A Survey of Forbidden Configuration Results

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Abstract

Let F be a $k \times \ell$ (0,1)-matrix. We say a (0,1)-matrix A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F. In the language of sets, a configuration is a *trace* and in the language of hypergraphs a configuration is a *subhypergraph*.

We define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Let F be a given $k \times \ell$ (0,1)-matrix. The matrix F need not be simple. We define $\text{forb}(m, F)$ as the maximum number of columns of any simple m-rowed matrix A which do not contain F as a configuration. Thus if A is an $m \times n$ simple matrix which has no submatrix which is a row and column permutation of F then $n \leqslant \text{forb}(m, F)$. Or alternatively if A is an $m \times (\text{forb}(m, F) + 1)$ simple matrix then A has a submatrix which is a row and column permutation of F. We call F a *forbidden configuration* and survey this idea [1].

The fundamental result is due to Sauer, Perles and Shelah, Vapnik and Chervonenkis. For K_k denoting the $k \times 2^k$ submatrix of all (0,1)-columns on k rows, then $\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots \binom{m}{0}$. We seek asymptotic results for $\text{forb}(m, F)$ for a fixed F and as m tends to infinity. A conjecture of Anstee and Sali predicts the asymptotically best constructions from which to derive the asymptotics of $\text{forb}(m, F)$. The conjecture has helped guide the research and has been verified for $k \times \ell$ F with $k = 1, 2, 3$ and for simple F with $k = 4$ as well as other cases including $\ell = 1, 2$. We also seek exact values for $\text{forb}(m, F)$.

Keywords: extremal set theory, extremal hypergraphs, (0,1)-matrices, forbidden configurations, trace, VC-dimension, subhypergraph, shattered set.

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[1]Previous version at https://www.combinatorics.org/files/Surveys/ds20/ds20v1-2013.pdf authored by R.P. Anstee

1 Introduction

The study of forbidden configurations is a problem in extremal set theory. It is convenient to use the language of matrix theory. We define a *simple* matrix as a (0,1)-matrix with no repeated columns. Such an $m \times n$ simple matrix A can be thought of a family $\mathcal{A}$ of n subsets of $[m] = \{1, 2, \ldots, m\}$ with the rows indexing the elements and the columns indexing the subsets. Let $\|A\|$ denote the number of columns in A (which is $|\mathcal{A}|$). Assume we are given a $k \times \ell$ (0,1)-matrix F. We say that a matrix A has a *configuration* F if a submatrix of A is a row and column permutation of F and so F is referred to as a *configuration* of A (sometimes called *trace* in the language of sets).

The reader may ask of the importance of the configuration idea in combinatorial investigations. I feel it is one of a few possible basic notions of substructure and it has arisen in applications though admittedly not as frequently as some other substructures. The investigations into the extremal problem of the maximum number of edges in an n vertex graph with no subgraph H originated with Erdős and Stone [ES46] and Simonovits [ES66] and has a large and illustrious literature. There are several ways to generalize to the hypergraph setting. Typically one considers *simple hypergraphs*, those with no repeated edges. One can consider a r-uniform (simple) hypergraph H and forbid a given *subhypergraph* H', itself a r-uniform (simple) hypergraph. Or one can extend to general hypergraphs and forbid a given *subhypergraph* where it is now natural to allow repeated edges in the forbidden object. This latter problem in the language of matrices is our focus [Ans81]. It is to be noted that hypergraphs are sometimes not allowed to have the empty edge whereas our simple matrices naturally allow the column of 0's.

There are interesting connections of results about forbidden configuration to other results. Some related problems (VC-dimension, forbidden submatrices, patterns, covering arrays etc.) are given in Section 2 as well as some relations between them.

Definition 1.1. For two $(0, 1)$-matrices F and A, we say that F is a *configuration* in A, and write $F \prec A$ if there is a row and column permutation of F which is a submatrix of A. We say A *has no configuration* F (or $F \nprec A$) if F is not a configuration in A.

We define

$$\text{Avoid}(m, F) = \{A \,:\, A \text{ is } m-\text{rowed, simple}, F \nprec A\}.$$

Our main extremal problem is to compute

$$\text{forb}(m, F) = \max_A\{\|A\| \;:\; A \in \text{Avoid}(m, F)\}.$$

Thus $\text{forb}(m, F)$ is the smallest value (depending on m and F) so that if A is a simple $m \times n$ matrix and A has no configuration F then $n \leqslant \text{forb}(m, F)$. Alternatively $\text{forb}(m, F)$ is the smallest value so that if A is an $m \times (\text{forb}(m, F)+1)$ simple matrix then A must have a configuration F. This survey mostly considers a single given fixed forbidden configuration F (though variations to forbidden families of configurations are in Section 2) and considers the asymptotics of $\text{forb}(m, F)$ as we let m grow. For structural investigations we define

$$\text{ext}(m, F) = \{A \in \text{Avoid}(m, F) \,:\, \|A\| = \text{forb}(m, F)\}.$$

Note that the definitions readily extend when replacing F by a family $\mathcal{F}$ of configurations.

One could define the equivalence class of matrices under row and column permutations. Let $\tilde{F}$ denote the equivalence class of matrices derived from F by taking all row and column permutations of F. Thus a matrix A has a configuration F if A has a submatrix in $\tilde{F}$. Here a submatrix is ordered. We often blur the distinction between a matrix F and the related equivalence class $\tilde{F}$. A matrix F is referred to as a *configuration* when we wish to consider whether another matrix A has F as a configuration.

Remark 1. Let A^c denote the 0-1-complement of A. Then $\text{forb}(m, F^c) = \text{forb}(m, F)$.

Remark 2. If $F' \prec F$ (i.e. F has a configuration F'), then $\text{forb}(m, F') \leqslant \text{forb}(m, F)$.

When giving results it is often convenient to note when we discover $\text{forb}(m, F') = \text{forb}(m, F)$ where $F' \prec F$. Typically one has a construction working for F' (a simple matrix A with no configuration F') which then necessarily works for F and we have a bound for $\text{forb}(m, F)$ which certainly applies to $\text{forb}(m, F')$. Equality (or asymptotic equality) of the construction for F' and the bound for F then yields equality (or asymptotic equality) for $\text{forb}(m, F')$ and $\text{forb}(m, F)$ as well as for any matrices F'' with $F' \prec F'' \prec F$). The following defines some standard configurations.

Definition 1.2. Let K_k be the $k \times 2^k$ simple matrix of all possible (0,1)-columns on k rows. Let K_k^s be the $k \times \binom{k}{s}$ simple matrix of all possible columns of column sum s. Let $\mathbf{1}_a\mathbf{0}_b$ denote the $(a+b) \times 1$ vector of a 1's on top of b 0's and for convenience we let $\mathbf{1}_a$ denote the $a \times 1$ vector of a 1's and $\mathbf{0}_b$ denote the $b \times 1$ vector of b 0's. Let I_k be the $k \times k$ identity matrix (equivalent to K_k^1). Let I_k^c be the (0,1)-complement of the $k \times k$ identity matrix (equivalent to K_k^{k-1}). Let T_k be the $k \times k$ triangular matrix T_k whose i, j entry is 1 if and only if $i \leqslant j$.

We have a number of results for 2-columned F and find the following notation useful.

Definition 1.3. We define $F_{a,b,c,d}$ as the $(a+b+c+d) \times 2$ matrix consisting of a rows $[1\,1]$, b rows $[1\,0]$, c rows $[0\,1]$ and d rows $[0\,0]$.

We use the notation $[A|B]$ to denote the matrix obtained from concatenating the two matrices A and B. We use the notation $k \cdot A$ to denote the matrix $[A|A|\cdots|A]$ consisting of k copies of A concatenated together. We give precedence to the operation $\cdot$ (multiplication) over concatenation so that for example $[2 \cdot A|B]$ is the matrix consisting of the concatenation of B with the concatenation of two copies of A.

Some useful set notation is:

$$[m] = \{1, 2, \ldots, m\}, \;\; 2^{[m]} = \{S \subseteq [m] \,:\, 0 \leqslant |S| \leqslant m\}, \;\; \binom{[m]}{k} = \{S \subseteq [m] \,:\, |S| = k\}.$$

Thus K_k corresponds to $2^{[k]}$ and K_k^s corresponds to $\binom{[k]}{s}$. Considering simple $m \times n$ matrix A as an element-set incidence matrix, A can be thought of as a family of sets:

$$\mathcal{A} \subseteq 2^{[m]}, \qquad |\mathcal{A}| = n.$$

For a subset of rows S, we define $A|_S$ to be the submatrix of A formed by the rows of S. Thus if F is k-rowed, then $F \prec A$ if there is some $S \in \binom{[m]}{k}$ with $F \prec A|_S$. We could also define

$$\mathcal{A}|_S = \{B \cap S \,:\, B \in \mathcal{A}\},$$

but note that you should choose between the set system $\mathcal{A}|_S$ and the multiset which would correspond to $A|_S$. Now A being simple yields that $\mathcal{A}$ is a set system but we do not expect either $A|_S$ or a configuration F to be simple. A *k-uniform* set system $\mathcal{F}$ has $\mathcal{F} \subseteq \binom{[m]}{k}$. The use of set notation is sometimes preferable. In that setting a forbidden configuration is called a *trace*.

There are alternate ways of describing simple matrices that could be considered. Another equivalent notation is to consider a square free integer $x = \prod_{i=1}^{m} p_i$ and then consider all possible divisors of x. This notation was used in [AA95]. One can generalize to all divisors of some given but arbitrary integer. See this multiset version in Section 2.

Definition 1.4. Let A be an $m_1 \times n_1$ simple matrix and let B be an $m_2 \times n_2$ simple matrix. Then $A \times B$ denotes the $(m_1 + m_2) \times (n_1 n_2)$ simple matrix each column consisting of a column of A placed on a column of B and this is done in all possible ways.

$$\text{e.g. } K_3^1 \times T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Many results have been obtained about $\text{forb}(m, F)$ but the following is the most fundamental.

Theorem 1.5. [Sauer [Sau72], Perles and Shelah [She72], Vapnik and Chervonenkis [VC71]] We have that

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Thus $\text{forb}(m, K_k)$ is $\Theta(m^{k-1})$.

There is mention in the paper [Sau72] that the problem is due to Erdős. Also there is an earlier citation in Russian for [VC71]. If a matrix A contains a copy of K_k in a k-set of rows S then we say that S is *shattered* by A. There are many results on shattered sets. We define a (0,1)-matrix A to have *VC-dimension* k if the largest cardinality of a shattered set is k (alternatively $K_k \prec A$ and $K_{k+1} \not\prec A$) and so $\|A\|$ is $O(m^k)$. There are many results on VC-dimension.

$$\text{Let } \text{ext}(m, F) = \{A \in \text{Avoid}(m, F) \,|\, \|A\| = \text{forb}(m, F)\}. \tag{1}$$

There are a multiplicity of matrices $A \in \text{ext}(m, K_k)$ including $[K_m^{k-1} \,|\, K_m^{k-2} \,|\, \cdots \,|\, K_m^0]$ or, for any $k \times 1$ (0,1)-column α, for A all columns with no *submatrix* α. There are interesting results about matrices in $\text{ext}(m, K_k)$ in [Ans88] and an interesting construction in [AS97] with all column sums in $\{t, t+1, t+2, \ldots, t+k-1\}$.

Theorem 1.5 has induction proofs (Section 11) using the standard induction [Sau72] and also with shattered sets (Thm 1.4,[Paj85]), a shifting proof (Section 12), and linear algebra proofs (Section 14) [FP83] and [Smo97]. The asymptotic growth of $\Theta(m^{k-1})$ was what interested Vapnik and Chervonenkis in Applied Probability. An easy consequence of Theorem 1.5 using Remark 2 is the following:

Corollary 1.6. Let F be a $k \times \ell$ simple matrix. Then $\text{forb}(m, F)$ is $O(m^{k-1})$. ■

It would seem reasonable to consider (0,1)-matrices F which are not simple as well. Füredi [Für83] noted the following general bound that can be proved using Theorem 1.5.

Theorem 1.7. [Für83] Let F be a $k \times \ell$ (0,1)-matrix. Then there is a constant c_F so that $\text{forb}(m, F) \leqslant c_F m^k$ i.e. $\text{forb}(m, F)$ is $O(m^k)$. ■

But what is the correct asymptotic growth as a function of F? We can obtain more detailed general results. The first result below (simultaneously and independently obtained by Füredi and Quinn (generalizing a result of Ryser[Rys72]) and the second result of Gronau are both exact and can be deduced by the existence of constructions since the bounds follows from Remark 2 in the first case using $F = K_k$ and in the second case using $F = K_{k+1}$.

Theorem 1.8. [FQ83] Let k, s be given positive integers with $0 \leqslant s \leqslant k$. Then

$$\text{forb}(m, K_k^s) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}. \ ■$$

Theorem 1.9. [Gro80] We have

$$\text{forb}(m, 2 \cdot K_k) = \binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{0}. \ ■$$

We use the Keevash result [Kee14] on the existence of designs. Use the notation a $t-(m, k, \lambda)$ *design* is a family of k-sets on m elements such that every t-set appears in λ sets. A *simple* design has no repeated blocks.

Theorem 1.10. [Kee14] Let t, k, λ be given. Assuming m is sufficiently large and satisfies $\binom{k-i}{t-i}$ divides $\lambda\binom{m-i}{t-i}$ for $i = 1, 2, \ldots, t-1$, then there is a simple $t-(m, k, \lambda)$-design.

The next result refines Füredi's result Theorem 1.7.

Theorem 1.11. [AF86] We have

$$\text{forb}(m, t \cdot K_k) = \text{forb}(m, t \cdot K_k^k) \leqslant \left(1 + \frac{t-2}{k+1}\right)\binom{m}{k} + \binom{m}{k-1} + \binom{m}{k-2} \cdots + \binom{m}{0},$$

with equality if a simple $k-(m, k+1, t-1)$-design exists. ■

The following four results are quite general refinements of Theorem 1.5 and Theorem 1.7. The following describes the boundary between $\Theta(m^{k-2})$ and $\Theta(m^{k-1})$ for simple $k \times \ell$ F.

Theorem 1.12. [AF10] Let k be given.
If F is a simple $k \times \ell$ matrix with the property that there is a pair of rows of F that do not contain K_2^0, a pair of rows of F that do not contain K_2^2 and a pair of rows of F that do not contain the configuration $K_2^1 = I_2$, then forb(m, F) is $O(m^{k-2})$.
If F is a simple $k \times \ell$ matrix with the property that either every pair of rows has K_2^0 or every pair of rows has K_2^2 or every pair of rows has K_2^1, then forb(m, F) is $\Theta(m^{k-1})$. ■

The maximal k-rowed simple matrices F with forb(m, F) being $O(m^{k-2})$are listed in Theorem 9.1. The following considers the boundary between $\Theta(m^{k-1})$ and $\Theta(m^k)$ for arbitrary $k \times \ell$ F. The result in Theorem 1.13 was first proved for $k = 3$ in [AS05],[AGS97] (there were two proofs originally, one for each of the two possible choices of a 3×4 B) and Theorem 1.14 was first proved for $k = 3$ in [AS05]. Theorem 1.13 was proven for general k in [AF11],[AFFS05] and Theorem 1.14 was proven for general k in [AF10].

Theorem 1.13. [AGS97][AFFS05][AS05] Let B be a simple $k \times (k + 1)$ matrix with the property that there is one column of each column sum. Let $K_k - B$ denote the $k \times (2^k - k - 1)$ matrix obtained from K_k by deleting the columns of B (row order matters here). Let t be given. Then forb$(m, [K_k \,|\, t \cdot [K_k - B]])$ is $\Theta(m^{k-1})$. ■

Theorem 1.14. [AS05][AF11] Let k be given and let D_{12} denote the simple k-rowed matrix of all columns of column sum at least 1 with no K_2^2 on rows 1 and 2. Then assuming $k \geqslant 3$ and $t \geqslant 2$ then forb$(m, [K_k^0 \,|\, t \cdot D_{12}])$ is $\Theta(m^{k-1})$. ■

Note that $t \cdot I_k \prec t \cdot D_{12}$.

Theorem 1.15. [AF10] Let F be a k-rowed matrix with maximum column multiplicity t. If $F \not\prec [K_k \,|\, (t-1) \cdot [K_k - B]])$ for any choice of B as in Theorem 1.13 and $F \not\prec [K_k^0 \,|\, t \cdot D_{12}])$ for D_{12} as in Theorem 1.14 then forb(m, F) is $\Theta(m^k)$. ■

This completely determines the boundary between $\Theta(m^k)$ and $\Theta(m^{k-1})$. The matrices that Conjecture 3.2 predicts to determine the boundary between $\Theta(m^{k-1})$ and $\Theta(m^{k-2})$ are described in Theorem 9.3. Theorem 9.1 helps in this analysis. There are complete asymptotic results for $k \times 2$ F in Section 7.

A large number of exact bounds are sprinkled throughout this survey including complete exact results for $1 \times \ell$ F in Section 4 and complete exact results for $k \times 1$ F in Section 7, a number of $2 \times \ell$ results in Section 4 and a number of general $k \times 2$ results in Section 7 as well as a number of 3×2, 3×3 and 3×4 results in Section 5 and a number of 4×2 and further 4-rowed results in Section 6. One gets an idea of what is typically driving the exact bounds for many F. In [AK10], we defined a *critical substructure* of a configuration F as a minimal configuration $F' \prec F$ with forb$(m, F') =$ forb(m, F). For K_4 we have the complete list of critical substructures but have not yet fully determined the list for K_5.

Theorem 1.16. [Rag11] The critical substructures of K_4 are $\mathbf{0}_4$, I_4, K_4^2, I_4^c, $\mathbf{1}_4$, $2 \cdot \mathbf{0}_3$ and $2 \cdot \mathbf{1}_3$. ■

We have verified (Prop. 4.3.8 [Rag11]) that the only k-rowed critical substructures of K_k are K_k^s for $s = 0, 1, \ldots, k$.

Problem 1.17. Show that $2\cdot\mathbf{1}_{k-1}$ and $2\cdot\mathbf{0}_{k-1}$ are the only (k-1)-rowed critical substructures of K_k.

It you show that $\text{forb}(m,[\mathbf{0}_{k-1}|2\cdot K^1_{k-1}|2\cdot K^2_{k-1}|\cdots|2\cdot K^{k-2}_{k-1}|\,\mathbf{1}^{k-1}]) < \text{forb}(m,K_k)$, then that yields yes answer ((Conj. 4.3.9 [Rag11])

The following result, while not best possible, indicates we can extend K_k quite a bit and still get the bound of Theorem 1.5. The extension in [AN22] had a nice proof to overcome the difficulty of not verifying base cases.

Theorem 1.18. [AM11][AN22] Assume $k \geqslant 4$ and $t \geqslant 1$. There exists an m_k so that for $m < m_k$, we have

$$\text{forb}(m,[K_k|t\cdot[K_2^T\times K_{k-4}]]) = \text{forb}(m,K_k),$$
$$\text{ext}(m,[K_k|t\cdot[K_2^T\times K_{k-4}]]) = \text{ext}(m,K_k).\ \blacksquare$$

Exact bounds often require a more complete understanding of what it means to forbid a configuration. In many cases we can also determine $\text{ext}(m,F)$ (see (1)). In trying to establish exact bounds we have found some interesting 'negative' results including Theorem 6.10 for the configuration $F_{2,1,1,0}$ [Duk15].

A purpose of this paper is to provide a single place to access existing results (Sections 4, 5, 6, 7, 8, 9) and the proof techniques employed (Sections 10, 11, 12, 13, 14, 15). In doing so, we are encouraging the gentle reader to consider ways to make progress in proving the conjecture described in Section 3 or perhaps obtaining exact bounds or exploring other related problems such as described in Section 2. Open problems are scattered throughout including Conjecture 3.2, Problem 3.4, Problem 7.5, Conjecture 8.1, Problem 15.2, Conjecture 2.13. Here are two very concrete problems, the first an update on the previous problem that has been solved (Theorem 6.4).

Problem 1.19. Show that

$$\text{forb}(m,t\cdot\begin{bmatrix}1&1&0&1\\1&0&1&0\\0&0&1&1\\0&1&0&0\end{bmatrix})\text{ is }O(m^2).\ \blacksquare$$

Problem 1.20. Show that for those m for which a triple system of multiplicity 2 exists,

$$\text{forb}(m,\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&0&0&0\end{bmatrix}) = \frac{5}{3}\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}.\ \blacksquare$$

I expect that I have missed many related results that have been stated in another context but have relevance here. I would be glad to hear about them; email me.

2 Variations including Forbidden Submatrices

Uniform Hypergraphs

In generalizing from graphs to hypergraphs, it is often the case that we restrict to r-uniform (simple) hypergraphs for a fixed r. In our setting, this is the requirement that all column sums are r. Frankl and Pach [FP94] considered Theorem 1.5 for r-uniform hypergraphs for which they established a basic bound of $\binom{m}{k-1}$. Ahlswede and Khachatrian [AK97b] obtain a construction of size $\binom{m-1}{k-1}+\binom{m-1}{k-3}$ while Mubayi and Zhao [MZ07a] obtain an improved upper bound of $\binom{m}{k-1}-\log_p m+k!k^k$. Other cases of forbidden configurations such as K_k^ℓ for r-uniform hypergraphs are considered [MZ07a]. Asymptotically sharp values for the maximum number of edges in a 3-uniform hypergraph containing no Fano plane are due to deCaen and Füredi [dCF00]. Some further examples in this direction are [Für91, FP84, FS05, KM10, MZ07b, Pat09, Pik08]. An asymptotically exact bound for Turán's problem remains elusive. Note that the problems are typically forbidding a r-uniform subhypergraph.

Families of forbidden configurations

The notion of some forbidden substructure often can be described by some family (often infinite) of forbidden configurations. Many problems in extremal combinatorics could be phrased that way but typically there is no special insight gained. We have forbidden families arise using inductive arguments in Corollary 5. We'll discuss a few other cases. The result of Balogh and Bollabás seems the most interesting result.

Theorem 2.1. [BB05] Let k be given. Then $\text{forb}(m,\{I_k,I_k^c,T_k\})$ is $O(1)$.

In some ways this seems to follow from Conjecture 3.2 since no linear construction (I_m, I_m^c or T_m) avoids all three forbidden configurations I_k, I_k^c, T_k. A less restrictive family of forbidden configurations also yielding a constant bound is in [BP09]. A meta version of Conjecture 3.2 namely that the product constructions yield the asymptotically best constructions is false in general (Theorem 2.8 and Theorem 2.9 below). An easy (not optimal) construction of an $m\times\binom{2k}{k}$ simple matrix A that has no configurations I_k, I_k^c, T_k is to take all columns of column sum $k-1$ in the (k-1)-fold product $T_{m/(k-1)}\times T_{m/(k-1)}\times\cdots T_{m/(k-1)}$. With Laura Dunwoody, we established some easy exact results.

Theorem 2.2. [AD] $\text{forb}(m,\{I_1,I_1^c,T_1\})=0$ and $\text{forb}(m,\{I_2,I_2^c,T_2\})=2$ and $\text{forb}(m,\{I_3,I_3^c,T_3\})=6$. ■

A result of Balin Fleming (related to results in [AF10], proof given in [Koc13]) yields a remarkably good bound:

Theorem 2.3. Let $F_a=\begin{bmatrix}1&0&t\cdot 1\\0&1&1\end{bmatrix}$, $F_b=\begin{bmatrix}1&0&t\cdot 0\\0&1&0\end{bmatrix}$, and $F_c=t\cdot\begin{bmatrix}0&1&1\\0&0&1\end{bmatrix}$. Then for $t\geqslant 2$, $\text{forb}(m,\{F_a,F_b,F_c\})\leqslant 6t-6$.

Theorem 2.1 yields as a corollary a useful boundary between $O(1)$ and $\Omega(m)$ noted in [Koc13].

Theorem 2.4. Let $\mathcal{F}$ be a finite forbidden family. Then $\text{forb}(m,\mathcal{F})$ is $O(1)$ if and only if there exist $F_i,F_j,F_k\in\mathcal{F}$ (not necessarily distinct configurations) such that for some

ℓ, $F_i \prec I_\ell$, $F_j \prec I_\ell^c$ and $F_k \prec T_\ell$. Moreover if the triple F_i, F_j, F_k doesn't exist then $\text{forb}(m, \mathcal{F})$ is $\Omega(m)$

The following three results follow from results of Balogh, Keevash and Sudakov [BKS05]. Somewhat different bounds occur if one adds $\mathbf{0}$ to I, adds $\mathbf{1}$ to I^c and adds $\mathbf{0}$ to T.

Theorem 2.5. Let $k \geqslant 2$ be given. Then $\text{forb}(m, \{I_k, I_k^c\})$ is $\Theta(m^{k-1})$.

Proof: We note that $\text{forb}(m, \{I_k\})$ is $O(m^{k-1})$ and hence $\text{forb}(m, \{I_k, I_k^c\})$ is $O(m^{k-1})$. The construction of the (k-1)-fold product $T_{m/(k-1)} \times T_{m/(k-1)} \cdots \times T_{m/(k-1)}$ show that $\text{forb}(m, \{I_k, I_k^c\})$ is $\Omega(m^{k-1})$ since if we take two rows from any one term of the product, we are unable to have I_2 and yet I_k and I_k^c have I_2 in every pair of rows. ∎

Theorem 2.6. Let $k \geqslant 2$ be given. Then $\text{forb}(m, \{I_k^c, T_k\})$ is $\Theta(m^{k-1})$.

Proof: We note that $\text{forb}(m, \{I_k^c\})$ is $O(m^{k-1})$ and hence $\text{forb}(m, \{I_k^c, T_k\})$ is $O(m^{k-1})$. The construction of the (k-1)-fold product $I_{m/(k-1)} \times I_{m/(k-1)} \cdots \times I_{m/(k-1)}$ show that $\text{forb}(m, \{I_k^c, T_k\})$ is $\Omega(m^{k-1})$ since if we take two rows from any one term of the product, we are unable to have $\binom{1}{1}$ and yet I_k^c and T_k have $\binom{1}{1}$ ∎

Theorem 2.7. Let $k \geqslant 2$ be given. Then $\text{forb}(m, \{I_k, T_k\})$ is $\Theta(m^{k-2})$.

Proof: We note that both I_k and T_k have a column with $k-1$ 0's and so neither can be found in the (k-2)-fold product $I^c_{m/(k-2)} \times I^c_{m/(k-2)} \cdots \times I^c_{m/(k-2)}$, hence $\text{forb}(m, \{I_k, T_k\})$ is $\Omega(m^{k-2})$. To prove the upper bound, we use induction on ℓ in the statement $\text{forb}(m, \{I_k, T_\ell\})$ is $O(m^{\ell-2})$, for $\ell \geqslant 2$. When $\ell = 2$, we note that forbidding T_2 means that any two sets (thinking of columns as sets) must be disjoint. Then the condition no configuration I_k means that there are at most $k-1$ disjoint nonempty sets (column sum at least 1) and the empty set (the column of 0's). Thus $\text{forb}(m, \{I_k, T_2\}) = k$ which is $\Theta(m^{2-2})$. Now we use induction on ℓ and the standard decomposition of (25) noting that applying Lemma 5 to $F = T_\ell$ yields $F_s = T_{\ell-1}$ for $s \neq 1$. Thus $\text{forb}(m, \{I_k, T_\ell) \leqslant \text{forb}(m-1, \{I_k, T_\ell\}) + \text{forb}(m-1, \{I_k, T_{\ell-1}\})$. Applying induction, we obtain the desired bound. ∎

The following result shows that our constructions of Conjecture 3.2 are no longer sufficient for asymptotics with families of forbidden configurations. General forbidden subgraph problems could be given this way.

Theorem 2.8. Let C_4 denote the 4×4 matrix that is the incidence matrix of a cycle of length 4. Then $\text{forb}(m, \{\mathbf{1}_3, C_4\})$ is $\Theta(m^{3/2})$.

Proof: Forbidding $\mathbf{1}_3$ makes this into a graph problem since apart from columns of sum 0 or 1, all remaining columns must have two 1's. A simple matrix with column sums 2 can be viewed as the vertex-edge incidence matrix of a graph on m vertices. Now the maximum number of edges in a graph on m vertices with no no cycle of length 4 is $\Theta(m^{3/2})$. ∎

We have obtained a stronger version of this by a complicated induction argument. Note that $I_2 \times I_2$ is C_4 as a configuration. We had started by forbidding $\{I_2 \times I_2, I_2 \times T_2, T_2 \times T_2\})$ but discovered $I_2 \times T_2$ wasn't required.

Theorem 2.9. [AKRS11] We have that $\text{forb}(m, \{I_2 \times I_2, T_2 \times T_2\})$ is $\Theta(m^{3/2})$. ■

While these result are 'negative' and suggests that handling families of forbidden configurations will be enormously more difficult than forbidding a single configuration, it is also the case that some of our inductive proofs for a single conjecture naturally consider families of forbidden configurations and perhaps in those cases our product constructions are still asymptotically optimal.

Assume t is given. Kleitman considered the maximum size of a set system $\mathcal{F} \subseteq 2^{[m]}$ with the property that for every pair $A, B \in \mathcal{F}$, $|A \backslash B| + |B \backslash A| \leqslant 2t$. The bound is $\text{forb}(m, K_{t+1})$. One can think of this as having forbidden the $(2t+1) \times 2$ configurations $F_{0,2t+1,0,0}, F_{0,2t,1,0}, \ldots, F_{0,t+1,t,0}$.

Balanced and *Totally Balanced* matrices are easily defined in terms of forbidden configurations. Let C_k denote the $k \times k$ matrix that is the incidence matrix of a cycle of length k. A matrix is *Balanced* if and only if it has no configuration C_k for $k \in \{3, 5, 7, 9, \ldots\}$. A matrix is *Totally Balanced* if and only if it has no configuration C_k for $k \in \{3, 4, 5, 6, \ldots\}$. The result that $\text{forb}(m, C_3) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ can be found in [Rys72] but also follows from Theorem 1.5 since C_3 is a configuration of K_3.

Theorem 2.10. [AF84] Let C_k denote the $k \times k$ matrix that is the incidence matrix of a cycle of length k. Then $\text{forb}(m, \{C_3, C_4, C_5, \ldots\}) = \text{forb}(m, C_3) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$. ■

One has the remarkable result that any $m \times (\binom{m}{2} + \binom{m}{1} + \binom{m}{0})$ simple matrix with no configuration C_3 is also totally balanced (Remark 3.1[Ans80]). Totally balanced matrices have been studied in many papers (e.g. [AF84]) with a survey contained in [Spi03].

Forbidden Submatrices: Fixed Row and Column Order

Another variation is to ask whether the row or column order is important. In most combinatorial investigations, permuting the row and column order is just a relabelling. Forbidding a configuration can be thought of as forbidding all submatrices in the equivalence class $\tilde{F}$. In other circumstances either the row order or the column order or both may be crucial. For example, there are algorithms that proceed by assuming you have a special ordering and then the algorithm exploits this special ordering [AF84]. It is a somewhat remarkable fact (due to Hoffman, Kolen and Sakarovitch [HKS86] as well as [AF84]) that a matrix is *Totally Balanced* if and only if the rows and columns can be ordered so that the resulting matrix has no submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Spinrad has a survey on some results in this area.

Results on Forbidden Submatrices can be found in [Ans85], [AF86], [FFP87], [Ans00], [Mer15]. One can restate Theorem 6.6 as a forbidden submatrix problem where we view $F_{0,k,0,0}$ as a matrix (not configuration).

Theorem 2.11. [FFP87] Let k, m be given and let $f(m, k)$ denote the maximum number of columns in a simple m-rowed matrix A such that A has no submatrix $F_{0,b,0,0}$ (we are viewing $F_{0,b,0,0} = [\mathbf{1}_k \,|\, \mathbf{0}_k]$ as a $k \times 2$ matrix and not as a configuration). Then $f(m, 2) = \binom{m}{2} + 2m - 1$ and $f(m, k) < \binom{m}{k} + 5k^2\binom{m}{k-1}$.

As noted above the theorem on bounding one-way differences yields a forbidden configuration result. The following is the general result for forbidden submatrices.

Theorem 2.12. [Ans00] Let F be a $k \times \ell$ (0,1)-matrix. Let A be an $m \times n$ (0,1)-matrix with no $k \times \ell$ submatrix of A being equal to F. Then

$$n \leqslant m^{2k-1-((k-1)/(13\log_2 \ell))}.$$

This was an improvement on the result that $n \leqslant m^{2k-1}$ proved in [FFP87] via a pigeonhole argument and on the first bound of $n \leqslant m^{13k\log_2 \ell}$ in [Ans85]. In any event the conjecture was made both in [AF86], [FFP87] that:

Conjecture 2.13. Let F be a $k \times \ell$ (0,1)-matrix. Let A be an $m \times n$ (0,1)-matrix with no $k \times \ell$ submatrix of A being equal to F. Then there exists a constant c_F depending only on F so that

$$n \leqslant c_F m^k.$$

Some recent progress is in [AC13] and [Mer15].

Theorem 2.14. [Mer15] Let F be a $k \times \ell$ (0,1)-matrix. Let A be an $m \times n$ (0,1)-matrix with no $k \times \ell$ submatrix of A being equal to F. For $k = 2$, there exists a constant c_F depending only on F so that $n \leqslant c_F m^{2+o(1)}$. For $k \geqslant 3$, we have $n \leqslant c_F m^{5k/3-1+o(1)}$

Fixed Row order for Configurations

There have been some investigations for cases where only column permutations of F are allowed. Some linear algebra proofs have this as an essential character [Ans95]. We note that a row permutation of K_k (or K_k^s) is a column permutation of K_k (or K_k^s). Thus in the standard cases we can avoid row permutations. Some induction proofs generalize, using this idea, to the idea of *order shattered sets* [ARS02].

Forbidding configurations on some selection of subsets of rows

There are cases where one might want to forbid a configuration of k rows on only some subset of the possible k-sets of rows or indeed on a collection of subsets of rows of varying sizes. Induction, shifting and linear algebra proofs continue to work. Theorem 2.15 of Alon [Alo83] is central to this. An exploration of the proof techniques and some generalizations are in [Ans88]. An application of the result is Theorem 2.12 [Ans00] to the problem of forbidden submatrices.

The paper [ARS02] explores shattered sets and so called *order shattered sets.* The number of shattered sets is lower bounded by the number of distinct columns Theorem 11.1 while the number of order shattered sets is equal to the number of distinct columns. Order shattered sets are special shattered sets.

Multivalued Matrices

Many results easily extend to allowing the elements of our family $\mathcal{A}$ to themselves be multisets, the usual approach being to allow element i (corresponding to row i) to have maximum multiplicity e_i. Thus rather than entries 0 or 1 the entries in row i of A are in $[e_i]$. The extension of Theorem 1.5 to multisets with $e_1 = e_2 = \cdots = e_m = e$ is in [Ste78] and the extension of Theorem 1.5 to multisets allowing different e_i's is in [KM78]. The extension to forbidding $K_{|S|}$ on rows S for a family of sets $S \in T \subseteq 2^{[m]}$ while having various element multiplicities is in [Alo83]. Define an $(m, e_1, e_2, \ldots, e_m)$-column as a column on m rows with the entry in the ith row of the column chosen from $\{0, 1, \ldots, e_i\}$ for $i = 1, 2, \ldots, m$. Define an m-rowed matrix A to be e-simple if each column is an $(m, e_1, e_2, \ldots, e_m)$-column and there are no repeated columns. In this context, we use K_S to denote the $k \times \left(\prod_{i \in S}(e_i + 1)\right)$ e-simple matrix.

Theorem 2.15. [Alo83]. Let $m, e_1, e_2, \ldots, e_m$ be given positive integers and let $\mathcal{S}$ be given with $\mathcal{S} \subseteq 2^{[m]}$. Let $f(m, \mathcal{S})$ be the number of $(m, e_1, e_2, \ldots, e_m)$-columns which do not have all 0's for the rows indexed by S for any $S \in \mathcal{S}$. Then if A is $m \times n$ e-simple matrix with $K_{|S|} \not\prec A|_S$ for any $S \in \mathcal{S}$, then

$$n \leqslant f(m, \mathcal{S}). \blacksquare$$

There are some forbidden configuration ideas in [AM85] that explore the natural generalization of K_k^s and Theorem 1.8 to multisets. The results in [AA95] are stated in terms of divisors of an integer $\prod_{i=1}^{m} p_i^{e_i}$.

A variation of Füredi and Sali [FS12] considers forbidding versions of K_k consisting of two symbols. Let $K_k(\{i, j\})$ denote the $k \times 2^k$ matrix consisting of all possible columns on the two symbols i, j. Let A be an $m \times n$ matrix with entries in $\{0, 1, 2, \ldots, e\}$ and no repeated columns. Assume that for each pair $i, j \in \{0, 1, 2, \ldots, m\}$ we have a bound $k(i, j)$. Assume A has no configuration $K_{k(i,j)}(i, j)$ for each pair $i, j \in \{0, 1, 2, \ldots, e\}$. Then we can obtain a polynomial bound on n (polynomial in m where e and the values $k(i, j)$ are viewed as constants) that reduces to Theorem 1.5 in the case $e = 1$ and $k(0, 1) = k$. In fact, we have a dichotomy here, that is it is shown in [FS12] that the only way to get a polynomial bound is to forbid an (i, j)-matrix (a matrix with entries from $\{i, j\}$) for every $1 \leqslant i < j \leqslant e$.

This result inspired research in two direction. The first is to find some families of forbidden multivalued configurations that have polynomial bound and determine the bound. Anstee and Lu [AL15] gave generalization of Theorem 2.1[BB05]. Define the generalized identity matrix $I_\ell(a, b)$ as the $\ell \times \ell$ matrix with a's on the diagonal and b's elsewhere. The standard identity matrix is $I_\ell(1, 0)$. Define the generalized triangular matrix $T_\ell(a, b)$ as the matrix with a's below the diagonal and b's elsewhere. The standard upper triangular matrix is $T_\ell(0, 1)$. Let

$$\mathcal{T}_\ell(e) = \{I_\ell(a, b) \colon a, b \in \{0, 1, \ldots, e-1\}, a \neq b\} \cup \{T_\ell(a, b) \colon a, b \in \{0, 1, \ldots, e-1\}, a \neq b\}.$$

Theorem 2.16. [AL15] Given e, ℓ, there is a constant $c(e, \ell)$ such that $\text{forb}(m, e, \mathcal{T}_\ell(e)) \leqslant c(e, \ell)$.

Here $\text{forb}(m, e, \mathcal{F})$ is the natural generalization of the concept of $\text{forb}(m, \mathcal{F})$ being the largest possible number of columns of a simple e-matrix A avoiding each $F \in \mathcal{F}$ as configuration. An important result is that $c(e,\ell)$ is $O(2^{c_e\ell^2})$ for some constant c_e. Anstee et.al. [ADSL23] consider forbidding the matrices $\mathcal{T}_\ell(e) \setminus \mathcal{T}_\ell(2)$. Their main idea is that if a matrix avoids those generalized identities and triangular matrices that contain an entry from $\{2,\dots,e-1\}$, then it sort of behaves (asymptotically) as a $(0,1)$-matrix. Note that any (0,1)-matrix $A \in \text{Avoid}(m,e,\mathcal{T}_\ell(e)\backslash\mathcal{T}_\ell(2))$ and so $\text{forb}(m,e,\mathcal{T}_\ell(e)\backslash\mathcal{T}_\ell(2)) = \Omega(2^m)$. To formulate the general result in this direction we need to introduce two more concepts. Define the matrix $T_\ell(a,b,c)$ to be the $\ell\times\ell$ matrix with a's below the diagonal, b's on the diagonal and c's above the diagonal. Also, let $\text{forbmax}(m,e,\mathcal{F}) = \max_{m'\leqslant m}\text{forb}(m',e,\mathcal{F})$.

Theorem 2.17. [AL15] Let $\mathcal{F}$ be a finite family of (0,1)-matrices. Then $\text{forb}(m,3,\mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2)\cup T_\ell(0,2,1)\cup\mathcal{F})$ is $O(\text{forbmax}(m,2,\mathcal{F}))$.

The main goal here is to get rid of the matrix $T_\ell(0,2,1)$ from the collection of forbidden configurations.

The paper [ADSL23] uses Ramsey theory extensively to prove partial results about the following general problem.

Problem 2.18. (i) In particular, if F is a (0,1)-matrix, then is it true that $\text{forb}(m,3,\mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2)\cup F)$ is $\Theta(\text{forbmax}(m,2,F))$?
(ii) Let F be an s-matrix. Is it true that $\text{forb}(m,r,\mathcal{T}_\ell(r)\backslash\mathcal{T}_\ell(s)\cup F)$ is $\Theta(\text{forbmax}(m,s,F))$?

Another way to provide polynomial bounds is to consider *symmetrization* of a $(0,1)$ configuration. That is, if F is a $(0,1)$-matrix, then $F(i,j)$ denotes the matrix with 0's replaced by i's and 1's replaced by j's in F, and $\text{Sym}(F) = \{F(i,j)\colon 0\leqslant i<j\leqslant e\}$. The paper [ELS20] contains several bounds on $\text{forb}(m,e,\text{Sym}(F))$ as a function of $\text{forb}(m,F)$. In particular, the order of magnitude is determined for all $2\times\ell$ and $3\times\ell$ $(0,1)$-matrices.

A recent direction of research is to consider $\text{forb}(m,e,F)$ for a $(0,1)$-matrix F as forbidden configuration. This was initiated by Dillon and Sali in [DS21], where they proved exact exponential bounds for several configurations. In fact, they showed that if F is simple and has *nested solutions* in the $(0,1)$-world, then $\text{forb}(m,e,F) = \sum_{k=0}^{m}\binom{m}{k}(e-2)^{m-k}\text{forb}(k,F)$. They also showed that the $e=3$ case behaves significantly different from the $e\geqslant 4$ case.

The configuration $M = \begin{bmatrix}0&1\\0&1\\1&0\end{bmatrix}$ has no nested solutions, and its investigation led Peaslee et. al. [PSY23], to an interesting optimization problem. Let G be a multigraph obtained by choosing exactly one edge from each triangle of the complete graph K_m. Let m_f^G denote the multiplicity of edge f in G. For each $\alpha\in\mathbb{R}$, what is $H(m,\alpha) = \max_G\sum_{f\in E(K_m)}\alpha^{m_f^G}$? Using upper and lower bounds for $H(m,\alpha)$ they significantly improved bounds on $\text{forb}(m,e,M)$.

The main result and its proof of [FS12] inspired Gao et.al. [GMRT23], to consider the k-Natarajan-dimension, a generalization of VC-dimension, that encompasses Natarajan-dimension [Nat89] and Steele-dimension [Ste78].

Interestingly, the Bixby and Cunningham [BC87] proof of the bound on the number of distinct columns for a totally unimodular matrix, a (-1,0,1)-matrix, uses Theorem 1.5

for $k = 2$. Further applications to matrices with more than just two possible entries are found in [Ans90a].

Results where we allow our family $\mathcal{A}$ to be a multiset are more problematic and we quickly would have $\text{forb}(m, F)$ be infinite by either repeating the column of 0's or the column of 1's. The design theoretic results of [AB] do use such an interpretation when the column sums are restricted.

VC-dimension

VC-dimension is defined in the Introduction. Vapnik and Chevonenkis [VC71] were interested in applied probablility when they developed the fundamental result Theorem 1.5. Applications to learning theory continue to be developed while raising interesting related questions [RSZ12]. There are other applications. Some have described VC-dimension as a good measure of the complexity of a hypergraph [ŁS10]. An important application is to transversals. For this concept, a column of 0's causes difficulties (or an empty edge in the hypergraph) so in what follows assume we do not have the column of 0's. Let $S \subseteq [m]$ be a *transversal* of A if each column of A has at least one 1 in a row of S. Seeking a minimum cardinality transversal, we let $\mathbf{x}$ be the (0,1)-incidence vector of S, and compute:

$$\tau = \min\left\{\mathbf{1} \cdot \mathbf{x} \text{ subject to } A^T\mathbf{x} \geqslant \mathbf{1}, \quad \mathbf{x} \in \{0,1\}^m\right\}.$$

The natural fractional problem is:

$$\tau^* = \min\left\{\mathbf{1} \cdot \mathbf{x} \text{ subject to } A^T\mathbf{x} \geqslant \mathbf{1}, \quad \mathbf{x} \geqslant \mathbf{0}\right\}.$$

Haussler and Welzl obtained a 'close' connection between τ and τ^*.

Theorem 2.19. (Haussler and Welzl [HW87]) Assume A is a (0,1)-matrix with no column of 0's. If A has VC-dimension k then $\tau \leqslant 16k\tau^* \log(k\tau^*)$.

An example of the use of this is by Łuczak and Thomassé [ŁS10] to solve a colouring problem.

VC-dimension has been used in Computational Geometry [Mat02]. Goldberg [ADF+11] used the low VC dimension of the map data to help efficiently (in suitable real-time) determine the shortest road paths. Alon et al. used it for sign rank [AMY16]. There are many applications.

Patterns

A problem which sounds very similar to forbidding a configuration is to consider how many 1's an $m \times n$ matrix can have subject to some 'forbidden configuration' of 1's sometimes called a *pattern*. There are several differences including that typically we do not allow row and column permutations and the fact we do not concern ourselves with 0's (if we think of patterns as subgraphs then our forbidden configurations are like induced subgraphs). If we choose to forbid a $k \times \ell$ submatrix of 1's then this is the problem of Zarankiewicz [KST54]. A number of papers study problems related to patterns: [Für90],[BG91],[FH92],[MT04],[Tar05]. Assume you have been given some $k \times \ell$ (0,1)-matrix F which we can call a *pattern*. We ask for the maximum number of 1's in an

$m \times n$ matrix A which has the property that there is no $k \times \ell$ submatrix B with $F \leqslant B$. Here we are using the definition that for two $a \times b$ matrices $C = (c_{ij})$ and $D = (d_{ij})$, we say $C \leqslant D$ if and only if $c_{ij} \leqslant d_{ij}$ for all choices $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, b$. Füredi and Hajnal [FH92] considers all patterns of 4 1's as well as other patterns. Marcus and Tardos [MT04] solve an important conjecture of Füredi and Hajnal and also a conjecture of Stanley and Wilf by considering a pattern corresponding to a permutation matrix [MT04]. Various bounds such as $m \log n$ arise for forbidden patterns so some results have quite different character from forbidden configuration bounds. Results from patterns have been useful in our investigations [AKRS11]. When applying Conjecture 3.2, it is natural to ask how many columns can we select from a large product (e.g. $T_{m/2} \times T_{m/2}$) while still avoiding some configuration (e.g. $T_2 \times T_2$). We may encode each chosen column of the product $T_{m/2} \times T_{m/2}$ as a 1 in an $m/2 \times m/2$ matrix A and the forbidden configuration $T_2 \times T_2$ forces A to avoid a pattern of a 4×4 permutation matrix (as well as other patterns). Results in [AKRS11] expand on this.

Covering Arrays

A *covering array of strength* k is a (0,1)-matrix such that every k-set of rows contains a copy of K_k (this is usually done for the transposed matrix). One would be interested in the minimum number of columns for which a covering array on m rows exist. The following result of Kleitman and Spencer answers most of the asymptotic questions.

Theorem 2.20. Kleitman and Spencer[KS73] Let k be given. Then there exists an m-rowed (0,1)-matrix A such that for every $S \in \binom{[m]}{k}$, that $K_k \prec A|_S$ such that $\|A\|$ is $\Theta(\log m)$.

A survey article on binary covering arrays by Lawrence et al [LKL$^+$11] is recommended. Note that for covering arrays, the greatest interest is in *exact* results. In [AM11] we defined

$$\mathrm{req}(m, F) = \min_A \{|A| \,:\, A \text{ is } m\text{-rowed and simple; for all } S \in \binom{[m]}{k}\ F \prec A|_S\}.$$

An application to forbidden configurations occurs when we consider what we must delete in order to avoid a configuration. The question is typically only relevant for the number of rows small with respect to the forbidden configuration.

Lemma 3. *[AM11] Let k, p, q be given with $p + q \leqslant k$. Let A be a k-rowed simple matrix with no configuration $F = \mathbf{1}_p\mathbf{0}_q \times K_{k-(p+q)}$. Then for every $S \subseteq \binom{[k]}{p+q}$ set of rows of the matrix $K^p_{p+q} \prec (K_k \backslash A)|_S$. Thus forb$(k, (\mathbf{1}_p\mathbf{0}_q) \times K_{k-(p+q)}) = 2^k -$ req(k, K^p_{p+q}).*

Erdős, Frankl and Füredi obtained the following, where containing I_k in every set of k-rows was equivalent to saying that you have a family of sets so that no set of the family is contained in the union of $k-1$ other sets. Here is one result.

Theorem 2.21. [EFF85]. Let m, k be given with $k \leqslant m < \binom{k+1}{2}$, then $\mathrm{req}(m, I_k) = m$.

Korner[Kör95] used $\mathrm{req}(m, I_3)$ (Problem C) and $\mathrm{req}(m, K_3)$ (problem D) in the context of Coding Theory.

Large Forbidden Configurations

The paper [AS16] considers a few problems with a forbidden configuration that grows with m. Main tools were Theorem 1.10 for the existence of designs and the multiplicity induction Lemma 6.

3 Main Conjecture for asymptotic bounds

Our investigations have led us to a conjecture on the asymptotic growth of $\text{forb}(m, F)$ for a fixed F as m goes to infinity. We had noted that all our results had $\text{forb}(m, F) = \Theta(m^e)$ for an integer e. Our conjecture involves the product construction (Definition 1.4). Let A_i be an $m_i \times n_i$ simple matrix for $1 \leqslant i \leqslant t$. The t-fold product $A = A_1 \times A_2 \times \cdots \times A_t$ is an $(\sum_{i=1}^t m_i) \times (\Pi_{i=1}^t n_i)$ simple matrix. Let I_h denote the $h \times h$ identity matrix and I_h^c denotes its (0,1)-complement. Let T_h denote the $h \times h$ *triangular* matrix

$$T_h = \begin{bmatrix} 1 & & & 1's \\ & 1 & & \\ & & \ddots & \\ 0's & & & 1 \end{bmatrix}.$$

The three matrices I, I^c, T are our proposed building blocks for product constructions. Note that if each A_i in the t-fold product above is of size $m/t \times m/t$ then the t-fold product has m rows and $\Theta(m^t)$ columns. Let F be a $k \times \ell$ (0,1)-matrix.

Definition 3.1. Let $X(F)$ be the smallest p so that F is a configuration in $A_1 \times A_2 \times \cdots \times A_p$ for every choice of A_i as either $I_{m/p}$, $I_{m/p}^c$ or $T_{m/p}$. Alternatively, assuming F is not a configuration in at least one of I, I^c, T, then $X(F) - 1$ is the largest choice of p so that F is not a configuration in $A_1 \times A_2 \times \cdots \times A_p$ for some choice of A_i as either $I_{m/p}$, $I_{m/p}^c$ or $T_{m/p}$.

We are assuming m is large and divisible by p, in particular that $m \geqslant (k+1)(k\ell+1)$ so that $m/p \geqslant k\ell + 1$. Divisibility by p does not affect the asymptotics since we can use a simple submatrix of a simple matrix that avoids F for construction purposes. We are also using the fact that we need only consider p-fold products for $p \leqslant k+1$, since F is a configuration in $\ell \cdot K_k$ and we can find $\ell \cdot K_k$ (and hence F) as a configuration in $A_1 \times A_2 \times \cdots \times A_{k+1}$ by taking one row from each of the first k products (each row has [01]) and then, since we are taking no rows from the final A_{k+1}, we get the configuration $(m/(k+1)) \cdot K_k$ in the product. Or we could appeal to Theorem 1.7 which has $\text{forb}(m, \ell \cdot K_k)$ being $O(m^k)$ and hence $\ell \cdot K_k$ must be in a $(k+1)$-fold product else this would yield ($\text{forb}(m, \ell \cdot K_k)$ is $\Omega(m^{k+1})$, a contradiction. If F is a configuration in the p-fold product $A_1 \times A_2 \times \cdots \times A_p$, assume that a_i rows of A_i are used with $\sum_{i=1}^p a_i = k$. If we form the submatrix of A_i of a_i rows, then we would be interested in at most ℓ copies of a given column on these rows (F has ℓ columns) if this is possible. Now for $t \geqslant k + \ell$, any a_i rows of K_t^1 contains ℓ columns of 0's as well as a copy of $K_{a_i}^1$. The analogous result is true for K_t^{t-1}. Also for $t \geqslant kl + l$, the a_i rows of T_t consisting of rows $\ell+1, 2\ell+1, 3\ell+1, \ldots, k\ell+1$ have ℓ columns of 0's and $\ell \cdot T_{a_i}$. Thus as long as $m \geqslant (k+1)(k\ell+1)$ we are able to use the matrices A_i as if they were arbitrarily large.

Conjecture 3.2. [AS05] We believe that

$$\text{forb}(m, F) = \Theta(m^{X(F)-1}). \blacksquare$$

Note that the definition of $X(F)$ ensures forb(m, F) is $\Omega(m^{X(F)-1})$, via the product construction, although for $X(F) = 1$ a little care must be taken. The use of the product construction for forbidden configurations is introduced in [AGS97] with non-trivial applications to Theorem 2.6 [AGS97] and Theorem 3.4 [AGS97] for cases with $k = 2$ and $k = 3$. The Conjecture 3.2 has been verified for $k = 2$ in Theorem 4.2, $k = 3$ in Theorem 5.1, $l = 2$ in Theorem 7.2, $k = 4$ and F simple in Theorem 6.1, and other cases. Moreover the Conjecture has motivated work such as in Conjecture 8.1.

It is important to note that the constant in front of the leading term $m^{X(F)-1}$ of forb(m, F) is not predicted by the Conjecture and so the Conjecture is little help with exact bounds. Also computing $X(F)$ is non-trivial (for large F).

Theorem 3.3. [Rag12] Computing $X(F)$ is NP-hard.

Perhaps the problem Partition into Cliques would be useful. We have yet to make a direct connection between our proofs of asymptotic bounds for forb(m, F) with the derivation of $X(F)$. We think of this problem as a configuration version of the Erdős-Stone-Simonovits Theorem [ES46] for the maximum number of edges in a graph avoiding some specified subgraph H where $\chi(H)$ is relevant.

Some consequences of the conjecture can be considered problems.

Problem 3.4. Let forb(m, F) be $\Theta(m^p)$. Is it true that forb$(m, t \cdot F)$ is $O(m^{p+1})$?
Let forb$(m, 2 \cdot F')$ be $\Theta(m^q)$. Is it true that forb$(m, t \cdot F')$ is $\Theta(m^q)$ for any $t \geqslant 2$?

4 F is a $1 \times \ell$ or $2 \times \ell$ (0,1)-matrix

For completeness we consider $1 \times \ell$ F (Theorem 5.1 and Corollary 5.2 from [AFS01]).

Theorem 4.1. Assume F is a $1 \times \ell$ (0,1)-matrix with p 1's and with $p \geqslant \ell - p \geqslant 0$ and let F' be the $1 \times p$ (0,1)-matrix with p 1's. Assume $m \geqslant p - 1 \geqslant 1$. Then

$$\text{forb}(m, F') = \text{forb}(m, F) = \lfloor \frac{pm}{2} \rfloor + 1. \blacksquare$$

For the case F is $2 \times \ell$, the asymptotic classification of forb(m, F) is completed in [AGS97]. We need some special matrices given below. Let us use the following notation for 2-rowed configurations (as opposed to notation for 2-columned configurations):

$$F_2(r, p, q, s) = \left[\begin{array}{cccc} \overbrace{00\cdots0}^{r} & \overbrace{11\cdots1}^{p} & \overbrace{00\cdots0}^{q} & \overbrace{11\cdots1}^{s} \\ 00\cdots0 & 00\cdots0 & 11\cdots1 & 11\cdots1 \end{array} \right].$$

Theorem 4.2. Let F be a $2 \times \ell$ (0,1)-matrix.
(Constant Cases) If $F = F_2(0, 1, 0, 0)$, then forb$(m, F) = \Theta(1)$.
(Linear Cases) If F has at least one configuration from K_2^0, K_2^1, K_2^2,

$F_2(0,2,0,0)$, and if F is a configuration in $F_2(1,t,1,1)$, $F_2(1,t,t,0)$, $(F_2(1,t,t,0))^c$ for some $t \geqslant 1$, then $\mathrm{forb}(m,F) = \Theta(m)$.
(Quadratic Cases) If F has at least one configuration from $2 \cdot K_2^0$, $[K_2^0|2 \cdot K_2^1|K_2^2]$, or $2 \cdot K_2^2$ then $\mathrm{forb}(m,F) = \Theta(m^2)$.
In addition, any $2 \times \ell$ (0,1)-matrix F will fall into one of the three Cases.

Proof: The linear bound for $\mathrm{forb}(m, F_2(1,t,1,1))$ is Theorem 2.2[AGS97]. The linear bound for $\mathrm{forb}(m, F_2(1,t,t,0))$ is Theorem 2.3[AGS97]. The quadratic construction for $[K_2^0|2 \cdot K_2^1|K_2^2]$ is Theorem 2.6[AGS97]. The quadratic bound in general for 2-rowed forbidden configurations follows from Theorem 1.7. All the lower bounds follow from the constructions given in Conjecture 3.2 but were developed in [AGS97]. For example a linear construction for $F_2(0,2,0,0)$ is I_m. ∎

A large number of exact or nearly exact bounds are available for 2-rowed F. Table 1 below considers cases $F_2(0,p,q,0)$.

Table 1.

configuration F	$\mathrm{forb}(m,F)$	reference
$\begin{bmatrix} \overbrace{0 \cdots 0}^{q} \\ 1 \cdots 1 \end{bmatrix}$	$\left\lfloor \frac{(q+1)m}{2} \right\rfloor + 2$, m large	Thm 4.6[AB]
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	2	[AFS01]
$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	$m+2$	[AFS01]
$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$2m+2$	[AFS01]
$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{5m}{2} \right\rfloor + 2$	[AK07]
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{3m}{2} \right\rfloor + 1$	[AGS97]
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{7m}{3} \right\rfloor + 1$	[AFS01]
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{11m}{4} \right\rfloor + 1$	[AK07]
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{15m}{4} \right\rfloor + 1$	[AK07]

configuration F	forb(m,F)	reference
$\begin{bmatrix}1&0&0&0&0&0&0\\0&1&1&1&1&1&1\end{bmatrix}$	$\lfloor\frac{21m}{4}\rfloor+1$	[Ede22]
$\begin{bmatrix}1&1&0&0&0\\0&0&1&1&1\end{bmatrix}$	$\lfloor\frac{8m}{3}\rfloor$	[AFS01]
$\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&1&1\end{bmatrix}$	$\lfloor\frac{10m}{3}-\frac{4}{3}\rfloor$	[AK07]
$\begin{bmatrix}1&1&0&0&0&0&0\\0&0&1&1&1&1&1\end{bmatrix}$	$4m$	[AK07]
$\begin{bmatrix}\overbrace{1\cdots1}^{p}\overbrace{0\cdots0}^{p}\\0\cdots0\,1\cdots1\end{bmatrix}$	$pm-p+2$	[AFS01]

An interesting case for which we do not know the exact bound is the following.

Theorem 4.3. [AK07], [AFS01] Let p,q be given with $p<q$. Then

$$(\frac{p+q}{2}+O(1))m \leqslant \text{forb}(m,\begin{bmatrix}\overbrace{1\cdots1}^{p}\overbrace{0\cdots0}^{q}\\0\cdots0\,1\cdots1\end{bmatrix}) \leqslant qm-q+2. \blacksquare$$

From Theorem 2.3 and Corollary 2.5 of [AFS01] we obtain:

Theorem 4.4.

$$\text{forb}(m,\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}) = \text{forb}(m,\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}) = \text{forb}(m,\begin{bmatrix}0&1&1&0&1\\0&0&0&1&1\end{bmatrix}) = \lfloor\frac{3m}{2}\rfloor+1. \blacksquare$$

From Theorem 2.6 and Corollary 2.7 of [AFS01] we obtain:

Theorem 4.5.

$$\text{forb}(m,\begin{bmatrix}0&1&1&1\\0&0&0&0\end{bmatrix}) = \text{forb}(m,\begin{bmatrix}0&1&1&1\\1&0&0&0\end{bmatrix}) = \text{forb}(m,\begin{bmatrix}0&1&1&1&0&1\\0&0&0&0&1&1\end{bmatrix}) = \left\lfloor\frac{7m}{3}\right\rfloor+1$$

We have the following exact bound (for large m) which is Theorem 1.3 in [AB]. A pigeonhole argument yields a bound that exceeds the bound below by a linear amount and for small m the larger pigeonhole bound can be achieved.

Theorem 4.6. [AB] Let $q\geqslant 3$ be given. Then for $m\geqslant\max\{5q-4,8q-18\}$,

$$\text{forb}(m,F_2(0,q,0,0)) = \overbrace{\begin{bmatrix}1&1&\cdots&1\\0&0&\cdots&0\end{bmatrix}}^{q}) = \lfloor\frac{q+1}{2}m\rfloor+2. \qquad \blacksquare \qquad (2)$$

Here is a table of bounds for 2-columned F with 1 or 2 rows (using notation for 2-columned configurations).

Configuration F	forb(m,F)	reference
$F_{1,0,0,0} = \begin{bmatrix}1 & 1\end{bmatrix}$	$\binom{m}{1}+\binom{m}{0}$	Thm 1.5
$F_{0,1,0,0} = \begin{bmatrix}1 & 0\end{bmatrix}$	$\binom{m}{0}$	Thm 1.5
$F_{2,0,0,0} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.5
$F_{1,1,0,0} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$	$\binom{m}{1}+\binom{m}{0}$	Thm 1.5
$F_{1,0,0,1} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$	$\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	[AFS01] or Thm 7.3
$F_{0,2,0,0} = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$	$\binom{m}{1}+\binom{m}{0}$	Thm 1.5
$F_{0,1,1,0} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\binom{m}{1}+\binom{m}{0}$	Thm 1.5

Theorem 4.7. (Thm 2.2 [AFS01]) Let $p \geqslant 1$ be given. Then forb$(m, F_2(1,p,p,0)) = pm - p + 2$. ■

Theorem 4.8. (Thm 2.3 [AFS01]) Let $p \geqslant 1$ be given. Then forb$(m, F_2(1,p,1,1)) \leqslant (p-\frac{1}{2})m + 1$. ■

Theorem 4.9. (Thm 3.4 [AFS01]) We have forb$(m, F_2(1,2,2,1)) = \lfloor \frac{m^2}{4} \rfloor + m + 1$. ■

Theorem 4.10. (Thm 3.3 [AFS01]) Let $p \geqslant 4$ be given. There exists an m_0 and a c so that for $m \geqslant m_0$ and $m \geqslant 4(p-1)^{3/2}$,

$$\frac{m^2}{4} + (p - 1\frac{1}{2} - \sqrt{p} - 1)m + O(p) \leqslant \text{forb}(m, F_2(1,p,p,1)) \leqslant \frac{m^2}{4} + (p-1)(m-2) + c. \quad ■$$

The following result follows from forbidding the $2 \times r$ block of 0's and so the bounds in [AF86] yield the result.

Theorem 4.11. (Thm 3.1 [AFS01]) Let r,p,q,s be given with $r \geqslant 2$, $r \geqslant p,q,s$. Then

$$\text{forb}(m, F_2(r,0,0,0)) = \text{forb}(m, F_2(r,p,q,s)) = \frac{r+1}{6}m^2 + O(m). \quad ■$$

The bounds do grow for larger p as the coefficient of m^2 increases from $\frac{r+1}{6}$ to $\frac{r-1}{2}$.

Theorem 4.12. (Thm 3.6 [AFS01]) Let r,p,q,s be given with $r,p,s \geqslant 2$ and $r \geqslant s$. Then

$$\text{forb}(m, F_2(r,p,p,s)) \leqslant \frac{r-1}{2}m^2 + O(m),$$

and for $r,s \geqslant 3$,

$$\lim_{p\longrightarrow\infty} \frac{\text{forb}(m, F_2(r,p,p,s))}{m^2} = \frac{r-1}{2}. \quad ■$$

The following (Theorem 3.5 [AFS01]) would be a useful (and somewhat surprising) tool in extending exact bounds.

Theorem 4.13. Let r, p, q, s be given with $2 \leqslant p < q$. If there exist a, b, c with forb$(m, F_2(r, p, p, s)) \leqslant am^2 + bm + c$ and $a, b > 0$, then there exists an m_0 (depending on r, p, q, s, a) so that for $m \geqslant m_0$ then forb$(m, F_2(r, p, q, s)) \leqslant am^2 + bm + c$.

5 $\boldsymbol{F}$ is a $\mathbf{3 \times \boldsymbol{\ell}}$ (0,1)-matrix

For the case F is $3 \times \ell$, the asymptotic classification of forb(m, F) is begun in [AGS97], [AFS01] and was completed in [AS05]. The following configurations are needed for Theorem 5.1. We will number them consecutively $F_1, F_2, \dots$ for this section etc. and will reuse the names applied to different matrices in Section 6 and Section 8.

$$F_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad F_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$F_4(t) = \begin{bmatrix} 0\,\overbrace{1\cdots1}^{t}\,\overbrace{0\cdots0}^{t}\,0\,\overbrace{1\cdots1}^{t}\,1\,\overbrace{0\cdots0}^{t}\,1 \\ 0\,0\cdots0\,1\cdots1\,0\,1\cdots1\,0\,1\cdots1\,1 \\ 0\,0\cdots0\,0\cdots0\,1\,0\cdots0\,1\,1\cdots1\,1 \end{bmatrix},$$

$$F_5(t) = \begin{bmatrix} 0\,\overbrace{1\cdots1}^{t}\,\overbrace{0\cdots0}^{t}\,0\,1\,\overbrace{1\cdots1}^{t}\,\overbrace{0\cdots0}^{t}\,1 \\ 0\,0\cdots0\,1\cdots1\,0\,1\,0\cdots0\,1\cdots1\,1 \\ 0\,0\cdots0\,0\cdots0\,1\,0\,1\cdots1\,1\cdots1\,1 \end{bmatrix},$$

$$F_6(t) = \begin{bmatrix} 0\,\overbrace{1\cdots1}^{t}\,\overbrace{0\cdots0}^{t}\,\overbrace{0\cdots0}^{t}\,\overbrace{1\cdots1}^{t}\,\overbrace{1\cdots1}^{t} \\ 0\,0\cdots0\,1\cdots1\,0\cdots0\,1\cdots1\,0\cdots0 \\ 0\,0\cdots0\,0\cdots0\,1\cdots1\,0\cdots0\,1\cdots1 \end{bmatrix}.$$

Theorem 5.1. Let F be a $3 \times \ell$ (0,1)-matrix.
(Linear Cases) If F has at least one column and if F is a configuration in F_2 then forb$(m, F) = \Theta(m)$.
(Quadratic Cases) If F has at least one configuration from K_3^0, K_3^1, K_3^2, K_3^3, $2 \cdot F_1$, $2 \cdot F_1^c$ or F_3 and if F is a configuration in $F_4(t)$, $F_5(t)$, $F_6(t)$ or $F_6(t)^c$ for some $t \geqslant 1$, then forb$(m, F) = \Theta(m^2)$.
(Cubic Cases) If F has at least one configuration from $2 \cdot K_3^0$, $[2 \cdot K_3^1 | K_3^2]$, $[2 \cdot K_3^1 | K_3^3]$, $[K_3^0 | 2 \cdot K_3^2]$, $[K_3^1 | 2 \cdot K_3^2]$ or $2 \cdot K_3^3$ then forb$(m, F) = \Theta(m^3)$.
In addition, any $3 \times \ell$ (0,1)-matrix F will fall into one of the three Cases.

Proof: The linear bound for forb(m, F_2) is Theorem 3.3[AGS97]. The quadratic bound for forb$(m, F_4(t))$ is Theorem 3.9[AGS97]. The quadratic bound for forb$(m, F_5(t))$ is Theorem 4.2 in [AS05] and the quadratic bound for forb$(m, F_6(t))$ is Theorem 4.1 in

[AS05]. The cubic bound for all 3-rowed F follows from Theorem 1.7 above. All the lower bounds follow from the constructions given in Conjecture 3.2 but had been developed as follows. Quadratic lower bounds for $\text{forb}(m, K_3^1), \text{forb}(m, K_3^2), \text{forb}(m, F_3)$ are in Corollary 3.5[AGS97], quadratic lower bound for $\text{forb}(m, K_3^3)$ (and hence $\text{forb}(m, K_3^0)$ by taking the 0-1-complement) is in Theorem 3.6[AGS97], quadratic lower bound for $\text{forb}(m, 2\cdot F_1)$ (and hence $\text{forb}(m, 2\cdot F_1^c)$) is in Theorem 3.7[AGS97]. A cubic lower bound for $\text{forb}(m, 2\cdot K_3^3)$ (and hence $\text{forb}(m, 2\cdot K_3^0)$) is in Theorem 3.9[AGS97] and cubic lower bounds for $\text{forb}(m, [2\cdot K_3^2|K_3^0])$ and $\text{forb}(m, [2\cdot K_3^2|K_3^1])$ (and hence also for $\text{forb}(m, [2\cdot K_3^1|K_3^3])$,$\text{forb}(m, [2\cdot K_3^1|K_3^2])$) are in Theorem 3.10[AGS97]. ■

There are a number of exact results.

Theorem 5.2. (Theorem 3.3 [AGS97]) $\text{forb}(m, F_2) = 2m$. ■

Theorem 5.3. $\text{forb}(m, F_3) = \lfloor m^2/4 \rfloor + m + 1$.

Proof: The construction of taking $[K^0_{m/2}\,|\,T_{m/2}] \times [K^0_{m/2}\,|\,T_{m/2}]$ is Theorem 3.4 [AGS97]. To prove the bound, one can use shifting (Section 12) and Theorem 12.1. The number of different columns of $A_{|S}$ on a given set S with $|S| = 3$ is at most 6 and so the same is true for the shifted matrix $T(A)$. But then since $T(A)$ is a downset, all columns in $T(A)$ have at most 2 1's and considering the columns of 2 1's as edges of a graph on a vertex set identified with the rows, we see that the graph has no triangles on any triple S (or $T(A)_{|S}$ would have 7 different columns). Thus by Mantel's bound (Turán) there are at most $\lfloor m^2/4 \rfloor$ columns of 2 1's and up to $m + 1$ additional columns of less than 2 1's. ■

$$\text{Let} \qquad F_7(k) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\\vdots\\1\end{matrix}}\right\} k-1 \\ \vphantom{1} \end{matrix}.$$

Note that $F_7(3)$ is in Table 2. The generality of this result for larger k costs nothing.

Theorem 5.4. [AK10] Let m be given.

$$\text{forb}(m, F_7(3)) = \text{forb}(3\cdot \mathbf{1}_2\mathbf{0}_0) \leqslant \frac{4}{3}\binom{m}{2} + \binom{m}{1} + \binom{m}{0},$$

with equality if $m \equiv 1, 3 (\text{mod } 6)$. Let k be given.

$$\text{then } \text{forb}(m, F_7(k)) = \text{forb}(m, 3\cdot \mathbf{1}_{k-1}) \leqslant \frac{k+1}{k}\binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}, \quad (3)$$

with equality if there exists a design on $[m]$ of blocks of size k such that for each subset $S \in \binom{[m]}{k-1}$, there is exactly one block of size k containing it. ■

We have the following exact bound (for large m) which is Theorem 1.5 in [AB]. A pigeonhole argument yields a bound that exceeds the bound below by a linear amount and for small m the larger pigeonhole bound can be achieved.

Theorem 5.5. [AB] Let $q > 2$ be given. There exists a constant M so that for $m > M$,

$$\text{forb}(m, q \cdot (\mathbf{1}_2\mathbf{0}_1) = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \end{matrix}}^{q} \\ \begin{matrix} 1 & 1 & \cdots & 1 \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} \end{bmatrix}) \leqslant m + 2 + \frac{q+1}{3}\binom{m}{2}, \tag{4}$$

with equality for $m \equiv 1, 3 (\text{mod } 6)$. ■

A number of exact results follow from the following result.

Theorem 5.6. [AK10] Let F be any one of the following three matrices:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then for $m \geqslant 3$, $\text{ext}(m, F) = \text{ext}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) = \text{ext}(m, \mathbf{1}_3\mathbf{0}_1)$. ■

The following two results were obtained with the assistance of a Genetic Algorithm. Here a genetic algorithm suggested both the bound and the structure of matrices that achieve the bound. Moreover it was used to help in the inductive steps by predicting the structures that would be encountered.

$$V = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Theorem 5.7. [AR11] Let $m \geqslant 2$. Then $\text{forb}(m, W) = \binom{m}{2} + 2m - 1$. ■

Theorem 5.8. [AR11] Let $m \geqslant 6$. Then $\text{forb}(m, V) = \binom{m}{2} + m + 4$. ■

3 × 2 Forbidden Configurations

Configuration F	forb(m, F)	reference
$F_{3,0,0,0} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	Thm 1.9
$F_{2,1,0,0} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	Thm 1.5
$F_{2,0,0,1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$	$\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}$	Thm 7.3
$F_{1,2,0,0} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$	$\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	Thm 1.5

Configuration F	forb(m,F)	reference
$F_{1,1,1,0} = \begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$2m$	Thm 3.3 in [AGS97]
$F_{1,1,0,1} = \begin{bmatrix}1&1\\1&0\\0&0\end{bmatrix}$	$\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	Thm 3.2 in [AGS97] (Thm 7.3)
$F_{0,3,0,0} = \begin{bmatrix}1&0\\1&0\\1&0\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.5
$F_{0,2,1,0} = \begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$	$\lfloor 3m/2\rfloor+1$	Thm 3.1 in [AGS97]

3 × 3 Forbidden Configurations

Configuration F	forb(m,F)	reference
$\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&0\\1&1&0\\1&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&0\\1&1&0\\1&0&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0\\1&0&0\\1&0&0\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.5
$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.8
$\begin{bmatrix}1&1&1\\1&0&0\\0&1&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0\\1&0&1\\0&1&0\end{bmatrix}$	$2m$	[AGS97]
$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	$\frac{5}{4}\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.11
$\begin{bmatrix}1&1&1\\1&1&1\\1&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1\\1&1&0\\1&1&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0\\1&1&0\\1&1&0\end{bmatrix}$	$\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.9

Configuration F	forb(m,F)	reference
$\begin{bmatrix}1&1&1\\1&1&1\\0&0&0\end{bmatrix}$	$\frac{4}{3}\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	Thm 5.5
$\begin{bmatrix}1&1&1\\1&1&1\\1&0&0\end{bmatrix}$	$\frac{4}{3}\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 5.4
$\begin{bmatrix}1&1&1\\1&0&0\\1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1\\1&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1\\1&1&0\\0&0&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	Thm 5.6

It is an exercise to verify that all 3×3 forbidden configurations (or their (0,1)-complements have been included in the table. We cannot complete the table for 3×4 matrices but perhaps it is instructive to see how many are solved by the general results. I've organized the cases by first considering the number of columns of 3 1's and then the number of columns $\mathbf{1}_2\mathbf{0}_1$.

3×4 Forbidden Configurations

Configuration F	forb(m,F)	reference
$\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\frac{6}{4}\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.11, $t=4$
$\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&0\end{bmatrix}$	$\frac{5}{4}\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.11, $t=3$
$\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&0&0&1\end{bmatrix}$, or $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&0&0&0\end{bmatrix}$	Exact bounds not known	

Configuration F	forb(m,F)	reference
$\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&0&0\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&1&0&0\end{bmatrix}$	$\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.9
$\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&0&0\\1&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&0&1&0\end{bmatrix}$, or $\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.5
$\begin{bmatrix}1&1&1&1\\1&1&1&1\\0&0&0&0\end{bmatrix}$	$\frac{5}{3}\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	Theorem 5.5

Configuration F	forb(m,F)	reference
$\begin{bmatrix}1&1&1&1\\1&1&1&0\\0&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\0&0&0&0\end{bmatrix}$ $\begin{bmatrix}1&1&1&0\\1&1&1&0\\0&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\0&0&0&0\end{bmatrix}$	Exact bounds not known	
$\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\0&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&1&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&0&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&0\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	Thm 5.6
$\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&1\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+m-2$	Thm 5.7
$\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+3$	Thm 5.8
$\begin{bmatrix}1&1&1&0\\1&1&0&1\\0&0&1&1\end{bmatrix}$	Exact bound not known	
$\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&0&0\end{bmatrix}$	$\binom{m}{2}+m+2$	Thm 6.11

Configuration F	forb(m,F)	reference
$\begin{bmatrix}1&1&0&1\\1&0&1&0\\0&1&1&0\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	Thm 1.8
$\begin{bmatrix}1&1&1&0\\1&0&0&1\\0&1&0&0\end{bmatrix}$	$2m$	Thm 5.2
$\begin{bmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\end{bmatrix}$	$\left\lfloor\frac{m^2}{4}\right\rfloor+\binom{m}{1}+\binom{m}{0}$	Thm 5.3

6 F is a $4\times\ell$ (0,1)-matrix

In this section we begin by considering F to be itself a simple matrix. For the case that F is simple and $4\times\ell$, the asymptotic classification of forb(m,F) was completed by Balin Fleming [AF10]. The main tools are Theorem 1.12 and Corollary 1.6.

We are able to establish the complete classification for the asymptotics of forb(m,F) for any $4\times\ell$ simple matrix F and the result is consistent with the conjecture. To state the result we need a number of matrices. We will number them consecutively $F_1,F_2,\ldots$ for this section etc. and will reuse the names for different matrices than those used in Section 5 and Section 8.

$$F_1=\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\quad F_2=\begin{bmatrix}1&1\\1&0\\0&1\\0&0\end{bmatrix},\quad F_3=\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\quad F_4=\begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix},\quad F_5=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{bmatrix},$$

$$F_6=\begin{bmatrix}1&0&1&1&1&0&1&1\\0&1&1&0&1&1&0&0\\0&0&1&1&0&1&0&1\\0&0&0&1&0&0&1&0\end{bmatrix},F_7=\begin{bmatrix}1&0&1&1&1&1&0&1\\0&1&1&1&1&0&1&0\\0&0&1&0&0&0&1&1\\0&0&0&1&0&1&0&0\end{bmatrix},$$

$$F_8=\begin{bmatrix}1&0&1&0&0&1\\0&1&0&1&1&0\\0&0&1&1&1&1\\0&0&1&1&0&0\end{bmatrix},$$

$$F_9=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&1\end{bmatrix},F_{10}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix},$$

$$F_{11} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, F_{12} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, F_{13} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Theorem 6.1. Let F be a $4 \times \ell$ simple matrix.
(Linear Cases) If F has a configuration F_1 and if F is a configuration in F_2 then $\text{forb}(m, F) = \Theta(m)$.
(Quadratic Cases) If F has a configuration F_3, F_3^c, F_4, F_5, or F_5^c and if F is a configuration in F_6, F_7, or F_8, then $\text{forb}(m, F) = \Theta(m^2)$.
(Cubic Cases) If F has a configuration K_4^0, F_9, F_9^c, F_{10}, F_{10}^c, F_{11}, F_{12}, F_{12}^c, F_{13}, or K_4^4 then $\text{forb}(m, F) = \Theta(m^3)$.
In addition, any $4 \times \ell$ simple matrix F will fall into one of the three Cases.

Proof: The lower bounds are established by constructions of Conjecture 3.2. For definiteness, note that $F_1 \notin I$, $F_3 \notin I \times I$, $F_4 \notin T \times I$, $F_5 \notin I^c \times I^c$, $K_4^0, F_9, F_{10} \notin I^c \times I^c \times I^c$, $F_{11}, F_{12}, F_{13} \notin T \times T \times T$. The arguments are not entirely trivial. We see that any two rows of I^c do not have $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and so a k-rowed matrix which has $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ on every pair of rows is not a configuration in the $k-1$-fold product $I^c \times I^c \times \cdots \times I^c$. Similarly, any two rows of T do not have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and so a k-rowed matrix which has $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on every pair of rows is not a configuration in the $(k-1)$-fold product $T \times T \times \cdots \times T$. This was noted following Corollary 3.5 in [AGS97]. Theorem 7.2, Theorem 1.12 and Theorem 1.5 establish the upper bounds.

To show that any $4 \times \ell$ matrix F is included in one of the three categories, assume that F is a matrix that falls into neither the linear case or the cubic case. For convenience, think of a column of column sum 2 as an edge (i, j) if the column has 1's in rows i, j. A matrix F falls into the linear case only if $F = F_1$ or $F = F_2$. Examining the configurations K_4^0, F_9, F_9^c, F_{10}, F_{10}^c, F_{11}, F_{11}, F_{12}, F_{12}^c, F_{13}, F_{13}^c or K_4^4, we deduce that F cannot have a column of all 0's (K_4^0)or a column of all 1's (K_4^4). F has at most two columns of column sum 1 and at most two columns of column sum 3 (using F_{10}, F_{10}^c). In addition four edges forming a four cycle yields F_{11} and so there are at most 4 edges in F which must be a subgraph of a triangle plus one edge from the triangle to the remaining vertex. (From this and Corollary 1.6 it follows that any 4-rowed configuration with a quadratic bound has at most 8 column types).

If F has no columns of either three 1's or three 0's then, assuming it is not F_1 or F_2, it must contain two disjoint edges and hence F_4 or have three columns of column sum 2 forming a triangle (F_5^c) or three columns of column sum 2 sharing a vertex (F_5). ■

For the general case F is $4 \times \ell$, the asymptotic classification of $\text{forb}(m, F)$ is not complete but we can use the conjecture to predict the answer. The following configurations are needed for Conjecture 6.3:

$$F_6(t) = \left[\begin{matrix}1&0&1&1\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{matrix}\; t\cdot\begin{bmatrix}1&0&1&1\\1&1&0&0\\0&1&0&1\\0&0&1&0\end{bmatrix}\right],$$

$$F_7(t) = \left[\begin{matrix}1&0&1&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{matrix}\; t\cdot\begin{bmatrix}1&1&0&1\\1&0&1&0\\0&0&1&1\\0&1&0&0\end{bmatrix}\right], F_8(t) = \left[\begin{matrix}1&0&1&0\\0&1&0&1\\0&0&1&1\\0&0&1&1\end{matrix}\; t\cdot\begin{bmatrix}0&1\\1&0\\1&1\\0&0\end{bmatrix}\right],$$

$$B_1 = \begin{bmatrix}0&0&0&0&1\\0&0&0&1&1\\0&0&1&1&1\\0&1&1&1&1\end{bmatrix}, B_2 = \begin{bmatrix}0&0&0&1&1\\0&0&0&1&1\\0&0&1&1&1\\0&1&1&0&1\end{bmatrix}, B_3 = \begin{bmatrix}0&0&0&1&1\\0&0&0&1&1\\0&0&1&0&1\\0&1&1&1&1\end{bmatrix},$$

$$B_4 = \begin{bmatrix}0&0&0&0&1\\0&0&1&1&1\\0&0&1&1&1\\0&1&0&1&1\end{bmatrix}, B_5 = \begin{bmatrix}0&0&0&1&1\\0&0&1&1&1\\0&0&1&0&1\\0&1&0&1&1\end{bmatrix}, B_6 = \begin{bmatrix}0&0&0&1&1\\0&0&1&1&1\\0&0&1&1&1\\0&1&0&0&1\end{bmatrix},$$

$$D_{12} = \begin{bmatrix}0&0&0&0&0&0&0&1&1&1&1\\0&0&0&1&1&1&1&0&0&0&0\\1&0&1&0&1&0&1&0&1&0&1\\0&1&1&0&0&1&1&0&0&1&1\end{bmatrix}.$$

Theorem 6.2. [ARS12] Let t be given. Then $\text{forb}(m, F_8(t))$ is $\Theta(m^2)$. ■

Conjecture 6.3. Let F be a $4 \times \ell$ (0,1)-matrix.
(Linear Cases) If F has F_1 as a configuration and if F is a configuration in F_2 then $\text{forb}(m, F) = \Theta(m)$.
(Quadratic Cases) If F has at least one configuration from F_3, F_3^c, F_4, F_5, F_5^c or $2 \cdot F_1$, and if F is a configuration in $F_6(t)$, $F_7(t)$ or $F_8(t)$ for some t , then $\text{forb}(m, F) = \Theta(m^2)$.
(Cubic Cases) If F has at least one configuration from K_4^0, $2 \cdot F_3$, F_9, F_9^c, F_{10}, F_{10}^c, F_{11}, F_{12}, F_{12}^c, F_{13}, $2 \cdot F_3^c$ or K_4^4 and if F is a configuration in $[K_4 \,|\, t \cdot [K_4 - B_i]]$ or $[K_4 \,|\, t \cdot [K_4 - B_i]]^c$ for $i = 1, 2, \ldots 6$ or $[K_4^0 \,|\, t \cdot D_{12}]$ then $\text{forb}(m, F) = \Theta(m^3)$.
(Quartic Cases) If F has at least one configuration from $2 \cdot K_4^0$, $[2 \cdot K_4^2]$, $[2 \cdot K_4^4]$ or $[2 \cdot K_4^1 | C]$ or $[2 \cdot K_4^1 | C]^c$ where C is one of K_4^2, F_{12}, F_9^c, F_{10}^c, K_4^4, then $\text{forb}(m, F) = \Theta(m^4)$.
In addition, any $4 \times \ell$ (0,1)-matrix F will fall into one of the four cases.

The boundary between linear and quadratic follows easily from Theorem 7.1 and Theorem 7.2. The boundary between quadratic and cubic is partly proven. The cubic lower bounds are from the constructions. The boundary between cubic and quartic is in Theorem 1.13 and Theorem 1.14. The quartic bound of Theorem 1.7 completes the cases. We would need to establish $\text{forb}(m, F_6(t))$ and $\text{forb}(m, F_7(t))$ are both quadratic to prove this conjecture, hence Problem 1.19.

The Conjecture 3.2 predicts The following is consistent with the Conjecture 3.2 and might be a helpful first step. The case $t = 2$ was proven in [Ans90b].

Theorem 6.4. [AL13] $\mathrm{forb}(m, t \cdot F_4)$ is $\Theta(m^2)$ for $t \geqslant 2$. ■

The multiplicity induction was used and the argument idea from [Ans90b] to obtain a generalization:

Theorem 6.5. [AL13] $\mathrm{forb}(m, t \cdot [\mathbf{1}_k\mathbf{0}_k \mid \mathbf{0}_k\mathbf{1}_k])$ is $\Theta(m^k)$ for $t \geqslant 2$. ■

An exact bound for $F_4 = F_{0,2,2,0}$ follows from a result of Frankl, Füredi, Pach [FFP87] who asked the following problem (which can be viewed as a forbidden submatrix problem).

Theorem 6.6. [FFP87] Let $f(n, k)$ denote the length of the longest sequence $\{S_1, S_2, \ldots\}$ of distinct subsets of $[m]$ such that $|S_i \backslash S_j| < k$ for all $i < j$. Then $f(m, 2) = \binom{m}{2} + 2m - 1$ and $f(m, k) < \binom{m}{k} + 5k^2\binom{m}{k-1}$.

Without loss of generality we may assume the sequence of sets is in non-decreasing order by cardinality. Now consider any simple m-rowed matrix which has no configuration $F_{0,k,k,0}$. If we reorder the columns so that columns sums are never decreasing from left to right then the resulting matrix has no submatrix $F_{0,k,0,0} = [\mathbf{1}_k \mid \mathbf{0}_k]$ and so interpreting the columns as subsets of $[m]$, we identify a sequence with the desired property. Moreover, interpreting the sequence as a simple matrix, the resulting matrix has no *submatrix* $F_{0,k,0,0} = [\mathbf{1}_k \mid \mathbf{0}_k]$ and hence no configuration $F_{0,k,k,0}$.

Corollary 6.7. [FFP87] We have $\mathrm{forb}(m, F_4) = \binom{m}{2} + m - 2$. ■

This can also be viewed as a variation of a result of Kleitman [Kle66]. In that result the condition was that pairs of sets B, C have $|B \backslash C| + |C \backslash B| \leqslant 2t$. The condition of forbidding F_4 is slightly weaker than the condition for $t = 1$ and so the bound for the result below is slightly larger than Kleitman's bound. The matrices in $\mathrm{ext}(m, F)$ are determined. This is modest progress for Problem 15.2.

We have the following exact bound (for large m) which is Theorem 1.6 in [AB]. A pigeonhole argument yields a bound that exceeds the bound below by a linear amount. For small m the larger pigeonhole bound can be achieved.

Theorem 6.8. Let $q > 2$ be given. There exists a constant M so that for $m > M$,

$$\mathrm{forb}(m, q \cdot F_1 = \begin{bmatrix} \overbrace{1 \;\; 1 \;\; \cdots \;\; 1}^{q} \\ 1 \;\; 1 \;\; \cdots \;\; 1 \\ 0 \;\; 0 \;\; \cdots \;\; 0 \\ 0 \;\; 0 \;\; \cdots \;\; 0 \end{bmatrix}) \leqslant 2 + 2m + \frac{q+3}{3}\binom{m}{2}, \tag{5}$$

with equality if in addition $m \equiv 1, 3 (\mathrm{mod}\ 6)$. Moreover, for $m > M$, if the bound is achieved by an m-rowed matrix A, then A has all columns of sum 0, 1, 2, $m-2$, $m-1$, m and for some integers a, b with $a + b = q - 3$, the columns of sum 3 correspond to a simple 2-$(m, 3, a)$-design and the columns of sum $m - 3$ correspond to the (0,1)-complement of a simple 2-$(m, 3, b)$-design and there are no other columns. ■

Using the notation $\mathbf{1}_p\mathbf{0}_q$ for the column of p 1's on top of q 0's we have obtained the result for cases with $p > q$.

Theorem 6.9. [ABP] Assume $p > q$.

$$\text{forb}(m, t \cdot \mathbf{1}_p\mathbf{0}_q) \leqslant \sum_{i=0}^{p-1} \binom{m}{i} + \left(1 + \frac{t-2}{p+1}\right)\binom{m}{p} + \sum_{i=m-q+1}^{m} \binom{m}{i},$$

with equality for the matrix

$$[K_m^0 K_m^1 \cdots K_m^p B K_m^{m-q+1} K_m^{m-q+2} \cdots K_m^m],$$

where B is the m-rowed incidence matrix of a p-$(m, p+1, t-2)$-design when it exists.

It is expected that this result might work when $p = q$. The existence of p-$(m, p+1, t-2)$ designs follows from Theorem 1.10. A more general result for $t \cdot (\mathbf{1}_p\mathbf{0}_1)$, and so involving p-designs, is proven by Niranjan Balachandran [Bal12]. The spirit of Theorem 6.9 is that the $t \times p$ matrix of 1's is the critical part of the forbidden configuration. It also indicates that the usual definition of a design as a certain number of columns of sum $p+1$ having no $t+1 \times p$ matrix of 1's can be improved to having no configuration of $(p+q) \times t$ matrix with p rows of 1's and q rows of 1's.

4 × 2 Forbidden Configurations

Configuration F	forb(m, F)	reference
$F_{4,0,0,0} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\binom{m}{4} + \binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	Thm 1.5
$F_{3,1,0,0} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	Thm 1.5
$F_{3,0,0,1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$	$\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}$	Thm 7.3
$F_{2,2,0,0} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$	$\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	Thm 1.5
$F_{2,1,0,1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}$	Thm 7.3

Configuration F	$\text{forb}(m,F)$	reference
$F_{2,1,1,0} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\geqslant (\frac{29}{21})\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ $\leqslant 2\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	[ABS11]
$F_{2,0,0,2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m-1} + \binom{m}{m}$	Thm 7.3
$F_{1,3,0,0} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$	$\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	Thm 1.5
$F_{1,2,1,0} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}$	[ABS11]
$F_{1,2,0,1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}$	[ABS11]
$F_{1,1,1,1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$	$4m - 4$	[ABS11]
$F_{0,4,0,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$	$\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$	Thm 1.5
$F_{0,3,1,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}$	[ABS11]
$F_{0,2,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\binom{m}{2} + 2m - 1$	Thm 6.6[FFP87]

The following suggests that an exact bound for $F_{2,1,1,0}$ would be difficult to obtain. In a similar way one expects that determining an exact bound for $F_{a,1,1,0}$ would be difficult.

Theorem 6.10. (Thm 9.1 [AK10]) Let m be a given integer and let c be a positive real number. Let A be an $m \times \left(c\binom{m}{2} + m + 2\right)$ simple matrix with no $F_{2,1,1,0}$. Then for some $M > m$, there is an $M \times \left((c + \frac{2}{m(m-1)})\binom{M}{2} + M + 2\right)$ simple matrix with no $F_{2,1,1,0}$. ■

Results of Peter Dukes [Duk15] give a fairly tight estimate on the coefficient of m^2 in the bound for forb$(m, F_{2,1,1,0})$. The following are in [AK10]. Theorem 6.12 would yield many exact bounds using Remark 2.

Theorem 6.11. Let $F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then for $m \geqslant 3$, $\text{forb}(m, F) = \binom{m}{2} + m + 2$. ■

Theorem 6.12. Let $m \geqslant 4$. Let F be one of the following three matrices.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then $\text{forb}(m, F) = \text{forb}(m, 2 \cdot \mathbf{1}_3\mathbf{0}_1) = \text{forb}(m, \mathbf{1}_4\mathbf{0}_1) = \binom{m}{3} + \binom{m}{2} + m + 2$. ■

Let

$$F_{11} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The following result (Thm 4.1 [AK10]) considers a k-rowed F with 2^{k-4} pairs of repeated columns. Note the difference and connection to Theorem 1.18. For a number of cases including $k = 4$, this result is generalized by the result in [AM11].

Theorem 6.13. [AK10] For $m \geqslant 5$, $\text{forb}(m, F_{11}) = \text{forb}(m, \mathbf{1}_4) = \text{forb}(m, K_4)$. Assume $k \geqslant 5$ and $m \geqslant k + 1$. Then

$$\text{forb}(m, K_{k-4} \times F_{11}) = \sum_{i=0}^{k-1} \binom{m}{i} = \text{forb}(m, \mathbf{1}_k) = \text{forb}(m, K_k). \tag{6}$$

Proof: The proof in the paper [AK10] is a little light on details for $k \geqslant 5$. We need the base case $\|A\| \leqslant \text{forb}(k + 1, K_k)$. Let $A \in \text{Avoid}(k + 1, K_{k-4} \times F_{11})$. Using the standard induction we have $\|A\| = \|[B(r)C(r)D(r)]\| + \|C(r)\|$ where $[B(r)C(r)D(r)]$ has no $K_{k-4} \times F_{11}$ and $C(r)$ has no $K_{k-3} \times F_{11}$. Now by induction $\|C(r)\| \leqslant \text{forb}(k, K_{k-1})$ and $\|[B(r)C(r)D(r)]\| \leqslant 2^k$. If $\|[B(r)C(r)D(r)]\| \leqslant 2^k - 1$, then we obtain $\|A\| \leqslant \text{forb}(k+1, K_k)$ establishing the base case. If not, then $[B(r)C(r)D(r)]$ contains all possible columns. Now this is true for every choice of r and so we deduce that $A \in \text{Avoid}(k + 1, (\mathbf{1}_2\mathbf{0}_2) \times K_{k-4})$ from which we find $\|A\| \leqslant \text{forb}(k + 1, K_k)$. ■

7 F is a $k \times 1$ or $k \times 2$ (0,1)-matrix.

Recall the definitions of the $(a+b)\times 1$ vector $\mathbf{1}_a\mathbf{0}_b$ and allow me to extend it to $\mathbf{1}_a\mathbf{0}_b\mathbf{1}_c\mathbf{0}_d$ to refer to the $(a+b+c+d)\times 1$ columns obtained as $\mathbf{1}_a\mathbf{0}_b$ placed on top of $\mathbf{1}_c\mathbf{0}_d$. We recall the $(a+b+c+d)\times 2$ matrix $F_{a,b,c,d} = [\mathbf{1}_{a+b}\mathbf{0}_{c+d}|\mathbf{1}_a\mathbf{0}_b\mathbf{1}_c\mathbf{0}_d]$ from Section 1. We first consider $k \times 1$ F.

Theorem 7.1. Let s, k be given positive integers with $s \leqslant k$. Then

$$\text{forb}(m, \mathbf{1}_s\mathbf{0}_{k-s}) = \sum_{i=0}^{s-1}\binom{m}{i} + \sum_{i=0}^{k-s-1}\binom{m}{i}. \ \blacksquare$$

For the case F is $k \times 2$, the asymptotic classification of $\text{forb}(m, F)$ is in [AK06]. By interchanging columns we see that $\text{forb}(m, F_{a,b,c,d}) = \text{forb}(m, F_{a,c,b,d})$, and by considering (0,1)-complements we see that $\text{forb}(m, F_{a,b,c,d}) = \text{forb}(m, F_{d,c,b,a})$. Therefore we may assume that $a \geqslant d$ and $b \geqslant c$. Our result for the function $\text{forb}(m, F_{a,b,c,d})$ is the following.

Theorem 7.2. [AK06] Suppose $a \geqslant d$ and $b \geqslant c$. Then $\text{forb}(m, F_{a,b,c,d})$ is $\Theta(m^{a+b-1})$ if either $b > c$ or $a, b \geqslant 1$. Also $\text{forb}(m, F_{a,0,0,d})$ is $\Theta(m^a)$ and $\text{forb}(m, F_{0,b,b,0})$ is $\Theta(m^b)$. $\blacksquare$

We should note that we have a sharper bound for $F_{0,b,b,0}$ from Theorem 6.6 [FFP87]. We prove Theorem 7.2 using the strong stability result Theorem 15.1 and induction such as Lemma 7. A number of exact results for $k \times 2$ F have been obtained.

Theorem 7.3. [ABS11]Assume a, d, m are given integers with $a \geqslant d$ and $m \geqslant a + d$, then

$$\text{forb}(m, 2\cdot\mathbf{1}_a\mathbf{0}_d) = \text{forb}(m, F_{a,0,0,d}) = \text{forb}(m, F_{a,1,0,d}) = \sum_{j=0}^{a}\binom{m}{j} + \sum_{j=m-d+1}^{m}\binom{m}{j}. \ \blacksquare$$

Theorem 7.4. [AK10] Let m, a, b be given integers. For $m \geqslant 1$, $a \geqslant 2$ and $b \geqslant 2$,

$$\text{forb}(m, F_{a,b,0,1}) = \text{forb}(m, F_{a,b,1,0}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1)$$

$$\text{and} \quad \text{forb}(m, F_{a,b,1,1}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_2).$$

Also for $a \geqslant 2$,

$$\text{forb}(m, F_{a,1,0,1}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1),$$

and for $b \geqslant 2$,

$$\text{forb}(m, F_{1,b,1,0}) = forb(m, \mathbf{1}_{1+b}\mathbf{0}_1),$$

$$\text{forb}(m, F_{1,b,1,1}) = \text{forb}(m, \mathbf{1}_{1+b}\mathbf{0}_2).$$

Also for $b \geqslant 3$ [ABS11],

$$\text{forb}(m, F_{0,b,1,0}) = \text{forb}(m, \mathbf{1}_b\mathbf{0}_1). \ \blacksquare$$

Problem 7.5. Assume we are given positive integers a, b, c, d with $a \geqslant d$ and $b \geqslant c$. Find some mild conditions on a, b, c, d so that $\text{forb}(m, F_{a,b,c,d}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$. $\blacksquare$

8 $\boldsymbol{F}$ is a simple $5 \times \boldsymbol{\ell}$matrix

For the case that F is a $5 \times \ell$ simple matrix, we can use Conjecture 3.2 to predict the results. The non-trivial calculations to achieve this are in [AR]. Some of the asymptotic bounds have proofs. The numbered matrices are given after the conjecture. We will number them consecutively $F_1, F_2, \ldots$ for this section etc. and will reuse the names for different matrices than those used in Section 5 and Section 6. Theorem 1.12 establishes the cubic bounds. The quadratic bounds for $F_3, F_4, \ldots F_{11}$ are an open problems except for F_7.

Conjecture 8.1. Let F be a $5 \times \ell$ simple matrix.
(Quadratic Cases) If F has a configuration of F_1 or F_2 and if F is a configuration in $F_3, F_4, \ldots, F_{11}$ then $\text{forb}(m, F)$ is $\Theta(m^2)$.
(Cubic Cases) If F has a configuration of one of $F_{12}, F_{13}, \ldots, F_{24}$ and if F is a configuration in $F_{25}, F_{26}, \ldots, F_{29}$ then $\text{forb}(m, F)$ is $\Theta(m^3)$.
(Quartic Cases) If F has a configuration one of $F_{30}, F_{31}, \ldots, F_{87}$ then $\text{forb}(m, F)$ is $\Theta(m^4)$.
In addition, any $5 \times \ell$ simple matrix F will fall into one of these three cases.

Minimal quadratics:

$$F_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Maximal quadratics (by Conjecture 3.2):

$$F_3 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_5 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$F_6 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad F_8 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$F_9 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad F_{10} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_{11} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Minimal Cubics (cubic constructions exist but minimal by Conjecture 3.2):

$$F_{12} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad F_{13} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad F_{14} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad F_{15} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_{16} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad F_{17} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad F_{18} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$F_{19} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad F_{20} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad F_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$F_{22} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad F_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad F_{24} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Maximal Cubics by Theorem 1.12:

$$F_{25} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{26} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$F_{27} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{28} = \begin{bmatrix} 1&0&1&1&1&1&0&0&1&1&1&1&1&0&1&1 \\ 0&1&1&0&0&0&1&1&1&1&0&0&0&1&1&0 \\ 0&0&0&1&0&0&1&0&1&0&1&1&0&1&1&1 \\ 0&0&0&0&1&0&0&1&0&1&1&0&1&1&1&1 \\ 0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&1 \end{bmatrix}$$

$$F_{29} = \begin{bmatrix} 1&0&1&1&0&0&1&1&0&0&1&0 \\ 0&1&0&0&1&1&0&0&1&1&0&1 \\ 0&0&1&0&1&0&1&1&1&1&1&1 \\ 0&0&0&1&0&1&1&0&1&0&1&1 \\ 0&0&0&0&0&0&0&1&0&1&1&1 \end{bmatrix}$$

Minimal quartics by Theorem 1.12:

$$F_{30} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad F_{31} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_{32} = \begin{bmatrix} 1&1&0 \\ 1&0&1 \\ 0&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix} \quad F_{33} = \begin{bmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix} \quad F_{34} = \begin{bmatrix} 0&1&1 \\ 0&1&1 \\ 1&1&0 \\ 1&0&1 \\ 1&1&1 \end{bmatrix}$$

$$F_{35} = \begin{bmatrix} 1&0&0 \\ 1&0&0 \\ 0&0&1 \\ 0&1&0 \\ 0&0&0 \end{bmatrix} \quad F_{36} = \begin{bmatrix} 1&0&0&1 \\ 0&1&0&1 \\ 0&0&1&1 \\ 1&1&1&0 \\ 1&1&1&1 \end{bmatrix} \quad F_{37} = \begin{bmatrix} 0&1&1&0 \\ 1&0&1&0 \\ 1&1&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$$

$$F_{38} = \begin{bmatrix} 1&0&1 \\ 0&1&1 \\ 0&1&1 \\ 1&0&1 \\ 1&1&0 \end{bmatrix} \quad F_{39} = \begin{bmatrix} 0&1&0 \\ 1&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \quad F_{40} = \begin{bmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 1&1&0&1&1&0 \\ 1&0&1&1&0&1 \\ 0&1&1&0&1&1 \end{bmatrix}$$

$$F_{41} = \begin{bmatrix} 0&0&0&1&1&1 \\ 1&1&1&0&0&0 \\ 0&0&1&0&0&1 \\ 0&1&0&0&1&0 \\ 1&0&0&1&0&0 \end{bmatrix} \quad F_{42} = \begin{bmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&1&0 \\ 0&1&0&1&0&1 \\ 0&0&1&0&1&1 \\ 1&1&1&1&1&1 \end{bmatrix} \quad F_{43} = \begin{bmatrix} 0&0&0&1&1&1 \\ 0&1&1&0&0&1 \\ 1&0&1&0&1&0 \\ 1&1&0&1&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

$$F_{44} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad F_{45} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F_{46} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$F_{47} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad F_{48} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad F_{49} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$F_{50} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad F_{51} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F_{52} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{53} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad F_{54} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad F_{55} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$F_{56} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad F_{57} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad F_{58} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$F_{59} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F_{60} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad F_{61} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$F_{62} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad F_{63} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad F_{64} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$F_{65} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad F_{66} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad F_{67} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$F_{68} = \begin{bmatrix} 0&0&1&1&1&0\\ 0&0&1&1&1&1\\ 1&0&1&0&0&1\\ 0&1&0&1&0&1\\ 1&1&0&0&1&0 \end{bmatrix} \quad F_{69} = \begin{bmatrix} 1&1&0&0&0&1\\ 1&1&0&0&0&0\\ 0&1&0&1&1&0\\ 1&0&1&0&1&0\\ 0&0&1&1&0&1 \end{bmatrix} \quad F_{70} = \begin{bmatrix} 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 1&1&0&1&0&0\\ 1&0&1&0&1&0\\ 0&1&1&0&0&1 \end{bmatrix}$$

$$F_{71} = \begin{bmatrix} 0&0&0&1&1&1\\ 1&0&0&1&0&0\\ 0&1&0&0&1&1\\ 0&0&1&1&1&0\\ 1&1&1&0&0&1 \end{bmatrix} \quad F_{72} = \begin{bmatrix} 1&1&1&0&0&0\\ 0&1&1&0&1&1\\ 1&0&1&1&0&0\\ 1&1&0&0&0&1\\ 0&0&0&1&1&0 \end{bmatrix} \quad F_{73} = \begin{bmatrix} 0&0&0&1&1&1\\ 1&0&0&1&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&1\\ 1&1&1&0&0&0 \end{bmatrix}$$

$$F_{74} = \begin{bmatrix} 0&0&1&1\\ 1&0&1&0\\ 0&1&0&1\\ 0&1&1&0\\ 1&0&0&1 \end{bmatrix} \quad F_{75} = \begin{bmatrix} 0&0&0&1&1&1\\ 1&0&0&1&0&1\\ 0&1&0&1&1&0\\ 0&1&1&0&1&1\\ 1&0&1&0&0&1 \end{bmatrix} \quad F_{76} = \begin{bmatrix} 1&1&1&0&0&0\\ 0&1&1&0&1&0\\ 1&0&1&0&0&1\\ 1&0&0&1&0&0\\ 0&1&0&1&1&0 \end{bmatrix}$$

$$F_{77} = \begin{bmatrix} 0&0&0&1&1&1\\ 1&0&0&1&1&0\\ 0&1&0&1&0&1\\ 0&1&1&0&0&1\\ 1&0&1&0&1&0 \end{bmatrix} \quad F_{78} = \begin{bmatrix} 1&0&0&0&1\\ 0&1&0&0&1\\ 0&0&1&0&1\\ 0&0&0&1&1\\ 1&1&1&1&0 \end{bmatrix} \quad F_{79} = \begin{bmatrix} 0&1&1&1&0\\ 1&0&1&1&0\\ 1&1&0&1&0\\ 1&1&1&0&0\\ 0&0&0&0&1 \end{bmatrix}$$

$$F_{80} = \begin{bmatrix} 1&0&0&1&1\\ 0&1&0&1&1\\ 0&0&1&0&1\\ 0&0&1&1&0\\ 1&1&0&0&1 \end{bmatrix} \quad F_{81} = \begin{bmatrix} 0&1&1&0&0\\ 1&0&1&0&0\\ 1&1&0&1&0\\ 1&1&0&0&1\\ 0&0&1&1&0 \end{bmatrix} \quad F_{82} = \begin{bmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&1&1&1\\ 0&1&1&0&1&0\\ 1&0&0&1&0&1 \end{bmatrix}$$

$$F_{83} = \begin{bmatrix} 0&1&1&1&0&0\\ 1&0&1&1&0&0\\ 1&1&0&0&0&0\\ 1&0&0&1&0&1\\ 0&1&1&0&1&0 \end{bmatrix} \quad F_{84} = \begin{bmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&1&0&1\\ 0&1&1&0&0&1\\ 1&0&0&1&1&0 \end{bmatrix} \quad F_{85} = \begin{bmatrix} 0&1&1&1&0&0\\ 1&0&1&1&0&0\\ 1&1&0&0&1&0\\ 1&0&0&1&1&0\\ 0&1&1&0&0&1 \end{bmatrix}$$

$$F_{86} = \begin{bmatrix} 1&0&0&0&0&1\\ 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{bmatrix} \quad F_{87} = \begin{bmatrix} 0&1&1&1&1&0\\ 1&0&0&1&1&0\\ 1&1&1&0&0&0\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1 \end{bmatrix} \quad \blacksquare$$

We would need to prove quadratic bounds for the 9 matrices $F_3, F_4, \ldots, F_{11}$ in order to complete the classification of the 5-rowed simple configurations. We have three results (a detailed inductive proof worked for each).

Theorem 8.2. [ARS11][Tan18] We have that $\text{forb}(m, F_6)$, $\text{forb}(m, F_7)$, $\text{forb}(m, F_{10})$ are all $\Theta(m^2)$. ■

9 Boundary Cases

This section considers some larger configurations where the conjecture suggests that they are at a boundary.

Consider those simple F for which $\text{forb}(m, F)$ is $\Omega(m^{k-1})$ (as large as possible) and those F for which $\text{forb}(m, F)$ is $O(m^{k-2})$. Conjecture 3.2 predicts which k-rowed F will have $\text{forb}(m, F)$ being $\Theta(m^{k-2})$. For the case of simple matrices we may use Theorem 1.12 directly and obtain the following 6 cases:

$$F_{1,k} = \begin{array}{c}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ \times \\ K_{k-3}\end{array}, \quad F_{2,k} = \begin{array}{c}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \times \\ K_{k-4}\end{array},$$

$$F_{3,k} = \begin{array}{c}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \\ \times \\ K_{k-4}\end{array}, \quad F_{4,k} = \begin{array}{c}\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \times \\ K_{k-5}\end{array},$$

$$F_{5,k} = \begin{array}{c}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\ \times \\ K_{k-5}\end{array}, \quad F_{6,k} = \begin{array}{c}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \times \\ K_{k-6}\end{array}.$$

Define $f(m, k)$ to be the bound determined by the recurrence

$$f(m, k) = f(m-1, k) + f(m-1, k-1), \qquad \text{with } f(m, 2) = 2 \text{ and } f(k, k) = 2^k - 2 \quad (7)$$

(so that $f(m, 3) = 2m$). There are other functions that satisfy this recurrence such as $\text{forb}(m, K_k) = \text{forb}(m-1, K_k) + \text{forb}(m-1, K_{k-1})$ but with different base cases namely $\text{forb}(m, K_1) = 1$ and $\text{forb}(k, K_k) = 2^k - 1$. Following the proof of Theorem 2.1 in [AF10], we have

Theorem 9.1. Let F be a simple k-rowed F for which forb(m, F) is $O(m^{k-2})$ and $k \geqslant 3$. Then F is a configuration in $F_{1,k}$, $F_{2,k}$, $F_{3,k}$, $F_{4,k}$, $F_{5,k}$, or $F_{6,k}$. Moreover forb$(m, F_{i,k}) \leqslant f(m, k)$ for $i = 1, 2, 3, 4, 5, 6$. ■

The proof of the bounds in [AF10] yields

$$\text{forb}(m, F_{i,k}) \leqslant f(m, k) \text{ for } i = 1, 2, \ldots, 6.$$

The following construction has $f(m, k)$ columns.

Lemma 4. *Let $A(k)$ be the m-rowed matrix $A(k)$ of all columns such that with for each k-tuple $i_1 < i_2 < \cdots < i_k$, the columns satisfy*

$$\begin{array}{c} \\ i_1 \\ i_2 \\ i_3 \\ \vdots \\ i_k \end{array} \begin{array}{c} \textit{no} \\ \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ \vdots \\ \end{array}\right] \end{array} \quad \textit{and} \quad \begin{array}{c} \\ i_1 \\ i_2 \\ i_3 \\ \vdots \\ i_k \end{array} \begin{array}{c} \textit{no} \\ \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ \vdots \\ \end{array}\right] \end{array} . \tag{8}$$

Then what is missing in each k-tuple of rows of $A(k)$ is a pair of two complementary columns where for k even, the complementary columns have $k/2$ 1's and for k odd, the columns have $\lceil k/2 \rceil$ 1's and $\lfloor k/2 \rfloor$ 1's. It can be verified that $\|A(k)\| = f(m, k)$.

In [ES22] it was verified $A(k) \in \text{Avoid}(m, F_{3,k})$ for $k = 4, 5, 6$ and $A(k) \in \text{Avoid}(m, F_{1,k})$ for $k = 3$. This yields the following.

Theorem 9.2. forb$(m, F_{3,k}) = f(m, k)$ for $k = 4, 5, 6$.

Computer explorations suggest that these are the only cases where forb$(m, F_{i,k}) = f(m, k)$.

On the other hand, coding theory helps to give a better construction for $F_{i,k}$ than the one given by the product construction given in Definition 3.1 for Conjecture 3.2. One has to observe that $F_{i,k}$ contains two columns of $k-1$ 1's for all $1 \leqslant i \leqslant 6$. Thus, if A is an m-rowed matrix consisting of all columns with at most $k-2$ 1's and columns of codewords of $A(m, 4, k-1)$, that is a constant weight code of minimum distance 4 and weight $k-1$, then $A \in \text{Avoid}(m, F_{i,k})$. Indeed, A does not have two columns with $k-1$ 1's on the same k-set of rows. By a theorem of Graham and Sloane [GS80] this gives the lower bound forb$(m, F_{i,k}) \geqslant \frac{1}{m}\binom{m}{k-1} + \sum_{i=0}^{k-2}\binom{m}{i}$. Thus, we have applying Theorem 9.1

$$\frac{1}{k-1}\binom{m-1}{k-2} + \sum_{i=0}^{k-2}\binom{m}{i} \leqslant \text{forb}(m, F_{i,k}) \leqslant \binom{m-1}{k-2} + \sum_{i=0}^{k-2}\binom{m}{i}. \tag{9}$$

.

When applying Conjecture 3.2 to determine non-simple k-rowed matrices with forb(m, F) being $\Theta(m^{k-2})$, the first guess would be to allow any column with column sum $\in \{2, 3, \ldots, k-2\}$ to be repeated t times. This has worked in some cases (e.g.

Theorem 6.2). For a k-rowed simple matrix A, Define $\text{Rep}(A,t)$ as the matrix obtained from A by repeating columns of column sum $\in \{2,3,\ldots,k-2\}$ t times while leaving the remaining columns of sum $0,1,k-1,k$ unchanged. Applying Conjecture 3.2, we obtain the following:

Conjecture 9.3. Let $t \geqslant 2$ be given. Then $\text{forb}(m,\text{Rep}(F_{1,k},t))$, $\text{forb}(m,\text{Rep}(F_{2,k},t))$, $\text{forb}(m,\text{Rep}(F_{3,k},t))$, $\text{forb}(m,\text{Rep}(F_{4,k},t))$, $\text{forb}(m,\text{Rep}(F_{5,k},t))$, $\text{forb}(m,\text{Rep}(F_{6,k},t))$ are all $\Theta(m^{k-2})$. ■

Problem 1.19 relates to $\text{Rep}(F_{1,4})$.

The conjecture also predicts that certain configurations have unusually small bounds. A configuration on $2k$ or $2k-1$ rows might have a predicted bound of $O(m^{k-1})$. Moreover any configuration F on $2k$ or $2k-1$ rows has a product construction showing $\text{forb}(m,F)$ being $\Omega(m^{k-1})$. It makes sense to study the column maximal examples as the hardest ones to prove. For $k=3$, the matrix F_1 of Section 4 is the example. For $k=4$, the 4×2 matrix K_2^T has a linear bound. For $k=5$, the 9 5×6 column maximal matrices that the conjecture predicts to have a quadratic bound are listed in Conjecture 8.1. Three of the cases have been proven. For $k=6$, there is a single 6×3 matrix $G_{6\times 3}$ predicted by the conjecture to have a quadratic bound and that is proven.

Interestingly using Theorem 8.2 and a straightforward induction, we can establish the 6-rowed F yielding quadratic bounds.

$$\text{Let } G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 9.4. [ARS11] We have that $\text{forb}(m,G_{6\times 3})$ is $\Theta(m^2)$. Moreover for any column α, then $\text{forb}(m,[G_{6\times 3}|\alpha])$ is $\Omega(m^3)$. In fact if F is not a configuration in $G_{6\times 3}$, then $\text{forb}(m,F)$ is $\Omega(m^3)$. ■

Theorem 9.5. [KLLW20] $\text{forb}(m,K_3^T)$ is $O(m^3)$.

Theorem 9.6. $\text{forb}(m,\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\times G_{6\times 3})$ is $O(m^3)$.

The second result follows by combining the quadratic bound for $G_{6\times 3}$ ([ARS11] and the induction of Theorem 7. There is however a unique 8-rowed matrix predicted by the conjecture to have a cubic bound [Din21][ES22] which then necessarily contains K_3^T and

$\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \times G_{6\times 3}$:

$$I_2 \times G_{6x3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$I_2 \times G_{6x3}$ can be written as $\left[\begin{smallmatrix}A\\A^c\end{smallmatrix}\right]$ with $A = [\mathbf{0}\,T_2] \times T_2$. Computer explorations were undertaken to find these matrices predicted by the conjecture for $k = 7, 8, 10, 12, 16$. These might be the best place to look for counterexamples to Conjecture 3.2. In analogy to Conjecture 8.1 for the Maximal Quadratics, the following are the 28 7-rowed examples which Conjecture 3.2 predicts to have bounds $O(m^3)$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix}1&1&1&0&0&1&1&1&0&0\\1&1&0&1&1&1&1&0&1&1\\1&0&1&1&0&1&1&1&1&1\\0&1&1&0&1&1&0&1&1&0\\0&0&0&1&1&0&0&0&1&1\\0&0&0&0&0&0&1&0&0&1\\0&0&0&0&0&0&0&1&0&0\end{bmatrix}, \quad \begin{bmatrix}1&1&1&0&0&1&1&0&0&0\\1&1&0&1&1&1&1&1&1&1\\1&0&1&1&0&1&1&1&1&1\\0&1&1&0&1&1&0&1&0&0\\0&0&0&1&1&0&0&1&1&1\\0&0&0&0&0&0&1&0&1&0\\0&0&0&0&0&0&0&0&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&1&1&0&0&1&1&1&0\\1&1&0&1&1&1&0&0&1\\1&0&1&1&0&1&1&1&1\\0&1&1&0&1&1&1&1&1\\0&0&0&1&1&0&0&0&1\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\end{bmatrix}, \begin{bmatrix}1&1&1&0&1&1&1&1\\1&1&0&1&1&1&1&1\\1&0&1&1&1&1&1&1\\0&1&1&0&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}, \begin{bmatrix}1&1&1&0&1&1&1&1\\1&1&0&1&1&1&1&1\\1&0&1&1&1&1&1&0\\0&1&1&0&1&0&0&1\\0&0&0&1&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&1&1&0&1&1&1&0\\1&1&0&1&1&1&1&1\\1&0&1&1&1&1&1&1\\0&1&1&0&1&0&0&0\\0&0&0&1&0&1&0&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}, \quad \begin{bmatrix}1&1&1&0&1&1&1&1\\1&1&0&1&1&1&1&1\\1&0&1&1&1&1&0&0\\0&1&1&0&1&0&1&1\\0&0&0&1&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}, \quad \begin{bmatrix}1&1&1&0&1&1&1&1\\1&1&0&1&1&1&1&0\\1&0&1&1&1&1&0&1\\0&1&1&0&1&0&1&1\\0&0&0&1&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&1&1&0&1&1&1&0\\1&1&0&1&1&1&1&1\\1&0&1&1&1&1&0&1\\0&1&1&0&1&0&1&0\\0&0&0&1&0&1&0&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}, \quad \begin{bmatrix}1&1&1&0&1&1&0&0\\1&1&0&1&1&1&1&1\\1&0&1&1&1&1&1&1\\0&1&1&0&1&0&0&0\\0&0&0&1&0&1&1&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}, \begin{bmatrix}1&1&1&0&1&1&1&0&0\\1&1&0&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\0&1&1&0&1&0&0&0&0\\0&0&0&1&0&0&0&1&1\\0&0&0&0&0&1&0&1&0\\0&0&0&0&0&0&1&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&1&1&1&0&0&1&1&1&1&1\\1&1&0&0&1&1&1&1&1&1&1\\1&0&1&0&1&0&1&1&1&0&0\\0&1&0&1&0&1&0&0&0&1&1\\0&0&1&1&0&0&1&0&0&1&0\\0&0&0&0&1&1&0&1&0&0&1\\0&0&0&0&0&0&0&0&1&0&0\end{bmatrix}, \quad \begin{bmatrix}1&1&1&1&0&0&1&1&1&0\\1&1&0&0&1&1&1&1&1&1\\1&0&1&0&1&0&1&1&0&1\\0&1&0&1&0&1&0&0&1&0\\0&0&1&1&0&0&1&0&1&0\\0&0&0&0&1&1&0&0&0&1\\0&0&0&0&0&0&0&1&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&1&1&1&0&1&1&1&1&1\\1&1&0&0&1&1&1&1&0&0\\1&0&1&0&1&1&1&1&1&1\\0&1&0&1&0&1&0&0&1&0\\0&0&1&1&0&0&0&0&1&1\\0&0&0&0&1&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&1\end{bmatrix},\quad\begin{bmatrix}1&1&1&0&1&1&1&1&0&0\\1&1&0&1&1&1&0&0&1&0\\1&0&1&0&1&1&1&1&1&1\\0&1&0&1&1&0&1&0&1&1\\0&0&1&0&0&0&1&1&0&1\\0&0&0&1&0&0&0&0&1&1\\0&0&0&0&0&1&0&1&0&0\end{bmatrix},$$

$$\begin{bmatrix}1&1&1&0&1&1&1&1\\1&1&0&1&1&1&1&1\\1&0&1&1&1&1&1&0\\0&1&0&0&1&0&0&1\\0&0&1&0&0&1&0&0\\0&0&0&1&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix},\quad\begin{bmatrix}1&1&1&0&1&1&1&0\\1&1&0&1&1&1&0&1\\1&0&1&1&1&0&1&1\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&0&1&1&1&1\end{bmatrix},\quad\begin{bmatrix}1&1&1&0&1&1&1&1\\1&1&0&1&1&1&1&1\\1&0&1&0&1&1&1&0\\0&1&0&1&1&0&0&1\\0&0&1&0&0&1&0&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&1&0\end{bmatrix},$$

$$\begin{bmatrix}1&1&1&0&1&1&1&1\\1&1&0&1&1&1&1&0\\1&0&1&0&1&1&0&1\\0&1&0&1&1&0&1&0\\0&0&1&0&0&1&0&1\\0&0&0&1&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix},\quad\begin{bmatrix}1&1&1&0&1&1&1&1\\1&1&0&1&1&1&1&1\\1&0&1&1&1&1&1&1\\0&1&0&0&1&0&0&0\\0&0&1&0&0&1&0&0\\0&0&0&1&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix},\quad\begin{bmatrix}1&1&1&0&1&1&1&0\\1&1&0&1&1&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&0&1&1&1&1\end{bmatrix}.$$

The case $k = 8$ has some proofs for subconfigurations. We establish a few properties of these matrices which were of great help in the computer search.

Theorem 9.7. [ES22]Let F be a $2k$-rowed matrix with $\text{forb}(m, F)$ being $O(m^{k-1})$. Then F has both a row of 0's and a row of 1's.

While these special matrices were all of the same size for 5-rowed cases and of course the 6-rowed case, this is not true in general. The 28 7-rowed examples have 8,9,10,11,12 columns! The following (obtained by computer search) are the three column maximal 10-rowed matrices with predicted bound $O(m^4)$. Necessarily they have column sums 5 and must be simple.

$$\begin{bmatrix}1&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&0&0\\1&1&1&1&1&0&0&1&0\\1&1&1&0&0&1&1&0&1\\1&1&0&1&0&1&0&1&1\\0&1&0&0&0&0&0&1&1\\0&0&1&0&1&0&1&0&0\\0&0&0&1&1&0&0&1&0\\0&0&0&0&0&1&1&0&1\\0&0&0&0&0&0&0&0&0\end{bmatrix},\quad\begin{bmatrix}1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&0\\1&1&1&1&1&0&0&1\\1&1&1&0&0&1&1&1\\1&1&0&1&0&1&0&0\\0&1&0&0&0&0&0&1\\0&0&1&0&1&0&1&1\\0&0&0&1&1&0&0&0\\0&0&0&0&0&1&1&0\\0&0&0&0&0&0&0&0\end{bmatrix},$$

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&1&0\\ 1&1&1&1&1&0&0&1\\ 1&1&1&0&0&1&0&0\\ 1&1&0&1&0&0&0&1\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&1&1&1&0\\ 0&0&0&1&1&0&1&1\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

The following (obtained by computer search) are the two column maximal 12-rowed matrices with predicted bound $O(m^5)$. Necessarily they have column sums 6 and must be simple.

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&1&1&1&1&0&0&1&1&1&1&0&0&1&1&1&1\\ 0&0&0&0&1&1&0&0&0&0&1&1&0&0&0&0&1&1\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&1&1&0&0&0&0&1&1&0&0&0&0\\ 1&1&1&1&0&0&1&1&1&1&0&0&1&1&1&1&0&0\\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0\\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \end{bmatrix}$$

More easily, if F is written as $\left[\begin{smallmatrix} A \\ A^c \end{smallmatrix}\right]$ then the two matrices A from above are $[\mathbf{0}\,T_2]\times[\mathbf{0}\,T_2]\times T_2$ and $[\mathbf{0}\,T_3]\times[\mathbf{0}\,T_1]\times T_2$. Note that those results were obtained by computer but they can be checked to be column maximal etc by checking product constructions. Checking if these lists are complete is a different matter.

10 What is missing if a configuration F is avoided?

Let F be a given $k \times \ell$ (0,1)-matrix. Let S be a subset of $[m]$, the rows of A. We are interested in what conditions on $A|_S$ must be satisfied so that A has no configuration F. The problem of forbidden configurations does not reduce to these conditions since the conditions do not refer to the simplicity of A but these conditions have been used successfully.

We say an $|S| \times 1$ column α on a set of rows S is in 'short supply' in A if $A|_S$ has at most some constant number of columns equal to α. In this circumstance row order is relevant. We are not considering columns of $A|_S$ up to row permutations. It is convenient to list what is missing on k-sets but sometimes one also lists what is missing on smaller or larger sets of rows.

A careful consideration is required to see what is missing from A when a configuration F is not a configuration in A. One can use the computer program of Miguel Raggi at http://www.math.ubc.ca/∼anstee/FConfThesisVersion.tar.gz
to solve small cases (say 4 or 5 rows). The following is an example from cases with $k = 3$. Let $\{i, j, k\}$ be a triple of rows of a matrix $A = (a_{rs})$. We say that we have

$$\text{no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{10}$$

if in every column q of A we do not have $a_{iq} = d, a_{jq} = e$ and $a_{kq} = f$ all occurring. As well, we say that there are

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{11}$$

if there are at most $t - 1$ columns q of A in which $a_{iq} = d, a_{jq} = e$ and $a_{kq} = f$ all occur.

Let S_p denote the symmetric group on p symbols.

Proposition 10.1. (Proposition 2.1[AS05]) Let A be a (0,1)-matrix with no configuration $F_6(t)$ of Section 5. Let a, b, c be a triple of rows of A. Then we either have a permutation $\pi_1 \in S_3$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$ (note that $\{a, b, c\}$ and $\{i, j, k\}$ are the same as sets) with

$$\text{no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{12}$$

or if we do not have (12), then we have a permutation $\pi_2 \in S_3$ with $\pi_2(a) = i$, $\pi_2(b) = j$, $\pi_2(c) = k$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{13}$$

or if we do not have (12),(13), then we have a permutation $\pi_3 \in S_3$ with $\pi_3(a) = i$, $\pi_3(b) = j$, $\pi_3(c) = k$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \tag{14}$$

Proof: (sample) If one of (12),(13),(14) is true we have no $F_6(t)$. Give (12) is false, we either have $t \cdot K_3^1$ in the triple of rows or not. If not, then (13) holds for some ordering. If we do have $t \cdot K_3^1$ in the triple of rows, then t copies of two columns of two 1's (in the triple of rows) will yield $F_6(t)$ and so at most one column of two 1's appears t or more times. Thus (14) holds. ■

Proposition 10.2. (Proposition 2.2[AS05]) Let A be a (0,1)- matrix with no configuration $F_5(t)$ of Section 5. Let a, b, c be a triple of rows of A. Then we either have a permutation $\pi_1 \in S_3$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$ with

$$\text{no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{15}$$

or if we do not have (15), then we have a permutation $\pi_2 \in S_3$ with $\pi_2(a) = i$, $\pi_2(b) = j$, $\pi_2(c) = k$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{16}$$

or if we do not have (15),(16), then we have a permutation $\pi_3 \in S_3$ with $\pi_3(a) = i$, $\pi_3(b) = j$, $\pi_3(c) = k$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tag{17}$$

or if we do not have (15),(16),(17), then we have a permutation $\pi_4 \in S_3$ with $\pi_4(a) = i$, $\pi_4(b) = j$, $\pi_4(c) = k$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \ \blacksquare \tag{18}$$

We can readily establish such results for various F but it does take some careful thought. We give below the specific result for $F(k) = [K_k^0 \,|\, t \cdot D_{12}]$ when $k = 4$ [AF10]. It was crucial in the proof of Theorem 1.14 that this notion for general k is considered. It used the fact that if you consider what is missing on a given set of k rows in a matrix A avoiding $F(k)$, then for any pair of rows rows p, q chosen from the k rows, there is a column in *short supply* in the submatrix of A formed by the k rows (either absent of occurring some bounded number of times) which is either 0 on row p or 0 on row q.

Proposition 10.3. Let A be a (0,1)-matrix with no configuration 4-rowed configuration $F_6(t) = [K_4^0 \,|\, t \cdot D_{12}]$ from Theorem 1.14. Let a, b, c, d be four of rows of A. Then we either have a permutation $\pi_1 \in S_4$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$, $\pi_1(d) = l$ (note that $\{a, b, c, d\}$ and $\{i, j, k, l\}$ are the same as sets) with

$$\text{no } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{19}$$

or if we do not have (19), then we have a permutation $\pi_2 \in S_4$ with $\pi_2(a) = i$, $\pi_2(b) = j$, $\pi_2(c) = k$, $\pi_2(d) = l$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{20}$$

or if we do not have (19),(20), then we have a permutation $\pi_3 \in S_4$ with $\pi_3(a) = i$, $\pi_3(b) = j$, $\pi_3(c) = k$, $\pi_3(d) = l$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{21}$$

or if we do not have (19),(20),(21), then we have a permutation $\pi_4 \in S_4$ with $\pi_4(a) = i$, $\pi_4(b) = j$, $\pi_4(c) = k$, $\pi_4(d) = l$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \tag{22}$$

or if we do not have (19),(20),(21),(22), then we have a permutation $\pi_5 \in S_4$ with $\pi_5(a) = i$, $\pi_5(b) = j$, $\pi_5(c) = k$, $\pi_5(d) = l$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{23}$$

or if we do not have (19),(20),(21),(22),(23), then we have a permutation $\pi_6 \in S_4$ with $\pi_6(a) = i$, $\pi_6(b) = j$, $\pi_6(c) = k$, $\pi_6(d) = l$ with

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \ \blacksquare \tag{24}$$

11 Induction

There are easy standard inductions based on either deleting the first row or perhaps the first two rows. The most attractive application is the bound Theorem 1.12 but there are many other applications.

The *standard induction* [AGS97] proceeds as follows. Let A be a simple $m \times n$ matrix with no configuration F or some such property. Then we can decompose A as follows. I have permuted the rows so row r of A appears at the top. When deleting row r from A, there may be repeated columns and we have permuted the columns to obtain the following where $[B(r)C(r)D(r)]$ is simple.

$$A = \begin{array}{c} \text{row } r \rightarrow \\ \\ \end{array} \left[\begin{array}{cccc} 00\cdots0 & 00\cdots0 & 11\cdots1 & 11\cdots1 \\ B(r) & C(r) & C(r) & D(r) \end{array}\right] \tag{25}$$

Now $[B(r)C(r)D(r)]$ is simple and has no configuration F. Also $\|A\| = \|[B(r)C(r)D(r)]\| + \|C(r)\|$. One can easily derive the upper bound of Theorem 1.5 this way by noting that if $K_k \nprec A$ then $K_{k-1} \nprec C(r)$. Then by induction $\|[B(r)C(r)D(r)]\| \leqslant \text{forb}(m-1, K_k)$ and $\|C(r)\| \leqslant \text{forb}(m-1, K_{k-1})$. Thus $\text{forb}(m, K_k) \leqslant \text{forb}(m-1, K_k) + \text{forb}(m-1, K_{k-1})$ and we obtain the desired bound. It may work to just use row $r = 1$ but in certain circumstances one should choose row r so that $C(r)$ is in some way minimal. A version describing what $C(r)$ avoids assuming A avoids F is stated in [AK06]. It also used for when forbidding families of configurations.

Lemma 5. *[AK10] Let k be given and let F be a k-rowed simple matrix. For each $s \in [k]$, decompose F as*

$$F = \begin{array}{c} \textit{row } s \rightarrow \\ \\ \end{array} \left[\begin{array}{cccc} 0\,0\cdots0 & 0\,0\cdots0 & 1\,1\cdots1 & 1\,1\cdots1 \\ B_s(F) & C_s(F) & C_s(F) & D_s(F) \end{array}\right], \tag{26}$$

where we have permuted the rows of F so row s is the first row and $C_s(F)$ consists of the repeated columns after deleting that row from F. Then if A is a simple matrix with no configuration F, then in the row decomposition of A of (25), we deduce that $C(r)$ has no configurations $F_s = [B_s(F)C_s(F)D_s(F)]$ for each $s \in [k]$. In particular if forb$(m, \{F_1, F_2, \ldots, F_k\})$ is $O(m^t)$ then forb(m, F) is $O(m^{t+1})$. ■

A slightly more careful argument can be used if F is not simple. We forbid from $C(r)$ any $(k\text{-}1)$-rowed F' that satisfies

$$F \prec \left[\begin{array}{cc} 0\,0\,\cdots\,0 & 1\,1\,\cdots\,1 \\ F' & F' \end{array}\right].$$

While this may look quite general, there is more that can be said about $C(r)$ in some cases. For example if $F = 2 \cdot F'$, then $C(r)$ avoids F'. This was used in [Ans90b]. We extend this idea further [AL13], extending the notion of *simple* to *t-simple* allowing columns to appear up to t times with the idea that this only affects this by a multiple of t. Let $\text{forb}(m, F, t)$ be the bounds obtained when you allow columns to appear up to

t times. Let $\mu(\alpha, A)$ be the multiplicity of column α in A. Define the *support* of F as $\text{supp}(F)$ so that $\mu(\alpha, \text{supp}(F)) = 1$ in $\text{supp}(F)$ if and only if $\mu(\alpha, F) \geqslant 1$.

Assume F is t-simple. Thus $F \prec t \cdot \text{supp}(F)$. Let $A \in \text{Avoid}(m, F, t-1)$. Given a row r we permute rows and columns of A to obtain

$$A = \begin{matrix} \text{row } r \rightarrow \\ \end{matrix} \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ G & H \end{bmatrix}. \tag{27}$$

Then $\mu(\alpha, G) \leqslant t-1$ and $\mu(\alpha, H) \leqslant t-1$. For those α for which $\mu(\alpha, [G\,H]) \geqslant t$, let C be formed with $\mu(\alpha, C) = \min\{\mu(\alpha, G), \mu(\alpha, H)\}$. Rewrite our decomposition of A as follows:

$$A = \begin{matrix} \text{row } r \rightarrow \\ \end{matrix} \begin{bmatrix} 0\,0 \cdots 0 & & 1\,1 \cdots 1 & \\ B & C & C & D \end{bmatrix}. \tag{28}$$

Then $[BCD]$ and C are both $(t-1)$-simple. The former follows from $\mu(\alpha, [B\,C\,D]) = \mu(\alpha, G) + \mu(\alpha, H) - \min\{\mu(\alpha, G), \mu(\alpha, H)\} \leqslant t-1$. Thus $F \not\prec [B\,C\,D]$ for $F \in \mathcal{F}$. Also for any $F' \prec C$ then $[0\,1] \times F' \prec A$ so define

$$\mathcal{G} = \{F' \,:\, \text{for } F \in \mathcal{F},\ F \prec [0\,1] \times F' \text{ and } F \not\prec [0\,1] \times F'' \text{ for all } F'' \prec F',\ F'' \neq F'\}. \tag{29}$$

Also since each column α of C has $\mu(\alpha, [G\,H]) \geqslant t$, we deduce that $\text{supp}(F) \not\prec C$ for each $F \in \mathcal{F}$. Our induction on m becomes:

Lemma 6. *[AL13]*

$$\begin{aligned} \textit{forb}(m, \mathcal{F}, t-1) &= \|A\| = \|[BCD]\| + \|C\| \\ &\leqslant \textit{forb}(m-1, \mathcal{F}, t-1) + (t-1) \cdot \textit{forb}(m-1, \mathcal{G} \cup \{\textit{supp}(F) \,:\, F \in \mathcal{F}\}). \end{aligned} \tag{30}$$

The power of this induction is the replacement in (29) of F by $\text{supp}(F)$ but we lose a factor of t in the bound.

Induction on the number of rows (elements) is a mainstay of studying set systems. Here is an application of a slight variant of standard argument. Recall that in an m-rowed matrix A, a set $S \subseteq [m]$ is *shattered* if $K_{|S|} \prec A|_S$.

$$\text{Let } \text{sh}(A) = \{S \subseteq [m] \,:\, K_{|S|} \prec A|_S\}$$

Theorem 11.1. [Paj85] Let A be given. Then $|\text{sh}(A)| \geqslant \|A\|$.

Proof: Decompose A as follows:

$$A = \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ A_0 & A_1 \end{bmatrix}.$$

Then $\|A\| = \|A_0\| + \|A_1\|$. By induction $|\text{sh}(A_0)| \geqslant \|A_0\|$ and $|\text{sh}(A_1)| \geqslant \|A_1\|$. Now $|\text{sh}(A_0) \cup \text{sh}(A_1)| = |\text{sh}(A_0)| + |\text{sh}(A_1)| - |\text{sh}(A_0) \cap \text{sh}(A_1)|$. If $S \in \text{sh}(A_0) \cap \text{sh}(A_1)$, then $1 \cup S \in \text{sh}(A)$. Thus the number of sets in $\text{sh}(A)$ that are not in $\text{sh}(A_0) \cup \text{sh}(A_1)$ is at least $|\text{sh}(A_0) \cap \text{sh}(A_1)|$. We conclude $|\text{sh}(A)| \geqslant |\text{sh}(A_0)| + |\text{sh}(A_1)|$. Hence $|\text{sh}(A)| \geqslant \|A\|$. ■

Note that this yields a proof of Theorem 1.5. In [ARS02] we use this induction where we always choose row 1.

Lemma 7. *Let F' be a $k \times \ell$ $(0,1)$-matrix for which forb(m, F') is $O(m^t)$. Then with*

$$F = \begin{bmatrix} 11 \cdots 1 \\ 00 \cdots 0 \\ F' \end{bmatrix},$$

we have forb(m, F) being $O(m^{t+1})$. ∎

Proof: Let $A \in \text{Avoid}(m, F)$. Ignoring the column of 0's and the column of 1's, we decompose the columns of A into blocks Z_i and J_i where Z_i consists of those columns of A whose first $i+1$ rows are $\mathbf{0}_i\mathbf{1}_1$ and J_i consists of those columns of A whose first $i+1$ rows are $\mathbf{1}_i\mathbf{0}_1$. Now $F \nprec Z_i$ implies that $\|Z_i\| \leqslant \text{forb}(m-i-1, F')$. Similarly $\|J_i\| \leqslant \text{forb}(m-i-1, F')$. The result follows by summing the bounds. ∎

An application of this induction is in Theorem 7.2. I would point out that we can extend these arguments (Lemma 5, Lemma 7) to families of forbidden configurations which may have relevance in the standard induction in view of Lemma 5. A two-rowed induction was used with success in [AS97] in the case that the columns of matrix A form an antichain as sets. Using that fact, we can deduce that C is empty in the decomposition (25) above and so we may write

$$A = \begin{bmatrix} 00 \cdots 0 & 00 \cdots 0 & 11 \cdots 1 & 11 \cdots 1 \\ 00 \cdots 0 & 11 \cdots 1 & 00 \cdots 0 & 11 \cdots 1 \\ C_1 & C_2 C_3 & C_3 C_4 & C_5 \end{bmatrix},$$

where $[C_1 C_2 C_3 C_4 C_5]$ is simple.

Induction, used cleverly, is the gift that keeps on giving. We used a new version of our standard induction where we consider the minimal set of rows in $C(r)$ for which the matrix of those rows is simple. This idea was used successfully in combination with the 'what is missing' idea (Section 10) to obtain structural results that led to proofs of Theorem 2.9[AKRS11] and Theorem 8.2[ARS11].

We have used repeated induction to obtain a number of results including extensions of Theorem 1.5 to Theorem 1.12 and also Theorem 1.18. It turns out that consideration of base cases becomes difficult. Theorem 4.1 in [AK10] considers the k-rowed configuration $F = [\mathbf{1}_k \,|\, 2 \cdot (\mathbf{1}_2\mathbf{0}_2) \times K_{k-4}]$ for which $\text{forb}(k, F) > \text{forb}(k, K_k)$ but we may verify $\text{forb}(k+1, F) = \text{forb}(k+1, K_k)$. Theorem 1.18 [AM11] also has $\text{forb}(m, [K_k \,|\, (\mathbf{1}_2\mathbf{0}_2) \times K_{k-4}]) > \text{forb}(m, K_k)$ for $m = k$ and perhaps $m = k+1$ and perhaps a additional small values of m. We succeed establishing $\text{forb}(k+1, [K_k \,|\, (\mathbf{1}_2\mathbf{0}_2) \times K_{k-4}]) = \text{forb}(k+1, K_k)$ for $k \leqslant 15$ which establishes $\text{forb}(m, [K_k \,|\, (\mathbf{1}_2\mathbf{0}_2) \times K_{k-4}]) = \text{forb}(m, K_k)$ for $m \geqslant k+1$. For larger k, new arguments are needed.

An essay on induction techniques is [Ved16].

12 Shifting proofs

Peter Frankl popularized the use of shifting arguments in extremal set theory. In this particular context there is a paper of Frankl [Fra83] and a paper of Alon [Alo83] using shifting techniques to generalize Theorem 1.5. I extended these arguments and used them in [Ans88]. Shifting is easily defined in set language. Let $\mathcal{F} \subseteq 2^{[m]}$. Let

$$T_j(B) = \begin{cases} B & \text{if } j \notin B \text{ or if } B \backslash j \in \mathcal{F} \\ B \backslash j & \text{if } j \in B \text{ and } B \backslash j \notin \mathcal{F} \end{cases}.$$

Then

$$T_j(\mathcal{F}) = \{T_j(B) \,:\, B \in \mathcal{F}\}.$$

Note that $|T_j(\mathcal{F})| = |\mathcal{F}|$. We can repeatedly apply T_j for each $j = 1, 2, \ldots, m$ to obtain the *shifted family* $T(\mathcal{F})$ which has the property that

$$T_j(T(\mathcal{F})) = T(\mathcal{F}) \text{ for } j = 1, 2, \ldots m.$$

Thus $|T(\mathcal{F})| = |\mathcal{F}|$ and $T(\mathcal{F})$ is a *downset* (namely for every $B \in T(\mathcal{F})$ and every $C \subseteq B$, we have $C \in T(\mathcal{F})$). Now let $S \subseteq [m]$ and let

$$\mathcal{F}|_S = \{B \cap S \,:\, B \in \mathcal{F}\}$$

Theorem 12.1. Let $S \subseteq [m]$. Then

$$\big|\mathcal{F}|_S\big| \geqslant \big|T(\mathcal{F})|_S\big|. \blacksquare$$

Using this one can prove Theorem 1.5 by noting that if $\mathcal{F}$ has no configuration K_k then for any $S \in \binom{[m]}{k}$, we have $|\mathcal{F}|_S| \leqslant 2^k - 1$ and hence $|T(\mathcal{F})|_S| \leqslant 2^k - 1$. Since $T(\mathcal{F})$ is a downset, the column of k 1's is absent. Thus we deduce $|\mathcal{F}| = |T(\mathcal{F})| \leqslant \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$ and hence prove Theorem 1.5.

Another application is for the forbidden matrix F_3 of Section 5, for which we note that a simple matrix A avoiding F_3 has at most 6 column types on any 3 rows. A consequence is the exact bound of Theorem 5.3. Alon [Alo83] refers to such possible results.

The shifting argument was utilized in [AA95] to obtain a forbidden configuration theorem associated with any ideal (downset) in the lattice of divisors. This led to the notion of order shattered sets in [ARS02]. These lead to multiset versions of Theorem 12.1.

13 Graph Theory

The use of graph theory is easiest to understand in considering a $2 \times \ell$ forbidden configuration F. In that case, it is natural to form a graph whose vertices are the rows of the matrix A. We consider what is missing if we forbid a 2-rowed F as in Section 10 and so columns in 'short supply' or absent can be noted in the graph perhaps using edge labels or directed edges (there are only 4 possible columns on 2 rows!). Results in that direction are repeatedly used in [AGS97],[AFS01],[AK07].

Results about cliques, connectivity, chromatic number are used. The following specialized result was obtained (a generalization of Rédei's Theorem that every tournament has a directed Hamiltonian path) in [AFS01] (some minor errors in published proof!) to obtain the exact bound of Theorem 4.7.

Lemma 8. *Let $D = (N, A)$ be a directed graph. There is an ordering of the vertices N as $1, 2, \dots m$ where $m = |N|$ and a subset $T \subseteq A$ consisting of a collection of vertex disjoint indirected trees with the following property. Each arc $p \to q$ of T has $p < q$ in the ordering. For each pair i, j, $1 \leqslant i < j \leqslant m$ either there is a directed path in T from i to j or there is a k with $i \leqslant k \leqslant m$ so that there is a directed path from i to k in T and there is no edge in D from k to j.* ■

Graph theory was successfully employed for larger F in [AS05] where the vertex set corresponds to $\binom{[m]}{k}$. The standard decomposition of a directed graph into strongly connected components with an acyclic graph between the strongly connected components was an essential tool. We used that a linear number (linear in the number of vertices) of directed edges is sufficient to assure strong connectivity. This idea was again employed in [AF10] along with indicator polynomials to establish Theorem 1.14.

14 Linear Algebra

Applications of linear algebra here include the proof of Theorem 1.5. Frankl and Pach obtained results for *null t-designs* [FP83]. One approach is the following. Given two columns β, γ, we say β *covers* γ if and only if $\beta \geqslant \gamma$. For an $m \times n$ simple matrix A and an $m \times 1$ (0,1)-vector γ, we can define $A(\gamma)$ as the $1 \times n$ (0,1)-row vector with a 1 in position j if column j of A covers γ.

Now the vector space $V = \text{span}\{A(\gamma) : \gamma \in \mathbf{R}^n\}$ is a vector space of dimension n and moreover $\{A(\gamma) : \gamma \text{ is a column of } A\}$ is a basis for V. Now if we take

$$\Gamma_{k-1} = \{\gamma : \text{ there exists an } s \text{ with } 0 \leqslant s \leqslant k-1 \text{ and } \gamma \text{ is a column of } K_m^s\}$$

we are able to verify the following.

Theorem 14.1. [FP83]([Rys72],[Ans85]). let A be an $m \times n$ simple matrix with no configuration K_k. Then n is the dimension of $V = \text{span}\{A(\gamma) : \gamma \in \Gamma_{k-1}\}$ for $\Gamma_{k-1} = \{\gamma : \text{ there exists an } s \text{ with } 0 \leqslant s \leqslant k-1 \text{ and } \gamma \text{ is a column of } K_m^s\}$. Hence

$$n \leqslant |\Gamma_{k-1}| = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \; ■$$

Another application of linear algebra is considering columns in short supply using *indicator polynomials*: Given an $m \times 1$ (0,1)-column α we can create a multilinear polynomial $p(\mathbf{x})$ of degree m in variables $x_1, x_2, \dots, x_m$ that evaluates to 1 for column α (where we take $x_i = \alpha_i$ for each $i = 1, 2, \dots, m$) and evaluates to 0 for all other (0,1)-columns.

Let $A \in \text{Avoid}(m, K_k)$. Then for each subset $S \subseteq [m]$ with $|S| = k$, we have that $A|_S$ has at least one missing column, say $\alpha(S)$, else A has K_k. Smolensky [Smo97] noted that the dimension of the space of real valued functions on the columns of A is $\|A\|$. Any real valued function on the columns of A can be given as a multilinear polynomial. Let $\mathbf{x} = (x_1, x_2, \dots, x_m)$. In particular for a column $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ we can form $\prod_{i=1}^m (x_i - 1 + \alpha_i)$ which will be non-zero only for column $\mathbf{x} = \alpha$ for all

$\mathbf{x} \in \{0,1\}^n$. A suitable linear combination of such multilinear polynomials (one for each column α of A) will yield any real valued function on columns of A. Smolensky showed that linear combinations of multilinear polynomials of degree at most $k-1$ suffice and so the dimension of that space is the bound of Theorem 1.5. Thus we have the bound of Theorem 1.5 for $\|A\|$. Assume $f(\mathbf{x})$ contains an expression $x_1x_2\cdots x_k$ and let $S = \{1,2,\ldots,k\}$. Then given the column $\alpha(S) = (\alpha_1, \alpha_2, \ldots, \alpha_k)^T$, we can form a polynomial $f_S(\mathbf{x}) = \prod_{i=1}^{k}(x_i - 1 + \alpha_i)$ of degree k with leading term $x_1x_2\cdots x_k$ so that for $\mathbf{x} \in \{0,1\}^n$, $f_S(\mathbf{x})$ is 0 if $\mathbf{x}|_S \neq \alpha(S)$ for $\mathbf{x} \in \{0,1\}^n$. We can now replace $x_1x_2\cdots x_k$ by $x_1x_2\cdots x_k - f_S(\mathbf{x})$. The new polynomial will evaluate to the same values on the columns of A and we will have replaced $x_1x_2\cdots x_k$ by terms of degree at most $k-1$. Do this for all choices of S and repeat until you obtain a polynomial of degree at most $k-1$.

This idea of indicator polynomials was exploited in [AFFS05] for cases where each k-set of rows has two missing columns and further exploited in [AF10]. Different ways to achieve a reduction in degree occur.

L. Rónyai noted how some of the inductive proofs and the concept of *order shattering* were suited to *Gröbner bases* [ARS02] and some nice applications followed. There are cases where the Gröbner basis can be computed in a general way [HR03]. There are application to Forbidden Configurations. Theorem 11.1 proved by Pajor states $|\mathrm{sh}(A)| \geqslant \|A\|$. Rónyai and Mészáros [RM11, RM14] use Gröbner bases to characterize family of sets $\mathcal{F}$ with $|\mathrm{sh}(\mathcal{F})| = |\mathcal{F}|$.

An MSc essay on applications of linear algebra is at [Pre17]

15 Strong Stability

Problems in extremal combinatorics are first concerned with bounds but considerations of stability are one of the next topics to explore. The idea of strong stability is to show that a set system satisfying some property (in our case having a forbidden configuration F) and having a number of sets close to the optimal value (for us $\mathrm{forb}(m,F)$) that the system of sets has much of its structure already determined. In the study of forbidden configurations, we consider $A \in \mathrm{Avoid}(m,F)$ (or $\mathrm{Avoid}(m,\mathcal{F})$) with $\|A\|$ sufficiently close to $\mathrm{forb}(m,F)$, and hope to deduce the structure of A.

The strong stability result used in proving Theorem 7.2 considers a k-uniform set system with no $F_{0,r+1,r+1,0}$ (the notation F_{abcd} is defined in Section 7) which is equivalent to having the set system be $(k-r)$-intersecting. As noted when introducing Theorem 6.6, having no configuration $F_{0,r+1,r+1,0}$ is the same as having no submatrix $F_{0,r+1,0,0}$ for a k-uniform family. Let numbers k, r_1, r_2 be given and suppose G and H are given disjoint sets with $|G| = k - r_1 + r_2$. We define $\mathcal{I}^k_{r_1,r_2}$ on the pair (H,G) to be the family consisting of all sets of size k in $G \cup H$ that intersect G in at least $k - r_1 = |G| - r_2$ points. Note that any two sets in $\mathcal{I}^k_{r_1,r_2}$ have at least $|G| - 2r_2 = k - r_1 - r_2$ points in common, i.e. $\mathcal{I}^k_{r_1,r_2}$ is $(k-r)$-intersecting, where $r = r_1 + r_2$. The Complete Intersection Theorem, conjectured by Frankl, and proved by Ahlswede and Khachatrian [AK97a], is that any k-uniform, $(k-r)$-intersecting family of maximum size on a given ground set is isomorphic to $\mathcal{I}^k_{r-p,p}$, for some $0 \leqslant p \leqslant r$, which depends on the size of the ground set. Note that

$|\mathcal{I}^k_{r_1,r_2}|$ is $O(m^r)$ ($\Theta(m^r)$ for $|G|$ and $|H|$ being $\Theta(m)$). The following was critical to prove Theorem 7.2.

Theorem 15.1. [AK06] Suppose $\mathcal{A}$ is a k-uniform $(k-r)$-intersecting set system on $[m]$ of size at least $(5r)^{5r}m^{r-1}$. Then $\mathcal{A} \subseteq \mathcal{I}^k_{r-p,p}$ for some $0 \leqslant p \leqslant r$.

We are also interested in the related family of sets $\mathcal{F}^k_{r_1,r_2}$ on the pair (H,G) to be the family consisting of all sets of size k in $G \cup H$ that intersect G in exactly $k - r_1 = |G| - r_2$ points. Note that $|\mathcal{F}^k_{r_1,r_2}|$ is also $O(m^r)$ and that $|\mathcal{I}^k_{r_1,r_2} \backslash \mathcal{F}^k_{r_1,r_2}|$ is $O(m^{r-2})$. A proof of Theorem 15.1 in the case $r = 1$ (where there are no asymptotics) is used in [Ans90b].

Strong stability is an interesting property and it is often a useful proof technique. Theorem 15.1 was used to prove bounds for all $k \times 2$ matrices, bounds that were consistent with the conjecture.

Problem 15.2. Can you to use Theorem 15.1 to prove some better bounds for F being the $2k \times 2$ matrix $[\mathbf{1}_k\mathbf{0}_k \,|\, \mathbf{0}_k\mathbf{1}_k]$ for which the bound is $O(m^k)$ by Theorem 6.6 [FFP87] (or Theorem 7.2)? Corollary 6.7 yields the case $k = 2$.

Totally Balanced Matrices are defines as $\text{Avoid}(m, \{C_3, C_4, C_5 \ldots\})$ where C_i is the $i \times i$ vertex edge incidence matrix of the cycle of length i.

Theorem 15.3. [Ans80] $\text{forb}(m, \{C_3, C_4, C_5 \ldots\}) = \text{forb}(m, C_3) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$

Matrices which are conformal and are hereditary with this property can be defined by a single forbidden configuration C_3 [Ans83a].

Theorem 15.4. [Ans83b] Let $A \in \text{Avoid}(m, \{C_3, C_4, C_5 \ldots\})$ then there is a matrix $A' \in \text{ext}(m, \{C_3, C_4, C_5 \ldots\})$ with $A \prec A'$.

One can describe this as 'maximal=maximum', namely a totally balanced matrix for which you can add no columns (maximal) will be in $\text{ext}(m, \{C_3, C_4, C_5 \ldots\})$ i.e. have the maximum number of columns. The consequence is that the structure of any totally balanced matrix is that of extremal totally balanced matrices. So we have very strong stability. The structure of $\text{ext}(m, \{C_3, C_4, C_5 \ldots\}) = \text{ext}(m, C_3)$ is described in [Ans80].

If we consider $\text{Avoid}(m, C_3)$, we don't get stability. We can have a matrix $A' \in \text{Avoid}(m, C_3)$ with $C_4 \in A'$ (hence $A' \not\prec A \in \text{ext}(m, C_3)$) and $\|A'\| = \text{forb}(m, C_3) - 1$. Consider the matrix $A \in \text{ext}(m, C_3)$ with the consecutive 1's property (columns of form 0*1*0*). Form a matrix A' by adding a column with 1's in rows 1,4 and deleting the two columns of sum 3 on rows 1,2,3 and rows 2,3,4. The resulting matrix A' has $\|A'\| = \|A\| - 1$. Assume $C_3 \prec A'$ on rows i, j, k. If $i, j, k \in \{1, 2, 3, 4\}$ then no C_3. If $i, j \in \{1, 2, 3, 4\}$ and $k \notin \{1, 2, 3, 4\}$ or $i \in \{1, 2, 3, 4\}$ and $j, k \notin \{1, 2, 3, 4\}$, then a C_3 in A' would imply a C_3 in A. This contradiction yields $A' \in \text{Avoid}(m, C_3)$ where A' doesn't have the structure of totally balanced matrices.

The bound $\text{forb}(m, F_{0,2,2,0}) = \binom{m}{2} + 2m - 1$ proven in [FFP87] was reproven in [ABS11] where it is shown that $\text{Avoid}(m, F_{0,2,2,0})$ always has a particular structure from which the bound follows. A different sort of stability but a happy case where $\text{Avoid}(m, F)$ is understood.

Another similarly powerful stability result considers $F = F(0,1,2,0)$ and its transpose $F^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, [AKL+24]. This is also interesting, because it is one of the extremely rare cases, when $\text{ext}(m, F) = \text{ext}(m, F^T)$.

Theorem 15.5. [AKL+24] Let $F = F(0,1,2,0)$. Then $A \in \text{Avoid}(m, F)$ or $A \in \text{Avoid}(m, F^T)$ if and only if A is a subset of the columns of a matrix of the following block form:

$$\begin{bmatrix} 1 & B_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & B_2 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & B_k & 0 \end{bmatrix} \tag{31}$$

where the B_i are either "empty" (their columns do not exist), or are some I_m or I_m^c for some $m \geqslant 2$ (and the B_i may contain additional repeated rows when $A \in \text{Avoid}(m, F) \setminus \text{Avoid}(m, F^T)$).

Our main bound Theorem 1.5 has $\text{ext}(m, K_k)$ containing varied matrices (e.g. Theorem 1.1[Ans83b], Theorem 4.2[Ans88], Theorem 3.1[AS97]). One can note that for any $A \in \text{ext}(m, K_k)$ and any k-set $S \in \binom{[m]}{k}$ there is a $k \times 1$ column $\alpha(S)$ so that $\alpha(S)$ is not a column in $A|_S$. This doesn't give the structure of matrices in $\text{ext}(m, K_k)$ since a specification of various $\alpha(S)$ would typically not yield a matrix in $\text{ext}(m, K_k)$ merely a matrix in $\text{Avoid}(m, K_k)$. For example A given by $\alpha_S = \mathbf{1}_k$ for all but one set $T \in \binom{[m]}{k}$ for which $\alpha_T = \mathbf{0}_k$ yields columns of at most $k-1$ 1's that have at least one 1 in rows of T and the column of k 1's on rows T. $A \in \text{Avoid}(m, K_k)$ with $\|A\| < \text{forb}(m, K_k)$. The what is missing idea of considering $\alpha(S)$ was helpful in proving Theorem 1.18. Consider Claim 3(k) proved in [AN22].

Lemma 9. *[AN22] Assume* $A \in \text{Avoid}(m, [K_k | t \cdot (K_2^T \times K_{k-4})])$*. Assume there is some* k*-tuple* S *of rows so that* $(t+1) \cdot (K_2^T \times K_{k-4}) \not\prec A|_S$*. Then there exist a constant* m_k *so that* $\|A\| \leqslant \text{forb}(m, K_k) - m + 4t$ *for* $m > m_k$*.*

This translates to a stability result:

Theorem 15.6. Assume $A \in \text{Avoid}(m, [K_k | t \cdot (K_2^T \times K_{k-4})])$. Use the constant m_k from Lemma 9. Assume for $m > m_k$ that $\|A\| > \text{forb}(m, K_k) - m + 4t$. Then $A \in \text{Avoid}(m, K_k)$. Thus for $m > m_k$, $\text{ext}(m, [K_k | t \cdot (K_2^T \times K_{k-4})]) = \text{ext}(m, K_k)$.

For some other stability results consult the paper [AKL+24].

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