation number precisely and characterized the set of minimum saturated graphs – spurred interest in saturation of trees and forests. Recall that the best known upper bound for $sat(n, G)$ stated in Theorem 2 in the Introduction depends upon an understanding of $(K_{1,k-1})$-saturated graphs. Most of the results in this section are of the form $sat(n, G)$ where G is a tree or a forest and will begin with trees, followed by forests, and finish with some miscellaneous results. Later sections of this survey, Saturation in Hypergraphs (Section 5), Saturation in Hosts other than K_n (Section 6), Saturation Spectrum of Graphs (Section 8), and Unique Saturation (Section 17) also contain results on trees.

The most intriguing question regarding saturation numbers and trees is in Section 14, Question 15, which essentially asks how to identify trees with small saturation number.

4.1 Trees

Let $S_k = K_{1,k-1}$ denote the star on k vertices. In [KT86], L. Kászonyi and Zs. Tuza established $sat(n, S_k)$, characterized $Sat(n, S_k)$, and proved that, of all the trees on k vertices, S_k, has the largest saturation number.

Both the value of $sat(n, S_k)$ and the characterization of $Sat(n, S_k)$ are proved simultaneously by observing that any S_k-saturated graph has maximum degree at most $k-2$ and that the set of vertices of degree less than $k-2$ must induce a complete graph. The number of edges in such a graph is bounded below by $f(s) = (n-s)(k-2)/2 + \binom{s}{2}$ where s is the number of vertices of degree less than $k-2$. All that is left is to show that f is minimized at the respective values and to construct the graphs that realize these lower bounds.

Similar results were given by K. Balińska, L. Quintas, and K. Zwierzyński [BQZ06]. They considered S_k-saturated graphs where the number of vertices of degree strictly less than $k-1$ is bounded.

Theorem 25. *[KT86] Let $S_k = K_{1,k-1}$ denote a star on k vertices. Then,*

$$sat(n, S_k) = \begin{cases} \binom{k-1}{2} + \binom{n-k+1}{2} & \text{if } k \leqslant n \leqslant \frac{3k-3}{2}, \\ \lceil \frac{k-2}{2} n - \frac{(k-1)^2}{8} \rceil & \text{if } \frac{3k-3}{2} \leqslant n, \end{cases}$$

$$Sat(n, S_k) = \begin{cases} K_{k-1} \cup K_{n-k+1} & \text{if } k \leqslant n \leqslant \frac{3k-3}{2}, \\ G^* & \text{if } \frac{3k-3}{2} \leqslant n, \end{cases}$$

where G^* is the disjoint union of a $(k-1)$-regular graph on $n - \lfloor k/2 \rfloor$ vertices and $K_{\lfloor k/2 \rfloor}$ if $k-1$ or $n - \lfloor k/2 \rfloor$ is even. Otherwise G^* consists of a nearly-$(k-1)$-regular graph on $n - \lfloor k/2 \rfloor$ vertices with an edge to a $K_{\lfloor k/2 \rfloor}$.

Furthermore, let T be a tree on k vertices such that $T \neq S_k$, then $sat(n, T) < sat(n, S_k)$.

J. Faudree, R. Faudree, R. Gould, and M. Jacobson [FFGJ09a] show that for a fixed $k \geqslant 5$ the tree on k vertices with the smallest saturation number is the tree obtained by subdividing a single edge $K_{1,k-2}$.

Theorem 26. *[FFGJ09a] Let T_0 be the tree obtained by subdividing a single edge $K_{1,k-2}$. For $n \geqslant k+2$, $sat(n, T_0) = n - \lfloor (n+k-2)/k \rfloor$ and $Sat(n, T_0)$ consists of K_2 along with a forest of $\lfloor (n-2)/k \rfloor$ stars on k or more vertices.*

Also in [KT86], Kászonyi and Tuza found $sat(n, P_k)$ for all k and n sufficiently large and again characterized the family of graphs in $Sat(n, P_k)$. They prove that all P_k-saturated trees contain a common subgraph which we will refer to as *a perfect, almost-binary tree* defined as follows. For $k = 2m$, this perfect, almost-binary tree can be constructed by starting with a perfect binary tree of depth m and duplicating, in its entirely, one branch from the root such that in the result the root has degree three. For $k = 2m+1$, the perfect, almost-binary tree can be constructed by starting with two copies of a perfect binary tree of depth m and adding an edge between their roots. Moreover, every P_k-saturated tree can be obtained by either multiplying branches of these perfect almost-binary trees or by adding a single pendant vertex to vertices of degree at least 3. In the theorem below, observe that a_k is the number of vertices in the perfect almost-binary tree. Note that the small order cases were handled in an ad hoc manner.

Theorem 27. *[KT86] Let P_k be a path on $k \geqslant 6$ vertices and let T_k be the almost binary tree defined above.*
Let $a_k = \begin{cases} 3 \cdot 2^{m-1} - 2 & \text{if } k = 2m, \\ 4 \cdot 2^{m-1} - 2 & \text{if } k = 2m+1. \end{cases}$ Then, for $n \geqslant a_k$, $sat(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$ and every graph in $Sat(n, P_k)$ consists of a forest with $\lfloor n/a_k \rfloor$ components each of which contains T_k as a subgraph.

Note that the result above only applies for $n \geqslant 3 \cdot 2^{(k-2)/2} - 2$ for even $k \geqslant 6$ and $n \geqslant 2^{(k+1)/2} - 2$ for odd $k \geqslant 7$. In fact, $sat(n, P_k)$ is known for all n for $k \leqslant 6$ (see [KT86] and [DW04b]). Dudek and Wojda [DW04b] found $sat(n, P_k) = n$ for $b_k \leqslant n < a_k$ where $b_k = 3 \cdot 2^{(k-4)/2}$ for even $k \geqslant 6$ and $n \geqslant 3 \cdot 2^{(k-3)/2}$ for odd $k \geqslant 7$. Graphs in $Sat(n, P_k)$ on this interval were also characterized. Upper bounds for $sat(n, P_k)$ for $n \geqslant k$ for all but a finite number of values of k can be found in [DKW06]. The examples used to establish these upper bounds make use of several special properties of snarks.

Problem 10. *Determine $sat(n, P_k)$ and $Sat(n, P_k)$ when $k \leqslant n \leqslant b_k$.*

When $n = k$ much more is known. Dudek et al. in [DKW06] exploited some properties of snarks established in [CE83, CES92, LJZY97] to construct graphs in $Sat(n, P_n)$ for $n \geqslant 54$ and several small order cases. Though the exact structure of all graphs in $Sat(n, P_n)$ seems to be quite complicated, this at least established an upper bound on $sat(n, P_n)$. The lower bound from [DKW06] was improved by M. Frick and J. Singleton [FS05]. There exist a finite number of small order cases for which $sat(n, P_n)$ remains unknown.

Theorem 28. *[DKW06] and [FS05] For $n \geqslant 54$, $sat(n, P_n) = \lceil \frac{3n-2}{2} \rceil$.*

Problem 11. *Determine $sat(n, P_n)$ for the remaining small order cases.*

Faudree et al. in [FFGJ09b] established $sat(n, T)$ precisely for several other infinite families of trees but largely without characterizations of $Sat(n, T)$. In the same paper, it is also established that any tree T with a sufficiently long sub-path (i.e. sub-tree with vertices of degree 2 ending with a vertex of degree 1) must have $sat(n, T) < n$ for sufficiently large n. For most trees T, $sat(n, T)$ is unknown.

4.2 Forests

Most of the results on the saturation of forests concern *linear* forests: forests such that each component is a path. The first result of this type concerned a forest consisting of m independent edges.

Theorem 29. *[Mad73, KT86] For $n \geqslant 3m - 3$*

$$sat(n, mK_2) = 3m - 3$$

and

$$Sat(n, mK_2) = (m-2)K_3 \cup (n - 3m + 3)K_1.$$

For $2m < n < 3m - 3$ and $k \geqslant 3$, the saturation number is unknown.

Problem 12. *Determine $sat(n, mK_2)$ and $Sat(n, mK_2)$ for $2m \leqslant n \leqslant 3m - 4$ and $k \geqslant 3$.*

G. Chen, R. Faudree, J. Faudree, R. Gould, and M. Jacobson (in [CFF+15a]) demonstrated that $sat(n, F)$ for a general linear forest $F = P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_t}$ is determined by the smallest path in the forest.

Theorem 30. *[CFF+15a] If $F = P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_t}$ where $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_t$, $q = \left(\sum_{i=1}^{t} k_t\right) - 1$ and a_{k_t} is defined as in Theorem 27, then*

$$sat(n, F) = \begin{cases} n - \left\lfloor \frac{n}{a_{k_t}} \right\rfloor + c(n) & \text{if } k \neq 4 \\ n - \left\lfloor \frac{n}{2} \right\rfloor + c(n) & \text{if } k = 4 \end{cases}$$

for some constant $c(n)$ such that $0 \leqslant c(n) \leqslant \binom{q}{2} - q + \left\lceil \frac{q}{a_{k_t}} \right\rceil$.

In [CFF$^+$15a], the Theorem above is improved in several special cases such as when $t=2$ and when $k_1=k_2=\cdots=k_t$.

There are a number of recent results on the saturation numbers of specific linear forests listed below:

Source	Saturation Number (for n sufficiently large)
[CFF$^+$15b]	$sat(n,tP_3)=\left\lfloor\frac{n+6t-6}{2}\right\rfloor$ for $t\in\{2,3\}$
[JF16, LH19]	$sat(n,tP_3\cup sK_2)=3(r+s-10)$, for $s\geqslant 3$
[CFF$^+$15b]	$sat(n,2P_4)=\left\lfloor\frac{n+13}{2}\right\rfloor$
[CFF$^+$15b]	$sat(n,P_4\cup sK_2)=3s+7$, for $s\geqslant 3$
[FW15]	$sat(n,P_5\cup sK_2)=\min\{\left\lceil\frac{5n-4}{6}\right\rceil,3s+12\}$ for $s\geqslant 3$
[Son17]	$sat(n,P_4\cup P_3\cup tP2)=3t+10$, for $t\geqslant 1$

Several conjectures concerning the saturation numbers of linear forests can be found in [CFF$^+$15b], the simplest of which is below.

Conjecture 3. *For $t\geqslant 2$ and n sufficiently large, $sat(n,tP_3)=\left\lfloor\frac{n+6t-6}{2}\right\rfloor$.*

Given a graph G on n vertices, a spanning subgraph that is a linear forest is called a *path cover.* An *m-path cover* of G is a spanning forest F such that all components of F are paths and F has at most m components. We say a graph G is *m-path cover saturated* if G does not contain an m-path cover but connecting any two nonadjacent vertices with an edge creates an m-path cover. The notion of an m-path cover saturated graph is one natural extension of Hamiltonian path saturated graphs to forest saturated graphs. Dudek et al. in [DKW03] established a lower bound on the saturation number of m-path cover saturated graphs.

Theorem 31. *[DKW03] Let n and m be positive integers such that $n\geqslant m+1$. Every m-path cover saturated graph on n vertices contains at least $\frac{3n}{2}-3(m+1)$ edges.*

Problem 13. *Determine the saturation number for m-path-cover-saturated graphs precisely.*

In addition, A. Jambulapati and R. Faudree ([JF16]) used results on Hamiltonian path saturated graphs to bound the saturation number for forests $P_n\cup P_2$ in graphs of order $n+2$.

Theorem 32. *For $n\geqslant 28$, $\frac{3n}{2}-4\leqslant sat(n+2,P_n\cup P_2)\leqslant\frac{3n}{2}+14$.*

4.3 Additional Topics

A *degree monotone path* is one in which the sequence of degrees of the vertices is monotonic when listed in the order in which they appear on the path. The number of vertices on a longest degree monotone path in graph G is denoted by $mp(G)$. A graph G is *monotone* *path saturated* if $mp(G) < mp(G+e)$ for every $e \in \overline{G}$. In [CLZ15], Caro et al. use $h(n,k)$ to denote the least number of edges in a graph G on n vertices such that $mp(G) < k$ but $k \leqslant mp(G+e)$ for every $e \in \overline{G}$.

Theorem 33. *[CLZ15] For $n \geqslant 3$, $k \geqslant 3$, and $n \leqslant h(n,k)$.*
If k is odd, then $h(n,k) \leqslant \frac{n(3k-1)}{12}$ for $n \equiv 0 \; mod \, 3(k-1)/2$.
If k is even, then $h(n,k) \leqslant \frac{n(3k+8)(k-2)}{4(3k-4)}$ for $n \equiv 0 \; mod \, (3k-4)/2$.

The authors obtain exact values for $h(n,k)$ for $k = 3, 4$. The end of their paper lists eight open problems, one of which is below.

Problem 14. *Determine $h(n,5)$ exactly. In particular, is it true that $h(n,5) = \frac{7n(1+o(1))}{6}$?*

One last observation is that a number of results have emerged concerning tree saturation in connected or unicyclic graphs. These results generally appear as lemmas and are used as tools in pursuit of larger goals such as finding $sat(n, P_k)$ for smaller values of n or for establishing the saturation spectrum of trees (see Section 8). For example, in [DW04b], there are a number of results on unicyclic P_k-saturated graphs. Results on connected P_k-saturated graphs can be found in [DKW06, Ash11] and [GTWZ12, BD18].

5 Hypergraphs

We now consider F-saturated graphs where F is a hypergraph. All of the results in this section, except one (Theorem 42), assume the graphs are k-uniform (all edges are of size k). Thus, unless stated otherwise, all graphs referenced in this section are k-uniform hypergraphs.

5.1 Complete hypergraphs

We introduce the following notation. Consider a vertex partition $S_1 \cup \ldots \cup S_p$ of $V(F)$ where $|S_i| = s_i$. For $k \leqslant p$, let $W^k_{s_1,\ldots,s_p}$ denote the k-uniform hypergraph consisting of all k-tuples that intersect k different parts (and call this the *weak* generalization of a complete graph). Let $S^k_{s_1,\ldots,s_p}$ denote the k-uniform hypergraph consisting of all k-tuples that intersect at least two parts (and call this the *strong* generalization of a complete graph). When $W^k_{1,1,\ldots,1} = S^k_{1,1,\ldots,1}$, we use K^k_n.

An early generalization of Theorem 1 was given by B. Bollobás [Bol65].

Theorem 34. *[Bol65]*

$$sat(n, K^k_p) = sat(n, W^k_{1,\ldots,1}) = sat(n, S^k_{1,\ldots,1}) = \binom{n}{k} - \binom{n-p+k}{k}$$

where p counts the number of classes in the partition. Furthermore, there exists a unique extremal graph.

Bollobás achieved this as the result of introducing a powerful weight inequality, the simplest version of which is given in the introduction as Theorem 3. This inequality is an extension of the Lubell-Yamamoto-Meshalkin inequality, itself an extension of Sperner's Lemma from 1928. Importantly, N. Alon [Alo85] generalized Bollobás' weight inequality; in fact, it is a special case of a corollary to his main result.

P. Erdős, Z. Furedi, and Zs. Tuza [EFT91] consider the saturation problem for families of hypergraphs with a fixed number of edges. Among these are the graphs $S^k_{1,k}$.

Theorem 35. *[EFT91] For $n \geqslant 4$, $sat(n, S^3_{1,3}) = \lfloor \frac{(n-1)^2}{4} \rfloor$. Moreover, there are two or one extremal hypergraphs according as n is odd or even.*

They also determined the asymptotic behavior of the function for the graph $S^k_{1,k}$ for $n > k \geqslant 2$. O. Pikhurko [Pik00] went further.

Theorem 36. *[Pik00] Let $m > k \geqslant 2$. Then*

$$\frac{m-k}{2}\binom{n}{k-1} \geqslant sat(n, S^k_{1,m-1}) \geqslant \frac{m-k}{2}\binom{n}{k-1} - O(n^{k-4/3}).$$

Later, Pikhurko [Pik04] posed the following.

Conjecture 4. *[Pik04] For $l \leqslant k-1$ and $l+m > k$,*

$$sat(n, S^k_{l,m}) = \frac{m+2l-k-1}{2(k-1)!} n^{k-1} + o(n^{k-1}).$$

5.2 Asymptotics

With the thought of extending Theorem 2 to hypergraphs, Zs. Tuza [Tuz86] (more readily available in [Tuz88]) conjectured that for any k-uniform hypergraph F, $sat(n, F) = O(n^{k-1})$. This was positively confirmed.

Theorem 37. *[Pik99a] For any finite family $\mathcal{F}$ of k-uniform hypergraphs, we have*

$$sat(n, \mathcal{F}) \;=\; O(n^{k-1}).$$

More generally, we can ask the following.

Question 4. *[Pik04] Does $sat(n, \mathcal{F}) = O(n^{k-1})$ for any infinite family of k-uniform hypergraphs?*

In light of the irregularity of the sat-function, which is discussed more fully in Section 14, Pikhurko asked: does there exist a finite family $\mathcal{F}$ of k-uniform hypergraphs, $k \geqslant 3$, for which the ratio $\frac{sat(n,\mathcal{F})}{n^{k-1}}$ does not tend to any limit? N. Behague [Beh18] answered in the affirmative.

Theorem 38. *[Beh18] For all $k \geqslant 3$ there exists a family $\mathcal{F}$ of four k-regular hypergraphs such that $\lim_{n\to\infty} \frac{sat(n,\mathcal{F})}{n^{k-1}}$ does not converge.*

It remains an open question whether the statement of the Theorem above holds if the family of k-regular hypergraphs is replaced by a single k-regular hypergraph.

Conjecture 5. *[Pik99b] The limit* $\lim_{n\to\infty} \frac{sat(n,F)}{n^{k-1}}$ *exists for every k-regular hypergraph F.*

For a hypergraph F and edges $E, E' \in F$, the *density of an edge* E, denoted by $D(E)$, is the largest natural number D such that there is an $E', E \neq E'$ with $|E \cap E'| \geqslant D$. The *local density of the hypergraph* F, $D(F)$, is $\min\{D(E) : E \in F\}$. Zs. Tuza [Tuz92] conjectured the following.

Conjecture 6. *[Tuz92] For a hypergraph F there exists a constant c depending on F such that $sat(n,F) = cn^{D(F)} + O(n^{D(F)-1})$.*

5.3 Berge-$\mathcal{F}$ Saturated Hypergraphs

For a graph $F = (V,E)$, a hypergraph H on vertex set V is called *Berge-F* if there is a bijection $\phi : E(F) \to E(H)$ such that $e \subseteq \phi(e)$ for all $e \in E(F)$.

A hypergraph G is *Berge-F-saturated* if G contains no Berge-F subgraph but adding any edge to G creates a copy of a Berge-F subgraph. The saturation number for Berge-F hypergraphs is the least number of edges in a Berge-F-saturated hypergraph. Generally, the host graph is required to be a k-uniform hypergraph; thus, we use the notation $sat(n, Berge^k\text{-}F)$ to emphasize this assumption.

Results on Berge-F-saturated graphs first appeared in the work of S. English, P. Gordon, N. Graber, A. Mehutku, and E. Sullivan [EGG+19]. English et al. established $sat(n, Berge^k\text{-}F)$ precisely when F is a K_3, a matching and for certain stars. In addition, the authors establish upper bounds when F is a cycle and both upper and lower bounds when F is a path. A subset of results are summarized below. The main result, on paths, references the constant $a_{m,k}$, determined by the k-uniformity and the number of vertices in the path P_m. This constant plays a similar role as the constant a_m in Theorem 27 on the saturation number of trees in graphs.

Theorem 39. *[EGG+19]*

- *For all $n \geqslant k(l-1)$, $sat(n, Berge^k\text{-}lK_2) = l-1$.*
- *For all $n \geqslant k+1$, $sat(n, Berge^k\text{-}K_3) = \lceil \frac{n-1}{k-1} \rceil$.*
- *If $k \geqslant m-1$ and $n < p(k-(p-2)) + (p-2)$, then $sat(n, Berge^k\text{-}C_p) \leqslant \lceil \frac{n-p+2}{k-p+2} \rceil$.*
- *If $k = p-2$ and $n \geqslant m^2$, then $sat(n, Berge^k\text{-}C_p) \leqslant \lfloor \frac{n-1}{p-2} \rfloor (p-1) + \frac{(n-1)\bmod(p-2)}{k-2}$.*
- *If $k \leqslant m-3$, $l = \max\{m/2+1, k+1\}$ and $n \geqslant l^2$, then $sat(n, Berge^k\text{-}C_p) \leqslant \lfloor \frac{n-1}{l-1} \rfloor \binom{l}{k} + ((n-1)\bmod(l-1))\binom{l}{k-1}$.*

- *Let $k \geqslant 3$ and $k \neq 5$. Let $m \geqslant 10$, and $n \geqslant (k-1)a_{m,k} + k - 1$. Then,*

$$\frac{1}{k-1}(n - \lfloor \frac{n-k-2}{(k-1)a_{m,k}+1} \rfloor) - k - 2 \leqslant sat(n, Berge^k\text{-}P_m)$$

  and

$$sat(n, Berge^k\text{-}P_m) \leqslant \lceil \frac{1}{k-1}(n - \lfloor \frac{n}{(k-1)a_{m,k}+1} \rfloor) \rceil$$

  where $a_{m,k}$, a constant depending on m and k, is the minimum number of edges of a k-uniform Berge-P_m saturated linear tree on at least $k+1$ vertices.

At present, there do not exist lower bounds for cycles or any results for complete graphs beyond K_3. The authors conjecture that the upper bound for $sat(n, Berge^k\text{-}P_m)$ is the correct one.

Conjecture 7. *[EGG+19] Let $k \geqslant 3$, $m \geqslant 10$, and $n \geqslant (k-1)a_m^{(k)}$. Then*

$$sat(n, Berge^k\text{-}P_m) \leqslant \lceil \frac{1}{k-1}(n - \lfloor \frac{n}{(k-1)a_{m,k}+1} \rfloor) \rceil.$$

The result on stars in Theorem 39 required that the number of end-vertices on the star be one more than the regularity of the host hypergraph. A more general result by B. Austhof and S. English [AE19] on stars soon followed.

Theorem 40. *[AE19] For all $k \geqslant 3$, $l \in \mathbb{N}$, and sufficiently large n,*

$$sat(n, Berge^k\text{-}K_{1,\ell}) = min_{a \in [n], \binom{a-1}{k-1} \leqslant l-2} \left\{ \left\lceil \frac{(l-1)(n-a)}{k} \right\rceil + \binom{a}{k} \right\}.$$

Austhof et al. [AE19] achieve these bounds by using *linear* k-uniform hypergraphs (a linear hypergraph satisfies the property that $|e_1 \cap e_2| \leqslant 1$ for all edges e_1 and e_2) that are *nearly d-regular*, meaning every vertex has degree either d or $d-1$ and fewer than k have the latter quality. It is not trivial that such hypergraphs exists for all numbers of vertices, so Austhof et al. [AE19] also prove following lemma:

Lemma 1. *[AE19] Let $d \geqslant 1$ and $k \geqslant 2$. Then, for sufficiently large n, there exists a nearly d-regular k-uniform hypergraph on n vertices.*

Since $sat(n, Berge^k\text{-}F)$ is known precisely only when F is a star or K_3, it is natural to consider other graphs.

Problem 15. *Determine $sat(n, Berge^k\text{-}F)$ for $F \not\cong K_{1,l}$ and $F \not\cong K_3$.*

English et al. [EGG+19] conjectured that for any fixed family of graphs $\mathcal{F}$ we have $sat(n, Berge^k\text{-}F) = O(n)$. This conjecture has been confirmed in several instances. English et al. [EGMT19] showed the conjecture holds for all graphs F when the uniformity of the hypergraph, k is 3, 4, or 5. Gerbner et al. [GPTV21] demonstrated the conjecture holds for complete multipartite graphs and graphs with the property that there exist two vertices of the smallest degree that are adjacent. The general conjecture remains open.

Conjecture 8. *[EGG+19] For any fixed finite family of graphs $\mathcal{F}$, $sat(n, Berge^k\text{-}\mathcal{F}) = O(n)$.*

English et al. [EGMT19] extended Conjecture 8 to hypergraph-based Berge hypergraphs. The authors show using paths that, if correct, the conjecture is best possible.

Conjecture 9. *[EGMT19] Let $3 \leqslant r \leqslant k$ be integers and let $F^{(r)}$ be any r-uniform hypergraph. Then,*

$$sat(n, Berge^k\text{-}F^{(r)}) = O(n^{r-1}).$$

Theorem 41. *[EGMT19] Let $3 \leqslant r \leqslant k < l$ be integers and let G be a k-uniform Berge-$P_l^{(r)}$-saturated hypergraph of order n. Then*

$$|E(G)| = \Theta(n^{r-1}).$$

All previous results concern Berge-F saturation in k-uniform hypergraphs; Theorem 42 does not make this restriction. For Berge-F saturation in arbitrary hypergraphs, saturation numbers are known.

Theorem 42. *[AW19] Let F be a graph with no isolated vertices, and let $n \geqslant |V(F)|$. Let H be a hypergraph on n vertices that is not necessarily k-uniform. Then the minimum number of edges in a Berge-F saturated hypergraph H is $|E(F)|$ if F is a star on at least 4 edges and $|E(F)| - 1$ otherwise.*

M. Axenovich and C. Winter [AW19] suggested that, in the context of hypergraphs, a vertex version of saturation could be investigated.

Problem 16. *[AW19] Investigate the properties of graphs saturated in that they are $\mathcal{F}$-free, but for all hyperedges e and vertices $v \notin e$, replacing e with $e \cup \{v\}$ creates a hypergraph that is not $\mathcal{F}$-free.*

5.4 A few specific problems

5.4.1 Hamiltonian Saturation

For $1 \leqslant \ell < k$, an *ℓ-overlapping Hamiltonian cycle* in a k-uniform n-vertex hypergraph G, denoted $C_n^{(k,\ell)}$, is a subgraph of G in which, for some cyclic ordering of $V(G)$, every edge consists of k consecutive vertices and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly ℓ vertices. Using this terminology, $C_n^{(2,1)}$ denotes the classic Hamiltonian cycle in a graph. In the special case that $\ell = k - 1$, the subgraph $C_n^{(k,k-1)}$ is often called a *Hamiltonian chain* or a *closed Hamiltonian chain.* A hypergraph that contains an ℓ-overlapping Hamiltonian cycle as a subgraph is called *ℓ-Hamiltonian* Observe that a necessary condition for H to be ℓ-Hamiltonian is for n to be divisible by $k - \ell$. Analogous terminology and notation can be defined for an ℓ-overlapping Hamiltonian path.

A k-uniform hypergraph H is called *ℓ-Hamiltonian saturated*, for $1 \leqslant \ell \leqslant k - 1$, if H is not ℓ-Hamiltonian but for every $e \in H^c$ the k-graph $H + e$ is such. For n divisible by $k-\ell$, let $sat(n, C_n^{(k,\ell)})$ be the smallest number of edges in an ℓ-Hamiltonian saturated k-uniform hypergraph on n vertices. Analogous notation and terminology applies to edge minimal k-uniform hypergraphs containing no ℓ-overlapping Hamiltonian path. Results on $sat(n, C_n^{(2,1)})$ and $sat(n, P_n^{(2,1)})$ can be found in Section 3 and Section 4, respectively.

Investigation of ℓ-overlapping Hamiltonian cycle (and path) saturation initially focused on the specific case where $\ell = k-1$ [KK99, Kat06, DZK10, DZ12] with results for general ℓ coming later [Ż13, RZ13, RZ16]. Most of the results focus on saturation with respect to cycles. Results on $sat(n, P_n^{(k,k-1)})$ can be found in [DZK10, DZ12].

A lower bound for $sat(n, C_n^{(k,\ell)})$ was quickly established via a simple counting argument based on the maximum number of ℓ-intersecting edges of a given edge.

Theorem 43. *[KK99, DZK10, RZ13] For integers $1 \leqslant \ell < k$, $sat(n, C_n^{k,\ell}) = \Omega(n^\ell)$.*

The bound above also applies to ℓ-overlapping Hamiltonian path saturated graphs.

As precise results for $sat(n, C_n^{(k,\ell)})$ (and $sat(n, P_n^{(k,\ell)})$) appear to be very difficult, the emphasis is on establishing the correct order of magnitude.

Conjecture 10. *[Kat06, RZ13] For all $k \geqslant 2$ and $1 \leqslant \ell < k$, $sat(n, C_n^{k,\ell}) = O(n^\ell)$.*

Results in several particular cases have been established.

Theorem 44.

- *[DZK10] For every $n \geqslant 12$, $sat(n, C_n^{(3,2)}) \leqslant O(n^{5/2})$.*
- *[RZ16] $sat(n, C_n^{4,2}) = O(n^{14/5})$.*
- *[RZ16] For $\ell \in \{\frac{2}{3}k, \frac{3}{4}k\}$ (and appropriately divisible k), $sat(n, C_n^{k,\ell}) = O(n^{\ell+1})$.*
- *[DZK10, DZ12] Let $k \geqslant 4$. Then $sat(n, C_n^{(k,k-1)}) = O(n^{k-1/2})$.*

The best nontrivial general upper bound which we now state is obtained via specialized construction methods refined from [Ż13, RZ13].

Theorem 45. *[RZ16] For all $k \geqslant 3$ and $2 \leqslant \ell < k$, $sat(n, C_n^{(k,\ell)}) = O(n^{(k+\ell)/2})$.*

5.4.2 Triangular family

Let $\mathcal{T}_k$ denote the family which consists of all k-uniform hypergraphs with three edges E_1, E_2, E_3 such that $E_1 \Delta E_2 \subseteq E_3$, where Δ denotes the symmetric difference. We call $\mathcal{T}_k$ a *triangular family.*

Theorem 46. *[Pik04] Let $k \geqslant 3$ be fixed. Then*

$$n - O(\log n) \leqslant sat(n, \mathcal{T}_k) \leqslant n - k + 1.$$

And, for $k = 3$ equality holds on the right.

Conjecture 11. *[Pik04] In Theorem 46 equality holds on the right.*

5.4.3 Intersecting hypergraphs

We will call a k-uniform hypergraph F *intersecting* (sometimes called *disjoint-edges-free*) if for every pair of edges of F the intersection of the pair is non-empty. (Some authors call such graphs k-cliques, however we refrain from doing so in light of how we wish to use this term elsewhere in this survey.) We say that such a graph is *maximal* if it cannot be extended to another intersecting hypergraph by adding a new edge and possibly new vertices. P. Erdős and L. Lovász [EL75] first investigated the minimum number and maximum number of edges in a maximal intersecting k-uniform hypergraph. In light of the topic of this survey, we are most interested in the minimum number of edges in a maximal intersecting k-uniform hypergraph, $m(k)$. Note that the function is independent of n for n sufficiently large.

Erdős and Lovász [EL75] gave a lower bound on $m(k)$ of $\frac{8k}{3} - 3$, while Z. Füredi gave an upper bound of $3k^2/4$ whenever $k = 2n$ for an integer n that is the order of a projective plane. We know from J.C. Meyer [Mey74] that trivially $m(1) = 1$ and $m(2) = 3$, and that $m(3) = 7$. S. Dow, D. Drake, Z. Füredi, and J. Larson [DDFL85] improved the previously mentioned lower bound and gave the following.

Theorem 47. *[DDFL85] For all* $k \geqslant 4, \quad m(k) \geqslant 3k$.

This result together with the upper bound of Füredi gives $m(4) = 12$.

Problem 17. *Determine the value of* $m(k)$ *for* $k > 4$.

5.4.4 Disjoint-union-free

We say that a k-uniform hypergraph F is *disjoint-union-free* if all disjoint pairs of elements of F have distinct unions; that is, if for all $E_1, E_2, E_3, E_4 \in E(F)$, $E_1 \cap E_2 = E_3 \cap E_4 = \emptyset$ and $E_1 \cup E_2 = E_3 \cup E_4$ implies that $\{E_1, E_2\} = \{E_3, E_4\}$. Should this implication fail, we say E_1, E_2, E_3, E_4 form a *forbidden union*. Let $\mathcal{D}_k$ denote the family of k-uniform hypergraphs such that each hypergraph is a set of 4 edges forming a forbidden union. (Note that $\mathcal{D}_2 \cong C_4$ and in this case we refer the reader to Section 3.)

P. Dukes and L. Howard [DH08] gave the following.

Theorem 48. *[DH08]*

$$sat(n, \mathcal{D}_3) = \frac{n^2}{12} + O(n).$$

They also suggested the following.

Problem 18. *[DH08] Determine* $sat(n, \mathcal{D}_k)$ *for* $k > 3$.

5.4.5 Weak Saturation in Multipartite Hypergraphs

Both of the subtopics of weak saturation and saturation in which the host graph is not K_n are given more comprehensive discussion later in Section 10 and Section 6, respectively; however, we include a hypergraph analog here. Recall that for $k \leqslant p$, $W^k_{s_1,\ldots,s_p}$ denotes the

k-uniform hypergraph consisting of all k-tuples that intersect k different parts. In this variant, the host graph, H, is a k-partite, k-uniform hypergraph where each partite set has n vertices and each edge intersects each partite set. The target graph is $W^k_{s_1,s_2,\dots,s_k}$ where $1 \leqslant s_i \leqslant n$ for all i. The fewest number of edges in a host hypergraph, H, such that it is possible to order the edges of the $W^k(n,n,\dots,n) - H$ such that the addition of each new edge, one after another, creates a new copy of the target graph, $W^k_{s_1,s_2,\dots,s_k}$ is denoted by $w\text{-}sat_k(H, W^k_{s_1,s_2,\dots,s_k})$. This idea in the context of multipartite k-regular hypergraphs was introduced by Balogh et al. [BBMR12] where a special case was proved. Moshkovitz and Shapiro [MS15] established the value in general using a multipartite version of the Two Families Theorem [Alo85]. A more detailed calculation shows that the magnitude of the saturation number in this case is determined by n and the size of the smallest partite set in the target graph.

Theorem 49. *[MS15] Let H be a k-partite, k-uniform hypergraph where each partite set has n vertices and each edge intersects each partite set. For all integers $1 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_k \leqslant n$, $w\text{-}sat_k(H, W^k_{s_1,s_2,\dots,s_k}) = n^k - q_n$ where q_n is the number of k-tuples $x \in [n]^k$ such that $x_{(i)} \geqslant p_i$ for every $1 \leqslant i \leqslant d$, where $x_{(i)}$ is the ith smallest element in the sorted k-tuple x.*

6 Host graphs other than K_n

Note that in our definition of an F-saturated graph in the introduction, we allowed G to be any subgraph of K_n. We now consider F-saturated graphs where G is restricted to being a subgraph of some graph other than K_n.

More formally, let J be an n-vertex graph. We say that $G \subseteq J$ is an *F-saturated graph of J* if G is F-free (i.e. has no subgraph isomorphic to F), but for every edge e not in $E(G)$ but in $E(J)$ the graph $G + e$ does contain a copy of F. We define the following:

$sat(J, F) = \min\{|E(G)| : V(G) = V(J), E(G) \subseteq E(J),$ and G is an F-saturated graph of $J\}$,

$Sat(J, F) = \{G : V(G) = V(J), E(G) \subseteq E(J), |E(G)| = sat(n, F),$ and G is an F-saturated graph of $J\}$.

Thus, $sat(K_n, F)$ and $Sat(K_n, F)$ are by definition $sat(n, F)$ and $\mathrm{Sat}(n, F)$, respectively. Of course, we are interested in determining $sat(J, F)$ and $Sat(J, F)$ for various choices of J and F. There has been quite a proliferation in the varieties of problems studied recently. Some of this proliferation was expansion to partite graphs of more and more parts, but one particularly interesting extension was to the Erdős-Rényi random graph, by Korándi and Sudakov [KS17].

The first types of problems considered in other host graphs were in multipartite graphs. Let $J_{(n_1,\dots,n_p)}$ be an n-vertex p-partite graph with n_i vertices in the i^{th} class. Let $F_{(r_1,\dots,r_p)}$ be a p-partite graph with $r_i \leqslant n_i$ vertices in the i^{th} class. Then $G \subseteq J_{(n_1,\dots,n_p)}$ is an *$F_{(r_1,\dots,r_p)}$-saturated graph of $J_{(n_1,\dots,n_p)}$* if G has no copy of $F_{(r_1,\dots,r_p)}$ with r_i vertices in the

i^{th} class, but $G+e$ has a copy of $F_{(r_1,\ldots,r_p)}$ for any edge e joining vertices from distinct classes and contained in $J_{(n_1,\ldots,n_p)}$. The difference between this definition and the previous one is that this is "sensitive" with respect to the partition. Analogously, we define $sat(J_{(n_1,\ldots,n_p)}, F_{(r_1,\ldots,r_p)})$ and $Sat(J_{(n_1,\ldots,n_p)}, F_{(r_1,\ldots,r_p)})$. Thus, the presence of parentheses in the subscript indicates that we are considering the partition "sensitive" problem, the absence of parentheses indicates we are considering the more general problem.

Problems of this type were first proposed in [EHM64]. Here the authors conjectured a value for $sat(K_{(n_1,n_2)}, K_{(r_1,r_2)})$, where $K_{(n_1,\ldots,n_p)}$ denotes the complete p-partite graph with n_i vertices in the i^{th} class. Their conjectured value was established to be correct by B. Bollobás ([Bol67b], [Bol67a]) and W. Wessel ([Wes66],[Wes67]). We thus have the following.

Theorem 50. *Let $2 \leqslant r_1 \leqslant n_1$ and $2 \leqslant r_2 \leqslant n_2$, then*

$$sat(K_{(n_1,n_2)}, K_{(r_1,r_2)}) = n_1 n_2 - (n_1 - r_1 + 1)(n_2 - r_2 + 1)$$

and $Sat(K_{(n_1,n_2)}, K_{(r_1,r_2)})$ consists of one graph, the n_1 by n_2 bipartite graph consisting of all edges incident with a fixed set of size $r_1 - 1$ of the n_1-set and all edges incident with a fixed set of size $r_2 - 1$ of the n_2-set.

Also, N. Alon [Alo85] reproved Theorem 50, generalizing it to complete k-uniform graphs in a k-partite setting – Alon's generalization is a consequence of an extremal problem on sets which was proved using multilinear techniques (exterior algebra). Unaware of some of these results, D. Bryant and H.-L. Fu [BF02] considered $K_{2,2}$-saturated graphs of K_{n_1,n_2} (which is the same as $K_{(2,2)}$-saturated graphs of $K_{(n_1,n_2)}$), showing how to construct such graphs (not just those of minimum size) using design theory. Another generalization of Theorem 50 can be found in the results on layered graphs of O. Pikhurko, for these we refer the reader to his Ph.D. thesis [Pik99b] (cf. page 14).

E. Sullivan and P. Wenger [SW17] extended the results of Theorem 50 to tripartite graphs.

Theorem 51. *[SW17] For positive integers l, m, p, n_1, n_2, n_3 with $p < m \leqslant l \leqslant n_3 \leqslant n_2 \leqslant n_1$,*

$$sat(K_{n_1,n_2,n_3}, K_{l,m,p}) \leqslant 2(m-1)(n_1+n_2+n_3) + (l-m)(n_2+2n_3) - 3l(m-1) + 3m - 3$$

For n_3 sufficiently larger than l and n_1 sufficiently larger than n_3, they conjecture that this is the exact value.

Conjecture 12. *[SW17] For positive integers l, m, p, n_1, n_2, n_3 with $p < m \leqslant l$, $n_3 \leqslant n_2 \leqslant n_1$, and n_3 sufficiently large compared with l, and n_1 sufficiently large compared with n_3,*

$$sat(K_{n_1,n_2,n_3}, K_{l,m,p}) = 2(m-1)(n_1+n_2+n_3) + (l-m)(n_2+2n_3) - 3l(m-1) + 3m - 3.$$

Sullivan and Wenger also find the saturation numbers for balanced and nearly-balanced tripartite graphs in sufficiently larger tripartite graphs, that is, specific cases of the above problems where the forbidden graph is such that $l = m = n$, $l = m = n + 1$, or $l = m = n + 2$ [SW17].

Balanced bipartite graphs have also been considered as the host graph rather than the forbidden subgraph. W. Gan, D. Korándi, and B. Sudakov's [GKS15] discovered a result analogous to the first theorem reported in the survey in this section, which Chakraborti, Chen, and Hasabnis [CCH21] improved upon. Gan, Korándi, and Sudakov found a lower bound for the balanced bipartite saturation number for all cases, and Chakraborti, Chen, and Hasabnis [CCH21] found the asymptotic value.

Theorem 52. *[CCH21] There exists some N depending on s and t such that $n \geqslant N$ implies*

$$sat(K_{n,n}, K_{s,t}) \geqslant (s+t-2)n - (t-1)(t-2) - \lfloor\frac{(s-1)^2}{4}\rfloor;$$

[GKS15] Let $1 \leqslant s \leqslant t$ both be fixed, and let $n \geqslant t$. Then,

$$sat(K_{n,n}, K_{s,t}) \geqslant (s+t-2)n - (s+t-2)^2.$$

Chakraborti, Chen, and Hasabnis's [CCH21] work resolved a conjecture of Moshkovitz and Shapira [MS15].

M. Ferrara, M. Jacobsen, F. Pfender, and P. Wenger [FJPW16] took this further, with balanced multipartite graphs of more than two parts. Notation-wise, instead of writing $K_{n,n,\ldots,n}$, we write K_k^n, if there are k parts. They found a first result, with the forbidden configuration being triangles.

Theorem 53. *[FJPW16] If $k \geqslant 4$ and $n \geqslant 100$, then*

$$sat(K_k^n, K_3) = \min\{2kn + n^2 - 4k - 1, 3kn - 3n - 6\}.$$

More recently, A. Girão, T. Kittipassorn, and K. Popielarz [GaKP19] extended this to all complete graphs. Their proof and result involves a function $\alpha(k, r)$. If we consider some k-partite subgraph $G \subseteq K_k^n$ that is K_r saturated within the host graph K_k^n, and an independent set of vertices X with exactly one vertex from each part, $\alpha(k, r)$ is the minimum number of edges between X and $\overline{X}$ in G, over all G and X [GaKP19].

Theorem 54. *[GaKP19] With $\alpha(k, r)$ defined as above,*

$$sat(K_k^n, K_r) = \alpha(k, r)n + o(n).$$

The function $\alpha(k, r)$ is the most interesting thing to study from here, and the authors ([GaKP19]) discovered the following about it already.

$$k(2r-4) \leqslant \alpha(k,r) \leqslant \begin{cases} (k-1)(4r-k-6) & \text{for } r \leqslant k \leqslant 2r-3. \\ (k-1)(2r-3) & \text{for } k \geqslant 2r-3. \end{cases}$$

They find some small specific cases also, mentioned in their paper, but classes of values for $\alpha(k, r)$ that remain unknown are stated by the following problem.

Problem 19. *[GaKP19] Determine* $\alpha(k,r)$ *for* $k \geqslant 2r-2$ *and* $r \equiv 1,3 \bmod 6$.

The problem of determining $sat(K_{n_1,n_2}, P_t)$, where P_t is the path of order t, was considered by A. Dudek and A. P. Wojda [DW04a]. They determined the saturation number precisely for $t \leqslant 6$ and for $t > 6$ they determined the value of the function under the added constraint that the graph contains no isolated vertices for n_1, n_2 sufficiently large.

Problem 20. *[DW04a] Determine* $sat(K_{n_1,n_2}, C_{2t})$ *for* $t > 2$*, where* C_{2t} *denotes the cycle of order* $2t$.

Some attention has also been given to determining $sat(Q_n, Q_2)$, where Q_i denotes the i-dimensional hypercube. S.-Y. Choi and P. Guan [CG08] give an asymptotic upper bound of $(\frac{1}{4}+\epsilon)n2^{n-1}$. Anthony Santolupo (a former undergraduate student of the fourth author) conjectured that $sat(Q_n, Q_2)$ is asymptotically $\frac{1}{4}n2^{n-1}$. Johnson and Pinto [JP17] disproved this conjecture working on the larger problem of $sat(Q_n, Q_m)$. Their upper bound was then improved by N. Morrison, J. Noel, and A. Scott [MNS17].

Theorem 55. *[JP17] [MNS17] Let* $m \geqslant 2$ *be fixed. Then*

$$\left(\frac{(m+1)}{2} - o(1)\right) \cdot 2^n \overbrace{\leqslant}^{[JP17]} sat(Q_n, Q_m) \overbrace{\leqslant}^{[MNS17]} (1+o(n)) \cdot 72m^2 \cdot 2^n.$$

Morrison, Noel, and Scott pose the following question, doubting the tightness of Johnson and Pinto's lower bound.

Question 5. *[MNS17] For fixed* $m \geqslant 2$*, does the limit*

$$\lim_{n\to\infty} \frac{sat(Q_n, Q_m)}{2^n}$$

exist?

Until [MNS17], only smaller-order hypercubes were considered as forbidden configurations in hypercubes. Morrison, Noel, and Scott, drawing inspiration from study of an analogous problem for the Turán function, also propose extending this to even cycles.

Problem 21. *[MNS17] For* $t \geqslant 2$ *and* $n \geqslant \log_2(2t)$*, determine* $sat(Q_n, C_{2t})$.

Following another type of problem initiated in the study of Turán's extremal number, Korándi and Sudakov [KS17] studied the saturation number in the Erdős-Rényi random graph. They started by looking at complete graphs as the forbidden configuration, and A. Mohammadian and B. Tayfeh-Rezaie [MTR18] expanded on this by studying stars as the forbidden configuration.

Theorem 56. *[KS17] Let* $0 < p < 1$ *be a constant probability and let* $s \geqslant 3$ *be an integer. Then,*

$$sat(G(n,p), K_s) = (1+o(1))n \log_{\frac{1}{1-p}}(n)$$

with high probability.

Theorem 57. *[MTR18] Let $0 < p < 1$ be a constant probability and let $s \geqslant 2$ be an integer. Then,*

$$sat(G(n,p), K_{1,s}) = \frac{(s-1)n}{2} - (1+o(1))(s-1)\log_{\frac{1}{1-p}}(n)$$

with high probability.

This leaves us with a natural following problem.

Problem 22. *Determine $sat(G(n,p), F)$ for F other than K_s and $K_{1,s}$.*

Finally, around the time of the first publication of this survey, the notion of minimal saturated matrices was introduced by A. Dudek, O. Pikhurko and A. Thomason [DPT13]. We omit introducing the required definitions and terminology here, and we refer the reader to [DPT13] for these and their results.

7 Colorings

7.1 Ramsey-minimal saturated graphs

We say F *arrows* a t-tuple $(F_1, \ldots, F_t)$ of graphs if any t-coloring of $E(F)$ contains a monochromatic F_i-subgraph of color i for some $i \in [t]$ and we denote this $F \to (F_1, \ldots, F_t)$. A graph G is $(F_1, \ldots, F_k)$-*Ramsey-minimal* if $G \to (F_1, \ldots, F_k)$ but for any proper subgraph G' of G, $G' \nrightarrow (F_1, \ldots, F_k)$. Let $sat_t(n, \mathcal{R}_{min}(F_1, \ldots, F_k))$ denote the family of all $(F_1, \ldots, F_k)$-*Ramsey-minimal* graphs.

The motivation for this subsection is the following conjecture due to D. Hanson and B. Toft [HT87].

Conjecture 13. *[HT87] Let $r = r(K_{t_1}, \ldots, K_{t_k})$ be the classical Ramsey number. Then*

$$sat_t(n, \mathcal{R}_{min}(K_{t_1}, \ldots, K_{t_k})) = \binom{r-2}{2} + (r-2)(n-r+2).$$

Notice that Conjecture 13 reduces to Theorem 1 in the case where either $k = 1$ or $t_2 = \cdots = t_k = 2$. G. Chen, M. Ferrara, R. Gould, C. Magnant, and J. Schmitt [CFG+11] confirmed this conjecture in the smallest open instance, that is, for $t_1 = t_2 = 3$ so long as $n \geqslant 56$.

A natural and obvious generalization to their problem is the following.

Problem 23. *Let $F_1, \ldots, F_k$ be graphs, each with at least one edge. Determine $sat(n, \mathcal{R}_{min}(F_1, \ldots, F_k))$.*

A first result in this more general direction was given in Chen et al. [CFG+11], who determined the saturation number for the family $\mathcal{R}_{min}(K_3, T_3)$, where T_3 is the (unique) tree on three vertices.

M. Ferrara, J. Kim and E. Yeager [FKY14] provided a solution to Problem 23 in the case where each F_i is of the form $m_i K_2$.

Theorem 58. *[FKY14] If $m_1, \ldots, m_k \geqslant 1$ and $n > 3(m_1 + \cdots + m_k - k)$, then*

$$sat(n, \mathcal{R}_{min}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \cdots + m_k - k).$$

Note that when $k = 1$, we are left with the theorem on matchings given by [Mad73, KT86] in Section 4.

Now let $\mathcal{T}_k$ denote the family of all trees on k vertices. M. Rolek and Z.-X. Song [RS18] found the saturation number for the family $\mathcal{R}_{min}(K_3, \mathcal{T}_4)$ for sufficiently large n (whereas the smaller case of $\mathcal{R}_{min}(K_3, \mathcal{T}_3)$ is covered by the above-mentioned result of [CFG+11] since $\mathcal{T}_3$ consists of only the one graph).

Theorem 59. *[RS18] For $n \geqslant 18$, $sat(n, \mathcal{R}_{min}(K_3, \mathcal{T}_4)) = \lfloor \frac{5n}{2} \rfloor$.*

They also obtained the following.

Theorem 60. *[RS18] For any integer $k \geqslant 5$ and $n \geqslant 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$, there exist constants $c = (\frac{1}{2}\lceil \frac{k}{2} \rceil + \frac{3}{2})k - 2$ and $C = 2k^2 - 6k + \frac{3}{2} - \lceil \frac{k}{2} \rceil (k - \frac{1}{2}\lceil \frac{k}{2} \rceil - 1)$ such that*

$$\left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c \leqslant sat(n, \mathcal{R}_{min}(K_3, \mathcal{T}_k)) \leqslant \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n + C.$$

Z.-X. Song and J. Zhang [SZ20] (see also [Zha19]) furthered the study of $(K_t, \mathcal{T}_k)$-Ramsey-minimal saturated graphs by generalizing the bounds proven for $(K_t, \mathcal{T}_k)$ in [RS18] to larger values of t and k.

Theorem 61. *[SZ20] Let $t, k \in \mathbb{N}$ with $t \geqslant 4$ and $k \geqslant \max\{6, t\}$. There exists a constant $\ell(t, k)$ such that, for all $n \in \mathbb{N}$ with $n \geqslant (t-1)(k-1) + 1$, then*

$$sat(n, \mathcal{R}_{min}(K_t, \mathcal{T}_k)) \geqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - \ell(t, k).$$

Theorem 62. *[SZ20] Let $t, k \in \mathbb{N}$ with $t \in \{4, 5, 6, 7\}$ and $k \geqslant \max\{3, 4t - 14\}$. There exists a constant $c(t, k)$ such that, for all $n \in \mathbb{N}$ with $n \geqslant (t-1)(k-1) + 1$, then*

$$sat(n, \mathcal{R}_{min}(K_t, \mathcal{T}_k)) \geqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c(t, k).$$

Song and Zhang also found upper bounds for certain values of n and t.

Theorem 63. *[SZ20] For each $t \in \{4, 5\}$ and $k \geqslant 3$ and $n \geqslant (2t-3)(k-1) + \lceil k/2 \rceil \lceil k/2 \rceil - 1$, then*

$$sat(n, \mathcal{R}_{min}(K_t, \mathcal{T}_k)) \leqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - C(t, k),$$

where

$$C(t, k) = \frac{1}{2}(t^2 + t - 5)k^2 - (2t^2 + 2t - 11)k - \frac{(t-2)(t-19)}{2} - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil ((2t-3)(k-1) - \left\lceil \frac{k}{2} \right\rceil).$$

This leaves the question of tightening these bounds for all values of t and k.

Problem 24. *Determine* $sat(n, \mathcal{R}_{min}(K_t, \mathcal{T}_k))$ *for* $t \geqslant 3$ *and* $k \geqslant 4$.

H.M. Davenport [Dav18] took inspiration from the work on $(K_3, \mathcal{T}_4)$-Ramsey-minimal saturated graphs: Davenport found the saturation number for $\mathcal{R}_{min}(K_3, K_{1,3})$. Note that $\mathcal{T}_4 = \{P_4, K_{1,3}\}$.

Theorem 64. *[Dav18] For all* $n \geqslant 13$, $sat(n, \mathcal{R}_{min}(K_3, K_{1,3})) = 3n - 4$.

Davenport furthered this result to stars of any order to obtain bounds for $t \geqslant 4$ and n sufficiently large.

Theorem 65. *[Dav18] For all integers* $t \geqslant 4$ *and* $n \geqslant 4t+2$, *there exists* $c = c(t)$ *such that* $sat(n, \mathcal{R}_{min}(K_3, K_{1,t})) = \left(\frac{3}{2} + \frac{t}{2}\right) n + c$. *Furthermore,* $t - t\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor^2 - 6 \leqslant c \leqslant t^2 + 3t - 3$.

7.2 Rainbow coloring

An edge-coloring of a graph F is *rainbow* if every edge of F receives a different color. Let $\Re(F)$ denote the set of all rainbow-colored copies of F. A t-edge colored graph G is $\Re_t(F)$-*saturated* if G does not contain a rainbow copy of F but for any edge $e \in \overline{G}$ and any color $i \in [t]$, the addition of e in color i to G produces a rainbow copy of F. We write $sat(n, \Re_t(F))$ to denote the minimum number of edges in a t-edge-colored $\Re_t(F)$-saturated graph of order n.

The line of investigation of $\Re_t(F)$-saturated graphs was initiated by M. Barrus, M. Ferrara, J. Vandenbussche and P. Wenger [BFVW17], who provided bounds for matchings and paths, as well as the asymptotic behavior for stars. They also conjectured that $sat(n, \Re_t(K_k)) = \Theta(n \log n)$. This conjecture was proven independently by M. Ferrara et al. [FJL+20], D. Korándi [Kor18], and A. Girão, D. Lewis and K. Popielarz [GaLP20].

Theorem 66. *[FJL+20],[Kor18],[GaLP20] For any integers* t *and* k *with* $t \geqslant \binom{r}{2}$,

$$sat(n, \Re_t(K_k)) = \Theta(n \log n).$$

For a given tree T, Barrus et al. [BFVW17] proved that $sat(n, \Re_t(T)) = O(n)$. More specifically, we have the following.

Theorem 67. *[BFVW17] Let* H *be a connected* k*-vertex graph with* $k \geqslant 5$. *If* H *has a vertex* v *with* $d(v) = 1$ *whose neighbor* v' *does not have degree* $k-1$, *there are two vertices* u *and* u' *in* $V(H) \backslash \{v, v'\}$ *that are not adjacent, and* $t \geqslant \binom{k-1}{2}$, *then*

$$sat(n, \Re_t(H)) \leqslant \left\lceil \frac{n}{k-1} \right\rceil \binom{k-1}{2}.$$

These authors also proved that for t sufficiently large, $sat(n, \Re_t(mK_2)) = O(1)$, but it would be interesting to have a precise value.

Problem 25. *Determine precisely* $sat(n, \Re_t(mK_2))$ *for* $m \geqslant 2$.

Girão, Lewis, and Popielarz [GaLP20] established the magnitudes for the function for various different structures of graphs, which notably includes the findings on complete graphs.

Theorem 68. *[GaLP20] Let F be a connected graph of order at least 3. Then, for every $t \geqslant e(F)$, $sat(n, \Re_t(F))$ equals:*

1. *$\Theta(n^2)$, if F is a star.*
2. *$\Theta(n \log n)$, if F has a conical vertex but is not a star.*
3. *$\Theta(n \log n)$, if every edge of F is in a triangle.*
4. *$\Theta(n)$, if F contains a nonpendant edge that does not belong to a triangle.*
5. *$\Theta(n)$, if F is a K_k with a rotated edge, for some even $k \geqslant 4$.*

Girão, Lewis and Popielarz [GaLP20] made the following conjecture when the palate of colors is infinite.

Conjecture 14. *[GaLP20] For any graph F, when the palate of colors is infinite the number of edges in an $\Re_\infty(F)$-saturated graph is $O(n)$.*

In addition to the well-structured graphs considered thus far, various authors have considered this problem for paths, resulting in the following statements.

Theorem 69. *[BFVW17]*

1. *$sat(n, \Re_t(P_\ell)) \geqslant n-1$, $\ell \geqslant 4$.[BFVW17]*
2. *$sat(n, \Re_t(P_4)) = n-1$, $t \geqslant 8$.[BFVW17]*
3. *$sat(n, \Re_t(P_\ell)) \leqslant \lceil \frac{n}{\ell-1} \rceil$, $t \geqslant \binom{\ell-1}{2}$.[BFVW17]*
4. *$sat(n, \Re_t(P_\ell)) \leqslant \lceil \frac{n}{\ell-1} \rceil \left(\binom{\ell-2}{2}+4\right)$, $\ell \geqslant 5$ and $t \geqslant 2\ell-5$. [CMT20]*

While Barrus et al. [BFVW17] have determined bounds for rainbow matchings, van Oostendorp has determined bounds for rainbow matchings in the $K_{n,n}$-setting.

Theorem 70. *[vO16] If $n > m$, then $2m \leqslant sat(K_{n,n}, \Re_t(mK_2)) \leqslant 4(m-1)$.*

In [BJR20], N. Bushaw, D. Johnston, and P. Rombach consider a slightly different rainbow saturation number that is analogous to the rainbow extremal number found in P. Keevash, D. Mubayi, B. Sudakov and J. Verstraëte [KMSV07]. Instead of what is considered above, one limits colorings to proper colorings, i.e. so that no two same colored edges are incident to the same vertex. Such a graph G is referred to as an F-*proper-rainbow-saturated graph*, if for all proper colorings of G there is no rainbow copy of F in G, but for all proper colorings of $G+e$ there is a rainbow F. Bushaw et al. [BJR20] found lower bounds for the minimum number of edges in such a graph when F is a complete graph, as well as the orders of magnitudes when F is P_4 or C_4.

8 Edge Spectra

Investigations of extremal numbers and saturation numbers led naturally to more general questions about saturated graphs such as whether there even exist H-saturated graphs with m edges for m in the interval $[sat(n,H), ex(n,H)]$. Hence the following definition.

The *edge spectrum for H-saturated graphs*, denoted by $ES(n,H)$, is defined as the set of values of m for which there exists an H-saturated graph on n vertices and m edges. This appears in the literature also as the *saturation spectrum of H*.

The study of the edge spectrum of graphs began with results on $ES(n,K_3)$ [BCF$^+$95a, Sid93], followed by results on $ES(n,K_4)$ ([Ami10, AFG12]), and a complete description of $ES(n,K_p)$ by K. Amin [Ami10] and K. Amin, J. Faudree, R. Gould and E. Sidorowicz [AFGS13]. The result was obtained by finding the saturation number and extremal number for K_p-saturated graphs that are not $(p-1)$-partite and then demonstrating that all integer values between these two numbers are possible. It is an easy calculation to see that the edge spectrum has gaps near $sat(n,K_p)$ and $ex(n,K_p)$.

Theorem 71. *[AFGS13, Ami10] Let p,q,r and n be integers such that $p \geqslant 3$, $n \geqslant 3p+4$, and $n=(p-1)q+r$, where $0 \leqslant r < p-1$. Then there exists a K_p-saturated graph G on n vertices and size m if and only if G is a complete $(p-1)$-partite graph or m satisfies*

$$(p-1)n - \frac{p(p-1)}{2} - 2 \leqslant m \leqslant \frac{n^2(p-2)}{2(p-1)} - \frac{n}{p-1} + \frac{r(r+2)}{2(p-1)} - \frac{r}{2} + 1.$$

For stars, the edge spectrum includes every integer value between $sat(n,K_{1,k})$ and $ex(n,K_{1,k})$ [FFG$^+$17, BD18]; however, gaps exist for all other graphs for which the edge spectrum has been studied. Paths have received the most study ([GTWZ12, FFG$^+$17, BD18, GHJT19]). The authors of [FFG$^+$17, BD18] independently demonstrated that $ES(n,P_k)$ contains all integer values from $sat(n,P_k)$ to near the extremal number, where [BD18] has a better bound on the upper end of this interval.

Theorem 72. *[BD18] Let $\epsilon > 0$, and let k and n be integers with $k \geqslant k_0(\epsilon)$ and $n \geqslant a_k$, where a_k is defined as in [KT86]. Then for any integer m such that $sat(n,P_k) \leqslant m \leqslant ex(n,P_k) - (\sqrt{2}+\epsilon)k^{3/2}$.*

The authors of [BD18, GHJT19] demonstrated that $ES(n,P_k)$ must have a gap just below the extremal number. In [BD18], the authors determine the size of the second largest P_k-saturated graphs on n vertices. In [GHJT19], the strategy is to show that every graph with sufficiently large average degree and sufficiently large minimum degree must contain certain trees as subgraphs. Consequently, the result in [GHJT19] implies a gap in the edge spectrum for paths and for a number of other types of trees.

Theorem 73. *[GHJT19] Let $k \geqslant 6$. Let T_k be a k-vertex path, broom, or non-star tree with a vertex adjacent to at least $\lfloor \frac{k-1}{2} \rfloor$ leaves. Let $n \equiv 0 \pmod{k-1}$ and $m \in \mathbb{Z}$ such that $1 \leqslant m \leqslant \lfloor \frac{k-3}{2} \rfloor - 1$. There is no T_k-saturated graph on n vertices with $\frac{n}{k-1}\binom{k-1}{2} - m$ edges.*

The edge spectrum of several small specific graphs have been studied. The edge spectrum of $K_4 - e$ is completely known, begun in [FG18] and completed by [GHM18]. $ES(n, K_4 - e)$ has gaps both above the saturation number and below the extremal number. The edge spectrum for $K_t - e$ remains open. The edge spectrum for two particular brooms was determined in [Tho16], where some partial results for the edge spectrum of general brooms can be found.

9 Saturation Games

Saturation games are yet another area of study in minimum graph saturation that have roots in the study of the extremal number. The original saturation game studied by Z. Füredi, D. Reimer, and A. Seress [FRS91] is based on Hajnal's triangle-free game, where two players alternate turns drawing edges in an empty graph on a given number of vertices, where the player that completes a triangle loses. Without other constraints, the players realize an extremal graph. In saturation games, the game instead leads to a saturated graph somewhere between the saturation number and the extremal number. The first saturation game was studied by Füredi, Reimer, and Seress [FRS91], and variants have been spun by other authors. Eventually, D. Hefetz, M. Krivelevich, A. Naor, and M. Stojaković [HKNS16] proposed playing saturation games for any monotone graph property, of which subgraph inclusion would be just one.

9.1 The Triangle-Free Game

To put a saturation twist on the triangle-free game, Füredi, Reimer, and Seress describe a game that proceeds as follows: starting with the empty graph on n vertices, two players alternate turns drawing one edge. Neither player is allowed to draw an edge that completes a triangle. The game ends when the players have drawn a triangle-saturated graph. The game's *score* is the number of edges, and it is of course bounded by the saturation and extremal numbers. A player named Max, who typically goes first, tries to maximize the score, while the name Min is given to the other player, who tries to minimize it. This game can be played for any forbidden subgraph F, and the *game saturation number*, written as $g\text{-}sat(n, F)$, is the score resulting from ideal play from each player.

Füredi et al. [FRS91] proved the following result:

Theorem 74. *[FRS91]*

$$g\text{-}sat(n, K_3) \geqslant \frac{n \log_2(n)}{2} - 2n \log_2(\log_2(n)) + O(n).$$

This result is a consequence of a simple strategy by Max to ensure that the graph contains a matching of size $\lfloor \frac{n}{2} \rfloor$ and another result from the same paper.

Theorem 75. *[FRS91] If a maximal triangle-free graph contains a matching of size v, then it contains at least*

$$v \log_2(v) - 4v \log_2(\log_2(v)) + \Theta(n)$$

edges.

It is interesting to note that this game actually motivated the study of the function $sat^{\Delta}(n, \mathcal{F})$ (where $\mathcal{F} = \{K_3\}$, originally); see Section 2.

Füredi, Reimer, and Seress reports a result that Erdős claimed in private communication, the proof of which has been lost. No alternate proof has been created either, leaving us with the problem of discovering one.

Conjecture 15. *(claimed result of Erdős reported in [FRS91])*

$$g\text{-}sat(n, K_3) \leqslant \frac{n^2}{5}$$

The work of [FRS91] remains open to improvement. The game saturation number for the original saturation game remains unknown.

Problem 26. *Determine $g\text{-}sat(n, K_3)$.*

C. Biró, P. Horn, and J. Wildstrom [BHW16] revived the study of the triangle-free game, and it was these authors that originated the term *game saturation number*. They obtained a lower bound for $g\text{-}sat(n, K_3)$ by following Max's strategy of creating copies of C_5. Their work does not improve on the lower bound of [FRS91], but it reveals a strategy: to search for unavoidable configurations that lead to a desired ending, and essentially find the game saturation number equivalent of the few T copies generalization. Their work leaves us with the following theorem.

Theorem 76. *[BHW16] In the triangle-avoiding saturation game, Max can always, regardless of Min's actions, play so as to create $\lfloor \frac{n-2}{11} \rfloor$ disjoint copies of C_5.*

The authors also give us an upper bound for the game saturation number:

Theorem 77. *[BHW16]*

$$g\text{-}sat(n, K_3) \leqslant \frac{26n^2}{121} + o(n^2).$$

9.2 Other forbidden graphs

D. Cranston, W. Kinnersley, S. O, and D. West [CKOW13] first generalized the saturation game to other forbidden subgraphs. Also, instead of making Max the first player to move, investigating either player starting. These authors focus on the P_3-saturation game, and they call the game saturation number for this graph the *game matching number*, as maximal matchings end the game. Analogously to alternate host graphs in the usual setting (see Section 6), in a host graph $G \not\cong K_n$, we write $g\text{-}sat(G, \mathcal{F})$. First off, it turns out that the choice of starting player makes little difference to the outcome of this game, although this does not hold true for many other games. Pursuant to their notation, a caret is added, and $g\text{-}\hat{s}at(n, \mathcal{F})$ indicates the game saturation number in the version of the game where Min starts.

Theorem 78. *[CKOW13] For any graph G and vertex $v \in V(G)$,*

1. $|g\text{-}sat(G, P_3) - g\text{-}\hat{s}at(n, P_3)| \leqslant 1$,
2. $g\text{-}sat(G, P_3) \geqslant g\text{-}sat(G - v, P_3)$, *and*
3. $g\text{-}\hat{s}at(G, P_3) \geqslant g\text{-}\hat{s}at(G - v, P_3)$.

Notice that the last two items tell us that the game matching number is monotone with respect to inclusion in the vertex set. Newer scholarship, discussed in a later subsection, extends games to any forbidden monotone property.

[CKOW13] gives a condition for the game matching number to simply be the size of the maximum matching.

Theorem 79. *[CKOW13] Let G be a graph on n vertices with a maximum matching M, which has size m, and let $u, u', v, v' \in V(G)$. If $uv \in E(G)$ implies $u'v' \in E(G)$ whenever $uu', vv' \in M$, then $g\text{-}sat(G, P_3) = g\text{-}\hat{s}at(G, P_3) = m$.*

This nice condition leads to a cleaner corollary, that follows from the definition of a graph cartesian product.

Corollary 1. *[CKOW13] For $n \geqslant 1$ and any graph H, Max can force a perfect matching in $K_{n,n} \square H$ regardless of who plays first.*

Moving on from our look into the game matching number, Spiro [Spi19] studies the problem where the forbidden subgraphs are odd cycles of order at least nine.

Theorem 80. *[Spi19] For $k \geqslant 4$,*

$$(\frac{1}{4} - \frac{1}{4k^2})n^2 + o(n^2) \leqslant g\text{-}sat(n, \mathcal{C}_{2k+1}) \leqslant (\frac{1}{4} - \frac{1}{20^6 k^4})n^2 + o(n^2),$$

where $\mathcal{C}_{2k+1}$ is the family of all odd cycles with length at most $2k + 1$.

Theorem 80 does not include C_5 and C_3 in the family, but this is because Spiro discoverd an even broader result. A quadratic lower bound is possible for any family of non-bipartite graphs that contains both C_3 and C_5.

Theorem 81. *[Spi19] For any family $\mathcal{C}$ of non-bipartite graphs with $\{C_3, C_5\} \subseteq \mathcal{C}$,*

$$g\text{-}sat(n, \mathcal{C}) \geqslant \frac{6}{25}n^2 + o(n^2).$$

Note that although this may look like an improvement on the first theorem of this section, it does not apply to the original triangle saturation game since C_5 must be included as a forbidden configuration as well. Spiro [Spi19] attacked the problem by dividing the game into phases where Min's best strategy is different than in the other phases of the game, building off of the thinking of [BHW16]. Just like Biró, Horn, and Wildstrom[BHW16], Spiro [Spi19] finds results limited to the opening, leaving uncertainty in the endgame.

Finally, Spiro leaves us with some suggestions to extend the research of his paper.

Conjecture 16. *[Spi19] For all $k \geqslant 1$, there exists a constant $c_k > 0$ such that*

$$g\text{-}sat(n, \mathcal{C}_{2k+1}) \leqslant (\frac{1}{4} - c_k)n^2 + o(n^2).$$

Conjecture 17. *[Spi19] For all $k \geqslant 2$ and n sufficiently large,*

$$g\text{-}sat(n, \mathcal{C}_{2k-1}) \leqslant g\text{-}sat(n, \mathcal{C}_{2k+1}).$$

We arrive at a problem inspired by surprising results: as shown by Carraher, Kinnersly, Reiniger, and West [CKRW17], and Spiro [Spi19], the inclusion of C_3 in the family of all odd cycles, $\mathcal{O}$, makes a big difference to the game saturation number.

Theorem 82. *[CKRW17] For $k \geqslant 1$,*

$$g\text{-}sat(n, \mathcal{O}) = \lfloor \frac{1}{4} n^2 \rfloor = ex(n, \mathcal{C}_{2k+1}).$$

Theorem 83. *[Spi19] For $k \geqslant 1$,*

$$g\text{-}sat(n, \mathcal{O} \setminus \{C_3\}) \leqslant 2n - 2.$$

The question arises from the natural wonder if the inclusion of other odd cycles in this family is similarly as important.

Question 6. *[Spi19] What is the order of magnitude of $g\text{-}sat(n, \mathcal{O} \setminus \{C_5\})$, where $k \geqslant 1$?*

Spiro has one more result, which is a bit narrower, focusing on one cycle at a time rather than omitting all or nearly all odd cycles. For any one odd cycle C_5 or larger omitted, the game saturation number is quadratic in n. We direct the reader to Spiro's paper [Spi19], which is more specific.

Moving on, Lee and Riet [LR15] prove several precise, specific results using various trees and paths as the forbidden configurations, building off of [CKOW13]. Building off that same paper, Carraher, Kinnersley, Reiniger, and West [CKRW17] studied similar topics, extending them to Min starting the game. We begin with the most basic results.

Theorem 84. *[LR15] For all $n \geqslant 5$,*

1.
$$\frac{4n}{5} - \frac{14}{5} \leqslant g\text{-}sat(n, P_4) \leqslant \frac{4n}{5} + 1.$$

2.
$$n - 1 \leqslant g\text{-}sat(n, P_5) \leqslant n + 2.$$

Theorem 85. *[CKRW17] For $n \geqslant 6$,*
$$g\text{-}sat(n, \mathcal{T}_n) = g\text{-}\hat{sat}(n, \mathcal{T}_n) = \binom{n-2}{2} + 1.$$

Smaller cases are resolved as well. Cases where $n \leqslant 3$ are trivial, and in the cases of $n = 4$ and $n = 5$, Max has an advantage starting since $g\text{-}sat(5, \mathcal{T}_5) = 6$ and $g\text{-}sat(4, \mathcal{T}_4) = 3$, but $g\text{-}\hat{sat}(n, \mathcal{T}_k)$ follows the pattern as above in those cases [CKRW17]. The authors invite the reader to note that Max is able to push these two cases all the way to the extremal number.

For more results involving trees and stars, we direct the reader to [LR15], [CKRW17].

Carraher et al. have a complete, precise result for $K_{1,3}$, as follows.

Theorem 86. *[CKRW17] For positive integer n,*
$$g\text{-}sat(n, K_{1,3}) = \begin{cases} n & \text{if } n \in \{3, 7\} \cup 2\mathbb{N} \setminus \{2\} \\ n - 1 & \text{otherwise,} \end{cases}$$
$$g\text{-}\hat{sat}(n, K_{1,3}) = \begin{cases} n - 1 & \text{if } n \in \{1\} \cup 2\mathbb{N} \setminus \{4\} \\ n & \text{otherwise.} \end{cases}$$

Finally, Lee and Riet introduced another variant of the game: Max is permitted to skip a turn (but, naturally Min does not gain what would be an overwhelming advantage). For this game saturation number, we write $sat_{g_{-P}}(n, \mathcal{F})$.

Theorem 87. *[LR15] For all $n \geqslant k$,*
$$\frac{n(k-2)}{4} \leqslant sat_{g_{-P}}(n, P_k) \leqslant \frac{n(k-1)}{2}.$$

Problem 27. *[LR15] Investigate $sat_{g_{-P}}(n, \mathcal{F})$.*

Carraher et al. [CKRW17] also studied the P_4-saturation game in the complete bipartite host graph, and determined the game saturation numbers exactly.

Theorem 88. *[CKRW17] For $m \geqslant n \geqslant 2$,*
$$g\text{-}sat(K_{m,n}, P_4) = \begin{cases} n & \text{when } n \text{ is even} \\ m & \text{when } n \text{ is odd but } m \text{ is even} \\ m + \lfloor \frac{n}{2} \rfloor & \text{when } mn \text{ is odd,} \end{cases}$$
$$g\text{-}\hat{sat}(K_{m,n}, P_4) = \begin{cases} m & \text{when } n \leqslant 2 \\ m + \lfloor \frac{n}{2} \rfloor & \text{when } n > 2 \text{ and } mn \text{ is even} \\ m + \lfloor \frac{n}{2} \rfloor - 1 & \text{when } n > 2 \text{ and } mn \text{ is odd.} \end{cases}$$

This is the first example we've seen of when the score of the game depends a lot on who plays first.

9.3 Further generalization to properties

A further generalization is to saturation with respect to other properties, such as connection number or chromatic number. D. Hefetz, M. Krivelevich, A. Naor, and M. Stojaković [HKNS16] devise notation for this: the saturation game starting of the empty graph on n vertices where the players, an edge minimizer and and edge maximizer, must avoid a monotone graph property $\mathcal{P}$, is simply written $(n, \mathcal{P})$. For example, if G is the graph the players are building at any point in the process, the game of Füredi, Reimer, and Seress would be written as $(n, K_3 \subseteq G)$. For the game $(n, \mathcal{P})$, the game saturation number here will be written $g\text{-}sat(n, \mathcal{P})$, with the Min-first variant written analogously. We present a few quick looks into some properties that have been studied.

9.3.1 Connection number

Hefetz et al. first proved some results on the vertex connection number version of the saturation game.

Theorem 89. *[HKNS16] For all $k \geqslant 5$ and sufficiently large n,*

$$g\text{-}sat(n, \mathscr{C}_k) \geqslant \binom{n}{2} - (k-1)(2k-4)(n-(k-1)(2k-3))$$

where $\mathscr{C}_k$ is the property of being k-vertex connected and spanning.

They also have a result where the subtracted term is linear in k but polynomial in n rather than polynomial in k and linear in n.

Theorem 90. *[HKNS16] For all positive integers k and sufficiently large n,*

$$g\text{-}sat(n, \mathscr{C}_k) \geqslant \binom{n}{2} - 5kn^{3/2}$$

where $\mathscr{C}_k$ is the property of being k-vertex connected and spanning.

9.3.2 Chromatic number

Colorability saturation games were studied by J. Carraher. W. Kinnersley, B. Reiniger and D. West [CKRW17], Hefetz et al. [HKNS16], and R. Keusch [Keu18]. This area of study is about $g\text{-}sat(n, \chi_{>k})$; the players are forbidden edges that would make the resulting graph require $k+1$ or more colors for a proper (vertex) coloring. It is important to note that the final graph in this game is always k-partite. For the 3-coloring version, Hefetz et al.[HKNS16] partition the vertices of the graph the players are building into 3 sets (although not necessarily the parts of the graph that make it tripartite): the "top," the vertices that are the same color as some (fixed) arbitrary vertex, "middle," the neighborhood of the top, and "bottom," which is everything else. Min's strategy is heavily influenced by the desire to prevent an edge from being drawn between vertices that are (at that moment) in the bottom set. For this version of the game, they find that although Max can push the score almost to the extremal number, but Min can keep the score smaller by a non-negligible fraction.

Theorem 91. *[HKNS16]*

$$g\text{-}sat(n, \chi_{>3}) \leqslant \frac{21n^2}{64} + O(n).$$

[HKNS16] also extended this work to all k, with the following result.

Theorem 92. *[HKNS16] There exists a real number C such that*

$$g\text{-}sat(n, \chi_{>3}) \geqslant (1 - C \log k/k)\binom{n}{2}$$

holds for every positive integer k and sufficiently large n.

9.3.3 Hamiltonicity

Finally, Hefetz et al. [HKNS16] suggest a new property for game study, Hamiltonicity. They use the symbol $\mathscr{H}$ to mean the property of admitting a Hamiltonian cycle, but for consistency, we will write the game saturation number for this case as $g\text{-}sat(n, C_n)$. The saturation number for Hamiltonicity is known, due to [LJZY97]. In light of this, Hefetz et al. [HKNS16] conjecture the following.

Conjecture 18. *[HKNS16]*

$$g\text{-}sat(n, C_n) = \Theta(n^2).$$

10 Weak saturation

We now discuss the related notion of weakly saturated graphs. To do so, we first introduce some definitions and terminology.

Let $k_F(G)$ count the number of copies of F in G; if $F = K_p$ we will write $k_p(G)$ in place of $k_{K_p}(G)$. We say that an n-vertex graph G is *weakly F-saturated* if there is a nested sequence of graphs $G = G_0 \subset G_1 \subset \ldots \subset G_l = K_n$ such that G_i has exactly one more edge than G_{i-1} for $1 \leqslant i \leqslant l$ and $k_F(G_0) < k_F(G_1) < \ldots < k_F(G_l)$. That is, G is weakly F-saturated if we can add the missing edges of G one at a time and each edge we add creates at least one new copy of F. Of course, we are interested in the minimum size of a weakly F-saturated n-vertex graph, $w\text{-}sat(n, F)$. Corresponding to this, an n-vertex graph that is weakly F-saturated and has $w\text{-}sat(n, F)$ edges is said to be a member of $W\text{-}sat(n, F)$. The notion of weak saturation appears to have been introduced by B. Bollobás [Bol68]. In his paper he states that the problem of determining the saturation number for k-uniform hypergraphs with $k \geqslant 3$ motivated the concept.

We first note that $w\text{-}sat(n, F) \leqslant sat(n, F)$ as any F-saturated graph is also weakly F-saturated. Of course, the first instance of the problem considered is when $F = K_p$. Bollobás [Bol68] showed that for $3 \leqslant p < 7$ we have $w\text{-}sat(n, K_p) = sat(n, K_p)$ (see Theorem 1). Bollobás also conjectured that equality holds for at least some larger values of p, and later conjectured [Bol78] (see page 362) that equality holds for all p. The conjecture was confirmed by L. Lovász [Lov77] using flats of matroids representable over fields.

Theorem 93. *For integers n and p, we have w-$sat(n, K_p) = sat(n, K_p)$.*

This result is of interest since the corresponding result in Turán extremal theory was such a key result, but it is also of great interest because of the many different proofs and mathematical tools used in the proofs. The different proofs found later included P. Frankl [Fra82], N. Alon [Alo85] and J. Yu [Yu93]. All of these proofs came from extremal results on pairs of families of sets with certain interesting properties that were then applied to obtain proofs of the conjecture of Bollobás on weakly K_p-saturated graphs. (For a statement of this more general result and discussion on how it implies the conjecture, see [GGL95] page 1274.)

Bollobás' conjecture was also proved via two additional and different methods of G. Kalai in [Kal84] and [Kal85]. The first proof, which we give below, is based on the fact that an embedding of a weakly K_p-saturated graph G in $\mathbb{R}^{p-2}$ with vertices in general position is rigid (continuous deformation of adjacent vertices preserving distance also preserves distance for all vertices). This, along with the fact that a graph G of order n with less than $(p-2)n - \binom{p-1}{2}$ edges embedded in $\mathbb{R}^{p-2}$ is flexible (not rigid), completes the proof.

To give G. Kalai's proof we need a few definitions. Given a graph G on vertex set $\{1, 2, \ldots, n\}$, a *d-embedding* $G(v)$ of G is a sequence of n points in $\mathbb{R}^d$, $v = (v_1, v_2, \ldots, v_n)$, together with the line segments $[v_i, v_j]$, for $\{i, j\} \in E(G)$. We say that $G(v)$ is *rigid* if any continuous deformation $(v_1(t), v_2(t), \ldots, v_n(t))$ of $(v_1, v_2, \ldots, v_n)$ that preserves the distance between every pair of adjacent vertices, preserves the distance between any pair of vertices. $G(v)$ is *flexible* if it is not rigid.

Proof of Theorem 93 as given by G. Kalai [Kal84]:

Suppose G is a weakly K_p-saturated graph, with $G = G_0 \subset G_1 \subset \cdots \subset G_l = K_n$ such that G_i has exactly one more edge than G_{i-1} for $1 \leqslant i \leqslant l$ and $k_p(G_0) < k_p(G_1) < \cdots < k_p(G_l)$. We first show that every embedding of G into $\mathbb{R}^{p-2}$, such that the vertices are in general position, is rigid. Suppose that $v_1, v_2, \ldots, v_n$ are n points in general position in $\mathbb{R}^{p-2}$, and consider $G(v)$. Note that $G_l(v) = K_n(v)$ is rigid. Assume $G_{i+1}(v)$ is rigid, and suppose that $G_i = G_{i+1} - e$, where $e = \{\mu, \nu\}$ belongs to a K_p of G_{i+1}. Every embedding of $K_p - e$ in $\mathbb{R}^{p-2}$, with vertices in general position, is rigid. Thus, any continuous deformation of $G_i(v)$ preserves the distance between v_μ and v_ν, and so is a continuous deformation of $G_{i+1}(v)$. By the assumption the deformation preserves the distances between any two vertices of $G_i(v)$. Repeated application of this argument shows that $G(v) = G_0(v)$ is rigid.

Now we take advantage of the fact that if G is a graph of order n and fewer than $(p-2)n - \binom{p-1}{2}$ edges, then $G(v)$ is flexible. □

The second proof [Kal85] introduced the notion of hyperconnectivity in matroids. Kalai defined a matroid $\mathcal{H}_k^n$ on the set of edges of the complete graph on n vertices. A graph G on the same vertex set is *k-hyperconnected* if the set of its edges span $\mathcal{H}_k^n$. Kalai showed that the rank of $\mathcal{H}_{p-2}^n$ equals $(p-2)n - \binom{p-1}{2}$ and that K_p corresponds to a circuit in $\mathcal{H}_{p-2}^n$. Now, let e be any non-edge in a (weakly) K_p-saturated graph G. Since the addition of e yields a copy of K_p, the edges of G span e and consequently the entire matroid. That is, G must be $(p-2)$-hyperconnected. Thus such a graph has at least $(p-2)n - \binom{p-1}{2}$ edges. The ideas introduced here were later used by O. Pikhurko [Pik01c], see Theorem 95.

In fact, we know even more — that is, we know that equality also holds for the complete k-uniform hypergraph [Lov77], [Fra82], [Kal84], [Kal85], [Alo85].

However, it is not the case that equality always holds! For instance, $sat(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$ (see Section 3) while $w\text{-}sat(n, C_4) = n$ (note that for n odd C_n is weakly C_4-saturated and for n even the graph obtained from C_{n-1} by appending an edge is weakly C_4-saturated). It is also interesting to note that while there exists a unique K_3-saturated graph of minimum size, this is not the case for weakly K_3-saturated graphs. Here, the set of all n-vertex trees comprise $W\text{-}sat(n, K_3)$. This pattern repeats itself for many graphs. This gives an indication that, here too, the determination of $w\text{-}sat(n, F)$ might be challenging. In addition, Zs. Tuza [Tuz88] points out that the behavior of $w\text{-}sat(n, F)$ and $sat(n, F)$ differ significantly if F is relatively sparse.

Question 7. *[Tuz88] Are there necessary and/or sufficient conditions for $w\text{-}sat(n, F)$ to equal $sat(n, F)$?*

Let $H_k(p, q)$ denote the family of all k-uniform hypergraphs with p vertices and q edges. Tuza [Tuz88] conjectured that $w\text{-}sat(n, H_k(k+1, q)) = \binom{n-k-2+q}{q-2}$. (Note that as $H_k(k+1, k+1)$ consists only of the complete k-uniform hypergraph on $k+1$ vertices, this instance of the conjecture is solved by Theorem 93.) As a first step towards this conjecture, P. Erdős, Z. Füredi, and Zs. Tuza [EFT91] gave the following result.

Theorem 94. *[EFT91] For $n > k \geqslant 2$, $w\text{-}sat(n, H_k(k+1, 3)) = n - k + 1$.*

They left open the problem of determining $W\text{-}sat(n, H_k(k+1, 3))$, but this was later solved by O. Pikhurko [Pik01b]. In a different paper [Pik01c], Pikhurko made further progress. To state these results we must introduce a new type of graph.

Let sequences $\mathbf{k} = (k_1, \ldots, k_t)$ of nonnegative integers and $P_1, \ldots, P_t$ of disjoint sets of sizes $\mathbf{p} = (p_1, \ldots, p_t)$ be given. Define $[t] = \{1, \ldots, t\}$ and, for $I \subseteq [t]$, we write k_I in place of $\sum_{i \in I} k_i$ and P_I in place of $\cup_{i \in I} P_i$; also, we assume $k_0 = 0, P_0 = \emptyset$, etc. Then the *pyramid* $\Delta = \Delta(\mathbf{p}; \mathbf{k})$ is the k-graph, $k = k_{[t]}$, on $P = P_{[t]}$ such that E is an edge of Δ if and only if, for every $i \in [t]$, we have $|E \cap P_{[i]}| \geqslant k_{[i]}$.

Theorem 95. *[Pik01c] Suppose we are given two non-empty sequences $\mathbf{p} = (p_1, \ldots, p_t)$ and $\mathbf{k} = (k_1, \ldots, k_t)$ of integers such that $p_i \geqslant k_i \geqslant 1$ for $i \in [t]$. Then*

$$w\text{-}sat(n, \Delta(\mathbf{p}; \mathbf{k})) = \sum_{\mathbf{k}'} \binom{n - p_{[t]} + k_t}{k'_{t+1}} \prod_{i \in [t]} \binom{p_i + k_{i-1} - k_i}{k'_i}, \quad n \geqslant p_{[t]},$$

where the summation is taken over all sequences of nonnegative integers $\mathbf{k}' = (k'_1, \ldots, k'_{t+1})$ such that $k'_{[t+1]} = k_{[t]}$ and, for some $i \in [t]$, $k'_{[i]} > k_{[i-1]}$.

Now, let us examine the many cases covered by this theorem. First, for $t = 1$ the graph $\Delta(p; k)$ is the k-uniform complete hypergraph on p vertices. Thus, this theorem confirms Bollobás' conjecture in the case $k = 2$ and its generalization for $k \geqslant 3$. In the case of $k = 2$, i.e. graphs, it gives a new result for split graphs (consider $\Delta(p_1, p_2; 1, 1)$. And, Theorem 95 confirms Tuza's conjecture as the only graph in $H_k(k+1, q)$ is the pyramid graph $\Delta(k-q+1, q; k-q+1, q-1)$.

In [Pik01b] Pikhurko gives a construction of an $H_k(p, q)$-saturated graph which he conjectures to be minimum. This conjecture remains open.

R. Faudree, R. Gould, and M. Jacobson [FGJ13] deal with sparse graphs, which is one set of cases where $w\text{-}sat(n, \mathcal{F})$ and $sat(n, \mathcal{F})$ act very differently.

Theorem 96. *[FGJ13] Let F be a graph on p vertices with q edges, and with $\delta(F) = \delta$. Then*

1.

$$q - 1 + \frac{(\delta-1)(n-p)}{2} \leqslant w\text{-}sat(n, F) \leqslant (\delta-1)n + \frac{(p-1)(p-2\delta)}{2}$$

for any $n \geqslant p$.

2.

$$\frac{\delta n}{2} - \frac{n}{\delta+1} \leqslant w\text{-}sat(n, F) \leqslant (\delta-1)n + \frac{(p-1)(p-2\delta)}{2}$$

for sufficiently large n.

Faudree, Gould, and Jacobson [FGJ13] also determine $w\text{-}sat(n, F)$ when F is $K_5 - e$ and $K_{2,3}$. Shortly thereafter, R. Faudree and R. Gould [FG14] determine the value of $w\text{-}sat(n, F)$ when F consists of disjoint copies of (usually) the same graph. This includes: $k(K_5 - 2K_2), k(K_{1,t}), k(K_p), k(C_l)$, and others, where $k(F)$ denotes the disjoint union of k copies of the graph F. In a set of two papers [CP16], [CP19], the authors Y. Cui and L. Pu extend the results of Faudree et al. and answer some of the problems they posed. Cui and Pu determined the value of $w\text{-}sat(n, F)$ when F is $k(K_p - 2K_2), k(K_p - K_{1,m})$ [CP16] and $K_{2,p}, K_p \sqcup K_q$ [CP19]. Together these authors leave us to consider the following problem.

Problem 28. *Investigate $w\text{-}sat(n, k(K_p - sK_2))$.*

We now turn to a beautiful method introduced in 2012 by J. Balogh, B. Bollobás, R. Morris and O. Riordan [BBMR12]. The method is a *dimension argument* and is reminiscent of the *linear algebra method* developed and used for many other combinatorial arguments. The statement we give here is not the full statement, but is sufficient for our purposes.

Lemma 2. *[J. Balogh, B. Bollobás, R. Morris, O. Riordan [BBMR12]] Let F be a graph and let W be a vector space. Suppose that there exists a set $\{h_e : e \in E(K_n)\} \subseteq W$ such that for every copy F' of F there are nonzero scalars $\{c_{e,F'} : e \in E(F')\}$ such that $\sum_{e \in E(F')} c_{e,F'} h_e = 0$. Then*

$$w\text{-}sat(n, F) \geqslant dim(span\{h_e : e \in E(H)\}).$$

To make use of this lemma to prove a lower bound, which is most often the more difficult bound to obtain, one must assign vectors to edges so that those on copies of F satisfy a particular dependence relation. Further, in order for this lower bound to be worthy of our consideration, the set of vectors should also have a large span. As noted by G. Kronenberg, T. Martins, and N. Morrison [KMM21], what this lemma does is that it turns the problem of computing a lower bound for the weak saturation number into a *constructive* problem.

The method allowed for the consideration of one of the most natural of questions that had gone unanswered for so long: what is $w\text{-}sat(n, K_{t,t})$? When G is a spanning graph, an exact value is given by G. Kronenberg, T. Martins, and N. Morrison [KMM21].

First they show that the graph G_n, which consists of an independent set of size $t-1$ joined to $K_t \cup (n-2t+1)K_1$, is a spanning weakly $K_{t,t}$-saturated graph and thus provides an upper bound on $w\text{-}sat(n, K_{t,t})$. To find lower bound for a spanning graph in $W\text{-}sat(n, K_{t,t})$ that implements an application of Theorem 2 they construct a family of vectors $\{h_e : e \in E(K_n)\}$ in a vector space such that: (1) for any copy F of $K_{t,t}$ in K_n, the vectors $\{h_e : e \in E(F)\}$ have a non-trivial dependence; (2) the subset of vectors $\{h_e : e \in E(G_n)\}$ are linearly independent.

Theorem 97. *[KMM21] Let $t \geqslant 2$ and $n \geqslant 3t-3$. If G is a spanning graph and weakly $K_{t,t}$-saturated, then*

$$|E(G)| = (t-1)(n+1-t/2).$$

As a consequence of Theorem 97, they also obtain the following.

Corollary 2. *[KMM21] Let $t \geqslant 2$ and $n \geqslant 3t-3$. If G is a spanning graph and weakly $K_{t,t+1}$-saturated, then*

$$|E(G)| = (t-1)(n+1-t/2) + 1.$$

Kronenberg et al. also prove the following.

Theorem 98. *[KMM21] Let $2 \leqslant s < t$ and $n \geqslant 4t$. If G is a spanning graph and weakly $K_{s,t}$-saturated, then*

$$|E(G)| = n(s-1) + c(s,t),$$

where $c(s,t)$ is an integer depending only on s and t.

Kronenberg, Martins and Morrison generalize the spanning graph G_n providing a graph that is weakly $K_{t,\dots,t}$-saturated and leave us with the problem of determining whether this is optimal. They also pose the following.

Question 8. *[KMM21] What is the minimum number of edges in a weakly $K_{t_1,\dots,t_k}$-saturated spanning graph?*

10.1 Host graphs other than K_n

Let H be a host graph. We say that an n-vertex graph G is *weakly F-saturated* if there is a nested sequence of graphs $G = G_0 \subset G_1 \subset \cdots \subset G_l = H$ such that G_i has exactly one more edge than G_{i-1} for $1 \leqslant i \leqslant l$ and $k_F(G_0) < k_F(G_1) < \cdots < k_F(G_l)$. That is, G is weakly F-saturated if we can add the missing edges of G in H one at a time and each edge we add creates at least one new copy of F. Of course, we are interested in the minimum size of a weakly F-saturated graph in H, $w\text{-}sat(H, F)$. Corresponding to this, an n-vertex graph that is weakly F-saturated and has $w\text{-}sat(H, F)$ edges is said to be a member of $W\text{-}sat(H, F)$.

Morrison, Noel, and Scott studied saturation in the hypercube, and when studying weak-saturation, found results for a more general type of graph: the grid graph. Useful in establishing their results, Morrison et al. make use of a generalization of Theorem 2.

Theorem 99. *[MNS17] For $n \geqslant m \geqslant 1$,*

$$w\text{-}sat(Q_n, Q_m) = (m-1))2^n - \sum_{i=0}^{m-2} (m-1-i)\binom{d}{i}.$$

Morrison, Noel, and Scott [MNS17] define a new parameter, $w\text{-}sat^*(H, F)$, which is the weak saturation number in the host graph H with the forbidden configuration F, but where the only graphs that may be considered must be spanning subgraphs; this is the minimum number of edges of a spanning subgraph $G \subset H$ that is F-free such that there exists an ordering of $E(H) \setminus E(G)$ where adding the edges to G in that order creates a new copy of F each time. This is relevant to this hypercube saturation problem because clearly no higher-dimensional hypercube could be created by the addition of an edge incident at a vertex with no other edges incident at it yet.

Theorem 100. *[MNS17] For $k \geqslant r \geqslant 2$ and $n \geqslant m \geqslant 1$,*

$$\begin{aligned} w\text{-}sat^*(P_k^n, P_r^m) = \quad & \sum_{j=0}^{n} \sum_{i=0}^{n-j} (m-1+i)\binom{n}{j}\binom{n-j}{i}(k-j(r-2)^i \\ & - \sum_{j=0}^{m-2} \sum_{i=0}^{n-j} (m-1-j)\binom{n}{j}\binom{n-j}{i}(k-r+1)^j(r-2)^i \end{aligned}$$

where $0^0 = 1$.

Morrison and Noel also obtained an exact expression in the hypercube host graph for weak saturation with stars [MN18].

Theorem 101. *[MN18] If $d \geqslant r$, then*

$$w\text{-}sat(Q_n, K_{1,r+1}) = r2^{r+1} + \sum_{j=1}^{r-1} \binom{d-j-1}{r-j} j2^{j-1}.$$

Also, Morrison and Noel provide a generalized result for the graph $P_{a_1}\square \dots \square P_{a_d}$.

Theorem 102. *[MN18] For $d \geqslant r \geqslant 1$ and $a_i \geqslant 2$ for all i, we have that w-sat$(P_1\square \dots \square P_d, K_{1,r+1})$ equals*

$$\sum_{S\subseteq\{i:a_i \text{ is defined}\},|S|\leqslant r-1} (\prod_{i\in S}(a_i-2))((r-|S|)2^{r-|S|-1}) + \sum_{j=1}^{r-|S|-1} \binom{d-|S|-j-1}{r-|S|-j} j2^{j-1}).$$

One of the implications of the work of Moshkovitz and Shapira [MS15] is to weakly $K_{t,t}$-saturated subgraphs of $K_{m,m}$. They use a skew version of Bollobás set-pair method discussed in the Introduction. Kronenberg, Martins, and Morrison [KMM21] generalized their result (and proof) to the case of weakly $K_{s,t}$-saturated subgraphs of $K_{\ell,m}$.

Theorem 103. *[MS15], [KMM21] Let $2 \leqslant s \leqslant \ell, t \leqslant m$. Then*

$$w\text{-}sat(K_{\ell,m}, K_{s,t}) = (m+\ell-s+1)(s-1)+(t-s)^2.$$

Kronenberg, Martins, and Morrison point out a connection to the complete graph setting.

Corollary 3. *[KMM21] When $t \geqslant 2$, $n \geqslant 3t-3$, and $\ell, m \geqslant 2$ such that $\ell+m=n$,*

$$w\text{-}sat(n, K_{t,t}) = w\text{-}sat(K_{\ell,m}, K_{t,t}) + \binom{t}{2}.$$

D. Korándi and B. Sudakov [KS17] began the study of weak saturation in Erdős-Rényi random graphs. Let $G(n,p)$ denote the Erdős-Rényi random graph on n vertices, with each pair of vertices having probability p that an edge will link them. Korándi and Sudakov's main result is as follows.

Theorem 104. *[KS17] Let $p \in (0,1)$ be a constant probability and let $s \geqslant 3$ be an integer. Then,*

$$w\text{-}sat(G(n,p), K_s) = (s-2)n - \binom{s-1}{2}$$

with high probability as n tends to infinity.

Korándi and Sudakov give the following problem.

Problem 29. *[KS17] Determine the exact probability range when w-sat$(G(n,p), K_s) = (s-2)n - \binom{s-1}{2}$.*

M. Bidgoli, A. Mohammadian, B. Tayfeh-Rezaie, and M. Zhukovksi [BMTRZ20] found a threshold function to address this question. They proved that there is a threshold probability, and established bounds.

Theorem 105. *[BMTRZ20] Let $s \geqslant 3$ be a fixed integer. Also, let*

$$q_s(n) = n^{-\frac{2}{s+1}(\ln(n)^{\frac{2}{(s-2)(s+1)}})}$$

and let

$$c_s = [2(1 - \frac{1}{s+1})(s-2)!]^{\frac{2}{(s+1)(s-2)}}.$$

There exists some probability a such that

$$w\text{-}sat(G(n,p), K_s) = (s-2)n - \binom{s-1}{2}$$

with high probability whenever $p > a$, and

$$w\text{-}sat(G(n,p), K_s) \neq (s-2)n - \binom{s-1}{2}$$

with high probability whenever $p < a$.
Furthermore, $c_s q_s(n) < a < n^{-\frac{1}{2s-3}}(\ln(n))^2$.

10.2 Asymptotics

When exact determination of the function $w\text{-}sat(n, F)$ is unknown, we may turn to the following result of Zs. Tuza [Tuz92] for an estimation. Prior to stating the estimation, we must introduce a graph invariant which as Tuza points out is a 'local' parameter of the graph F. This is in contrast to the chromatic number, a global parameter, which controls the asymptotic behavior of the extremal number as told to us by the theorem of Erdős-Stone-Simonovits.

We assume that F is a k-uniform hypergraph with at least two edges. For an edge $E \in E(F)$ the *sparseness of an edge* $s(E)$ is the smallest natural number s for which there is an $E^* \subseteq E$ with $|E^*| = s + 1$ such that $E^* \subseteq E' \in E(F)$ implies $E' = E$ (i.e. a set which uniquely determines the edge); if E is a subset of some other edge of F, then we put $s(F) := |F|$. The *local sparseness of the hypergraph F* $s(F)$ is the $\min\{s(E) : E \in E(F)\}$. Note that $1 \leqslant s(F) \leqslant k - 1$ for all k-uniform hypergraphs.

Theorem 106. *[Tuz92] For every k-uniform hypergraph F, $w\text{-}sat(n, F) = \Theta(n^{s(F)})$.*

Tuza suggests that this statement might be refined, and thus offers the following.

Conjecture 19. *[Tuz92] For some positive constant $c = c(F)$, we have*

$$w\text{-}sat(n, F) = cn^{s(F)} + O(n^{s(F)-1}).$$

The results in [Alo85] yield $w\text{-}sat(n, F) = cn + o(n)$ in the case of $s(F) = 1$.

10.3 Other results

When $\mathcal{F}$ is the set of all minimal forbidden subgraphs of some hereditary property P, some results for $w\text{-}sat(n, \mathcal{F})$ have been obtained. Such hereditary properties include k-degeneracy and bounded maximum degree. For results of this type, we refer the reader to work by G. Semanišin [Sem97], M. Borowiecki and E. Sidorowicz [BS02], and E. Sidorowicz [Sid07].

11 Bootstrap Percolation

Sometimes, papers on weak saturation will also have results on a mechanism called bootstrap percolation, which we formally introduce here. A *bootstrap percolation* on a graph G is a sequence of functions (η_t) where $\eta : V(G) \to \{0, 1\}$, and η_t is determined entirely by η_{t-1}: if and only if its *(neighborhood) parameter* r or more neighbors of a vertex v map to 1 by η_{t-1}, or if v maps to 1 itself under that function, then v maps to 1 under η_t. Thus, the entire sequence is determined by the initial state η_0 and the graph G. Weak saturation is a form of edge bootstrap percolation. A bootstrap percolation is defined to be *spanning* (or it *spans* the graph), or it is said to *percolate* if every vertex eventually maps to 1. A vertex, or *site* is "occupied," "active," or "infected" if and only if it maps to 1, and likewise "empty," "vacant," "passive," "inactive," or "healthy" otherwise. Bootstrap percolation is often studied in lattices, of which the hypercube is a special case. A *random bootstrap percolation* refers to a bootstrap percolation where vertices are randomly infected at the beginning with a given probability, independent of the status of other vertices. Random bootstrap percolations are the norm to study, so the "random" quality is often left unspecified.

Addressing intuition, it is indeed the case in any graph, with any neighborhood parameter r, that the probability that a random bootstrap percolation is spanning is strictly increasing with p, the probability of a vertex being initially infected. Weak saturation has been reimagined as a bootstrap process on the edges of graphs, and is sometimes written about in the language of bootstrap percolation, often using the words r-bond bootstrap percolation, which translates to weak saturation with stars, and H-bootstrap percolation, which is weak saturation with a graph H, in that whenever a new copy of H can be made, it is. In both of these cases, however, the edges are not necessarily added one at a time, although this is irrelevant to most results.

Bootstrap percolation was first coined as a term by J. Chalupa, P. Leath, and G. Reich [CLR79], who studied the Bethe lattice to inform research on metamagnets. In the original model, sites could become uninfected, making it more like Conway's Game of Life than bootstrap percolation as in this survey. Bootstrap percolation as a topic was studied in a computer science setting as well [AL88]. Finally, J. Balogh, B. Bollobás, and R. Morris [BBM12] point out that a sociological modeling of mass decision-making is relevant to bootstrap percolation [Gra78]. In this particular model, each site is a potential mob member, and the assumption is that the more mob-ready the people around someone are, the more mob-ready they will become themselves. This model proposed the idea of different vertices having different individual neighborhood parameters, to take into account that some people will more readily join a mob than others, but a process like that has not been studied.

11.1 Threshold and Critical Probabilities

We start by introducing the founding works of bootstrap percolation. J. Balogh and B. Bollobás [BB06] studied the threshold functions of random bootstrap percolations. Specifically, they seek a function Φ for each graph G and parameter r that gives the probability that a random bootstrap percolation with probability p is spanning. When unable, they seek thresholds probability functions p_0 and p_1: a lower threshold p_0 where, with high probability, such a random bootstrap percolation is not spanning, and an upper threshold p_1, where, with high probability, such a random bootstrap percolation is spanning.

Theorem 107. *[BB06] For bootstrap percolation on $G \cong P_2^n \cong \{0,1\}^n \cong Q_n$, with neighborhood parameter $r = 2$, $\frac{2^{-2\sqrt{n}}}{150n^2}$ is a lower threshold function, and $\frac{5000 \cdot 2^{-2\sqrt{n}}}{n^2}$ is an upper threshold function.*

Balogh and Bollobás [BB06] used initially spanned subcubes of the hypercube to prove their thresholds. This result shows us the order of the threshold functions in the commonly-studied hypercube; they are both $\Theta(\frac{2^{\sqrt{n}}}{n^2})$.

Instead of the threshold probabilities, the *critical probability*, where the probability that a vertex will be initially infected is such that the probability that the ensuing bootstrap percolation is spanning is 0.5, has also been studied. J. Balogh, B. Bollobás, and R. Morris [BBM09] write this as $p_{\frac{1}{2}}$; and in general write $P(G, r, p)$ for the probability that a random initially infected set percolates (completely) in a graph G with parameter r and probability p, and then that $p_\alpha(G, r) = \inf\{p : P(G, r, p) \geqslant \alpha\}$. Confusingly, $p_{\frac{1}{2}} = p_c$ is also sometimes termed a threshold. Balogh, Bollobás, and Morris [BBM09] extended this to the 3-dimensional case; in P_n^3, and with neighborhood parameter $r = 3$.

Theorem 108. *[BBM09] Let $\lambda \approx 0.4039$.*

$$p_c(P_n^3, 3) = \frac{\lambda + o(1)}{\ln \ln n}.$$

This is a smaller case of a more general result, with λ defined in r and d, where d is the dimension of the grid; the case above uses $\lambda(3, 3)$. This more general result is due to J. Balogh, B. Bollobás, H. Duminil-Copin, and R. Morris [BBDCM12]. First, a definition for $\lambda(d, r)$ is necessary. First, let

$$\beta_k(u) = \frac{1 - (1-u)^k + \sqrt{1 + (4u-2)(1-u)^k + (1-u)^{2k}}}{2}.$$

Using that, let

$$g_k(w) = -\ln(\beta_k(1 - e^{-w})).$$

Finally, for $2 \leqslant r \leqslant d$, let

$$\lambda(d,r) = \int_0^\infty g_{r-1}(z^{d-r+1})dz.$$

Now, we can state the theorem.

Theorem 109. *[BBDCM12] For $2 \leqslant r \leqslant d$,*

$$p_c(P_n^d, r) = (\frac{\lambda(d,r) + o(1)}{\ln_{r-1}(n)})^{d-r+1}$$

where $\ln_a$ indicates the natural logarithm iterated a times; $\ln_1(n) = \ln(n)$ and $\ln_{a+1}(n) = \ln(\ln_a(n))$.

They used cartesian products of grids to set up their proof by induction on r.

Balogh, Bollobás, and Morris [BBM10] examined the asymptotics of bootstrap percolation in grids. First, we need an equation of theirs.

$$\sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{2^{k^2-k} k!} = 0 \tag{2}$$

The smallest positive root of Equation (2) is $\lambda \approx 1.16577$.

Theorem 110. *[BBM10] For sufficiently large d and λ as above,*

$$\frac{16\lambda}{d^2}(1 + \frac{\log_2 d}{\sqrt{d}})2^{-2\sqrt{d}} \leqslant p_c(P_2^d, 2) \leqslant \frac{16\lambda}{d^2}(1 + \frac{5(\log_2 d)^2}{\sqrt{d}})2^{-2\sqrt{d}}.$$

Here, $\log_2$ indicates not iterated evaluation of the logarithm but simply the base 2 logarithm. They also present a more specific result, where $r = 2$:

Theorem 111. *[BBM10] Let $n = n(d)$ be a function with $1 \leqslant \log_2(n) \leqslant d$ as $d \to \infty$. Then, with $\lambda \approx 1.166$ as above,*

$$p_c(P_n^d, 2) = (4\lambda + o(1))(\frac{n}{d(n-1)})^2(2^{-2\sqrt{d \log_2(n)}}).$$

J. Balogh and B. Pittel [BP07] used Markov chains and modeling with differential equations to find threshold results for r-neighbor bootstrap percolation in the random d-regular graph on n vertices, which they write as $G_{n,d}$. Their work uses a concept applied to this narrow situation with the following definition:

$$p^* = 1 - \inf_{y \in (0,1)} \frac{y}{\mathbb{P}(\mathrm{Bin}(d-1, 1-y) < r)},$$

where $\mathrm{Bin}(d-1, 1-y)$ is the binomially distributed random variable with parameters $d-1$ and $1-y$. p still refers to the probability that a vertex was initially infected.

Theorem 112. *[BP07] Suppose that $2 \leqslant k < d-1$. Let $\omega = \omega(n) \to \infty$ slowly enough that* $\log(\omega(n)) = o(\log(n))$.

1. *If $p > p^* + \omega n^{-1/2}$, then with high probability as n grows, a set of size pn percolates in $G_{n,d}$.*
2. *If $p \leqslant p^* - \omega n^{-1/2}$, then with high probability as n grows, a set of size pn does not percolate in $G_{n,d}$.*

Balogh and Pittel questioned whether the general number of vertices that remain uninfected can be determined in the $G(n, d)$ setting.

Theorem 113. *[BP07]*

1. *Suppose $p - p^* \geqslant n^{-\epsilon}$, where $\epsilon = \epsilon(n) \to 0$ and $\epsilon \log(n) \to \infty$. Then, with high probability as n grows, the number of vertices that remain uninfected in $G_{n,d}$ is*
$$O((p-p^*)^{-3/2}).$$
2. *Let $p \leqslant p^* - \omega n^{-\sigma}$, where $\sigma = \frac{1}{2d+10}$, and let $\omega = \omega(n) \to \infty$ slowly enough that* $\log(\omega(n)) = o(\log(n))$. *Then, with high probability as n grows, a set of size pn will not percolate in $G_{n,d}$.*

11.2 Minimum percolating sets

A natural problem with a more saturation-like flavor is the study of the smallest possible percolating set. Balogh and Bollobás introduced this topic with a conjecture which N. Morrison and J. Noel [MN18] proved. This topic uses $m(G, r)$ to indicate the smallest possible percolating set in a graph G using neighborhood parameter r.

Theorem 114. *[MN18] Proving the conjecture of [BB06], let $r \geqslant 3$ be fixed. Then, as $d \to \infty$,*
$$m(Q_d, r) = (\frac{1}{r} + o(1))\binom{d}{r-1}.$$

M. Przykucki and T. Shelton [PS19] provide another basic result, giving the smallest percolating set in any grid.

Theorem 115. *[PS19] For all integer $n, d \geqslant 1$, $m(P_n^d, d) = n^{d-1}$.*

Balogh, Bollobás, Morris, and Riordan use a different generalization of hypercubes, a sort of rook graph that they write K_n^d. To avoid possible confusion with balanced complete bipartite graphs, we will write $*K_n^d$ to mean the rook graph which has the same vertex set as P_n^d. For vertices u and v, uv is an edge not only if they differ in exactly 1 coordinate any amount, not just a difference of 1. They introduce generalizations of $*K_n^d$ and P_n^d to the hypergraph setting; $\mathcal{K}(n, d, t, r)$, which has the same vertex set as $*K_n^d$, and its hyperedges are all sets $S = I_1 \times \cdots \times I_d$ where r of the sets I_j are of size t and the remaining $d - r$ of them are singletons. $\mathcal{P}(n, d, t, r)$ is defined similarly, except the r non-singleton sets are intervals in the grid. Note briefly that $\mathcal{P}(n, d, t, r) \subset \mathcal{K}(n, d, t, r)$.

Theorem 116. *[BBMR12] For every $n \geqslant t \geqslant 2$ and $d \geqslant r \geqslant 1$,*

$$m(\mathcal{K}(n,d,t,r)) = m(\mathcal{P}(n,d,t,r)) = \sum_{s=0}^{r-1} \binom{d}{s} (t-1)^{d-s}(n+1-t)^s.$$

The reader may have noticed that a bootstrap percolation with parameter r functions similarly to weak saturation with a star of $r+1$ points, the only essential difference being that in weak saturation, only one vertex is "added" at a time (by being connected as part of a star), but all applicable vertices are infected at once in the bootstrap process on the vertices. Morrison and Noel [MN18] point out an important link between $m(G,r)$ and $w\text{-}sat(G, K_{1,r+1})$.

Lemma 3. *[MN18]*

$$m(G,r) \geqslant \frac{w\text{-}sat(G, K_{1,r+1})}{r}.$$

It was this that allowed them to prove their results in both weak saturation and r-neighbor bootstrap percolation.

As mentioned at the beginning of the subsection, Morrison and Noel proved the suspicion of Balogh, Bollobás, Morris, and Riordan. They propose several questions to follow up on their results.

Question 9. *[MN18] For fixed $r \geqslant 4$, does*

$$\lim_{d\to\infty} \frac{m(Q_d, r) - \frac{d^{r-1}}{r!}}{d^{r-2}}$$

converge? If so, what is the limit?

Question 10. *[MN18] For fixed $r \geqslant 4$, is it true that*

$$m(Q_d, r) = 2^{r-1} + \left\lceil \sum_{j=1}^{r-1} \binom{d-j-1}{r-j} \frac{j2^{j-1}}{r} \right\rceil$$

for sufficiently large d?

Morrison and Noel furnish an exact result as well for $r = 3$:

Theorem 117. *[MN18] For $d \geqslant 3$, $m(Q_d, 3) = \lceil \frac{d(d+3)}{6} \rceil + 1$.*

Indeed, even just the next case is open:

Problem 30. *[MN18] Determine $m(Q_d, 4)$ for all $d \geqslant 4$.*

Problem 31. *[MN18] Determine $m(Q_d, r)$ for all $d \geqslant 4$ and $r \geqslant 4$.*

L. Hambardzumyan, H. Hatami, and Y. Qian [HHQ20] study weak saturation, but it can be linked by Morrison and Noel's bridge [MN18]. Their results and proof are outlined in detail below, with r-bond bootstrap percolation.

11.3 Bootstrap Percolation on Edges and Graphs

Morrison and Noel's bridge shows how bootstrap percolation and weak saturation are closely related. In fact, sometimes a process is studied that is effectively weak saturation, which is called in general H-bootstrap percolation. By this process, wherever in some graph, there is a copy of $H - e$ for any edge e, that edge is added to the graph. A special case of this, termed r-bond bootstrap percolation, is defined as the bootstrap process on edges where, when r adjacent edges to an edge are infected, it becomes infected; this is $K_{1,r+1}$-bootstrap percolation. The only way these processes are different from weak saturation is that the definition we use of weak saturation describes an ordering of edges to be added in, but in these processes, the ordering does not matter. We may as well be adding many edges at once. What sets these apart from study on weak saturation is that the focus of study is on topics more related to bootstrap percolation, such as critical probabilities, especially in random graphs.

The results and proofs use a common characteristic of the configuration H that percolates. The symbol λ, in the context of H-bootstrap percolation, is defined as follows.

$$\lambda = \frac{|E(H)| - 2}{|V(H)| - 2}.$$

Balogh, Boolobás, and Morris [BBM12] found bounds on the critical probability for K_r-bootstrap percolation, which Bartha and Kolesnik [BK21] improved on.

Theorem 118. *[BK21] For $r \geqslant 5$,*

$$p_c(n, K_r) = \Theta(n^{-1/\lambda}).$$

The result of Balogh, Bolobás, and Morris included a bound for when $r = 4$, on the order $O(n^{n^{-1/\lambda}\log(n)})$. The same authors also bounded $p_c(n, K_4)$; it is $\Theta((n \log n)^{-\frac{1}{2}})$. Some simple results bounding the critical probability are known, specifically on cycles, graphs with leaves, and r-clique trees, which have stars at the center with mutually disjoint copies of K_r at each vertex. These are due again to [BBM12].

Finally, we have some open problems, all due to Balogh, Bollobás, and Morris.

Problem 32. *[BBM12] Determine $\lim_{n\to\infty} \frac{\log(p_c(n,H))}{\log n}$ for every graph H.*

Problem 33. *[BBM12] Characterize graphs H for which the probability p that H percolates in $G(n, p)$ has a sharp threshold.*

Problem 34. *[BBM12] Determine $p_c(n, K_{s,t})$, up to a poly-logarithmic factor.*

Problem 35. *[BBM12] Find bounds for $p_c(n, G(k, \frac{1}{2}))$ which hold with high probability as $k \to \infty$.*

11.3.1 Weak Saturation, Bootstrap Percolation, and the Polynomial Method

L. Hambardzumyan, H. Hatami, and Y. Qian [HHQ20] used the polynomial method and algebraic techniques to study r-bond bootstrap percolation, which is also applicable to weak saturation with stars. The parameter they study, the smallest percolating set of edges, $m_e(G, r)$, to mirror the analogous parameter for vertices, is the same as $w\text{-}sat(G, K_{1,r+1})$.

Before we examine the proof, we need a key definition.

Definition 1. *Let r be a non-negative integer, let G be a graph, and let $c : E(G) \to \mathbb{R}$ be a proper edge coloring of G. Let $W^r_{G,c}$ be the vector space consisting of functions $\Phi : E(G) \to \mathbb{R}$ that meet the following requirement: there exists a set of univariate polynomials $\{p_v \in \mathbb{R}[x] : v \in V(G)\}$ such that*

1. *$\forall v \in V(G)$, $deg(p_v) \leqslant r - 1$;*
2. *$\forall uv \in E(G)$, $p_u(c_{uv}) = p_v(c_{uv}) = \Phi(uv)$.*

Then, the set of polynomials $\{p_v : v \in V(G)\}$ is said to recognize Φ.

Let's now state the theorem and investigate the proof:

Theorem 119. *[HHQ20] Let $c : E(G) \to \mathbb{R}$ be a proper edge coloring of a graph G, and let $r \geqslant 0$ be an integer. Then,*

$$m_e(G, r) = w\text{-}sat(G, K_{1,r+1})) \geqslant \dim(W^r_{G,c}).$$

The theorem gives a lower bound for weak saturation number with stars of any size in any host graph from only a proper edge coloring of the host graph. By Morrison and Noel's [MN18] bridge, mentioned above, we can adapt this to a statement about r-neighbor bootstrap percolation.

Corollary 4. *[HHQ20]*

$$\frac{\dim(W^r_{G,c})}{r} \leqslant m(G, r)$$

with $W^r_{G,c}$ defined as above for any graph G with non-negative integer r equipped with a proper edge coloring $c : E(G) \to \mathbb{R}$.

Proof of Theorem 119: [HHQ20] Let $F \subseteq E(G)$ be a percolating set for the r-bond bootstrap process in G. The idea is that if $\Phi \in W^r_{G,c}$ satisfies $\Phi(e) = 0$ for all $e \in E(G)$, then $\Phi \equiv 0$, which implies

$$W^r_{G,c} \cap \{\Phi : E(G) \to \mathbb{R} \mid \Phi(e) = 0 \,\forall e \in F\} = \{0\},$$

which implies

$$\dim(W^r_{G,c}) \leqslant |E(G)| - \dim(\{\Phi : E(G) \to \mathbb{R} \mid \Phi(e) = 0 \,\forall e \in F\}),$$

$$= |E(G)| - (|E(G)| - |F|) = |F|$$

which is the lower bound that was sought. Since F is a percolating set, there must be elements of F in every nontrivial component of G. By the definition of $W^r_{G,c}$, $\Phi(e_1) = \Phi(e_2)$ for any pair of neighboring edges, so in fact all edges in the same component must have the same image under Φ. Since $\exists e : e \in F$ and $e \in H$ for every nontrivial component $H \subseteq G$, every edge of G must map to 0 under Φ if F is a percolating set, which means $\Phi[E(G)] = 0$ and so $\Phi \equiv 0$. Then $\dim(W^r_{G,c}) \leqslant |E(G)|$ since it is a subspace of $\{\Phi | \Phi : E(G) \to \mathbb{R}\}$, which has dimension $|E(G)|$. $\{\Phi : E(G) \to \mathbb{R} \mid \Phi(e) = 0 \, \forall e \in F\}$ has dimension $|E(G)| - |F|$ since a basis for it would be the set of characteristic functions for each edge (mapping each edge to 0 except one of them), of which there are $|E(G)| - |F|$. This is the example the authors use to show the bound is sharp.□

If there are no edges with $r-1$ or more other coincident edges, there could be no nontrivial percolation r-bond bootstrap percolation and so it is appropriate that the lower bound turns out to be the indeterminate form 0, as there is no way to guarantee solutions to the equation $p_v = 0$ for any $v \in V(G)$, since even with a proper edge coloring, there may not be enough colors for the domain of $p_v = 0$. This is why r does not need to be given an upper bound for which the theorem is valid. Otherwise, of course, if there is an edge with $r-1$ or more coincident edges, then the proper edge coloring c must have $|c[E(G)]| \geqslant r$, and the Nullstellensatz guarantees that finding such set of polynomials that recognize Φ is possible.

What the polynomial method gives us in this proof is the helpful assurance that this lower bound is not always 0; whenever there exist such univariate polynomials, the lower bound is nonzero.

The authors used this process to find the smallest percolating sets for cartesian products of cycles and paths, tori and grids, respectively.

11.4 Percolation Time

One aspect of bootstrap percolation that we have not examined is the number of time steps that it would take for percolation to play out. This is one of the aspects of bootstrap percolation that makes it different than just weak saturation with stars or weak saturation with vertices instead of edges; many things can happen at once (instead of just one edge being added at a time with weak saturation), and the number of steps is an interesting part to study. Przykucki and Shelton [PS19] examined this aspect. In bootstrap percolation, the *closure* of a set of vertices with parameter r is the set of all vertices that are eventually infected by the r-neighbor bootstrap percolation starting from that set, denoted $\langle A \rangle_r$ for $A \subseteq V(G)$ in a graph G. The percolation time for a set A is written $T(A)$. Przykucki and Shelton write $m_d(n)$ to indicate $\min\{T(A) : \langle A \rangle_d = P_n^d, |A| = n^{d-1}\}$.

Theorem 120. *[PS19] Where $G \cong P_n^d$*

1. $m_{G,1}(n) = \lceil \frac{n}{2} \rceil$,

2. $m_{G,2}(n) = n - 1$, *and*

3. *for all* $d \geqslant 3$,

$$\frac{dn}{2} + O(1) \leqslant m_{G,d}(n) \leqslant (d+2)n^2 + n.$$

Based on numerical simulations, the authors expect that $m_{P_n^d,d}(n) = \Theta(n^2)$ for $d \geqslant 3$.

Question 11. *[PS19] Is* $m_{P_n^d,d}(n) = \Theta(n^2)$ *for* $d \geqslant 3$*?*

They also wonder about tori $\mathbb{T}_n^d$.

Question 12. *[PS19] What is the value of* $m_{\mathbb{T}_n^d,d}$ *for* $d \geqslant 3$*?*

12 Few Copies of H in an F-saturated graph

A generalization of the extremal number, $ex(n, F)$, introduced by N. Alon and C. Shikhelman [AS16], counts the maximum number of copies of a graph H in an n-vertex F-free graph. Observe that when $H = K_2$ this count is $ex(n, F)$. Motivated by Alon and Shikhelman [AS16], J. Kritschgau, A. Methuku, M. Tait, and C. Timmons [KMTT20] investigated the analogous generalization of the saturation number.

We write $sat(n, H, F)$ for the minimum number of copies of the graph H in an n-vertex F-saturated graph. Thus, $sat(n, F) = sat(n, K_2, F)$ and if $F \subseteq H$, then trivially $sat(n, H, F) = 0$.

This initial paper by Kritschgau et al. [KMTT20] contains 16 theorems and propositions, most of which concern various combinations of complete graphs and cycles and would be a good place for the interested reader to begin. Unlike the classic saturation number, the generalized saturation number is not necessarily bounded above by a linear function of n, (Theorem 121), though for complete graphs a linear bound does hold (Proposition 1).

Theorem 121. *[KMTT20] Let* $s \geqslant 5$ *and* $r \leqslant 2s - 4$ *in the case that* r *is even, or* $r \leqslant \frac{4s-4}{3}$ *in the case that* r *is odd. Then,* $sat(n, C_r, K_s) = \Theta(n^{\lfloor r/2 \rfloor})$.

Proposition 1. *[KMTT20] Let* $n \geqslant 1$ *and* $r \geqslant 2$ *be integers. For any graph* F*, there is a constant* $C = C(r, F)$ *such that* $sat(n, K_r, F) < Cn$.

It is an easy observation that a sufficiently large complete bipartite graph is C_{2k+1}-saturated and contains no odd cycle; thus, $sat(n, C_{2r+1}, C_{2s+1}) = 0$ for integers r, s and n sufficiently large. Kritschgau et al. [KMTT20] proved $sat(n, C_3, C_k) = 0$ for any $k \geqslant 5$ and sufficiently large n. Following this, Timmons [Tim19] constructed C_r-free graphs that are C_{2k}-saturated for any odd $r \geqslant 5$ giving the following Theorem.

Theorem 122. *[Tim19] For any odd integer* $r \geqslant 5$ *and any* $2k \geqslant r+5$, $sat(n, C_r, C_{2k}) = 0$ *for all* $n \geqslant 2kr$.

Curiously, this leaves the case of $sat(n, C_3, C_4)$ still unresolved. Additionally, instances where the target cycle, $C_r r$ is even are not well understood. Some general bounds and some results on small order cases can be found in [KMTT20].

Problem 36. *[KMTT20] Determine if $sat(n, C_3, C_4) > 0$ for infinitely many n.*

Problem 37. *[KMTT20] Determine if $sat(n, C_4, C_6) > 0$ for infinitely many n.*

Upper and lower bounds for $sat(n, K_r, K_s)$ were established in [KMTT20] along with a conjecture that has already been settled in the affirmative by D. Chakraborti and P.-S. Loh [CL20].

Theorem 123. *[KMTT20, CL20] For every $s > r \geqslant 2$, there exists a constant $n_{r,s}$ such that for all $n \geqslant n_{r,s}$, we have $sat(n, K_r, K_s) = (n - s + 2)\binom{s-1}{r-1} + \binom{s-2}{r}$. Moreover, there exists a constant $c_{r,s} > 0$, such that the only K_s-saturated graph with up to $sat(n, K_r, K_s) - c_{r,s}$ many copies of K_r is $K_{s-2} + \overline{K_{n-s+2}}$.*

The previous theorem requires n to be sufficiently large; for 'small' n the value of $sat(n, K_r, K_s)$ remains open. The second statement also raises the question of the range of values for $sat(n, K_r, K_s)$.

Problem 38. *[CL20] For $3 \leqslant r < s$ and all values of n, determine the exact value of $sat(n, K_r, K_s)$.*

Problem 39. *[CL20] For $3 \leqslant r < s$, determine the range of possible numbers of copies of K_r in an n-vertex K_s-saturated graph.*

Several results for specific small-order cases can be found in [KMTT20]; the one below, in particular, has motived results analogous to that of [Day17] in Section 2.

Theorem 124. *[KMTT20] For $n \geqslant 7$, $sat(n, K_3, K_4) = n - 2$. Furthermore, the only n-vertex K_4-saturated graph with $n-2$ triangles is the join of an edge and an independent set of $n - 2$ vertices.*

Observe that the unique extremal graph from the previous theorem has minimum degree 2. Motivated by this observation and building on the work relating minimum degree and saturation number by [Day17], C. Timmons, B. Cole, A. Curry, and D. Davini [TCCD20] began investigating the minimum number of copies of a graph H in an F-saturated n-vertex graph with minimum degree t, denoted $sat_t(n, H, F)$. A selection of the many results on $sat_t(n, K_r, K_s)$ from [TCCD20] follow.

Theorem 125. *[TCCD20] For $n \geqslant 14$, $sat_4(n, K_3, K_4) = 2n - 4$. For $t \geqslant 4$ and $n \geqslant 2t$, $sat_t(n, K_3, K_4) \leqslant 2n + 2t - 12$.*

Conjecture 20. *[TCCD20] For an integer $t \geqslant 4$, there is an integer n_t such that for all $n \geqslant n_t$, $sat_t(n, K_3, K_4) = 2n + 2t - 12$.*

In the case that $t = 4$, Cole et al. have identified the unique family of extremal graphs and conjectured a similar type of extremal family for a general t. They also established an upper bound for arbitrary t, K_r and K_s; no general lower bound exists.

Theorem 126. *[TCCD20] Let $3 \leqslant r < s$ and $t \geqslant 2(s-2)+1$ be integers. For $n \geqslant 2(s-2)+2t$, $sat_t(n, K_r, K_s) \leqslant \binom{s-2}{r-1}2^{r-1}n + C_{s,r,t}$ where $C_{s,r,t}$ is a constant depending only on s, r, and t.*

At the end of a paper on rainbow saturation appearing in Section 7, N. Bushaw, D. Johnston, and P. Rombach [BJR20] defined a generalized rainbow saturation number to be the minimum number of copies of the graph F in any rainbow H-saturated graph and suggested the following problem.

Problem 40. *[BJR20] Determine the minimum number of copies of the graph F in any rainbow H-saturated graph.*

13 Induced Saturation

This section considers generalizations of classical graph saturation to saturation on induced graphs. A natural first attempt would be to say a graph G is F-induced saturated if G contains no *induced* copy of F but the addition of any edge gives an *induced* copy of F in G. Unfortunately, this definition has the problem that there exist target graphs F and values of n such that every n-vertex graph G either fails to be induced-F-free or contains an edge $e \in \overline{G}$ such that $G + e$ has no induced copy of F. (A simple example is $F = K_{1,3}$ and $n = 4$.)

As as way around this issue, R. Martin and J. Smith [MS12] adopted the use of *trigraphs*.

A *trigraph* T is a quadruple $(V(T), Eb(T), Ew(T), Eg(T))$ where $V(T)$ is the vertex set and the other three sets form a partition of $\binom{V(T)}{2}$ into a set of black edges ($Eb(T)$), a set of white edges ($Ew(T)$), and a set of gray edges ($Eg(T)$). These are viewed as edges in T, nonedges in T and potential edges of T, respectively.

Observe that if $Eg(T) = \emptyset$, then T is a graph.

A *realization* of a trigraph T is any graph G with vertex set $V(T)$ such that $E(G) = Eb(T) \cup S$, where $S \subseteq Eg(T)$. Intuitively, a realization of T is a graph obtained from T by starting with the edge set of T (black edges) and adding any subset of the gray edges to form G.

A trigraph T is *F-induced-saturated* if no realization of T contains F as an induced subgraph, but F occurs as an induced subgraph of some realization whenever any black or white edge is changed to gray.

The induced saturation number of F with respect to n, $indsat(n, F)$, is defined to be the minimum number of gray edges in an n-vertex trigraph that is F-induced-saturated. It should be clear that given any graph F, the n-vertex trigraph T such that all edges are gray contains a realization of F provided $|V(F)| \leqslant |V(T)|$; thus, $indsat(n, F)$ is well-defined.

Moreover, if $indsat(n, F) = 0$, it follows that there does indeed exist an F-induced-saturated *graph* on n vertices. To be clear, such a graph would have the property that it contains no induced copy of F but adding any new edge or deleting any existing edge would result in a graph containg an induced copy of F. In this case, it makes sense to look for F-induced-saturated graphs with a minimum number of edges. S. Beherens, C. Erbes, M. Santana, D. Yager, and E. Yeager defined the parameter $indsat^*(n, F)$ to be the minimum number of edges in an F-induced-saturated graph under the condition that $indsat(n, F) = 0$.

Martin and Smith, who began the investigation into induced saturation in [MS12], established several foundational results. For example, $indsat(n, F) \leqslant sat(n, F)$ for any graph F and positive integers $n \geqslant |V(F)|$ since, given graph $G \in Sat(n, F)$, replacing all edges of G with gray edges forms an F-induced-saturated trigraph T. Since cliques are always induced, $indsat(n, K_p) = sat(n, K_p)$. Finally, they used $F \cong K_p - e$ to establish that there are some F for which $indsat(n, F) \neq sat(n, F)$.

Perhaps because it is an easy observation that there exist P_k-induced-saturated graphs for $k = 2, 3$, paths have received much attention, with the first result from Martin et al.

Theorem 127. *[MS12] For all $n \geqslant 4$, $indsat(n, P_4) = \lceil \frac{n+1}{3} \rceil$.*

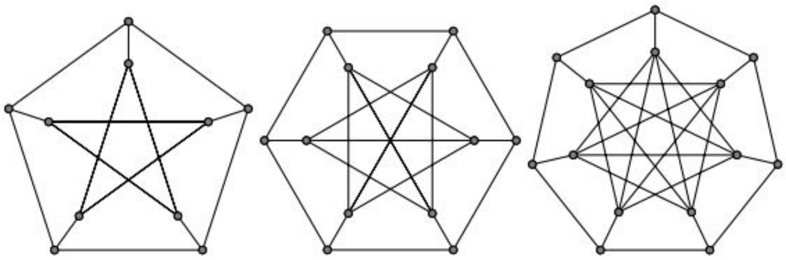
Observe that this result establishes that there do not exist P_4-induced-saturated graphs which raised the question of whether there exist P_k-induced-saturated graphs for paths on more than 4 vertices. The combined work of E. Räty [R2̈0], E.-K. Cho, I. Choi, and B. Park [CCP21], M. Bonamy, C. Groenland, N. Morrison and A. Scott [BGMS20], and V. Dvořák [Dvo20] showed that there are induced P_k-saturated graphs for $k \geqslant 5$.

Räty begun work on this problem by mapping a graph to a finite field of order 16 to prove the result for $k = 6$. Cho, Choi, and Park extended Räty's method to all paths on $3k$ vertices after observing that the Petersen graph is induced P_6-saturated. This observation prompted Dvořák to construct induced P_6-saturated for all $n \geqslant 6$ with larger graphs that resemble the Petersen graph. Bonamy, Groenland, Morrison, and Scott used brute-force to find induced P_5-saturated graphs, none of which are as small or structured as the Petersen-like graphs of Dvořák might lead one to expect.

Theorem 128. *[R2̈0] [CCP21] [BGMS20] [Dvo20] For all integers $k \geqslant 2$ with $k \neq 4$, there exists an induced P_k-saturated graph.*

Problem 41. *Investigate $indsat(n, \mathcal{F})$ for $\mathcal{F} \neq P_k$.*

Another interesting problem is if Dvořák's Petersen graph-like constructions are edge-minimal for any n. Here are examples of Dvořák's constructions, directly from the paper: [Dvo20]



The leftmost of these is induced P_6-saturated; the next is induced P_7-saturated, and the last is induced P_8-saturated. Notice that graphs of this type that are induced P_k-saturated can be constructed by starting with a clique on $k-1$ vertices, removing a cycle, adding exactly one pendant vertex adjacent to each vertex of what was the clique, and then linking the pendant vertices with a cycle in the order of the cycle among their neighbors that was deleted at the beginning. These graphs have $\frac{(k-1)(k-2)}{2} + k - 1$ edges.

Question 13. *Is* $indsat^*(2(k-1), P_k) = \frac{(k-1)(k-2)}{2} + k - 1$ *for* $k \geqslant 6$*?*

Thanks to Behrens et al. [BES+16], we know the graphs $K_{1,n}$, nK_2, C_3, C_4, C_{2k-1}, C'_{2k} and $\tilde{C}_{2k}$ have induced saturation number zero for $n \geqslant 2$ and $k \geqslant 3$, meaning that induced saturated graphs exist for all of those as forbidden configurations. Axenovich and Csikos [AC19] introduced two new classes of graphs such that for a graph F in one of these classes, there exists an F-induced-saturated graph G that is a Cartesian product of cliques. The first, $\mathcal{F}(n)$ with $n \geqslant 3$, is all graphs F with a cut vertex v of degree $n+1$ such that $F - v$ has $n+1$ connected components, each of which is an induced subgraph of $\overset{n}{\square} K_k$ for some $k \leqslant n-1$. The other is $\mathcal{J}(n)$, graphs J that can be obtained as the union of graphs F'' and T such that F'' is an induced subgraph of $\overset{n}{\square} K_k$ for some $k \leqslant n-1$, T is the union of $n+1$ paths P_{n+2} that all intersect at an endpoint, and $|V(F'') \cap V(T)| = V'$, where V' is the set of leaves of T [AC19].

Now, knowing that a graph F of one of these types has $indsat(n, F) = 0$, a natural next step is to determine $indsat^*(n, F)$. Behrens et al. have proven bounds for many of these, and solved the smallest case of $K_{1,3}$ for $n \equiv 0, 1 \mod 3$.

Problem 42. *[AC19] Determine* $indsat^*(n, F)$ *for* F *chosen from the above classes of graphs.*

C. Tennenhouse [Ten16], on the other hand, studied induced saturation as formulated in the first paragraph of this section. Given a target graph F and a parent graph G, we say G is F-induced-saturated if G has no induced copy of F but the addition of any edge $e \in \overline{G}$ to G results in an induced copy of F. Unlike the earlier formulations where edges were added or deleted, in this formulation edges can only be added but never removed, to create the forbidden configuration. Also, recall that one is not guaranteed such a graph G exists. Thus, the work in [Ten16] consists mostly of establishing the existence of induced F-saturated graphs under this definition. The author finds examples of F-induced-saturated graphs when F consists of certain path, cycles and $K_{1,3}$. In particular, Tennenhouse used an infinite family of cubic hamiltonian graphs discovered by Lederberg, Coxeter, and Frucht, denoted as $[x, -x]^a$, to show that there exist induced P_{9+6k}-saturated graph for any integer $k \geqslant 0$. As for other values of k, Tennenhouse gives examples of graphs that are P_k-induced saturated up to $k = 30$. Results for cycles and $K_{1,3}$ follow.

Theorem 129. *[Ten16] If $k \geqslant 3$ and $n \geqslant 3(k-2)$, then there exists a (non-edge-removing) induced C_k-saturated graph of order n.*

The lower bound for n is due to Tennenhouse's construction. Joining arbitrarily large cliques to the graph creates a new non-removing induced saturated graph on arbitrarily many more vertices.

Question 14. *[Ten16] Does there exist an induced C_k-saturated graph of order n when $n < 3(k-2)$?*

Theorem 130. *[Ten16] For all $n \geqslant 12$, there is a graph on n vertices that is (non-edge-removing) induced $K_{1.3}$-saturated*

14 Irregularity of the *sat*-function

The function $sat(n, \mathcal{F})$, in general, is *not* monotone with respect to n or $\mathcal{F}$. Turán's extremal function is monotone with respect to n and $\mathcal{F}$. That is, for $F' \subseteq F$ and $\mathcal{F}' \subseteq \mathcal{F}$ the following inequalities hold for every n.

$$ex(n, F') \leqslant ex(n, F) \tag{3}$$

$$ex(n, \mathcal{F}) \leqslant ex(n, \mathcal{F}') \tag{4}$$

$$ex(n, \mathcal{F}) \leqslant ex(n+1, \mathcal{F}) \tag{5}$$

If we replace ex by sat in each of the above inequalities, then for $F' \subseteq F$ and $\mathcal{F}' \subseteq \mathcal{F}$ we need not have a true statement. Prior to giving examples that illustrate when these inequalities fail, we note that the failure to be monotone makes proving statements about $sat(n, \mathcal{F})$ difficult. In particular, inductive arguments generally do not work — this may also be due to the non-uniqueness of the extremal graphs; for example, see the result on $K_{2,2}$ [Oll72] or [Che11]. The failure to be monotone also may explain the scarcity of results for $sat(n, \mathcal{F})$, but in the authors' collective opinion makes the function an interesting study.

To see that the sat-function is not, in general, monotone with respect to subgraphs, consider the 'irregular pair' as given by O. Pikhurko [Pik04], and that answered a question of Zs. Tuza [Tuz92] about the existence of a connected spanning subgraph F' of subgraph F. Let $F' = K_{1,m}$ and $F = K_{1,m} + e$, where e joins two vertices in the m-set. Then $sat(n, F) \leqslant n - 1$ as $K_{1,n-1}$ serves as an extremal graph. However, $sat(n, F')$ is strictly larger for n large enough as seen by Theorem 25. Even in the class of trees, this monotone property fails at a very high level, and was observed in Section 4 (see [FFGJ09a]).

To see that the *sat*-function is not, in general, monotone with respect to subfamilies, consider $\mathcal{F} = \{K_{1,m}, K_{1,m} + e\}$ and $\mathcal{F}' = \{K_{1,m} + e\}$. Then $sat(n, \mathcal{F}) = sat(n, K_{1,m}) > n - 1$, but $sat(n, \mathcal{F}') \leqslant n - 1$. (Note that for any $F' \subset F, F, F' \in \mathcal{F}$ then $sat(n, \mathcal{F}) = sat(n, \mathcal{F} \setminus F)$.)

To see that the *sat*-function is not, in general, monotone in n, consider when $F = P_4$. By a result in [KT86], we have $sat(2k-1, P_4) = k+1 > sat(2k, P_4) = k$.

As a result of this 'irregularity', Zs. Tuza [Tuz86] (more readily available in [Tuz88]) made the following conjecture.

Conjecture 21. *[Tuz86],[Tuz88] For every graph F, the limit $\lim_{n\to\infty} \frac{sat(n,F)}{n}$ exists.*

Some progress towards settling this conjecture has been made, both in the positive and negative direction. However, the conjecture still remains open. We first give statements in the positive direction.

Theorem 131. *[TT91] Let F be a graph. If $\liminf_{n\to\infty} \frac{sat(n,F)}{n} < 1$, then $\lim_{n\to\infty} \frac{sat(n,F)}{n}$ exists and is equal to $1 - \frac{1}{p}$, for some positive integer p.*

A characterization of graphs for which $\lim_{n\to\infty} \frac{sat(n,F)}{n} = 1 - \frac{1}{p}$ for any given p is given in terms of connected components. Unfortunately, this characterization 'grows' with p. In the characterization tree components of F play a role. Thus, Tuza gave the following problem.

Question 15. *[Tuz88] Which trees T satisfy $\lim_{n\to\infty} \frac{sat(n,T)}{n} < 1$?*

Towards the negative direction of settling Conjecture 21, O. Pikhurko [Pik99a] showed that there exists an infinite family $\mathcal{F}$ of graphs for which $\lim_{n\to\infty} \frac{sat(n,\mathcal{F})}{n}$ does not exist. Later, in [Pik04] he improved this to show that for every integer $m \geqslant 4$ there exists a family $\mathcal{F}$ consisting of m graphs (i.e. a family of size m) for which $\lim_{n\to\infty} \frac{sat(n,\mathcal{F})}{n}$ does not exist, and suggested that his approach might be altered to yield a smaller family. Indeed, D. Chakraborti and P.-S. Loh [CL20] found a smaller family.

Theorem 132. *[CL20] There exist infinitely many families $\mathcal{F}$ containing three graphs such that $\lim_{n\to\infty} \frac{sat(n,\mathcal{F})}{n}$ does not exist.*

In light of these results, one might ask the following.

Question 16. *[CL20] Does there exist a singleton family of graphs $\mathcal{F} = \{F\}$ such that $\lim_{n\to\infty} \frac{sat(n,\mathcal{F})}{n}$ does not exist?*

An affirmative answer is equivalent to disproving Conjecture 21.

Their work also extends into the few H copies generalization. See Section 12 for more details and relevant definitions.

Theorem 133. *[CL20] There exist infinitely many families $\mathcal{F}$ containing exactly three graphs such that* $\lim_{n\to\infty} sat(n, K_p, \mathcal{F})$ *does not exist.*

Theorem 134. *[CL20] For all $r \geqslant 5$, there exists a graph F such that $sat(n, C_r, F) = 0$ for infinitely many n and also $sat(n, C_r, F) > 0$ for infinitely many n.*

N. Behague [Beh18] shows how the work on Tuza's conjecture can be generalized to hypergraphs.

Theorem 135. *[Beh18]*

1. *For all $k \geqslant 2$ there exists a family $\mathcal{F}$ of k-uniform hypergraphs and a constant $c_k \in \mathbb{N}$ such that*
$$sat(n, \mathcal{F}) = \begin{cases} O(n) & \text{if } c_k | n \\ \Omega(n^{k-1}) & \text{if } c_k \nmid n \end{cases}$$
2. *$c_k = 2$ is an option for any k.*

The irregular behavior this shows is that $sat(n, \mathcal{F})$ does not necessarily have a limit as n grows large for any family $\mathcal{F}$.

Behague also generalizes Pikhurko's results to hypergraphs.

Theorem 136. *[Beh18] For all $r \geqslant 3$, there exists a family $\mathcal{F}$ of four k-uniform hypergraphs such that* $\lim_{n\to\infty} \frac{sat(n,\mathcal{F})}{n^{k-1}}$ *does not exist.*

This result makes progress towards the following generalization of Conjecture 21 due to O. Pikhurko [Pik99b].

Conjecture 22. *[Pik99b] For every k-uniform hypergraph F, the limit* $\lim_{n\to\infty} \frac{sat(n,F)}{n^{k-1}}$ *exists.*

N. Behague (see Section 5 of [Beh18]) proposes a graph that might serve as a counterexample to this conjecture.

We now make some additional comments unrelated to Conjecture 21.

In Section 7 we encounter the following notions. An edge-coloring of a graph F is *rainbow* if every edge of F receives a different color. Let $\Re(F)$ denote the set of all rainbow-colored copies of F. A t-edge colored graph G is $\Re_t(F)$*-saturated* if G does not contain a rainbow copy of F but for any edge $e \in \overline{G}$ and any color $i \in [t]$, the addition of e in color i to G produces a rainbow copy of F. We write $sat(n, \Re_t(F))$ to denote the minimum number of edges in a t-edge-colored $\Re_t(F)$-saturated graph of order n. Ferrara et al. [FJL$^+$20] find that $sat(n, \Re_t(F))$ is not monotone with respect to inclusion as a subgraph in F. These authors investigate other notions of monotonicity with respect to $\Re_t(F)$-saturated graphs.

Lastly, we mention that G. Semanišin [Sem97] has given certain instances under which the *sat*-function *is* monotone and uses these to prove some inequalities and estimations.

15 Graphs with directed edges

We now briefly focus our attention on graphs with directed edges. (Our focus is brief as the number of results is fairly limited. We restrict our attention to graphs; the only result for hypergraphs that we are aware of is given in [Pik99a],[Pik99b].)

Investigation in this direction began with Zs. Tuza [Tuz86] (he presented further results and a summary of earlier results in the more readily available [Tuz88]). We begin with some definitions found in [Pik99b]. Let $\mathcal{C}$ be a class of objects, with a binary relation $\subseteq$. A member G of the class $\mathcal{C}$ is *$\mathcal{F}$-admissible* if, for every $F \in \mathcal{F}$, G does not contain F as a sub-object. Then we denote the family of maximal $\mathcal{F}$-admissible objects of order n by $SAT(n, \mathcal{F})$. G, of order n, is called *$\mathcal{F}$-saturated* if $G \in SAT(n, \mathcal{F})$, and if, in addition, G has minimum size, we say it has size $sat(n, \mathcal{F})$.

O. Pikhurko [Pik99b] (cf. Section 4) asked if the order estimates given above (see Theorem 2 and Theorem 37) remain valid for the class of directed graphs. That is, for directed graphs do we have $sat(n, \mathcal{F}) = O(n)$? He pointed out that, in general, the answer is no. As an immediate consequence to the main result of Z. Füredi, P. Horak, C. Pareek, and X. Zhu [FHPZ98], we have that $sat(n, C_3) \geqslant O(n \log n)$ (where C_3 has directed edges $12, 23$ and 31); that is, the order estimate is super-linear! The results of these authors do not provide an upper bound (their constructions contain copies of C_3), and so we pose the following.

Problem 43. *In the class of directed graphs determine a good upper bound for $sat(n, C_3)$.*

O. Pikhurko [Pik99b] did show that the order estimates do remain valid under certain conditions. He considered the class of cycle-free directed graphs. A graph is *cycle-free* if it does not contain a cycle, in other words, there is no alternating sequence of vertices and edges $(x_1, e_1, x_2, e_2, \ldots, x_l, e_l, x_{l+1} = x_1)$ such that $x_i x_{i+1} = e_i$. So within the class of cycle-free directed graphs, a graph G is $\mathcal{F}$-saturated if it contains no $F \in \mathcal{F}$ but the addition of any directed edge creates a copy of some $F \in \mathcal{F}$ *or* a directed cycle.

Theorem 137. *[Pik99b] In the class of cycle-free directed graphs $sat(n, \mathcal{F}) = O(n)$ for any family $\mathcal{F}$ of cycle-free graphs.*

In addition, M. Jacobson and C. Tennenhouse [JT12] considered $sat(n, F)$ and showed that $SAT(n, F)$ is non-empty for any F. They also give values and estimates for $sat(n, P_k)$, where all arcs of P_k point in the 'same direction'. Similar results were given earlier by S. van Aardt, J. Dunbar, M. Frick, and O. Oellermann [vAFDO09].

16 Saturation in partially ordered sets

A *partially ordered set* (or poset) consists of a set P and a binary relation $\leqslant$ which is reflexive, transitive, and antisymmetric and is denoted $\mathcal{P} = (P, \leqslant)$. A poset $\mathcal{P}' = (P', \leqslant')$ is a *subposet* of $\mathcal{P} = (P, \leqslant)$ if there exists an injective function $f : P' \to P$ such that for every $u', v' \in P'$ if $u' \leqslant' v'$ then $f(u') \leqslant f(v')$ in P.

The n-dimensional Boolean Lattice, $\mathcal{B}_n$, will denote the poset $(2^{[n]}, \subseteq)$ where $[n] = \{1, 2, \cdots, n\}$ and the elements of $2^{[n]}$ are ordered by inclusion. If a subposet $\mathcal{F}$ of $\mathcal{B}_n$ does not contain some poset $\mathcal{P}$ as a subposet, we say that $\mathcal{F}$ is $\mathcal{P}$-free.

Let $\mathcal{P}$ and $\mathcal{Q}$ be subposets of $\mathcal{B}_n$ where we think of $\mathcal{P}$ as the host poset and $\mathcal{Q}$ as the target poset. We say $\mathcal{P}$ is *$\mathcal{Q}$-semisaturated* if for every $S \in 2^{[n]} - P$, the poset $\mathcal{P} \cup S$ (with order inherited from $\mathcal{B}_n$) contains additional copies of $\mathcal{Q}$. We say $\mathcal{P}$ is *$\mathcal{Q}$-saturated* if $\mathcal{P}$ is semi-saturated and $\mathcal{Q}$-free.

Let $s\text{-}sat(\mathcal{B}_n, \mathcal{Q})$ denote the smallest size of $|P|$ for $P \subseteq \mathcal{B}_n$ such that $\mathcal{P} = (P, \subseteq)$ is $\mathcal{Q}$-semi-saturated. Let $sat(\mathcal{B}_n, \mathcal{Q})$ denote the smallest size of $|P|$ such that $\mathcal{P} = (P, \subseteq)$ is $\mathcal{Q}$-saturated.

The definitions and properties presented in this subsection could be translated into the language of hypergraphs provided edges of cardinality 0 and 1 are allowed. The notation and language used in this section is quite different from that used in the cited papers for several reasons. The use of the modifier *weak* is used many different ways in the papers referenced here; for this survey we have reserved its use to that in Section 10. Also, we rephrase the notion of *inheriting a k-Sperner property* in the more tradition language of saturation theory for the sake of simplification.

16.1 Saturation and Semisaturation in Posets

The study of saturation on posets began by D. Gerbner, B. Keszegh, N. Lemons, C. Palmer, D. Pálvölgyi, and B. Patkós ([GKL$^+$13]) and focused on the saturation numbers and semi-saturation numbers of chains and flat antichains.

Let $\mathcal{P}_{k+1}$ denote a $(k+1)$-element chain (i.e. a set of $k+1$ elements any two of which are comparable). The authors found several basic results that set the stage for more to come.

Theorem 138. *[GKL$^+$13] For integers $6 \leqslant k \leqslant n$, $s\text{-}sat(\mathcal{B}_n, \mathcal{P}_{k+1}) = O(\frac{\log(k)}{k} 2^k)$.*

Theorem 139. *[GKL$^+$13] For nonnegative integers k, c, n,*
$2^{\frac{k}{2}-1} \leqslant s\text{-}sat(\mathcal{B}_n, \mathcal{P}_{k+1}) \leqslant sat(\mathcal{B}_n, \mathcal{P}_{k+1}) \leqslant 2^{k-1}$ whenever $k \leqslant n$ and n sufficiently large.

The authors conjectured that for all k, $\lim_{n\to\infty} sat_{\mathcal{B}_n}(n, \mathcal{P}_{k+1}) = 2^{k-1}$. In [MNS14], N. Morrison, J. Noel and A. Scott proved that the conjecture is correct for $k \leqslant 5$, but for $k \geqslant 6$, $\lim_{n\to\infty} sat_{\mathcal{B}_n}(n, \mathcal{P}_{k+1}) < 2^{k-1}$.

The authors extended this to find the following theorem:

Theorem 140. *[MNS14] There exists an $\epsilon > 0$ such that for all k,*

$$\lim_{n\to\infty} sat(\mathcal{B}_n, \mathcal{P}_{k+1}) = 2^{(1+o(1))ck}$$

where c is some constant in the interval $[\frac{1}{2}, 1-\epsilon]$

The precise value of c is an open question.

Question 17. *[MNS14] What is the constant $c \in [\frac{1}{2}, 1-\epsilon]$ such that*

$$\lim_{n\to\infty} sat(\mathcal{B}_n, \mathcal{P}_{k+1}) = 2^{(1+o(1))ck}?$$

$\mathcal{P}_{k+1}$-semisaturated graphs have a similar asymptotic behavior, answering another basic question of [GKL$^+$13].

Theorem 141. *[MNS14] For integer k,*

$$\lim_{n\to\infty} s - sat_{\mathcal{B}_n}(n, \mathcal{P}_{k+1}) = 2^{(1/2+o(1))k}.$$

Observe that $\mathcal{P}_2$-saturated families are maximal antichains and it is in this way that $\mathcal{P}_{k+1}$-saturated families were initially studied, particularly motivated by the *Flat Antichain Theorem.* A *flat* anti-chain is an anti-chain in which all members of the family have cardinality ℓ or $\ell+1$. The Flat Anti-Chain Theorem says that for every anti-chain of $2^{[n]}$, there exists a flat anti-chain of the same cardinality. In [GHK$^+$09] using our present terminology, the authors address the question of the minimum size of a flat $\mathcal{P}_2$-saturated family where $\ell = 2$. The authors provided a lower bound and determined all anti-chains in which this bound holds. In [GKL$^+$13] the authors build on this work producing the following theorem.

Theorem 142. *[GHK$^+$09] [GKL$^+$13] Let $\mathcal{F}$ be an anti-chain such that $\mathcal{F} \subseteq \binom{[n]}{2} \cup \binom{[n]}{3}$. Then*

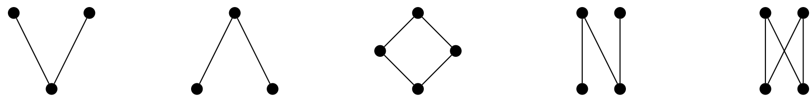
$$\binom{n}{2} - \left\lceil \frac{(n+1)^2}{8} \right\rceil \leqslant |\mathcal{F}| \leqslant \left(\frac{3}{8} - o(1)\right) n^2.$$

16.2 Induced Saturation in Posets

A poset $\mathcal{P}' = (P', \leqslant')$ is an *induced subposet* of $\mathcal{P} = (P, \leqslant)$ if there exists an injective function $f : P' \to P$ such that for every $u', v' \in P'$, $u' \leqslant' v'$ in P' if and only if $f(u') \leqslant f(v')$ in P. If a subposet $\mathcal{P}$ of $\mathcal{B}_n$ does not contain some poset $\mathcal{Q}$ as an induced subposet, we say that $\mathcal{P}$ is induced-$\mathcal{Q}$-free. We say $\mathcal{P}$ is induced-$\mathcal{Q}$-saturated if $\mathcal{P}$ is $\mathcal{Q}$-free and for every $S \in 2^{[n]} - P$, the poset $\mathcal{P} \cup S$ (with order inherited from $\mathcal{B}_n$) contains additional induced copies of $\mathcal{Q}$. Note that if the target poset, $\mathcal{Q}$, is not itself an induced subposet of $\mathcal{B}_n$, then the only induced-$\mathcal{Q}$-saturated poset is $\mathcal{B}_n$ itself. Let *indsat*$(\mathcal{B}_n, \mathcal{Q})$ denote the smallest size of $|P|$ such that $\mathcal{P} = (P, \subseteq)$ is induced-$\mathcal{Q}$-saturated.

The notion of induced saturation and posets occurred later by Ferrara et al. [FKK$^+$17]. However, since any two members of a chain are comparable, the saturation number and the induced saturation number for chains must be the same. In [FKK$^+$17], the authors establish several upper and lower bounds for various posets of 4 or fewer elements (Figure 16.2). Some lower bounds were recently improved [MSW20]. In addition, the Ferrara et al. make several conjectures.

Theorem 143. *[FKK$^+$17, MSW20]*



$\mathcal{V}_2$ Λ $\mathcal{D}_2$ $\mathcal{N}$ $\bowtie$

- *If* $n \geqslant 2$, *then* $indsat(\mathcal{B}_n, \mathcal{V}_2) = indsat(\mathcal{B}_n, \Lambda) = n + 1$.
- *If* $n \geqslant 3$, *then* $\lceil \log_2 n \rceil \leqslant indsat(\mathcal{B}_n, \bowtie) \leqslant \binom{n}{2} + 2n - 1$.
- *If* $n \geqslant 2$, *then* $\lceil \sqrt{n} \rceil \leqslant indsat(\mathcal{B}_n, \mathcal{D}_2) = n + 1$.
- *Let* $\mathcal{A}_{k+1}$ *be an antichain of* $k+1$ *elements. If* $n > k \geqslant 3$ *and* n *sufficiently large, then*

$$\left(1 - \frac{1}{\log_2 k}\right) \frac{kn}{\log_2 k} \leqslant indsat(\mathcal{B}_n, \mathcal{A}_{k+1}) \leqslant (n-1)k - \left(\frac{1}{2}\log_2 k + \frac{1}{2}\log_2 \log_2 k + O(1)\right).$$

Conjecture 23. *[FKK⁺17]*

- $indsat(\mathcal{B}_n, \bowtie) = \Theta(n^2)$.
- *For* $n \geqslant 2$, $indsat(\mathcal{B}_n, \mathcal{D}_2) = \Theta(n)$.
- $indsat(\mathcal{B}_n, \mathcal{A}_{k+1}) = kn(1 + o(1))$.

16.3 Poset Saturation with respect to Vapnik-Chervonenkis dimension

For a family $\mathcal{F}$ and a set X, let $\mathcal{F}|_X = \{F \cap X \ : \ F \in \mathcal{F}\}$ be the *projection* of $\mathcal{F}$ onto X. We say that $\mathcal{F}$ shatters X if $\mathcal{F}|_X = 2^X$. The *Vapnik-Chervonenkis dimension* (or *VC-dimension*) is the largest X shattered by $\mathcal{F}$ and is denoted by $VC(\mathcal{F})$.

We say that $\mathcal{F} \subset 2^{[n]}$ is *d-VC-saturated* if $VC(\mathcal{F}) < VC(\mathcal{F}')$ for every $\mathcal{F}' \subset 2^{[n]}$ such that $\mathcal{F} \subset \mathcal{F}'$ and $VC(\mathcal{F}) = d$. We define $sat_{VC}(n, d)$ to be the minimum size of a d-VC-saturated family $\mathcal{F} \subseteq 2^{[n]}$. Observe that every subset of $2^{[n]}$ is 0-VC-saturated; hence $sat_{VC}(n, 0) = 1$. Dudley showed [Dud85] that $sat_{VC}(n, 1) = n + 1$. The authors of [FKKP20] demonstrate that for larger values of d the saturation number is bounded by a function of d.

Theorem 144. *[FKKP20] For any* $d \geqslant 3$, $sat_{VC}(n, d-1) \leqslant 4^d$ *for any* $n \geqslant 2d$. *Moreover, if* d *is odd or* $d \geqslant 14$, *then* $2^d \leqslant sat_{VC}(n, d-1) \leqslant \frac{1}{2}\binom{2d}{d}$.

Problem 44. *[FKKP20] Show that* $sat_{VC}(n, d-1) \leqslant C^d$ *for some* $C > 2$.

17 Other Saturation Related Measures

In this section we consider the saturation function in relation to other functions in graph theory.

17.1 Saturation numbers and extremal numbers

As noted in the Introduction, P. Turán [Tur41] determined $ex(n, K_p)$ and raised the question of determining $ex(n, W^k_{\underbrace{1,\ldots,1}_{p}})$. Despite a strong understanding of the function $ex(n, F)$ for graphs (see [ES66]), his question remains unanswered for $3 \leqslant k < p$. In the case $k = 3, p = 4$, Turán conjectured that $ex(n, W^3_{1,1,1,1}) = (\frac{5}{9} + o(1))\binom{n}{3}$ — more commonly known as *Turán's* $(3, 4)$*-conjecture*.

In a series of papers O. Pikhurko [Pik99a] (cf. Section 3), [Pik01a] gave results that could be thought of as generalizing Theorem 25. Note that the star on $m + 1$ vertices is isomorphic to $W^2_{1,m}$. That is, Pikhurko first determined in [Pik99a] the asymptotic behavior of $sat(n, W^3_{1,1,m})$. Pikhurko [Pik01a] also gave a constructive upper bound for $sat(n, W^4_{1,1,1,m})$, while also considering the more general problem and giving a lower bound $sat(n, W^k_{\underbrace{1,\ldots,1}_{p-1<k},m})$ in terms of the extremal number. Specifically, we know that if Turán's (3, 4)-conjecture is true, then $sat(n, W^4_{1,1,1,m}) = (\frac{m}{9} + o(1))\binom{n}{3}$.

17.2 Saturation numbers and potential numbers

M. Ferrara and J. Schmitt [FS09] considered the following problem and related it to the saturation number. For a given graph F, an integer sequence π is said to be *potentially* F*-graphic* if there is some realization of π that contains F as a subgraph. Let $\sigma(\pi)$ denote the sum of the terms of π. Define $\sigma(n, F)$ to be the smallest integer m so that every n-term graphic sequence π with $\sigma(\pi) \geqslant m$ is potentially F-graphic. It is assumed that F has no isolated vertices and that n is sufficiently large relative to $|V(F)|$. Define the quantities $u(F) = |V(F)| - \alpha(F) - 1$, and $s(F) = min\{\Delta(H) : H \subseteq F, |V(H)| = \alpha(F) + 1\}$. The following is an immediate consequence to Theorem 2 and the lower bound they establish for $\sigma(n, F)$.

Theorem 145. *[FS09] Let d be defined as in Theorem 2. Given a graph F, if there exists an $F' \subseteq F$ with $2u(F') + s(F') \geqslant 2u(F) + d(F)$, then for n sufficiently large we have*

$$2(sat(n, F)) < \sigma(n, F). \tag{6}$$

In particular, this result holds if $d(F) = s(F)$.

These authors believe that the conclusion of Theorem 145 holds in general, even though the hypothesis does not. They conjecture the following.

Conjecture 24. *Let F be a graph and let n be a sufficiently large integer. Then*

$$2(sat(n, F)) < \sigma(n, F).$$

17.3 Saturation numbers and guessing numbers

In the *guessing game* on a graph G, each vertex is assigned one of s colors uniformly at random. Then each vertex simultaneously guesses its assigned color using a predetermined strategy that depends only on knowing the assigned colors of its neighbors. The players win if every vertex correctly guesses its assigned color.

For a given number of colors, s, the *guessing number* of a graph G, denoted $gn(G, s)$, is the largest number a such that there exists a strategy for the guessing game on G such that with probability at least $\frac{1}{s^{n-a}}$ every vertex guesses its assigned color (i.e. the probability of winning is at least $\frac{1}{s^{n-a}}$).

This formulation of the definition of the guessing number of a graph and much foundational work appears in a paper by D. Christofides and K. Markström [CM11].

We say a graph G is *($gn_s \geqslant a$)-saturated* if $gn_s(G) < a$ but the addition of any edge $e \in \overline{G}$ has the property that $gn_s(G+e) \geqslant a$. The saturation number $sat(n, gn_s \geqslant a)$ is the minimum number of edges over all graphs on n vertices that are ($gn_s \geqslant a$)-saturated. In [MR20], J. Martin and P. Rombach defined and investigated the extremal and saturation numbers of a graph with respect to guessing number. They find $sat(n, gn_s \geqslant a)$ for $a = 2$ and $a = n - 1$, and demonstrate that for guessing number at least three the saturation number is bounded by a constant.

Theorem 146. *[MR20] Let $a \leqslant 2$ and s be positive integers. For $n \geqslant 2a + 1$, we have $sat(n, gn_s \geqslant a + 1) \leqslant a^2 + 1$.*

The authors demonstrate that the guessing number of a graph is related to forbidden subgraphs and thus the saturation number of a graph with respect to guessing number is in fact the classic saturation number for a particular family of forbidden graphs. However, the forbidden family is not, in general, known.

Theorem 147. *[MR20] For every $s \in \mathbb{N}$ and $a \in \mathbb{R}$, there exists a unique finite family of minimal forbidden subgraphs $\mathcal{F}_{s,a}$ such that for any graph G,*

$$gn_s(G) < a \Leftrightarrow G \text{ is } \mathcal{F}_{s,a}\text{-free.}$$

18 A small potpourri

We collect here an assortment of burgeoning topics on the topic of saturation.

18.1 Saturation and (0, 1)-Matrices

Given a $(0, 1)$-matrix P, $(0, 1)$-matrix M is *P-saturated* if M does not contain a submatrix that can be turned into P by changing some 1 entries to 0 entries, and changing an arbitrary 0 into a 1 in M introduces such a submatrix. In order to avoid trivial cases, it is customary to assume that if P is an $r \times s$ matrix and M is an $m \times n$ matrix, then $r \leqslant m$ and $s \leqslant n$. The fewest number of 1 entries in a P-saturated $m \times n$ matrix M is denoted $sat(M_{m\times n}, P)$. The numbers s-$sat(M_{m\times n}, P)$ and $ex(M_{m\times n}, P)$ can be defined similarly.

This matrix version of saturation can be translated into the language of labeled bipartite graphs with some additional ordering restrictions applied to the vertices.

While extremal problems on $(0,1)$-matrices have been extensively studied, R. Brualdi and L. Cao [BC21] initiated the study of saturation problems in $(0,1)$-matrices though using different terminology: "pattern avoiding" as opposed to saturation.

Theorem 148. *[BC21] Let I_k be the $k \times k$ identity matrix and $m, n \geqslant k$. Then*

$$sat(M_{m\times n}, I_k) = ex(M_{m\times n}, I_k) = (k-1)(m+n-(k-1)).$$

In addition to finding the saturation number of the identity matrix exactly, several structural properties of I_k-saturated matrices are described and J_k-saturated graphs are investigated, where J_k is the matrix obtained from I_k by moving the top row to the bottom row.

Theorem 149. *[BC21] Let J_k be the $k \times k$ matrix obtain from the identity matrix by moving the top row to the bottom row.*
$sat(M_{m\times n}, J_3) = ex(M_{m\times n}, J_3) = 2(m+n-2)$,
$sat(M_{m\times n}, J_k) \leqslant (k-1)(m+n-(k-1))$, *for* $k \geqslant 3$.

R. Fulek and B. Kaszegh [FK21] later established a lower bound for the saturation number of J_k making progress on a conjecture by Brualdi and Cao.

Theorem 150. *[FK21]* $sat(M_{m\times n}, J_k) \geqslant (k-2)\max\{m,n\} + m + n - 1 - \frac{(k-2)(k-1)}{2}$.

Conjecture 25. *[BC21]* $sat(M_{m\times n}, J_k) = ex(M_{m\times n}, J_k) = (k-1)(m+n-(k-1))$.

In [FK21], Fulek and Kaszegh establish many other general results. For example, unlike $ex(M_{n\times n}, P)$ which can have growth rates such as $\Theta(n^{3/2})$ or $\Theta(n \log n)$, they demonstrate that $sat(M_{n\times n}, P) \in \{\Theta(1), \Theta(n)\}$ and find a particular P for which $sat(M_{n\times n}, P) < 400$. Thus, they also show that despite the two theorems above, the extremal numbers and saturation numbers are not always the same. The authors completely characterize which matrices have semisaturation numbers that grow linearly with n and which are constant. There are several open questions at the end of the paper, with one example given below.

Question 18. *[FK21] Do there exist matrices P other than I_1 and the permutation matrix associated with* (25314) *where* $sat(M_{n\times n}, P) = \Theta(1)$*?*

18.2 Saturation and k-Planar Drawings

A drawing D of a loopless, multigraph G in the plane is called *k-planar* if each edge in the drawing is crossed at most k times. A drawing D is *k-planar saturated* if D is k-planar but the addition of any edge results in a drawing that is no longer k-planar. S. Chaplick, J. Rollin and T. Ueckerdt [CRU21] investigated the minimum number of edges in k-planar saturated drawings for a variety of classes of k-planar graphs where different classes are determined by restrictions on the nature of the drawing. For example, let $\mathcal{I}$ be the set of k-planar drawings with the added restriction that incident edges are not allowed to cross (though an edge may cross itself) and let $\mathcal{S}$ be the set of k-planar drawings such that no edge crosses itself. Let $sat(n, k\text{-planar}(\mathcal{I} \cap \mathcal{S}))$ denote the least number of edges in a k-planar saturated drawing on n vertices restricted to the set $\mathcal{I} \cap \mathcal{S}$. Chaplick et al. established a catalog of results of the type below, including constructing infinite families of drawings achieving the given bounds.

Theorem 151. *[CRU21] For infinitely many values of n,*

$$sat(n, k\text{-}planar(\mathcal{S})) = sat(n, k\text{-}planar(\mathcal{I} \cap \mathcal{S})) = \frac{2}{k-1}(n-1).$$

Their technique depends upon the family having the property that deleting an edge or vertex preserves its class. For families without this property the k-planar saturation number is open.

Problem 45. *[CRU21] Let $\mathcal{H}$ be all graphs that can be drawn in the plane such that no two parallel edges are homotopic (when vertices are viewed as holes.) Determine $sat(n, k\text{-}planar(\mathcal{H}))$.*

18.3 Unique Saturation

Given a graph F, a graph G is *uniquely F-saturated* if G is F-saturated and has the additional property that adding any edge of the complement to G produces a unique copy of F. The K_r-saturated graphs of minimum size consisting of an $(r-2)$-clique joint to an independent set of order $n-r+2$ (from [EHM64]) is also uniquely K_r-saturated. On the other hand, P. Wenger [Wen10] proved no nontrivial uniquely P_k-saturated graphs exists for $k \geqslant 5$. Hence, questions about uniquely F-saturated graphs focus on their existence.

J. Cooper, J. Lenz, T. LeSaulnier, P. Wenger and D. West [CLL$^+$12] initiated the study of uniquely saturated graphs by characterizing all uniquely C_k-saturated graphs for $k = 3$ and $k = 4$. They also proved there exist an infinite number of uniquely C_5-saturated graphs.

Later, P. Wenger and D. West [WW17] characterized uniquely C_5-saturated graphs, proved there are no uniquely C_k-saturated graphs for $k = 6, 7$. Finally, they demonstrated that there can exist only a finite number of uniquely C_k-saturated graphs for $k \geqslant 8$ and they conjecture that there are none.

Theorem 152. *[CLL$^+$12, WW17] A summary of what is known about the family of uniquely C_k-saturated graphs.*

- *For $k = 3$, the family consists of all stars and Moore graphs of diameter of which there are only a finite number.*
- *For $k = 4$, there exist 10 graphs.*
- *For $k = 5$, all nontrivial members consist of edge disjoint triangles all of which share a single vertex (so-called* friendship graphs*).*

- *For $k \geqslant 6$, only finitely many can exist.*

Conjecture 26. *[WW17] For $k \geqslant 8$, there are no nontrivial uniquely C_k-saturated graphs.*

L. Berman, G. Chappell, J. Faudree, J. Gimbel, and C. Hartman [BCF$^+$16] proved that the only tree T for which there exists an infinite family of uniquely T-saturated graphs is when T is a balanced double star. They conjecture that double stars are the only trees for which nontrivial uniquely T-saturated graphs exist. They prove the existence of nontrivial uniquely double-star-saturated graphs but the conjecture below remains open.

Conjecture 27. *[BCF$^+$16] Let T be a tree. If there exists a nontrivial uniquely T-saturated graph, then T is a double star.*

Building on uniquely C_3-saturated graphs, S. Hartke and D. Stolee [HS12] began investigating uniquely K_r-saturated graphs and observed that if G is uniquely K_r-saturated and has a dominating vertex v, then $G-v$ is uniquely K_{r-1}-saturated. To avoid this issue, they define a graph to be *r-primitive* if it is uniquely K_r-saturated and has no dominating vertex.

Using the technique of orbital branching, an exhaustive computer-aided search for uniquely K_r-saturated graphs on n vertices for $r \leqslant 8$ and $n \leqslant 20$ found ten new primitive uniquely K_r-saturated graphs, two of which were Cayley graphs [HS12]. With these Cayley graphs as models, the authors constructed two new infinite families of uniquely K_r-saturated graphs.

In the statement of the theorem below, for a finite group Γ and generating set $S \subseteq \Gamma$, let $G(\Gamma, S)$ be the undirected Cayley graph with vertex set Γ and edge set defined by action from S. Let $\overline{G}(\Gamma, S)$ be the complement of Γ.

Theorem 153. *[HS12]*

- *Let $t \geqslant 2$, $n = 4t^2 + 1$, and $r = 2t^2 - t + 1$. Then, $\overline{G}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.*
- *Let $t \geqslant 2$, $n = 9t^2 - 3t + 1$, and $r = 3t^2 - 2t + 1$. Then, $\overline{G}(\mathbb{Z}_n, \{1, 3t-1, 3t\})$ is r-primitive.*

Question 19. *[HS12] For each $r \geqslant 3$, are there a finite number of r-primitive uniquely K_r-saturated graphs?*

A. Gyárfás, S. Hartke and C. Viss [GHV18] generalized the notion of uniquely K_r-saturated graphs to hypergraphs as follows. Let $K_r^{(k)}$ denote the complete k-uniform hypergraph on r vertices. For integers k, r, n, such that $2 \leqslant k < r < n$ a k-uniform hypergraph H on n vertices is *uniquely $K_r^{(k)}$-saturated* if H does not contain $K_r^{(k)}$ but adding to H any k-set that is not a hyperedge of H results in exactly one copy of $K_r^{(k)}$. A uniquely $K_r^{(k)}$-saturated hypergraph is called *primitive* if it has no dominating vertex meaning that for every vertex the hypergraph does not contain all $\binom{n-1}{k-1}$ edges containing that vertex.

In [GHV18] Gyárfás et al. describe two distinct approaches to constructing primitive $K_r^{(k)}$-saturated hypergraphs for large ranges of n, an outcome remarkably different from the case when $k = 2$. They also prove that as when $n - r$ is fixed and n is large enough, there can exist no primitive uniquely $K_r^{(k)}$-saturated hypergraphs. For $n - r = 1$, the authors determine precisely the range of n for which primitive uniquely $K_r^{(k)}$-saturated hypergraphs exist. As part of the proof techniques appearing in the paper, the authors establish interesting equivalences between primitive uniquely k-uniform hypergraphs and both a type of complementary hypergraph and a class of τ-critical hypergraphs. Two typical results are below.

Theorem 154. *[GHV18]*

- *Let $k \geqslant 4$ be given. Then for any r such that $k < r \leqslant 2k-3$, there exists a primitive uniquely $K_r^{(k)}$-saturated hypergraph on n vertices for every $n > r$.*
- *Fix $k \geqslant 2$. Let $\ell = n - r \geqslant 1$ be a fixed constant. Then no primitive uniquely $K_r^{(k)}$-saturated hypergraph on n vertices exists if n is at least*

$$\binom{k+\ell}{k-1} + \binom{k+\ell-1}{k-1}.$$

The authors employed some computational techniques to determine whether primitive uniquely $K_r^{(k)}$-saturated hypergraph on n vertices exist for certain specific choices of k, r and n. A clear understanding in the case of $K_4^{(3)}$-saturated hypergraphs remains open.

Question 20. *[GHV18] Are there only finitely many primitive uniquely $K_4^{(3)}$-saturated hypergraphs?*

18.4 Cover Saturation

Cover saturation combines semi-saturation with a stronger requirement that every edge in the host graph be in some copy of the target graph. This idea was introduced by D. Rorabaugh [Ror19] where the author proves several structural results and several results on specific classes of graphs. Unlike most saturation measures, the cover saturation measure has some inherited properties (Theorem 155).

Recall that given a graph F, a graph G is *F-semi-saturated* if for any edge $e \in \overline{G}$ the graph $G + e$ contains additional copies of F. The requirement that G be F-free is omitted. Rorabaugh defines a graph G as *F-covered* provided every edge of G is in a subgraph of G isomorphic to F and G is *F-cov-sat* if G is both F-semi-saturated and F-covered. Assuming $|V(F)| \leqslant |V(G)|$, the *cov-sat number* of F, denoted $csat(n, F)$ is the minimum number of edges in an n-vertex F-cov-sat graph G.

Theorem 155. *[Ror19] If G is F-covered then $csat(n, F) \leqslant csat(n, G)$ for all $n \geqslant |V(G)|$.*

Most of the results in [Ror19] give values or bounds on the *asymptotic cov-sat density* of F defined as $csat(F) = \lim_{n\to\infty} \frac{csat(n,F)}{n}$ when the limit exists. The author demonstrates that like the regular saturation number, the cov-sat number can grow no faster than some constant multiple of n.

Theorem 156. *[Ror19] For $r \geqslant 3$, $csat(K_r) = r - \frac{2}{3}$.*

18.5 Connectedness and Saturation

Let $\mathcal{F}_k$ be the family of k-connected graphs and let $\mathcal{F}'_k$ be the family of k-edge-connected graphs. P. Wenger [Wen14] established the saturation number for $\mathcal{F}_k$ and demonstrated that while the unique graph in $Sat(n, K_k)$ is also a member of $Sat(n, \mathcal{F}_k)$ it is not unique. In the theorem below a *k-tree* is any graph obtained from K_k by iteratively introducing a new vertex whose neighborhood in the previous graph consists of k pairwise adjacent vertices. A complete characterization of the graphs in $Sat(n, \mathcal{F}_k)$ remains open.

Theorem 157. *[Wen14] Let $\mathcal{F}_k$ be the family of k-connected graphs. For $n \geqslant k+1$, $sat(n, \mathcal{F}_k) = (k-1)n - \binom{k}{2}$. Furthermore, every $(k-1)$-tree on n vertices has this many edges and is $\mathcal{F}_k$-saturated.*

Later, H. Lei, S. O, Y. Shi, D. West, and X. Zhu [LOS$^+$19] proved both the saturation number and the extremal number for $\mathcal{F}'_k$. In addition, the authors characterize both families of extremal graphs and give bounds on the spectral radius of $\mathcal{F}'_k$-saturated graphs. More on the spectral radius of a graph can be found in [KKKO20].

Theorem 158. *[LOS$^+$19] For $n \in \mathbb{N}$ and $t = \left\lfloor \frac{n}{k+1} \right\rfloor$, $sat(n, \mathcal{F}') = (k-1)(n-1) - t\binom{k-1}{2}$.*

In addition to results bounding the cov-sat number based on structural properties (such as the existence of a bridge), the cov-sat number is calculated (or bounded) for paths, cycles and stars. There are several open questions at the end of the paper. As an example, the relationship between the cov-sat number and other saturation numbers (semi-saturation number, saturation number, weak saturation number) have not been investigated.

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Table 1: Notation

Notation	**Read as**	**Definition or explanation**
$\overline{G}$	the complement of G	a graph G' with $V(G') = V(G)$ and $E(G') = E(K_{\lvert V(G)\rvert})\backslash E(G)$. This may be seen in other literature as G^c. The complement T' of a trigraph T has $V(T') = V(T)$, $EB(T') = EW(T)$, $EW(T') = EB(T)$, and $EG(T') = EG(T)$.
$\subseteq$	is a subgraph or subset of	-
$\subset$	is a proper subgraph or subset of	-
K_n	the complete graph on n vertices	-
P_n	the path on n vertices	(so $\lvert E(P_n)\rvert = n-1$)
C_n	the cycle on n vertices	-
Q_n	the hypercube in n dimensions	-
$K_{n_1,n_2,\ldots,n_k}$	-	The complete k-partite graph with n_i vertices in the ith part.
$K_{k\times n}$	-	The complete k-partite graph where each part has n vertices; $\lvert V(K_k^n)\rvert = nk$. This is sometimes written as in the literature as K_k^n.
$\delta(G)$	-	The minimum degree over all vertices of G. This can be shortened to simply δ.
$\Delta(G)$	-	The maximum degree over all vertices of G. This can be shortened to simply Δ.
$W^k_{s_1,\ldots,s_p}$	the weak hypergraph generalization of a complete graph	The k-uniform hypergraph consisting of all k-tuples that intersect exactly k different parts, where $k \leqslant p$ and $s_i = \lvert S_i\rvert$, and S_i is the i^{th} part in a partition of vertices.
$S^k_{s_1,\ldots,s_p}$	the strong hypergraph generalization of a complete graph	The k-uniform hypergraph consisting of all k-tuples that intersect at least 2 different parts, where $k \leqslant p$ and $s_i = \lvert S_i\rvert$, and S_i is the ith part in a partition of vertices of the hypergraph.
K_p^k	the instance when $W^k_{1,1,\ldots,1} = S^k_{1,1,\ldots,1}$	The k-uniform hypergraph consisting of all possible k-tuples of p vertices.

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Table 1 – *Continued from previous page*

Notation	**Read as**	**Definition or explanation**
$\mathcal{T}_n$	The family of all trees of order n	-
$\mathcal{C}_n$	The family of all cycles of order (and size) n	-
$\mathcal{C}_{\geqslant n}$	The family of all cycles of order (and size) at least n	-
$G - e$	-	For a graph G and an edge $e \in G$, $V(G-e) = V(G)$ and $E(G) = E(G) \setminus \{e\}$
$G \square H$	The graph cartesian product of G and H	$V(G \square H) = V(G) \times V(H)$ and $E(G) = \{(u,v)(x,y) : u = x \text{ and } vy \in E(H) \text{ or } v = y \text{ and } ux \in E(G)\}$
$G + H$	the join of G and H	$V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{gh : g \in V(G) \text{ and } h \in V(H)\}$. This is sometimes written in other literature as $G \vee H$.
$[t]$	a t-element integer system	A set of t elements (where $t \in \mathbb{Z}$). These are usually used for coloring as a set of t colors.
$G(n,p)$	The Erdős-Rényi random graph on n vertices with probability p	$V(G(n,p)) = \{1, 2, \ldots, n\}$ and each pair of vertices $i, j \in V(G(n,p))$ has probability p to be an edge.
$\sqcup$	the disjoint union	-
kG	the (generally disjoint) union of k copies of G	-
P_n^d	the d dimensional grid of size n	$\square_{i \in [d]} P_n$. This is also notated in other literature as $[n]^d$.
-	G is $\mathcal{F}$-saturated	G does not contain any member of $\mathcal{F}$ as a subgraph but the addition of any edge between vertices of G creates a graph G' that does contain a member of $\mathcal{F}$ as a subgraph. When $\mathcal{F} = \{F\}$, substitute F for $\mathcal{F}$.
$sat(n, \mathcal{F})$	The saturation number	The minimum number of edges in a graph on n vertices that is $\mathcal{F}$-saturated.

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Table 1 – *Continued from previous page*

Notation	**Read as**	**Definition or explanation**
$Sat(n, \mathcal{F})$	-	The set of graphs G (up to isomorphism) on n vertices that have $\lvert E(G)\rvert = sat(n, \mathcal{F})$
$ex(n, \mathcal{F})$	The extremal number	The maximum number of edges in a graph on n vertices that is $\mathcal{F}$-saturated.
$Ex(n, \mathcal{F})$	-	The set of graphs G (up to isomorphism) on n vertices that have $\lvert E(G)\rvert = ex(n, \mathcal{F})$
$sat(H, \mathcal{F})$	The saturation number in H	The minimum number of edges in a graph G with $V(G) = V(H)$ and $E(G) \subseteq E(H)$ that does not contain a member of $\mathcal{F}$ as a subgraph such that $G + e$ contains a member of $\mathcal{F}$ as a subgraph when $e \in E(H) \setminus E(G)$. In this style, the typical saturation number is expressed $sat(K_n, \mathcal{F})$. $sat(K_k^n, K_r)$ may be seen in other literature as $sat(n, k, r)$. In other literature, the order of the arguments may be reversed or interchangeable, but it will always be as in this table in the survey.
$sat(n, J, \mathcal{F})$	-	The minimum number of subgraphs $J_i \subseteq G$ with $J_i \cong J$ over all G where G is an $\mathcal{F}$-saturated graph on n vertices. Note that $sat(n, K_2, \mathcal{F}) = sat(n, \mathcal{F})$.
$s\text{-}sat(n, \mathcal{F})$	The semi-saturation number	The minimum number of edges in a graph G on n vertices such that adding any edge $e \in E(K_n) \setminus E(G)$ increases the number of copies of members of $\mathcal{F}$ as subgraphs. In the setting of an alternate host graph, substitute the host H for K_n. Just as with saturation number, write $S\text{-}Sat(n, \mathcal{F})$ for the set of graphs with this quality. Note $s\text{-}sat(n, \mathcal{F}) \leqslant sat(n, \mathcal{F})$. This is also referred to in other literature as strong saturation, in contrast with weak saturation. Just simply saturation is called this as well, rarely, and so neither are used as such in this paper.

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Table 1 – *Continued from previous page*

Notation	Read as	Definition or explanation
$w\text{-}sat(n, \mathcal{F})$	The weak saturation number	The minimum number of edges in a graph G on n vertices such that edges $e_i \in E(K_n) \setminus E(G)$ can be added one by one, each time increasing the number of copies of members of $\mathcal{F}$ contained as subgraphs. In the setting of an alternate host graph, substitute the host H for K_n. Just as with saturation number, write $W\text{-}Sat(n, \mathcal{F})$ for the set of graphs with this quality. Note $w\text{-}sat(n, \mathcal{F}) \leqslant s\text{-}sat(n, \mathcal{F}) \leqslant sat(n, \mathcal{F})$.
$u\text{-}sat(n, \mathcal{F})$	The unique saturation number	The minimum number of edges in an $\mathcal{F}$-free graph G such that $\forall e \in \overline{G}$, $G + e$ contains exactly one copy of a member of the family $\mathcal{F}$ as a subgraph.
$\Re(F)$	-	The set of all rainbow-colored copies of a graph F
$ES(n, \mathcal{F})$	the edge spectrum of $\mathcal{F}$	The set of all possible sizes of $\mathcal{F}$-saturated graphs.
-	G is $(F_1, ..., F_t)$-saturated	There exists a coloring C of $E(G)$ in t colors $1, 2, ..., t$ such that there is no monochromatic copy of F_i in color i : $1 \leqslant i \leqslant t$, but the addition of any new edge with color i to G creates a monochromatic F_i in color i
$sat^{\Delta}(n, \mathcal{F})$	-	The minimum number of edges in an $\mathcal{F}$-saturated graph with maximum degree at most Δ
$sat_{\delta}(n, \mathcal{F})$	-	The minimum number of edges in an $\mathcal{F}$-saturated graph with minimum degree at least δ
-	a trigraph T	A quadruple of sets, $T = (V(T), EB(T), EW(T), EG(T))$, where $V(T)$ is the vertex set of T, $EB(T)$ contains the known black, or present edges of T, $EW(T)$ contains the known white, or absent edges of T, and $EG(T)$ contains the gray edges of T; they could be either black or white.

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Table 1 – *Continued from previous page*

Notation	Read as	Definition or explanation
$EB(T)$	the black edges of a trigraph T	-
$EW(T)$	the white edges of a trigraph T	-
$EG(T)$	the gray edges of a trigraph T	Note that if $EG(T) = \emptyset$, T can be thought of as a normal graph with $E(T) = EB(T)$ and $E(\overline{T}) = EW(T)$
-	a realization of a trigraph T	A graph G with $V(G) = V(T)$ and, for some $S \subseteq EG(T)$, $E(G) = EB(T) \cup S$
-	T is induced F-saturated	No realization of the trigraph T contains a copy of F as an induced subgraph, but changing any white or black edge to gray results in some realization that does contain a copy of F as an induced subgraph. An induced subgraph copy of F in the trigraph setting is one where both the black and white edges are the same.
$indsat(n, F)$	the induced saturation number	The minimum number of gray edges of a trigraph T on n vertices such that T is induced F-saturated
$indsat^*(n, F)$	-	$min\{\lvert E(G)\rvert : \lvert V(G)\rvert = n,\ G \text{ is induced } F\text{-saturated}\}$; the minimum number of edges in a graph such that adding any edge in $E(\overline{G})$ or removing any edge in $E(G)$ creates a copy of F as an induced subgraph
$indsat^+(n, F)$	-	The minimum number of edges in an F-free graph G such that the addition of any edge creates a copy of F as an induced subgraph. Notice $indsat^*(n, F) \geqslant indsat^+(n, F)$.
$F \rightarrow (F_1, ..., F_t)$	F arrows $(F_1, ..., F_t)$, a t-tuple of graphs	Any t-coloring of $E(F)$ contains a monochromatic F_i of color $i : 1 \leqslant i \leqslant t$
-	G is Ramsey-minimal	$G \rightarrow (F_1, ..., F_t)$ and yet $\forall G' \subset G$, $G' \nrightarrow (F_1, ..., F_t)$
$\mathcal{R}_{min}(F_1, ..., F_t)$	The family of all $(F_1, ..., F_t)$ Ramsey-minimal subgraphs	-

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Table 1 – *Continued from previous page*

Notation	Read as	Definition or explanation
-	$\Re_t(F)$-saturated	A graph G does not contain a rainbow (in t colors) copy of F but $\forall e \in E(\overline{G}), i \in [t]$, $G+e$ contains a rainbow copy of F when e is in color i
$sat(n, \Re_t(F))$	-	The minimum number of edges in a t-edge-colored $(\Re(F), t$-saturated graph of order n
$(n, \mathscr{P})$	the $\mathscr{P}$ saturation game	A game starting in $\overline{K_n}$ in which two players, one trying to prolong the game and one trying to end the game, alternate turns drawing edges until a graph saturated relative to the property $\mathscr{P}$ is reached.
Max	the maximizer	The player in Füredi, Reimer, and Seress's F-free game and its variants who tries to prolong the game, making the number of edges in the final graph as close to $ex(n, F)$ as possible. If not otherwise noted, Max is assumed to play first. Max is also known as Prolonger, Maximizer, and Maxi.
Min	the minimizer	The player in Füredi, Reimer, and Seress's F-free game and its variants who tries to shorten the game, making the number of edges in the final graph as close to $sat(n, F)$ as possible. If not otherwise noted, Min is assumed to play second. Min is also known as Minimizer and Mini.
$sat_g(n, F)$	the game saturation number	The final number of edges in Füredi, Reimer, and Seress's F-free game after optimal play from both players. Note that $sat_g(n, F) \in ES(n, F)$.
$\hat{sat}_g(n, F)$	the Min-first game saturation number	The final number of edges in Füredi, Reimer, and Seress's F-free game after optimal play from both players in the game variant where Min plays first. Note that $\hat{sat}_g(n, F) \in ES(n, F)$.

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Table 1 – *Continued from previous page*

Notation	Read as	Definition or explanation
$sat_{g-P}(n, F)$	the move-optional game saturation number	The final number of edges in Füredi, Reimer, and Seress's F-free game after optimal play from both players in the game variant where Max can skip turns. Note that $sat_g(n, F) \in ES(n, F)$.
$\langle A \rangle_r$	the closure of A with parameter r	Where for some graph G, $A \subseteq V(G)$, the subset of vertices of G that are eventually infected by the r-neighbor bootstrap percolation with A initially infected.
$\langle A \rangle_r = G$	A percolates; or, A is a percolating set	For a graph G, $A \subseteq V(G)$ percolates (or, is a percolating set) if every vertex eventually becomes infected.
$m(G, r)$	-	The size of the minimum percolating set in a graph G undergoing r-neighbor bootstrap percolation. Note that $m(G, r) \geqslant \frac{w\text{-}sat(G, K_{1,r+1})}{r}$.
$T(A)$	the percolating time	For a graph G, the number of time steps for a set $A \subseteq V(G)$ to infect every vertex.
$p_a(G, r)$	-	The probability that a vertex will be initially infected such that the probability that the ensuing r-neighbor bootstrap percolation on a graph G is spanning is a. $a = \frac{1}{2}$ is called the critical probability.
Berge-F	-	Given a graph F, a hypergraph H is Berge-F if there exists an injection $\phi : E(F) \to E(H)$ where for any $e \in E(F)$, $e \in \phi(e)$. Informally, a hypergraph H is Berge-F if it could be created by adding vertices to the edges of F. A hypergraph H is Berge-F-saturated if H does not contain a Berge-F subhypergraph, but the addition of any nontrivial hyperedge creates a Berge-F subhypergraph in H.
$sat(n, Berge^k\text{-}F)$	the $Berge^k$-F saturation number	The minimum number of hyperedges in a Berge-F-saturated k-uniform hypergraph.

References

[AC19] Maria Axenovich and Mónika Csikós. Induced saturation of graphs. *Discrete Math.*, 342(4):1195–1212, 2019.

[AE19] Bethany Austhof and Sean English. Nearly-regular hypergraphs and saturation of Berge stars. *Electron. J. Combin.*, 26(4):Paper No. 4.49, 11, 2019.

[AEHK96] Noga Alon, Paul Erdős, Ron Holzman, and Michael Krivelevich. On k-saturated graphs with restrictions on the degrees. *J. Graph Theory*, 23(1):1–20, 1996.

[AFG12] Kinnari Amin, Jill Faudree, and Ronald Gould. The edge spectrum of K_4-saturated graphs. *J. Combin. Math. Combin. Comput.*, 81:233–242, 2012.

[AFGS13] Kinnari Amin, Jill Faudree, Ronald J. Gould, and Elżbieta Sidorowicz. On the non-$(p-1)$-partite K_p-free graphs. *Discuss. Math. Graph Theory*, 33(1):9–23, 2013.

[AL88] M. Aizenman and J. L. Lebowitz. Metastability effects in bootstrap percolation. *J. Phys. A*, 21(19):3801–3813, 1988.

[Alo85] Noga Alon. An extremal problem for sets with applications to graph theory. *J. Combin. Theory Ser. A*, 40(1):82–89, 1985.

[Ami10] Kinnari Amin. *On K_t-saturated Graphs.* PhD thesis, Emory University, 2010.

[AS16] Noga Alon and Clara Shikhelman. Many T copies in H-free graphs. *J. Combin. Theory Ser. B*, 121:146–172, 2016.

[Ash11] Yehuda Ashkenazi. Path saturated graphs. *Int. J. Pure Appl. Math.*, 69(4):389–397, 2011.

[AW19] Maria Axenovich and Christian Winter. A Note on Saturation for Berge-G Hypergraphs. *Graphs Combin.*, 35(4):933–939, 2019.

[BB06] József Balogh and Béla Bollobás. Bootstrap percolation on the hypercube. *Probab. Theory Related Fields*, 134(4):624–648, 2006.

[BBDCM12] József Balogh, Béla Bollobás, Hugo Duminil-Copin, and Robert Morris. The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.*, 364(5):2667–2701, 2012.

[BBM09] József Balogh, Béla Bollobás, and Robert Morris. Bootstrap percolation in three dimensions. *Ann. Probab.*, 37(4):1329–1380, 2009.

[BBM10] József Balogh, Béla Bollobás, and Robert Morris. Bootstrap percolation in high dimensions. *Combin. Probab. Comput.*, 19(5-6):643–692, 2010.

[BBM12] József Balogh, Béla Bollobás, and Robert Morris. Graph bootstrap percolation. *Random Structures Algorithms*, 41(4):413–440, 2012.

[BBMR12] József Balogh, Béla Bollobás, Robert Morris, and Oliver Riordan. Linear algebra and bootstrap percolation. *J. Combin. Theory Ser. A*, 119(6):1328–1335, 2012.

[BC21] Richard A. Brualdi and Lei Cao. Pattern-avoiding $(0, 1)$-matrices and bases of permutation matrices. *Discrete Appl. Math.*, 304:196–211, 2021.

[BCE+96] Curtiss A. Barefoot, Lane H. Clark, Roger C. Entringer, Thomas D. Porter, László A. Székely, and Zsolt Tuza. Cycle-saturated graphs of minimum size. *Discrete Math.*, 150(1-3):31–48, 1996. Selected papers in honour of Paul Erdős on the occasion of his 80th birthday (Keszthely, 1993).

[BCF+95a] Curtiss Barefoot, Karen Casey, David Fisher, Kathryn Fraughnaugh, and Frank Harary. Size in maximal triangle-free graphs and minima, graphs of diameter 2. *Discrete Math.*, 138:93–99, 1995.

[BCF+95b] Curtiss Barefoot, Karen Casey, David Fisher, Kathryn Fraughnaugh, and Frank Harary. Size in maximal triangle-free graphs and minimal graphs of diameter 2. *Discrete Math.*, 138(1-3):93–99, 1995. 14th British Combinatorial Conference (Keele, 1993).

[BCF+16] Leah Wrenn Berman, Glenn G. Chappell, Jill R. Faudree, John Gimbel, and Chris Hartman. Uniquely tree-saturated graphs. *Graphs Combin.*, 32(2):463–494, 2016.

[BD18] Paul Balister and Ali Dogan. On the edge spectrum of saturated graphs for paths and stars. *J. Graph Theory*, 89(4):364–385, 2018.

[Beh18] Natalie C. Behague. Hypergraph saturation irregularities. *Electron. J. Combin.*, 25(2):#P2.11, 2018.

[BES+16] Sarah Behrens, Catherine Erbes, Michael Santana, Derrek Yager, and Elyse Yeager. Graphs with induced-saturation number zero. *Electron. J. Combin.*, 23(1):#P1.54, 2016.

[BF92] László Babai and Péter Frankl. Linear algebra methods in combinatorics with applications to geometry and computer science. unpublished book manuscript, 1992.

[BF02] Darryn E. Bryant and Hung-Lin Fu. C_4-saturated bipartite graphs. *Discrete Math.*, 259(1-3):263–268, 2002.

[BFP10] Tom Bohman, Maria Fonoberova, and Oleg Pikhurko. The saturation function of complete partite graphs. *J. Comb.*, 1(2):149–170, 2010.

[BFVW17] Michael D. Barrus, Michael Ferrara, Jennifer Vandenbussche, and Paul S. Wenger. Colored saturation parameters for rainbow subgraphs. *J. Graph Theory*, 86(4):375–386, 2017.

[BGMS20] Marthe Bonamy, Carla Groenland, Tash Morrison, and Alex Scott. Induced saturation for P_5. https://tomjohnston.co.uk/blog/2020-05-22-induced-saturation-for-paths.html, 2020.

[BHW16] Csaba Biró, Paul Horn, and D. Jacob Wildstrom. An upper bound on the extremal version of Hajnal's triangle-free game. *Discrete Appl. Math.*, 198:20–28, 2016.

[BJR20] Neal Bushaw, Daniel Johnston, and Puck Rombach. Rainbow saturation. arXiv:2003.13200, 2020.

[BK21] Zsolt Bartha and Brett Kolesnik. Weakly saturated random graphs. arXiv:2007.14716, 2021.

[BMTRZ20] M. R. Bidgoli, A. Mohammadian, B. Tayfeh-Rezaie, and M. Zhukovskii. Threshold for weak saturation stability. arXiv:2006.06855, 2020.

[Bol65] Béla Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar*, 16:447–452, 1965.

[Bol67a] Béla Bollobás. Determination of extremal graphs by using weights. *Wiss. Z. Techn. Hochsch. Ilmenau*, 13:419–421, 1967.

[Bol67b] Béla Bollobás. On a conjecture of Erdős, Hajnal and Moon. *Amer. Math. Monthly*, 74:178–179, 1967.

[Bol68] Béla Bollobás. Weakly k-saturated graphs. In *Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967)*, pages 25–31. Teubner, Leipzig, 1968.

[Bol78] Béla Bollobás. *Extremal graph theory*, volume 11 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[Bol04] Béla Bollobás. *Extremal graph theory*. Dover Publications Inc., Mineola, NY, 2004. Reprint of the 1978 original.

[Bon72] John Adrian Bondy. Variations on the Hamiltonian theme. *Canad. Math. Bull.*, 15:57–62, 1972.

[BP07] József Balogh and Boris G. Pittel. Bootstrap percolation on the random regular graph. *Random Structures Algorithms*, 30(1-2):257–286, 2007.

[BQZ06] Krystyna T. Balińska, Louis V. Quintas, and Krzysztof T. Zwierzyński. Extremal size in graphs with bounded degree. *Ars Combin.*, 81:235–246, 2006.

[BS02] Mieczysław Borowiecki and Elżbieta Sidorowicz. Weakly $\mathcal{P}$-saturated graphs. *Discuss. Math. Graph Theory*, 22(1):17–29, 2002. Conference on Graph Theory (Elgersburg, 2000).

[CCES86] Lane H. Clark, Ronni P. Crane, Roger C. Entringer, and Henry D. Shapiro. On smallest maximally non-Hamiltonian graphs. In *Proceedings of the seventeenth Southeastern international conference on combinatorics, graph theory, and computing (Boca Raton, Fla., 1986)*, volume 53, pages 215–220, 1986.

[CCH21] Debsoumya Chakraborti, Da Qi Chen, and Mihir Hasabnis. Minimizing the Number of Edges in $K_{(s,t)}$-Saturated Bipartite Graphs. *SIAM J. Discrete Math.*, 35(2):1165–1181, 2021.

[CCP21] Eun-Kyung Cho, Ilkyoo Choi, and Boram Park. On induced saturation for paths. *European J. Combin.*, 91:103204, 12, 2021.

[CE83] Lane Clark and Roger Entringer. Smallest maximally non-Hamiltonian graphs. *Period. Math. Hungar.*, 14(1):57–68, 1983.

[CES92] Lane H. Clark, Roger C. Entringer, and Henry D. Shapiro. Smallest maximally non-Hamiltonian graphs. II. *Graphs Combin.*, 8(3):225–231, 1992.

[CFF+15a] Guantao Chen, Jill R. Faudree, Ralph J. Faudree, Ronald J. Gould, Michael S. Jacobson, and Colton Magnant. Results and problems on saturation numbers for linear forests. *Bull. Inst. Combin. Appl.*, 75:29–46, 2015.

[CFF+15b] Guantao Chen, Jill R. Faudree, Ralph J. Faudree, Ronald J. Gould, Michael S. Jacobson, and Colton Magnant. Results and problems on saturation numbers for linear forests. *Bull. Inst. Combin. Appl.*, 75:29–46, 2015.

[CFG08] Guantao Chen, Ralph J. Faudree, and Ronald J. Gould. Saturation numbers of books. *Electron. J. Combin.*, 15(1):#R118, 2008.

[CFG+11] Guantao Chen, Michael Ferrara, Ronald J. Gould, Colton Magnant, and John Schmitt. Saturation numbers for families of Ramsey-minimal graphs. *J. Comb.*, 2(3):435–455, 2011.

[CG08] Sul-young Choi and Puhua Guan. Minimum critical squarefree subgraph of a hypercube. In *Proceedings of the Thirty-Ninth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 189, pages 57–64, 2008.

[Che09] Ya-Chen Chen. Minimum C_5-saturated graphs. *J. Graph Theory*, 61(2):111–126, 2009.

[Che11] Ya-Chen Chen. All minimum C_5-saturated graphs. *J. Graph Theory*, 67(1):9–26, 2011.

[Che14] Ya-Chen Chen. Minimum $K_{2,3}$-saturated graphs. *J. Graph Theory*, 76(4):309–322, 2014.

[CKOW13] Daniel W. Cranston, William B. Kinnersley, Suil O, and Douglas B. West. Game matching number of graphs. *Discrete Appl. Math.*, 161(13-14):1828–1836, 2013.

[CKRW17] James M. Carraher, William B. Kinnersley, Benjamin Reiniger, and Douglas B. West. The game saturation number of a graph. *J. Graph Theory*, 85(2):481–495, 2017.

[CL20] Debsoumya Chakraborti and Po-Shen Loh. Minimizing the numbers of cliques and cycles of fixed size in an F-saturated graph. *European J. Combin.*, 90:103185, 20, 2020.

[CLL+12] Joshua Cooper, John Lenz, Timothy D. LeSaulnier, Paul S. Wenger, and Douglas B. West. Uniquely C_4-saturated graphs. *Graphs Combin.*, 28(2):189–197, 2012.

[CLR79] J. Chalupa, P.L. Leath, and G.R. Reich. Bootstrap percolation on a bethe lattice. *J. Phys. C*, 12:L31–L35, 1979.

[CLZ15] Yair Caro, Josef Lauri, and Christina Zarb. The saturation number for the length of degree monotone paths. *Discuss. Math. Graph Theory*, 35(3):557–569, 2015.

[CM11] Demetres Christofides and Klas Markström. The guessing number of undirected graphs. *Electron. J. Combin.*, 18(1):#P192, 2011.

[CMT20] Shujuan Cao, Yuede Ma, and Zhenyu Taoqiu. A note on rainbow saturation number of paths. *Appl. Math. Comput.*, 378:125204, 2020.

[Cox81] Harold S. M. Coxeter. A systematic notation for the Coxeter graph. *C. R. Math. Rep. Acad. Sci. Canada*, 3(6):329–332, 1981.

[CP16] Yajuan Cui and Liqun Pu. Weak saturation number for multiple copies of the complete graph with a star removed. *Discrete Appl. Math.*, 204:216–220, 2016.

[CP19] Yajuan Cui and Liqun Pu. Weak saturation numbers of $K_{2,t}$ and $K_p \bigcup K_q$. *AKCE Int. J. Graphs Comb.*, 16(3):237–240, 2019.

[CRU21] Steven Chaplick, Jonathan Rollin, and Torsten Ueckerdt. Edge-minimum saturated k-planar drawings. arXiv:2012.08631, 2021.

[Dav18] Hunter M. Davenport. On Saturation Numbers of Ramsey-minimal Graphs, 2018. Honors Undergraduate Thesis.

[Day17] A. Nicholas Day. Saturated graphs of prescribed minimum degree. *Combin. Probab. Comput.*, 26(2):201–207, 2017.

[DDFL85] Stephen J. Dow, David A. Drake, Zoltán Füredi, and Jean A. Larson. A lower bound for the cardinality of a maximal family of mutually intersecting sets of equal size. In *Proceedings of the sixteenth Southeastern international conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1985)*, volume 48, pages 47–48, 1985.

[DH86] Dwight A. Duffus and Denis Hanson. Minimal k-saturated and color critical graphs of prescribed minimum degree. *J. Graph Theory*, 10(1):55–67, 1986.

[DH08] Peter Dukes and Lea Howard. Small maximally disjoint union-free families. *Discrete Math.*, 308(18):4272–4275, 2008.

[DKW03] Aneta Dudek, Gyula Y. Katona, and A. Paweł Wojda. m-path cover saturated graphs. In *2nd Cologne-Twente Workshop on Graphs and Combinatorial Optimization*, volume 13 of *Electron. Notes Discrete Math.*, page 4 pp. (electronic). Elsevier, Amsterdam, 2003.

[DKW06] Aneta Dudek, Gyula Y. Katona, and A. Paweł Wojda. Hamiltonian path saturated graphs with small size. *Discrete Appl. Math.*, 154(9):1372–1379, 2006.

[DPT13] Andrzej Dudek, Oleg Pikhurko, and Andrew Thomason. On minimum saturated matrices. *Graphs Combin.*, 29(5):1269–1286, 2013.

[Dud85] R. M. Dudley. The structure of some Vapnik-Červonenkis classes. In *Proceedings of the Berkeley conference in honor of Jerzy Neyman and Jack Kiefer, Vol. II (Berkeley, Calif., 1983)*, Wadsworth Statist./Probab. Ser., pages 495–508. Wadsworth, Belmont, CA, 1985.

[Dvo20] Vojtěch Dvořák. P_n-induced-saturated graphs exist for all $n \geqslant 6$. *Electron. J. Combin.*, 27(4):#P4.43, 2020.

[DW04a] Aneta Dudek and A. Paweł Wojda. P_m-saturated bipartite graphs with minimum size. *Discuss. Math. Graph Theory*, 24(2):197–211, 2004.

[DW04b] Aneta Dudek and A. Paweł Wojda. P_m-saturated graphs with minimum size. *Opuscula Math.*, 24(1):43–55, 2004.

[DZ12] Aneta Dudek and Andrzej Żak. On hamiltonian chain saturated uniform hypergraphs. *Discrete Math. Theor. Comput. Sci.*, 14(1):21–28, 2012.

[DZK10] Aneta Dudek, Andrzej Żak, and Gyula Y. Katona. Hamilton-chain saturated hypergraphs. *Discrete Math.*, 310(6-7):1172–1176, 2010.

[EFT91] Paul Erdős, Zoltán Füredi, and Zsolt Tuza. Saturated r-uniform hypergraphs. *Discrete Math.*, 98(2):95–104, 1991.

[EG61] Paul Erdős and Tibor Gallai. On the minimal number of vertices representing the edges of a graph. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 6:181–203, 1961.

[EGG+19] Sean English, Pamela Gordon, Nathan Graber, Abhishek Methuku, and Eric C. Sullivan. Saturation of Berge hypergraphs. *Discrete Math.*, 342(6):1738–1761, 2019.

[EGMT19] Sean English, Dániel Gerbner, Abhishek Methuku, and Michael Tait. Linearity of saturation for Berge hypergraphs. *European J. Combin.*, 78:205–213, 2019.

[EH94] Paul Erdős and Ron Holzman. On maximal triangle-free graphs. *J. Graph Theory*, 18(6):585–594, 1994.

[EHM64] Paul Erdős, András Hajnal, and John W. Moon. A problem in graph theory. *Amer. Math. Monthly*, 71:1107–1110, 1964.

[EL75] Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[ES66] Paul Erdős and Miklós Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1:51–57, 1966.

[FFG+17] Jill Faudree, Ralph J. Faudree, Ronald J. Gould, Michael S. Jacobson, and Brent J. Thomas. Saturation spectrum of paths and stars. *Discuss. Math. Graph Theory*, 37(3):811–822, 2017.

[FFGJ09a] Jill Faudree, Ralph J. Faudree, Ronald J. Gould, and Michael S. Jacobson. Saturation numbers for trees. *Electron. J. Combin.*, 16(1):#R91, 2009.

[FFGJ09b] Ralph Faudree, Michael Ferrara, Ronald Gould, and Michael Jacobson. tK_p-saturated graphs of minimum size. *Discrete Math.*, 309(19):5870–5876, 2009.

[FFL95] David C. Fisher, Kathryn Fraughnaugh, and Larry Langley. On C_5-saturated graphs with minimum size. In *Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995)*, volume 112, pages 45–48, 1995.

[FFL97] David C. Fisher, Kathryn Fraughnaugh, and Larry Langley. P_3-connected graphs of minimum size. *Ars Combin.*, 47:299–306, 1997.

[FG13] Ralph J. Faudree and Ronald J. Gould. Saturation numbers for nearly complete graphs. *Graphs Combin.*, 29(3):429–448, 2013.

[FG14] Ralph J. Faudree and Ronald J. Gould. Weak saturation numbers for multiple copies. *Discrete Math.*, 336:1–6, 2014.

[FG18] Jessica Fuller and Ronald J. Gould. On $(K_t - e)$-saturated graphs. *Graphs Combin.*, 34(1):85–95, 2018.

[FGJ13] Ralph J. Faudree, Ronald J. Gould, and Michael S. Jacobson. Weak saturation numbers for sparse graphs. *Discuss. Math. Graph Theory*, 33(4):677–693, 2013.

[FHPZ98] Zoltán Füredi, Peter Horak, Chandra M. Pareek, and Xuding Zhu. Minimal oriented graphs of diameter 2. *Graphs Combin.*, 14(4):345–350, 1998.

[FJL+20] Michael Ferrara, Daniel Johnston, Sarah Loeb, Florian Pfender, Alex Schulte, Heather C. Smith, Eric Sullivan, Michael Tait, and Casey Tompkins. On edge-colored saturation problems. *J. Comb.*, 11(4):639–655, 2020.

[FJM+12] Michael Ferrara, Michael Jacobson, Kevin G. Milans, Craig Tennenhouse, and Paul S. Wenger. Saturation numbers for families of graph subdivisions. *J. Graph Theory*, 71(4):416–434, 2012.

[FJPW16] Michael Ferrara, Michael S. Jacobson, Florian Pfender, and Paul S. Wenger. Graph saturation in multipartite graphs. *J. Comb.*, 7(1):1–19, 2016.

[FK13] Zoltán Füredi and Younjin Kim. Cycle-saturated graphs with minimum number of edges. *J. Graph Theory*, 73(2):203–215, 2013.

[FK21] Radoslav Fulek and Balázs Keszegh. Saturation Problems about Forbidden 0-1 Submatrices. *SIAM J. Discrete Math.*, 35(3):1964–1977, 2021.

[FKK+17] Michael Ferrara, Bill Kay, Lucas Kramer, Ryan Martin, Benjamin Reiniger, Heather Smith, and Eric Sullivan. The saturation number of induced subposets of the boolean lattice. `arXiv:1701.03010`, 2017.

[FKKP20] Nóra Frankl, Sergei Kiselev, Andrey Kupavskii, and Balázs Patkós. Vc-saturated set systems. `arXiv:2005.12545`, 2020.

[FKY14] Michael Ferrara, Jaehoon Kim, and Elyse Yeager. Ramsey-minimal saturation numbers for matchings. *Discrete Math.*, 322:26–30, 2014.

[Fra82] Peter Frankl. An extremal problem for two families of sets. *European J. Combin.*, 3(2):125–127, 1982.

[FRS91] Zoltán Füredi, Dave Reimer, and Ákos Seress. Hajnal's triangle-free game and extremal graph problems. In *Proceedings of the Twenty-second Southeastern Conference on Combinatorics, Graph Theory, and Computing (Baton Rouge, LA, 1991)*, volume 82, pages 123–128, 1991.

[FS09] Michael J. Ferrara and John R. Schmitt. A general lower bound for potentially H-graphic sequences. *SIAM J. Discrete Math.*, 23(1):517–526, 2008/09.

[FS94] Zoltán Füredi and Ákos Seress. Maximal triangle-free graphs with restrictions on the degrees. *J. Graph Theory*, 18(1):11–24, 1994.

[FS05] Marietjie Frick and Joy Singleton. Lower bound for the size of maximal nontraceable graphs. *Electron. J. Combin.*, 12:#R32, 2005.

[FW15] Qiong Fan and Chunxiang Wang. Saturation numbers for linear forests $P_5 \cup tP_2$. *Graphs Combin.*, 31(6):2193–2200, 2015.

[GaKP19] António Girão, Teeradej Kittipassorn, and Kamil Popielarz. Partite saturation of complete graphs. *SIAM J. Discrete Math.*, 33(4):2346–2359, 2019.

[GaLP20] António Girão, David Lewis, and Kamil Popielarz. Rainbow saturation of graphs. *J. Graph Theory*, 94(3):421–444, 2020.

[GGL95] Ronald L. Graham, Martin Grötschel, and László Lovász, editors. *Handbook of combinatorics. Vol. 1, 2.* Elsevier Science B.V., Amsterdam, 1995.

[GHJT19] Ronald J. Gould, Paul Horn, Michael S. Jacobson, and Brent J. Thomas. Gaps in the saturation spectrum of trees. *Discuss. Math. Graph Theory*, 39(1):157–170, 2019.

[GHK+09] Martin Grüttmüller, Sven Hartmann, Thomas Kalinowski, Uwe Leck, and Ian T. Roberts. Maximal flat antichains of minimum weight. *Electron. J. Combin.*, 16(1):#R69, 2009.

[GHM18] Jun Gao, Xinmin Hou, and Yue Ma. The edge spectrum of K_4^--saturated graphs. *Graphs Combin.*, 34(6):1723–1727, 2018.

[GHV18] András Gyárfás, Stephen G. Hartke, and Charles Viss. Uniquely $K_r^{(k)}$-saturated hypergraphs. *Electron. J. Combin.*, 25(4):#P4.35, 2018.

[GKL+13] Dániel Gerbner, Balázs Keszegh, Nathan Lemons, Cory Palmer, Dömötör Pálvölgyi, and Balázs Patkós. Saturating Sperner families. *Graphs Combin.*, 29(5):1355–1364, 2013.

[GKS15] Wenying Gan, Dániel Korándi, and Benny Sudakov. $K_{s,t}$-saturated bipartite graphs. *European J. Combin.*, 45:12–20, 2015.

[GŁS06] Ronald Gould, Tomasz Łuczak, and John Schmitt. Constructive upper bounds for cycle-saturated graphs of minimum size. *Electron. J. Combin.*, 13(1):#R29, 2006.

[GPTV21] Dániel Gerbner, Balázs Patkós, Zsolt Tuza, and Máté Vizer. On saturation of Berge hypergraphs. arXiv:2103.08437, 2021.

[Gra78] M. Granovetter. Threshold models of collective behavior. *Am. J. Sociol.*, 83:1420–1443, 1978.

[GS07] Ronald J. Gould and John R. Schmitt. Minimum degree and the minimum size of K_2^t-saturated graphs. *Discrete Math.*, 307(9-10):1108–1114, 2007.

[GTWZ12] Ronald J. Gould, Wenliang Tang, Erling Wei, and Cun-Quan Zhang. The edge spectrum of the saturation number for small paths. *Discrete Math.*, 312(17):2682–2689, 2012.

[Haj65] András Hajnal. A theorem on k-saturated graphs. *Canad. J. Math.*, 17:720–724, 1965.

[HHQ20] Lianna Hambardzumyan, Hamed Hatami, and Yingjie Qian. Lower bounds for graph bootstrap percolation via properties of polynomials. *J. Combin. Theory Ser. A*, 174:105253, 12, 2020.

[HKNS16] Dan Hefetz, Michael Krivelevich, Alon Naor, and Miloš Stojaković. On saturation games. *European J. Combin.*, 51:315–335, 2016.

[HLSZ19] Shenwei Huang, Hui Lei, Yongtang Shi, and Junxue Zhang. The saturation number of $K_{3,3}$. arXiv:1910.04967, 2019.

[HŠ86] Peter Horák and Jozef Širáň. On a construction of Thomassen. *Graphs Combin.*, 2(4):347–350, 1986.

[HS12] Stephen G. Hartke and Derrick Stolee. Uniquely K_r-saturated graphs. *Electron. J. Combin.*, 19(4):#P6, 2012.

[HT87] Denis Hanson and Bjarne Toft. Edge-colored saturated graphs. *J. Graph Theory*, 11(2):191–196, 1987.

[Isa75] Rufus Isaacs. Infinite families of nontrivial trivalent graphs which are not Tait colorable. *Amer. Math. Monthly*, 82:221–239, 1975.

[JF16] Arun Jambulapati and Ralph Faudree. A collection of results on saturation numbers. *J. Combin. Math. Combin. Comput.*, 99:167–185, 2016.

[JP17] J. Robert Johnson and Trevor Pinto. Saturated subgraphs of the hypercube. *Combin. Probab. Comput.*, 26(1):52–67, 2017.

[JT12] Michael Jacobson and Craig Tennenhouse. Oriented graph saturation. *J. Combin. Math. Combin. Comput.*, 80:157–169, 2012.

[Kal84] Gil Kalai. Weakly saturated graphs are rigid. In *Convexity and graph theory (Jerusalem, 1981)*, volume 87 of *North-Holland Math. Stud.*, pages 189–190. North-Holland, Amsterdam, 1984.

[Kal85] Gil Kalai. Hyperconnectivity of graphs. *Graphs Combin.*, 1(1):65–79, 1985.

[Kat06] Gyula Y. Katona. Hamiltonian chains in hypergraphs. In *CTW2006—Cologne-Twente Workshop on Graphs and Combinatorial Optimization*, volume 25 of *Electron. Notes Discrete Math.*, pages 81–87. Elsevier Sci. B. V., Amsterdam, 2006.

[Keu18] Ralph Keusch. Colorability saturation games. arXiv:1606.09099, 2018.

[KK99] Gyula Y. Katona and H. A. Kierstead. Hamiltonian chains in hypergraphs. *J. Graph Theory*, 30(3):205–212, 1999.

[KKKO20] Jaehoon Kim, Seog-Jin Kim, Alexandr V. Kostochka, and Suil O. The minimum spectral radius of K_{r+1}-saturated graphs. *Discrete Math.*, 343(11):112068, 4, 2020.

[KMM21] Gal Kronenberg, Taísa Martins, and Natasha Morrison. Weak saturation numbers of complete bipartite graphs in the clique. *J. Combin. Theory Ser. A*, 178:105357, 15, 2021.

[KMSV07] Peter Keevash, Dhruv Mubayi, Benny Sudakov, and Jacques Verstraëte. Rainbow Turán problems. *Combin. Probab. Comput.*, 16(1):109–126, 2007.

[KMTT20] Jürgen Kritschgau, Abhishek Methuku, Michael Tait, and Craig Timmons. Few H copies in F-saturated graphs. *J. Graph Theory*, 94(2):320–348, 2020.

[Kor18] Dániel Korándi. Rainbow saturation and graph capacities. *SIAM J. Discrete Math.*, 32(2):1261–1264, 2018.

[KS17] Dániel Korándi and Benny Sudakov. Saturation in random graphs. *Random Structures Algorithms*, 51(1):169–181, 2017.

[KT86] László Kászonyi and Zsolt Tuza. Saturated graphs with minimal number of edges. *J. Graph Theory*, 10(2):203–210, 1986.

[LH19] Min Liu and Zhiquan Hu. Saturation number for linear forest $2P_3 \bigcup tP_2$. *Wuhan Univ. J. Nat. Sci.*, 24(4):283–289, 19.

[LJZY97] Xiaohui Lin, Wenzhou Jiang, Chengxue Zhang, and Yuansheng Yang. On smallest maximally non-Hamiltonian graphs. *Ars Combin.*, 45:263–270, 1997.

[LOS+19] Hui Lei, Suil O, Yongtang Shi, Douglas B. West, and Xuding Zhu. Extremal problems on saturation for the family of k-edge-connected graphs. *Discrete Appl. Math.*, 260:278–283, 2019.

[Lov77] László Lovász. Flats in matroids and geometric graphs. In *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*, pages 45–86. Academic Press, London, 1977.

[LR15] Jonathan D. Lee and Ago-Erik Riet. $\mathcal{F}$-saturation games. *Discrete Math.*, 338(12):2356–2362, 2015.

[LSZ15] Song Luo, Maiko Shigeno, and Mingchao Zhang. On the number of edges in a minimum C_6-saturated graph. *Graphs Combin.*, 31(4):1085–1106, 2015.

[Łuc] Tomasz Łuczak. Personal communication.

[Mad73] Wolfgang Mader. 1-faktoren in graphen. *Math. Ann.*, 201:269–282, 1973.

[Mey74] Jean-Claude Meyer. Quelques problèmes concernant les cliques des hypergraphes h-complets et q-parti h-complets. In *Hypergraph Seminar (Proc. First Working Sem., Ohio State Univ., Columbus, Ohio, 1972; dedicated to Arnold Ross)*, pages 127–139. Lecture Notes in Math., Vol. 411. Springer, Berlin, 1974.

[MHHG21] Yue Ma, Xinmin Hou, Doudou Hei, and Jun Gao. Minimizing the number of edges in $\mathcal{C}_{\geqslant r}$-saturated graphs. *Discrete Math.*, 344(11):Paper No. 112565, 14, 2021.

[MN18] Natasha Morrison and Jonathan A. Noel. Extremal bounds for bootstrap percolation in the hypercube. *J. Combin. Theory Ser. A*, 156:61–84, 2018.

[MNS14] Natasha Morrison, Jonathan A. Noel, and Alex Scott. On saturated k-Sperner systems. *Electron. J. Combin.*, 21(3):#P3.22, 2014.

[MNS17] Natasha Morrison, Jonathan A. Noel, and Alex Scott. Saturation in the hypercube and bootstrap percolation. *Combin. Probab. Comput.*, 26(1):78–98, 2017.

[MR20] Jo Martin and Puck Rombach. Guessing numbers and extremal graph theory. arXiv:2009.04529, 2020.

[MS12] Ryan R. Martin and Jason J. Smith. Induced saturation number. *Discrete Math.*, 312(21):3096–3106, 2012.

[MS15] Guy Moshkovitz and Asaf Shapira. Exact bounds for some hypergraph saturation problems. *J. Combin. Theory Ser. B*, 111:242–248, 2015.

[MSW20] Ryan R. Martin, Heather C. Smith, and Shanise Walker. Improved bounds for induced poset saturation. *Electron. J. Combin.*, 27(2):#P2.31, 2020.

[MTR18] A. Mohammadian and B. Tayfeh-Rezaie. Star saturation number of random graphs. *Discrete Math.*, 341(4):1166–1170, 2018.

[Oll72] L. Taylor Ollmann. $K_{2,2}$ saturated graphs with a minimal number of edges. In *Proceedings of the Third Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1972)*, pages 367–392, Boca Raton, Fla., 1972. Florida Atlantic Univ.

[Pik99a] Oleg Pikhurko. Extremal hypergraphs. *Ph.D Thesis, Cambridge University*, 1999.

[Pik99b] Oleg Pikhurko. The minimum size of saturated hypergraphs. *Combin. Probab. Comput.*, 8(5):483–492, 1999.

[Pik00] Oleg Pikhurko. Asymptotic evaluation of the sat-function for r-stars. *Discrete Math.*, 214(1-3):275–278, 2000.

[Pik01a] Oleg Pikhurko. Constructing designs straightforwardly: worst arising cases. *J. Combin. Des.*, 9(2):100–106, 2001.

[Pik01b] Oleg Pikhurko. Uniform families and count matroids. *Graphs Combin.*, 17(4):729–740, 2001.

[Pik01c] Oleg Pikhurko. Weakly saturated hypergraphs and exterior algebra. *Combin. Probab. Comput.*, 10(5):435–451, 2001.

[Pik04] Oleg Pikhurko. Results and open problems on minimum saturated hypergraphs. *Ars Combin.*, 72:111–127, 2004.

[PS08] Oleg Pikhurko and John R. Schmitt. A note on minimum $K_{2,3}$-saturated graphs. *Australasian J. Combin.*, 40:211–215, 2008.

[PS19] Michał Przykucki and Thomas Shelton. Smallest percolating sets in bootstrap percolation on grids. arXiv:1907.01940, 2019.

[R2̈0] Eero Räty. Induced saturation of P_6. *Discrete Math.*, 343(1):111641, 3, 2020.

[Ror19] Danny Rorabaugh. Graph cover-saturation. *Graphs Combin.*, 35(5):1225–1237, 2019.

[RS18] Martin Rolek and Zi-Xia Song. Saturation numbers for Ramsey-minimal graphs. *Discrete Math.*, 341(12):3310–3320, 2018.

[RZ13] Andrzej Ruciński and Andrzej Żak. Hamilton saturated hypergraphs of essentially minimum size. *Electron. J. Combin.*, 20(2):#P25, 2013.

[RZ16] Andrzej Ruciński and Andrzej Żak. Upper bounds on the minimum size of Hamilton saturated hypergraphs. *Electron. J. Combin.*, 23(4):#P4.12, 2016.

[Sem97] Gabriel Semanišin. On some variations of extremal graph problems. *Discuss. Math. Graph Theory*, 17(1):67–76, 1997.

[Sid93] Elżbieta Sidorowicz. Size of C_3-saturated graphs. *Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz.*, (17):61–66, 1993.

[Sid07] Elżbieta Sidorowicz. Size of weakly saturated graphs. *Discrete Math.*, 307(11-12):1486–1492, 2007.

[Son17] Feifei Song. Saturation numbers for linear forests $P_4 \cup P_3 \cup tP_2$. *Util. Math.*, 104:175–186, 2017.

[Spi19] Sam Spiro. Saturation games for odd cycles. *Electron. J. Combin.*, 26(4):#P4.11, 2019.

[Sta96] Ladislav Stacho. Maximally non-Hamiltonian graphs of girth 7. *Graphs Combin.*, 12(4):361–371, 1996.

[Sta98] Ladislav Stacho. Non-isomorphic smallest maximally non-Hamiltonian graphs. *Ars Combin.*, 48:307–317, 1998.

[SW17] Eric Sullivan and Paul S. Wenger. Saturation numbers in tripartite graphs. *J. Graph Theory*, 84(4):428–442, 2017.

[SZ20] Zi-Xia Song and Jingmei Zhang. On the size of $(K_t, \mathcal{T}_k)$-co-critical graphs. arXiv:1904.07825, 2020.

[TCCD20] Craig Timmons, Benjamin Cole, Albert Curry, and David Davini. Triangles in Ks-saturated graphs with minimum degree t. *Theory Appl. Graphs*, 7(1):Art. 2, 24, 2020.

[Ten16] Craig M. Tennenhouse. Induced subgraph saturated graphs. *Theory Appl. Graphs*, 3(2):Art. 1, 14, 2016.

[Tho74] Carsten Thomassen. Hypohamiltonian and hypotraceable graphs. *Discrete Math.*, 9:91–96, 1974.

[Tho16] Brent Thomas. *Saturation Spectrum for Trees.* PhD thesis, University of Colorado Denver, 2016.

[Tim19] Craig Timmons. C_{2k}-saturated graphs with no short odd cycles. *Graphs Combin.*, 35(5):1023–1034, 2019.

[TT91] Mirosław Truszczyński and Zsolt Tuza. Asymptotic results on saturated graphs. *Discrete Math.*, 87(3):309–314, 1991.

[Tur41] Paul Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[Tut60] William T. Tutte. A non-Hamiltonian graph. *Canad. Math. Bull.*, 3:1–5, 1960.

[Tuz86] Zsolt Tuza. A generalization of saturated graphs for finite languages. *Tanulmányok—MTA Számitástech. Automat. Kutató Int. Budapest*, (185):287–293, 1986.

[Tuz88] Zsolt Tuza. Extremal problems on saturated graphs and hypergraphs. *Ars Combin.*, 25(B):105–113, 1988. Eleventh British Combinatorial Conference (London, 1987).

[Tuz89] Zsolt Tuza. C_4-saturated graphs of minimum size. *Acta Univ. Carolin. Math. Phys.*, 30(2):161–167, 1989. 17th Winter School on Abstract Analysis (Srní, 1989).

[Tuz92] Zsolt Tuza. Asymptotic growth of sparse saturated structures is locally determined. *Discrete Math.*, 108(1-3):397–402, 1992. Topological, algebraical and combinatorial structures. Frolík's memorial volume.

[Tuz94] Zs. Tuza. Applications of the set-pair method in extremal hypergraph theory. In *Extremal problems for finite sets (Visegrád, 1991)*, volume 3 of *Bolyai Soc. Math. Stud.*, pages 479–514. János Bolyai Math. Soc., Budapest, 1994.

[Tuz96] Zs. Tuza. Applications of the set-pair method in extremal problems. II. In *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, volume 2 of *Bolyai Soc. Math. Stud.*, pages 459–490. János Bolyai Math. Soc., Budapest, 1996.

[vAFDO09] Susan A. van Aardt, Marietjie Frick, Jean E. Dunbar, and Ortrud Oellermann. Detour saturated oriented graphs. *Util. Math.*, 79:167–180, 2009.

[vO16] Christine van Oostendorp. *Colored Saturation Parameters for Bipartite Graphs.* PhD thesis, Rochester Institute of Technology, 2016.

[Wen10] Paul Shannahan Wenger. *Three existence problems in extremal graph theory.* ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign.

[Wen14] Paul S. Wenger. A note on the saturation number of the family of k-connected graphs. *Discrete Math.*, 323:81–83, 2014.

[Wes66] Walter A. Wessel. Über eine klasse paarer graphen, i: Beweis einer vermutung von Erdös, Hajnal, und Moon. *Wiss. Z. Tech. Hochsch. Ilmenau*, 12:253–256, 1966.

[Wes67] Walter Wessel. Über eine Klasse paarer Graphen. II. Bestimmung der Minimalgraphen. *Wiss. Z. Techn. Hochsch. Ilmenau*, 13:423–426, 1967.

[WW17] Paul S. Wenger and Douglas B. West. Uniquely cycle-saturated graphs. *J. Graph Theory*, 85(1):94–106, 2017.

[Yu93] Jie Tai Yu. An extremal problem for sets: a new approach via Bezoutians. *J. Combin. Theory Ser. A*, 62(1):170–175, 1993.

[Ż13] Andrzej Żak. Growth order for the size of smallest Hamiltonian chain saturated uniform hypergraphs. *European J. Combin.*, 34(4):724–735, 2013.

[Zha19] Jingmei Zhang. *Two Ramsey-related Problems.* PhD thesis, University of Central Florida, 2019.