The Graph Crossing Number and its Variants: A Survey

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Submitted: Dec 20, 2011; Accepted: Apr 4, 2013
First Edition: Apr 17, 2013
Second Edition: May 15, 2014
Third Edition: Dec 22, 2017
Fourth Edition: Feb 14, 2020
Fifth Edition: Sep 4, 2020
Sixth Edition: May 21, 2021
Seventh Edition: Apr 8, 2022
Eighth Edition: May 17, 2024
Mathematics Subject Classifications: 05C62, 68R10

Abstract

The crossing number is a popular tool in graph drawing and visualization, but there is not really just one crossing number; there is a large family of crossing number notions of which the crossing number is the best known. We survey the rich variety of crossing number variants that have been introduced in the literature for purposes that range from studying the theoretical underpinnings of the crossing number to crossing minimization for visualization problems.

1 So, Which Crossing Number is it?

The *crossing number*, $\text{cr}(G)$, of a graph G is the smallest number of crossings required in any drawing of G. Or is it? According to a popular introductory textbook on combinatorics [743, page 40] the crossing number of a graph is "the minimum number of pairs of crossing edges in a depiction of G". So, which one is it? Is there even a difference?[1] To start with the second question, the easy answer is: yes, obviously there is a difference, the difference between counting all crossings and counting pairs of edges that cross. But maybe these different ways of counting don't *make* a difference and always come out

[1]For a recent story of confusion on this issue, see [518].

the same? That is a harder question to answer. Pach and Tóth in their paper "Which Crossing Number is it Anyway?" [613] coined the term *pair crossing number*, pcr, for the crossing number in the second definition. One of the big open problems in the theory of crossing numbers is whether $\mathrm{pcr}(G) = \mathrm{cr}(G)$ for all graphs G. If we don't know whether they are the same, why do we see both notions called crossing number in the literature?

One potential source for the confusion between pcr and cr may be the famous crossing number inequality which states that for any graph G on n vertices and m edges we have $\mathrm{cr}(G) \geqslant c \cdot m^3/n^2$ for $m \geqslant 4n$ and some constant c. The original proofs of this result are due independently to Ajtai, Chvátal, Newborn, Szemerédi [32] and Leighton [528]. Leighton defines cr as pcr; since $\mathrm{pcr}(G) \leqslant \mathrm{cr}(G)$, he is making a stronger claim; his proof is analyzed in the section on crossing lemma variants below. The importance and influence of Leighton's paper may explain why some later papers using the crossing number inequality work with the pair crossing number [41, 735]. The danger, of course, is that the two notions get confused; for example, Leighton [529, Theorem 1] proves that $\mathrm{cr}(G)+n \geqslant \Omega(\mathrm{bw}(G)^2)$, where $\mathrm{bw}(G)$ is the bisection width of G (and G has bounded degree); his construction is fine for the standard crossing number, but does not work for pcr, the definition of crossing number he chose.[2]

Another influential crossing number result is Garey and Johnson's proof that the crossing number problem is **NP**-complete [348]; Garey and Johnson first mentioned the problem as an open problem in their book on **NP**-completeness, where they write: "Open problems for other generalizations of planarity include 'Does G have crossing number K or less, i.e. can G be embedded in the plane with K or fewer pairs of edges crossing one another?' " [347, OPEN3]. Clearly, they are defining what we now call the pair crossing number; in their later **NP**-completeness paper they write that K is the least integer so that "G can be embedded in the plane so that there are no more than K pair-wise intersections of curves representing edges (not counting the required intersections at common endpoints)" [348]. This is already somewhat ambiguous: does "pair-wise" mean that they only count the pairs, or that crossings count for each pair they belong to (which is relevant if more than two edges cross in a crossing). When they show that the crossing number problem lies in **NP**, it becomes clear that they mean the standard crossing number and not the pair crossing number (for which membership in **NP** is not trivial [679]).

This last example suggests another possible explanation for confusion among crossing numbers: when trying to make precise what it means to count crossings, it is natural to speak of pairwise crossings (to avoid problems with three edges crossing in the same point), and from there it is a short step to "pairs of edges crossing".

However, the main reason for confusion is most likely one identified by Székely [724] in his discussion of drawing conventions. In a drawing D of G minimizing $\mathrm{cr}(G)$ we have $\mathrm{cr}(D) = \mathrm{pcr}(D)$ since every pair of edges crosses at most once. This does not imply that $\mathrm{pcr}(G) = \mathrm{cr}(G)$ but it may have mistakenly suggested it; the subtle confusion is between a cr-minimal drawing, in which every pair of edges crosses at most once, and a pcr-minimal

[2]Kolman and Matoušek [502] show that Leighton's result can be extended to pcr, but with slightly weaker bounds.

drawing, for which we do not know whether this is true.[3] This confusion may have been exacerbated by the fact that $\mathrm{cr}(G)$ as defined above from the beginning coexisted with what we now call the *rectilinear crossing number*, $\overline{\mathrm{cr}}(G)$, in which drawings of G are restricted to straight-line drawings.[4] In a straight-line drawing D of G we again have $\mathrm{cr}(D) = \mathrm{pcr}(D)$ since every pair of edges can cross at most once, so it is natural to define the crossing number for straight-line drawings as the number of pairs of edges that cross in a straight-line drawing (e.g. [765]); later authors may have dropped the straight-line requirement without changing the way crossings are counted.[5]

Remark 1. As far as we know there are currently only three crossing number variants for which it is known that counting pairs of crossing edges as opposed to all crossings decreases the value of the crossing number: the constrained crossing number [582], the local crossing number (see that entry), and the geodesic crossing number (on a pseudosurface, see Footnote 101). ♦

Adjacent Crossings

There is some independent corroboration to Székely's thesis that cr-minimal drawings are at the root of the confusion between different crossing number notions; cr-minimal drawings also have the property that adjacent edges do not cross, and sure enough there are several instances in which researchers have ignored (sometimes at their peril) crossings between adjacent edges. Tutte, in a slightly different context, famously remarked that "adjacent crossings are trivial and easily got rid of" [747].

To show that adjacent edges do not cross in a cr-minimal drawing, one typically refers to two pictures, like the left and middle pictures of Figure 1.

While this works fine for the standard crossing number (though even there one needs an additional argument that shows how to remove self-crossings that can be introduced when swapping arcs), this need not be the case for other crossing number notions. For example, consider the pair crossing number in the scenario depicted in the right picture of Figure 1; swapping the arcs, or even just rerouting one of the arcs along the adjacent edge will lead to an increase in the pair crossing number, so the simple local redrawing moves common for cr do not seem to work. It is open whether a pcr-minimal drawing may have crossings between adjacent edges (this question is equivalent to whether $\mathrm{pcr} < \mathrm{pcr}_+$, see the entry on pair crossing number in Section 3).

Even for the standard crossing number this is not the end of the story for adjacent crossings. Here is a quote from a paper on Albertson's conjecture: if G has chromatic number at least r, then $\mathrm{cr}(G) \geqslant \mathrm{cr}(K_r)$.

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[3]Székely [724] writes: "How is it possible that decades in research of crossing numbers passed by and no major confusion resulted from these foundational problems? The answer is the following: the conjectured optimal drawings are usually normal and nice and the lower bounds (...) usually also apply for all kinds of crossing numbers.

[4]The first paper to define crossing number for arbitrary graphs also defined rectilinear crossing number [389].

[5]Recent examples defining crossing number as pcr include textbooks in combinatorics [735, 743, 758], and books in algorithms and complexity [69, 73, 77, 438].

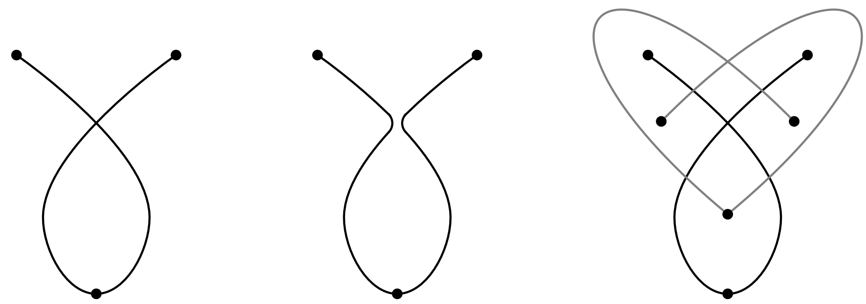


Figure 1: *(left)* adjacent crossing, *(middle)* removing adjacent crossing, *(right)* adjacent crossing that's hard to remove by local redrawing.

> "A crossing of two edges e and f is *trivial* if e and f are adjacent or equal, and it is *non-trivial* otherwise. A drawing is *good* if it has no trivial crossings. The following is a well-known easy lemma.
>
> **Lemma 1.1.** A drawing of a graph can be modified to eliminate all of its trivial crossings, with the number of non-trivial crossings remaining the same." [592]

The *independent crossing number*, $\text{cr}_-(G)$, only counts crossings occurring between independent edges. If Lemma 1.1 were true, it would imply that $\text{cr}_- = \text{cr}$, a question that's open to the best of our knowledge.[6] Fortunately, the use of Lemma 1.1 could be eliminated in this case [591], but wouldn't it be nice if we could establish $\text{cr}_- = \text{cr}$ and not have to worry about adjacent crossings anymore? The left and middle picture of Figure 1 explain why Lemma 1.1 looks so convincing: crossings between adjacent edges can easily be removed by local redrawing, but the right picture shows that this can create crossings between non-adjacent pairs of edges. A proof of a result like Lemma 1.1 will require a more global approach.

Question 1. Here are two simple-looking problems that illustrate our lack of understanding of adjacent crossings. (i) Can subdividing an edge change cr_- of a graph? (ii) Suppose a graph can be drawn on a surface so that all crossings in the drawing are between adjacent edges. Can the graph be embedded in that surface? An answer to the second question is known for the plane and the projective plane by virtue of the Hanani-Tutte theorems for those surfaces [618], but not for any other surface.[7] The first question is

[6] Start with a cr_--minimal drawing. By the lemma, all trivial crossings can be eliminated, only leaving "non-trivial" crossings, that is, crossings that count towards cr, so cr of the resulting drawing is at most cr_-. In the other direction, $\text{cr}_- \leqslant \text{cr}$ follows from the definition.

[7] The Hanani-Tutte theorem for a surface Σ is true if every graph which can be drawn on Σ so that no two independent edges cross an odd number of times is embeddable in Σ. The Hanani-Tutte theorem is known to be true for the plane (sphere) [205, 747] and the projective plane [618]. It is not known to be true for any other surface, and it has been announced that it fails for surfaces of genus 4 and higher [336]. In terms of crossing numbers, the Hanani-Tutte condition can be expressed as saying that $\text{iocr}_\Sigma(G) = 0$ implies that $\text{cr}_\Sigma(G) = 0$ for all graphs G.

open.

While not nearly as common as the pcr versus cr problem, cr is occasionally defined as the smallest number of independent crossings; this may again be due to the fact that for straight-line drawings, adjacent edges do not cross. For example, Moon [573] in one of the earliest papers on crossing numbers defines what amounts to the independent (geodesic) spherical crossing number which equals the geodesic spherical crossing number, since geodesics representing adjacent edges do not cross on the sphere. Nahas [584] defines the crossing number of $K_{m,n}$ as $\mathrm{cr}_-(K_{m,n})$. Papers on crossing minimization via linear programming also often ignore variables that encode crossings between adjacent edges. This is fine, of course, if the resulting program enforces that adjacent edges do not cross; otherwise, they compute cr_-.

Remark 2. As far as we know there are only two crossing number notions for which the independent variant is known to differ from the regular variant, namely the odd and the algebraic crossing number: there are graphs G for which $\mathrm{iocr}(G) < \mathrm{ocr}(G)$ and $\mathrm{iacr}(G) < \mathrm{acr}(G)$ [338]. The same paper also shows that prohibiting crossings between adjacent edges in monotone drawings can lead to an increase in the monotone odd crossing number. The same is true for the local crossing number, see Footnotes 118 and 121, and the simultaneous crossing number, see Footnote 156. For directed graphs, the bimodal crossing number may require crossings between adjacent edges in an optimal drawing. ♦

Crossing Lemma Variants

The crossing lemma, or crossing number inequality, established independently by Ajtai, Chvátal, Newborn, Szemerédi [32] and Leighton [528], is one of the most celebrated (and famous) results on crossing numbers.[8] In its original form, it shows that $\mathrm{cr}(G) \geqslant c \cdot m^3/n^2$, where $n = |V|$, and $m = |E|$. How does it fare for other crossing number variants, and pair and odd crossing number in particular? Crossing lemmas for other variants are listed in the compendium below.

The usual probabilistic proof of the crossing lemma for a crossing number γ proceeds in three steps: first, we observe that if $\gamma(G) = 0$, then G is planar, so Euler's formula applies, and $m \leqslant 3n - 6$, where $n = |V(G)|$, $m = |E(G)|$. In a second step, we argue that we can remove at most $\gamma(G)$ edges from G to reduce γ to 0, so $m - \gamma(G) \leqslant 3n - 6$, and, hence, $\gamma(G) \geqslant m - 3n$. In a third step, we consider a random subgraph G' of G, keeping each vertex with probability p. The expected number of vertices and edges in $G' = (V', E')$ are $\mathbb{E}(|V'|) = pn$ and $\mathbb{E}(|E'|) = p^2 m$. Fix a γ-minimal drawing D of G. Assuming each crossing in D which contributes to γ is caused by two independent edges, a crossing is associated with four endpoints. For the crossing to survive in D', the induced drawing of G', all four endpoints have to be kept, so $\mathbb{E}(\gamma(G')) \leqslant p^4 \gamma(G)$. Now G' fulfills $\gamma(G') \geqslant |E'| - 3|V'|$ (by the second step), so, taking expected values, we get $p^4\gamma(G) \geqslant p^2 m - pn$, or $\gamma(G) \geqslant mp^{-2} - np^{-3}$ (assuming $p \geqslant 0$). Choosing $p = 4n/m$ implies that $\gamma(G) \geqslant 1/64\ m^3/n^2$, as long as $m \geqslant 4n$ (which we need so $p \leqslant 1$).

[8] For a very readable introduction, see Terence Tao's blog entry [734], which also discusses applications to incidence geometry and sum-product estimates.

For $\gamma = \text{cr}$, this proof works just fine, and it's been claimed in the literature (e.g. [600]) that this proof also works for pair and odd crossing numbers. But there are two subtle problems. Consider the case $\gamma = \text{pcr}$, the case claimed by Leighton [528]: in the second step, the pcr-minimal drawing D may contain crossings between adjacent edges, and those contribute to pcr. Since we do not know how to remove adjacent crossings in general without increasing $\text{pcr}(D)$, we have to take adjacent crossings into account; since those survive with probability p^3, we would get a substantially worse bound than $\Omega(m^3/n^2)$ on $\text{pcr}(G)$. Alon [41], and Tao and Vu in their book on additive combinatorics [735] circumvent this problem by working with pcr_-, the independent pair crossing number, in which only the number of crossings of independent pairs of edges are counted. However, for that crossing number the second step is no longer obvious: if we have a drawing D with k independent pairs of edges crossing, then removing k edges yields a drawing in which all remaining crossings are between adjacent edges. Is that graph planar? The answer is yes, but it requires the Hanani-Tutte theorem (see Footnote 7) to prove so (at least we are not aware of a direct proof).

Pach and Tóth [613] work with $\gamma = \text{ocr}$, the odd crossing number, which only counts pairs of edges crossing an odd number of times. They use Hanani-Tutte in the first and second steps, but in the third step again assume that a crossing is associated with four endpoints, which may not be the case for ocr. However, their proof is essentially correct if read for $\gamma = \text{iocr}$, the independent odd crossing number, which counts the number of independent pairs of edges crossing an odd number of times. For iocr, the Hanani-Tutte theorem guarantees that we can remove $\text{iocr}(G)$ edges from G to make G planar, ensuring the correctness of the first and second steps. And since iocr *by definition* only counts independent pairs, the argument in the third step also works. We conclude that $\text{iocr}(G) \geqslant 1/64\ m^3/n^2$, as long as $m \geqslant 4n$. Since ocr, pcr, and pcr_- (as well as acr and iacr) are all bounded below by iocr, this immediately proves the crossing lemma for all these variants. The constant $c = 1/64$ in these cases is weaker than what is currently known for cr, but seems hard to improve [600, Remark 4.2], though it was shown $c = 1/34.2$ works for pcr_+ [20], and $c = 1/54$ for $\text{ocr}_\pm$ [478].

Remark 3 (Crossing Lemma on Surfaces). For the standard crossing number, extensions of the crossing lemma to arbitrary surfaces are known [698]. Does this imply crossing lemmas for pcr, pcr_-, or iocr on higher-order surfaces? Since iocr lower-bounds pcr and pcr_- (on any surface), we can focus on iocr_Σ. Since we do have a Hanani-Tutte theorem for the projective plane, N_1, the proof of the crossing lemma just sketched can be completed for the projective plane, and we obtain a crossing lemma for iocr_{N_1} and the other crossing numbers on the projective plane. Since we do not yet know whether the Hanani-Tutte theorem (or even the weaker version suggested in Question 1 (*ii*), which would be sufficient) holds for the torus, we appear to be stuck. And, since Fulek and Kynčl [336] showed that the Hanani-Tutte theorem fails for surfaces of genus 4 and higher, this approach will not extend to arbitrary surfaces. There is a solution which works for arbitrary surfaces, but has one issue, it relies on a major unpublished folklore result.[9]

[9]This idea is outlined in an answer to a mathoverflow question given by Kynčl [518].

Assuming the folklore result to be true, one can show that for every surface Σ, there is a surface Σ' so that $\text{iocr}_\Sigma(G) = 0$ implies that $\text{cr}_{\Sigma'}(G) = 0$ for every graph G [337]. Applying the crossing lemma for $\text{cr}_{\Sigma'}$, then yields a crossing lemma for iocr_Σ, for any surface Σ.[10] ♦

Conclusion

We are forewarned that there is some subtlety to defining the crossing number, but rather than seeing this as an issue, this gives us an opportunity. János Pach once said, in effect, "we don't need more crossing numbers, we need fewer crossing numbers". As a look at the compendium will show it may be too late for that. Some crossing number variants may have arisen by mistake, but most were defined with a specific purpose in mind. This purpose may be theoretical, aimed at developing a theory of crossing number (as Tutte [747] did with his crossing chains and iacr) or it may be practical, aimed at improving the layout of graphs (as in the Metro-line crossing minimization problem). The recent growth of graph drawing research and crossing minimization problems for very specific visualization tasks is important evidence for that. Some variants, such as the local crossing number or the maximum rectilinear crossing number, are so fundamental that they have been rediscovered over and over again under various names.

This survey of crossing number variants follows two main goals: to collect as many different crossing number variants from the literature as possible (unifying presentations and names), and to attempt a systematic description of what makes a crossing number. The results of this second step are presented first, in Section 2. The results of the first step are collected in the Compendium in Section 3. There also is an index of crossing number variants after the bibliography.

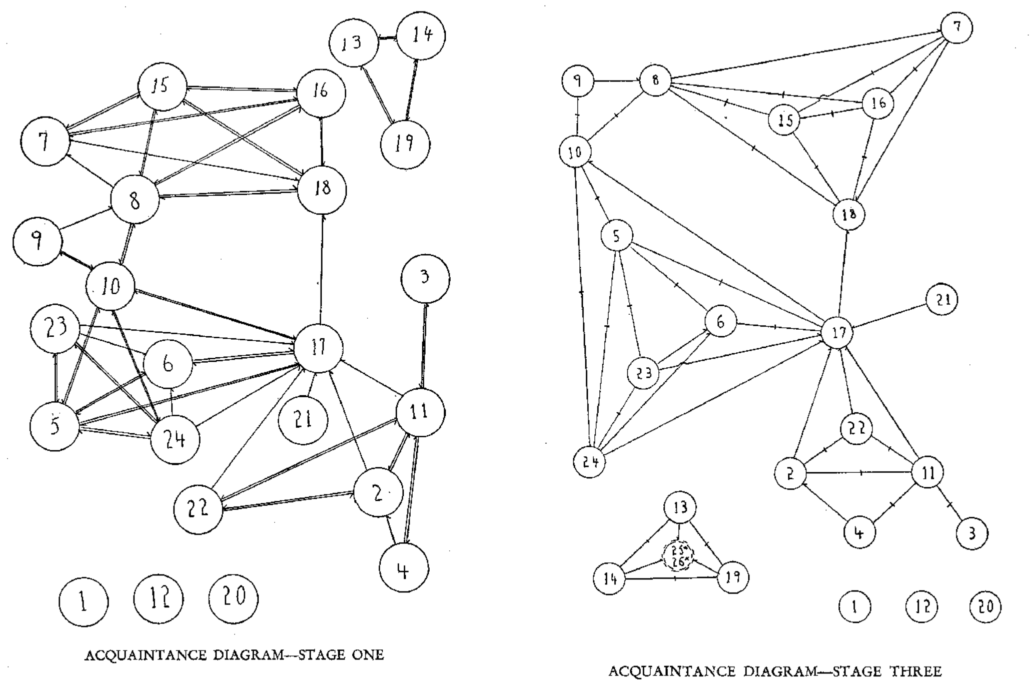
Originally, this survey was to contain a section on the history of the crossing number, but Beineke and Wilson's "Early History of the Brick Factory problem" [103] and Székely's "Turán's brick factory problem: the status of the conjectures of Zarankiewicz and Hill" [728] make this part mostly superfluous. We add two remarks on forerunners found after the papers by Beineke and Wilson, and Székely were published.

Remark 4 (Forerunners of Crossing Minimization in Sociology). David Eppstein [289] discovered the earliest known references to (general) crossing minimization.[11] They come from sociology, more specifically the area of sociometry which is concerned with measuring (and depicting) social relationships: in discussing sociograms (essentially graphs), Bronfenbrenner [148] in 1945 writes that "The arrangement of subjects on the diagram, while haphazard in part, is determined largely by trial and error with the aim of minimizing the number of intersecting lines". Sociograms were introduced in J.L. Moreno's "Who Shall Survive" [574] in 1934, however, the first edition of that book, while containing many interesting graph visualizations, does not seem to discuss crossing minimization. In

[10]Since this approach requires graph minor machinery, one should not expect explicit bounds on the crossing lemma constant, even for a fixed surface such as the torus.

[11]There are earlier references to crossing minimization when it comes to specific families of graphs [208, 497, 716], but none that are as general as these.

the later, 1953, edition [575], there is an interesting paragraph which reads: "A readable sociogram is a good sociogram. To be readable, the number of lines crossing must be minimized." This mantra occurs repeatedly in the literature on sociograms, and at least once in an earlier paper by Borgatta [139] who writes: "A readable diagram is a good diagram. To be readable, the number of lines crossing must be minimized. This may be taken as a primary principle in the construction of inter-action diagrams; the fewer the number of lines crossing, the better the diagram. The problem, then, is to find the procedure which best minimizes the number of lines that cross in a diagram." Borgatta then outlines a multi-stage heuristic for crossing minimization (start with a small number of high-degree vertices, drawn far apart, add vertices by decreasing degree, redraw diagram to improve drawings of subgroups), and illustrates his method by working out an example on 26 vertices and 43 edges, shown in Figure 2; his final drawing uses two crossings (which is optimal, since his graph contains two disjoint copies of K_5).



(a) (b)

Figure 2: Maybe the first published instance of a crossing minimization, reducing 16 crossings in (a) to the optimal 2 crossings in (b). Taken (with permission) from a 1951 article in the journal "Group Psychotherapy" by Edgar F. Borgatta [139].

The earliest reference (found so far) on crossing minimization seems to be a 1940 paper by Northway [589] in which she suggests the use of radial layouts; vertices (school children) are placed at various distances from a center based on some quantity (their scores); directed edges between them are drawn as straight-line arrows. She writes that "it has been convenient to use counters [...]. These are moved in the circles to which their score belongs and arranged to get the best "fit" among the individuals, i.e., to have as few long lines and crossing lines as possible." She also suggests that grouping vertices by some characteristic (in her example, sex), simplifies this task. These quotes are quite remarkable, and one wonders whether there is more early material on crossing minimization that is unknown in the mathematical literature. ♦

Remark 5 (A Forerunner of Crossing Minimization in Circuit Design). Electronic circuit design has long been a motivating source for crossing number studies, with Leighton's thesis [528] being one of the most famous examples connecting the two areas. Crossing minimization by practitioners is older, though, and the paper "Formulation and Solution of Circuit Card Design Problems through Use of Graph Methods" [501] from 1962 by Uno R. Kodres anticipates several later developments in the mathematical literature. This paper would be fifth on Vrťo's (chronological) list of crossing number papers, having been published just before the paper by Harary and Hill [389]. Kodres describes and implements a heuristic process for reducing crossings in biplanar (and k-planar) drawings using linear integer programming. His paper contains several other interesting ideas such as placing the vertices in a drawing at their barycentric coordinates (a year before Tutte's famous paper [745]), a discussion of bend-minimization when embedding a graph at prescribed vertex locations, a criterion for recognizing a minimal set of edges whose removal makes the graph bipartite, and a proof that $K_{7,7}$ is not biplanar. It seems unfortunate that this paper did not come to the attention of the growing graph theory community at the time. ♦

One aspect that remains to be studied, is the history of knot crossing numbers and their influence (or not) on graph crossing numbers. When it comes to methods of counting crossings, it seems that knot crossing numbers led the way; e.g. Tutte's theory of crossing numbers is based on counting crossings algebraically, as one would for the algebraic crossing number in knot theory, and as Gauß would have done hundreds of years ago [352, page 271–279].

Remark 6 (Axioms). What makes a crossing number a crossing number? We have chosen a descriptive/extensional approach for this survey, however, the material collected here may at some point make a basis for a prescriptive/intensional approach. As far as we know there has never been an attempt to axiomatize the notion of crossing number, either as the standard crossing number or as the family of crossing number variants. Although not plentiful, there are some candidate axioms based on common crossing number properties.

Embeddability Crossing numbers are generally considered to be "measures of non-planarity" or non-embeddability. It seems natural then to require that if $\gamma_\Sigma(G) = 0$ for some crossing number γ in surface Σ, then G is embeddable in Σ. Let us call this the *embeddability axiom*. For the standard crossing number this is true by definition (on any surface). For the independent odd crossing number it amounts to the Hanani-Tutte theorem (which is only known for the plane and the projective plane, see Footnote 7). For the confluent crossing number, the string crossing number and the quasi-crossing number, the embeddability axiom fails (complete graphs have confluent embeddings, there are non-planar string graphs, and K_{10} has a quasi-planar drawing). A stronger, quantitative version of this axiom would require that the removal of at most $\gamma(G)$ edges from G makes G planar. The intuition behind this strengthened version is that each crossing is caused by two edges, so a crossing can be eliminated by removing one of the participating edges. This axiom holds for the standard crossing number by definition (on any surface), and for the pair crossing number. It also holds for the independent odd crossing number in the plane and the projective plane, by the Hanani-Tutte theorem (Footnote 7), but, by [336] it fails on surfaces of genus 4 and higher. It also fails for the degenerate crossing number, in which more than two edges can cross in a crossing, and for any of the crossing numbers based on maximization.

Embedding By the same "measure of non-planarity" argument, a graph G that can be embedded in a surface Σ should have crossing number $\gamma_\Sigma(G) = 0$. Let us call this the *embedding axiom*. This axiom is trivially true for most crossing number variants, although there are some notable exceptions including crossing numbers defined via maximization (maximum crossing number, maximum rectilinear crossing number) and crossing numbers that require certain drawing conventions (e.g. bimodal, bipartite, convex, and orchard crossing numbers). For the rectilinear crossing number, the axiom amounts to Fary's (Steinitz's, Koebe's, Wagner's, or Stein's) theorem. It appears to be an open problem whether the axiom holds for the geodesic crossing number on other surfaces.[12]

Subgraph Monotonicity The *subgraph monotonicity axiom* requires that if G is a subgraph of H, then $\gamma(G) \leqslant \gamma(H)$. This is true (and trivial) for nearly all crossing number variants. We are aware of only two provable exceptions, the triple crossing number, for which triple-cr$(K_{5,3}) = \infty$ while triple-cr$(K_{6,3}) = 2$ [733], and the confluent crossing number (all complete graphs have confluent crossing number 0). For the maximum crossing number, monotonicity is a well-known open problem even if G is required to be an induced subgraph of H [652]. A stronger requirement is *topological minor monotonicity*: if G is a subdivision of a subgraph of H, then $\gamma(G) \leqslant \gamma(H)$. This is still true for a large number of crossing numbers, but is not known to hold for any of the independent crossing number variants, like cr_-, and typically fails for alternative representations (like the confluent crossing number). In contrast, most crossing numbers do *not* satisfy *minor-monotonicity* which has led to the definition of the minor (or minor-monotone) and the genus crossing numbers.

Surface Monotonicity The *surface monotonicity axiom* requires that if surface Σ has smaller genus than surface Γ, then $\gamma_\Sigma \geqslant \gamma_\Gamma$. We are not aware of any crossing number that does not fulfill this axiom. One could imagine sharper quantitative versions of this axiom, for example if Σ has smaller genus than Γ, then $\gamma_\Sigma(G) > \gamma_\Gamma(G)$ unless $\gamma_\Sigma(G) = 0$.

[12] Results in this direction seem to work with metrics of non-positive curvature [224, 450, 564].

One can imagine further axioms, for example based on what could be called the *spectrum* of the crossing number of a graph G: $\{\gamma(D) : D$ is a (simple) drawing of $G\}$. This notion has occasionally been studied for the crossing number [158, 159, 220, 371, 397, 403, 630], the rectilinear crossing number [158, 339, 717], the convex crossing number [158], and the edge crossing number [405]. Harborth [402] showed that the spectrum of K_{14} under cr is not a subset of the spectrum of K_{14} under the 2-page crossing number bkcr_2, and conjectured that K_{14} is the smallest complete graph for which the spectra of cr and bkcr_2 differ.[13]

It is probably unreasonable to expect an axiomatization of the (standard) crossing number; however, it may be reasonable to attempt to axiomatize sufficiently many standard properties of the crossing number that would show why many of them allow a crossing lemma. Or why many of them can be bounded within each other. ♦

2 A Systematic Approach

In this section we want to take a systematic approach to crossing number variants. The discussion is based on the crossing number notions collected from the literature and presented in Section 3, and the reader is asked to look for definitions there if they are not given in this section. Before reviewing crossing numbers, we begin with a discussion of crossings themselves.

What is a crossing? Typically, a crossing is defined to be a common interior point of two edges; hence, a shared endpoint (of two adjacent edges) is not considered a crossing. This distinguishes a crossing from an intersection of two edges.[14]

The definition as given also distinguishes a crossing from the point in the plane at which the crossing occurs (and this is good). The definition does, however, include points in which two curves touch; this is of no consequence for the standard crossing number since in crossing-minimal drawings no touching points occur, but for other variants, e.g. the odd crossing number, counting touching points as crossings would trivialize the notion. For Kleitman [492] a crossing requires that the two edges involve actually cross. This requirement leads to other issues if not handled carefully: take a drawing of K_5 with a single crossing and replace the crossing with a short line segment (so the two edges involved in the crossing run parallel for a short stretch). According to Kleitman's definition this drawing is free of crossings (even though it has an infinite number of intersection points). This suggests the importance of restricting drawings to drawings with a finite number of intersection points (which is what we will do) which causes a slight inconvenience when dealing with confluent drawings: in confluent drawings of graphs edges seem to overlap heavily. We resolve this by looking at confluent drawings not as drawings of the edges and vertices of the graph, but as a drawing of branches and switches that represent the underlying graph.

[13] Harborth mentions an unpublished paper that seems to establish significant parts of this conjecture.

[14] One subtlety already: it excludes from the notion of crossing any intersection occurring when an edge passes *through* a vertex, as opposed to ending there. Such intersections are typically prohibited, but what happens if we allow them?

We return to a more formal definition of crossing in Section 2.2.1 after discussing basic drawing conventions.

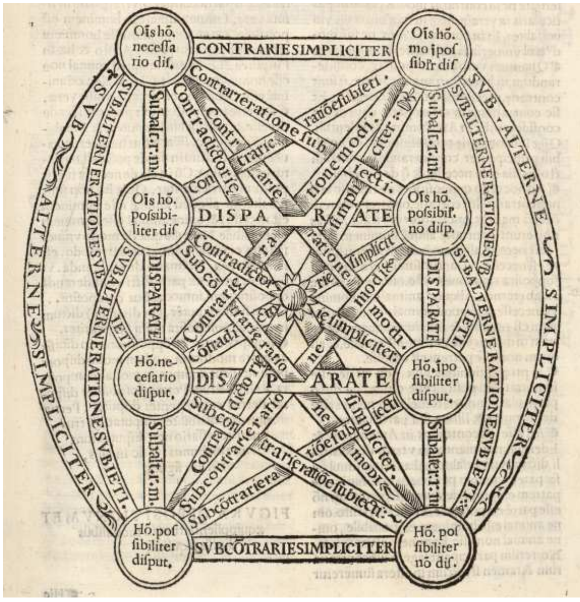


Figure 3: Drawing of K_8 from de la Vera Cruz' *Recognitio Summularum* with ribbons crossing through each other. The image is taken from the online (public domain) version of the book available through Primeros Libros at http://www.primeroslibros.org/. Page 36 contains the drawing of K_8, page 57 contains a drawing of $K_{4,4} - e$.

Remark 7 (Drawing Crossings). How do we draw a crossing? The most common way is to simply let the curves representing the edges cross, preferably at a large angle (RAC drawings require right angles); alternatively one can draw crossings as bridges or by using edge casing; see "Edges and switches, tunnels and bridges" by Eppstein, van Kreveld, Mumford and Speckmann [294]. There may be more options in alternative styles; for example, if vertices are represented by disks and edges as ribbons with boundary, then crossings can be visualized by ribbons passing above or below each other, see for example the 16th century drawing of K_{12} in [517, Figure 6] which has both vertex and edge labels (illustrating a modal square of opposition). Alonso de la Vera Cruz uses an interesting twist to visualize K_8 (in his 1554 *Recognitio Summularum*, again for a square of opposition). He not only has ribbons passing above and below each other, but also through each other, see Figure 3; for background on the book, see [157]. ⧫

For a survey of graph layout in the presence of crossings, see [442, Chapter 11].

Most of the research on crossing numbers seems to have been done in English, but there are terms for crossings and crossing numbers in other languages. In German there is *Kreuzung*, *Schnitt* and *Doppelpunkt* for crossing and *Kreuzungszahl* for crossing number.[15]

[15]Steinitz [717] uses the term Doppelpunkt; it stems from the algebraic tradition and is now used for

In French, we have *points d'intersection* [750] and *croisement* for crossings[16] and *nombre de croisement* for crossing number. In Italian there is *incrocio* for crossing and *numero d'incrocio* for crossing number. Spanish uses *número de cruce* for crossing number.

2.1 A General Notion of Crossing Number

There are (at least) three main dimensions which influence the specific notion of crossing number one ends up with: the *drawing style*, the *method of counting*, and the *mode of representation*. Within each dimension multiple decisions can be made, both global and local. Global decisions in the drawing style include: underlying surface, straight-line edges, monotone edges, centrally symmetric drawing; local decisions include: no three edges sharing the same interior point, no edge passing through a vertex; for method of counting, again we have global decisions such as: do we count crossings between adjacent edges or edges that cross evenly and local decisions: each crossing counts 1 or ± 1 (depending on orientation), etc.; mode of representation is typically global; in the standard mode a curve carries exactly one edge, but there are alternative models like bundled crossing drawings, confluent graph drawing and simultaneous graph drawing in which a curve can carry more than one edge.

Many of these decisions have rarely been made explicitly; they were either assumed implicitly or not considered at all. Even as one surveys the surprisingly large collection of different crossing number variants that exist, one often finds that they differ from the standard crossing number in at most one of the three dimensions (although there are exceptions such as the local toroidal crossing number, the book edge crossing number, or the monotone independent odd crossing number).

Within this framework we can attempt a general definition of a crossing number ψ: given a graph G consider a particular drawing D representing G (via some mode of representation). Assign to each crossing in D a value (typically 1, but could be -1, e.g. for algebraic crossing number; values in $\mathbb{Q}$, $\mathbb{C}$ or some group may be interesting). Now calculate the crossing number $\psi(e, f)$ for each pair of edges.[17] This is typically done as the sum (or absolute sum) of the values of the crossings shared by e and f.[18] Finally, $\psi(D)$ is calculated by combining all the values of $\psi(e, f)$, typically by summing them up (over all unordered pairs). Then $\psi(G)$ is the minimum (sometimes maximum) over all $\psi(D)$ where D is an admissible drawing (depending on the drawing style) that represents G. This generic definition of crossing number describes nearly all crossing number variants reviewed in this paper. In any case, we are trying to be descriptive, not prescriptive.

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crossings in knots. *Schnittzahl* typically means intersection number from algebraic geometry rather than crossing number.

[16] Leclerc and Monjardet [525] use *points non signifiants* (as opposed to the points representing vertices).

[17] One can also define the crossing number by counting crossings along each edge (and dividing the total by 2) but pairwise counting is the standard. This would seem to exclude some variants, like the local crossing number or the triple crossing number, but see the discussion in Example 1. It does exclude the quasi-crossing number, which requires counting triples of edges.

[18] One could consider multiplication or maximization instead of addition.

Example 1. Let us check some of the crossing number variants to test the bounds of our general crossing number notion. For definitions, see the compendium.

Natural fits. The degenerate crossing number fits the general definition above: a crossing shared by k edges is weighted as $1/\binom{k}{2}$. Independent crossing numbers can be captured by assigning values of 0 to crossings between adjacent edges. The Rule + variants introduced by Pach and Tóth [612] are captured in the drawing style: adjacent edges are not allowed to cross (alternatively, we could assign a value of ∞ to each adjacent crossing). The triple crossing number (in which all crossings have to be triple crossings) can be captured by pairwise counts (each triple crossings gives three double crossings; since only triple crossings are allowed we can divide by 3 to get the triple crossing number). The pair crossing number maximizes (rather than adds) the number of crossings along each pair of edges.

Acceptable fits. The local crossing number would be a more natural fit for counting crossings edge-wise (as opposed to pairwise), but it can be made to fit the general definition. It is expressible as $\max_{e \in E} \sum_{f \in E} \mathrm{cr}(e, f)$.

Forced fits. The minor crossing number can be made to fit the general description of crossing number above, albeit with some force: say a drawing D represents G if D is a drawing of a graph containing G as a minor. One could question whether this is a natural interpretation, but we decided to include this notion. The degenerate and bundled crossing numbers can also be made to fit the definition by defining an intermediate notion of drawing.

Not a fit. The *skewness* of a graph, the smallest number of edges that need to be removed from a graph to make it planar, does not fit the general definition of crossing number given above. One can debate whether skewness is a crossing number variant, but we decided to include it.[19] It is easy to abbreviate the standard definition of crossing number to the point where it incorrectly defines a notion similar to skewness, e.g. "Is the crossing number of $G \leqslant K$? i.e. can G be embedded in the plane in such a way that no more than K edges cross?" [416], see the *edge crossing number*. Similar in spirit to skewness is the *splitting number* of a graph, that is, the smallest number of vertex splits required to make the graph planar. This notion does not fit our general definition of crossing number, and we do not include it in the survey. The *nodal crossing number* which is similar to the local crossing number, but looks at the total number of crossings with any edge incident to a vertex, and then maximizes over all vertices. One could think of it as a local crossing number for hypergraphs. Even though it does not fit our general model, we decided to include it because of its ties to the local crossing number.

Let us next review some of the options available for creating a crossing number within the three dimensions we identified; we start with a discussion of drawing styles, followed by methods of counting, and modes of representation.

[19] This is a change in version 7 of the survey.

2.2 Drawing Styles

In this section we discuss different drawing styles; we make a rather rough distinction between basic drawing properties that are often taken to be part of the very definition of a drawing, sometimes called a good drawing and what may more properly be called a style of drawing (Section 2.2.2). We treat drawing surfaces separately in Section 2.2.3.

2.2.1 The Basics

A drawing stripped of any mystic ballast is just a mapping of a graph (vertices and edges) to a surface. With this generous definition of drawing, the whole graph could map to a single point, losing all structure. There has not been much discussion of what assumptions to make on a drawing, Eggleton's thesis [280] is one of the rare places in which some of these issues are brought up. We first discuss issues related to drawing **vertices and edges**.

An edge is represented by a curve. But what type of curves do we allow? Do we want a curve to be connected? In the work on odd and algebraic crossing numbers edges are often split into multiple components temporarily. Becker, Eick and Wilks [102] suggested "line shortening" for geometric drawings: only the ends of edges are drawn (without further restrictions this removes all crossings, see [150] for a recent paper). If we require the curve to be connected (but not path-connected), we can get some anomalies, for example Kratochvíl [513] notes that every graph is a string graph if strings are allowed to be arbitrary connected curves (string graphs are intersection graphs of simple curves in the plane). So we should require edges to be simple plane curves, which are homeomorphic images of the unit interval. This is the typical choice when defining a drawing. However, it does preclude edges from crossing themselves which may be desirable in some contexts. We discuss the issue of self-intersections below. For practical reasons, it may make sense to "fatten up" edges, we discuss this possibility below together with vertex representations.

Vertices are endpoints of the edge. Often edges are defined as open arcs at which point one has to specify that the points representing the vertices of the edge occur at (opposite) ends of the arc. One could easily imagine a drawing of K_5 with the 5 endpoints as isolated points and 10 parallel arcs representing the edges (maybe with the ends of the arcs labeled by the names of the vertices). One could also consider this a special case of allowing a vertex to be represented by multiple points (see below).

Vertices are represented by points. Suppose we represent vertices by disks and only require edges to attach at the boundary of the disk. This idea was (ab)used by Dudeney in his original solution to the Gas, Water, Electricity problem [262, Problem 251] which essentially asks for a crossing-free drawing of $K_{3,3}$: Dudeney has the final path—which would cause a crossing—pass through one of the houses (vertices) which he drew as rectangles. Suppose we do allow edges to pass through vertices.

If we allow such crossings for free (as Dudeney suggests) we trivialize the notion of crossing number: every graph can be represented so that a vertex is a disk, edges end on the boundary of the disk representing their endpoint, edges are allowed to pass through the disk, and no two edges cross. However, we could consider allowing edges to pass through vertices for a cost. As far as we know no such notion has been investigated, although there are crossing numbers which count crossings other than edge crossings (e.g. the spine crossing number).

One reason to relax the requirement that vertices be points may be that the vertices represent objects with internal structure that has to be captured. Eades and Lai [277, 520] called these *practical graphs*, and suggested a two-step approach: first use a general layout algorithm for the abstract graph, and then, in a second step, lay out the graph with vertices having various shapes; the goal of the second step is to avoid or remove overlap between vertices and vertices with edges. Kodres [501] studied a similar problem in the context of electronic circuit design allowing multiple planes. Waddle [756] discusses port diagrams (in which vertices are rectangles, and edges attach at a port) to visualize data structures; his goal is to find drawings that avoid crossings within vertices, also see [490, 686]. Duncan, Efrat, Kobourov and Wenk [269] investigated planar drawings with "fat edges", where vertices are disks and edges have thickness.[20] Van Kreveld [516] suggested the notion of bold drawings in which vertices are disks and edges are rectangles. In computational biology, such drawings have been suggested for visualizing chromosomes [322]. Medieval scholars used a similar style (vertices as disks, edges as ribbons) to visualize squares of opposition (in logic) as we saw in Figure 3. Other choices for representing vertices include curves—the string crossing number is based on that idea—and graphs: If we minimize the crossing number by allowing vertices to be replaced by arbitrary connected graphs, we obtain the minor crossing number.

Each vertex is represented by a single point. One can easily imagine a vertex being represented by multiple points. For example, how would the standard crossing number be affected if every vertex could be represented by two points (which together are incident to all the edges incident to the original vertex), we could call this the *duplicate crossing number*.[21] This seems nearly the same (is it?) as asking for the crossing number of the graph on an n-*spindle*, the pseudosurface resulting from a sphere by pinching (identifying) n pairs of distinct points. If $n = |V(G)|$, then the duplicate crossing number of G is at most the crossing number of G on the n-spindle, since we can simply pinch every vertex with its duplicate. The duplicate crossing number also resembles the biplanar crossing number: here too every vertex is represented by two points, but the duplicate points live on a different sphere, so there cannot be an edge between the original and the duplicate vertices. There is re-

[20] The discussion of edges with width and points with extension is much older in "practical geometry"; Hjlemslev [418, 419] attempted an axiomatization, which earned him the scorn of Wittgenstein [772, Gesichtsraum, p.59].

[21] Bertin [116, Figure 19, p.270] suggests using diagrams in which every vertex is duplicated.

search on whether graphs can be planarized by multiplying vertices, following ideas of Fellows and Negami from the 1980s on planar emulators and covers, see [193] for a more recent overview. Eades and Mondança looked at the splitting number, the smallest number of vertex splits required to make a graph planar, and its relation to graph layout [275]. Unfortunately, the splitting number is **NP**-complete [304]. Finally, one can turn a cyclic layout into a linear layout by repeating one of the layers (for example, turning a cyclic level crossing number problem into a k-layer crossing number problem).[22]

Different vertices are mapped to different locations. This is generally assumed for graph drawings though there are some exceptions. For example, when speaking of realizing a linkage one does not care about vertex overlap, and the definition of a Euclidean graph similarly allows multiple vertices at the same location. For crossing numbers, this has not been a major issue; the only crossing number that allows vertex overlap is the diagonal crossing number introduced by Negami (though one could argue that the simultaneous crossing number also is an instance). For visualization purposes one could imagine a model in which different vertices are allowed at the same location as long as edges adjacent to a particular vertex are consecutive in the rotation. Buchheim, Jünger, Menze, Percan [154] suggest the notion of bimodal crossing number which has some similarity.

Edges are not allowed to pass through vertices. Again this restriction is naturally violated by linkages and Euclidean graphs. For example, a triangle with side-lengths 1, 1 and 2 can only be realized if we allow the edge of length 2 to pass through the vertex it is not incident on. Edges may also pass through vertices while redrawing the graph, e.g. see [624, Theorem 4.6]. We are not aware of any crossing number variant that allows edges to pass through vertices (although it would probably lead to a non-trivial notion if we do not allow edges to make sharp turns while passing through a vertex), unless one interprets the minor crossing number or Metro-line crossing number in this way.[23] Passing through a vertex may be more palatable if vertices are represented not by points but by disks (or disk-homeomorphs), as discussed earlier.

We next turn to issues regarding **intersections between edges**.

Edges are not allowed to touch. Without becoming too technical, let us agree that a touching point is a common point of two edges so that at least locally (close to the point), the two edges can be separatedrectilinear separated by a curve. Allowing touching points leads to undesirable effects. For example, we already mentioned that allowing touching points trivializes the odd crossing number: take any drawing of a graph, if two edges cross oddly, then add a touching point between them close

[22] This is beautifully illustrated by an example from Bertin [116, Figure 4, p.109].

[23] We should mention a paper[26], that repeatedly uses the term $m + \overline{\text{cr}}$ to denote the total number of crossings in a geometric drawing including m crossings of edges through vertices.

to one of the crossings, so all pairs of edges "cross" evenly (since a touching point would count as a crossing), showing that every graph has odd crossing number 0 if touching points are allowed and counted. Another variant that would be affected is the maximum crossing number; if we allow touching points, C_4 can be drawn with 2 "crossings", but it is known that C_4 is not thrackleable, so its maximum crossing number (under the standard definition) is 1.

The real reason touching points are undesirable, however, is that they lead to ambiguous drawings. While a drawing is defined as a mapping, we only see the result of the mapping, which is a subset of the plane (or some surface). Even if we assume that we know where the vertices are located we may not be able to distinguish a crossing point from a touching point just by looking at the drawing: imagine four curves entering a point, two from the left and two from the right, all with one common tangent. Then the drawing does not tell us whether we are looking at a crossing or touching point. The problem remains even if the curves don't meet at a common tangent: when we see an intersection looking like an x we automatically assume that it's a crossing, however, if touching points are allowed that need not be the case since we generally do not assume that the curves used to represent edges are smooth (polygonal arcs are common in representing edges, so a restriction to smooth curves would exclude a popular way of drawing edges).

No self-intersections. Do we allow edges to intersect themselves (either crossing or touching)? This issue is rarely discussed (if one thinks of an edge as adjacent to itself then a prohibition on adjacent crossings will automatically exclude self-intersections). The presence or absence of self-intersection is the difference between Pach and Tóth's degenerate crossing number, $\mathrm{dcr}(G)$, and Mohar's genus crossing number [567], $\mathrm{gcr}(G)$. Mohar conjectures that $\mathrm{dcr}(G) = \mathrm{gcr}(G)$, but this seems far from obvious. Similarly, it is not clear whether allowing self-intersections reduces acr_+, one of the algebraic crossing numbers. Since edges are equipped with directions for algebraic crossing numbers, the standard trick for removing self-intersections does not work, see [338].[24]

The number of intersections in the drawing is finite. We do not allow two edges to overlap in more than a finite number of points. If some drawing style (like confluent drawings) seems to require this, we introduce an intermediate representation (train tracks consisting of branches and switches in confluent drawings), and define the crossing numbers for that representation instead of for the underlying graph.

So even at this basic level there is reasonable room for disagreement on what makes a drawing. Different crossing numbers have different demands, and a single definition will not do all of them justice, but let us try. We will generally understand a *drawing* to fulfill the following requirements: each vertex will be represented by a unique point.

[24]Quite possibly the first description of how to remove a self-intersection from a (closed) curve can be found in Clifford's "Common Sense of the Exact Sciences" [222, Chapter III(12)]; he calls closed curves "tangles", and crossings "knots".

An edge e in a drawing is a homeomorphic mapping from $[0,1]$ to the topological space of the drawing so that $e(0)$ and $e(1)$ are the endpoints of the edge, and $e(0,1)$ does not contain any vertices. An *intersection* of two edges e and f is a point $(s,t) \in [0,1]^2$ so that $e(s) = f(t)$; two edges are not allowed to touch. If $(s,t) \in (0,1)^2$ we call the intersection a *crossing*. By definition, any intersection that is not a crossing must be a common endpoint. We require that the total number of intersections in a graph is finite.

This notion of drawing will work for most crossing numbers we will see below. There are two conditions we will occasionally relax: we will allow edges to touch for some variants, and an edge will sometimes just be a continuous mapping from $[0,1]$ to allow self-intersections. A *self-intersection* of an edge e is $0 \leqslant s < t \leqslant 1$ so that $e(s) = e(t)$, it is a *self-crossing* if $0 < s < t < 1$. The only self-intersection which is not a self-crossing is an endpoint of a loop (in multigraphs).

At the next level we consider additional assumptions that are sometimes made on drawings. Drawings with these additional properties are typically called normal or good. It is often the case that crossing number optimal drawings, that is, drawings which minimize the value of a crossing number for a given graph have all of these properties, so sometimes they are assumed automatically. This assumption is fair for the standard crossing number,[25] but it does fail for some other variants (e.g. in a constrained crossing number optimal drawing two edges may have to cross more than once [582]). So we will not generally require these additional properties. They have been discussed in detail by Székely [724], but also by Winterbach [771].

Every two edges cross at most once. Drawings in which every two edges cross at most once are often called *simple*, but this term has at least three identifiable meanings. The original definition may go back to Ringel [655] who used simple to mean that every two edges intersect at most once (so adjacent edges cannot cross). This is more restrictive than only requiring that every two edges cross at most once. If we want to make this distinction, we will use *intersection-simple* (for Ringel's notion) versus *crossing-simple* or just *simple* (since this usage is more common these days). The third meaning of simple is to only allow each edge to cross at most one other edge. We will avoid using simple with this third meaning (unfortunately, the simple crossing number is named for this stricter notion of simplicity). We follow tradition in denoting crossing number variants that assume their drawings are simple by placing a $*$ in the super-index; requiring drawings to be simple does not affect most crossing numbers, e.g. $\mathrm{cr}^* = \mathrm{cr} = \mathrm{pcr}^* = \mathrm{ocr}^* = \mathrm{acr}^*$ and $\mathrm{ecr} = \mathrm{ecr}^*$.[26] There are some exceptions, however. A drawing realizing the constrained crossing number, the degenerate crossing number or the local crossing number of a graph may require non-simple crossings, and the notion of min-k-planarity depends on whether simplicity is required [425].

Adjacent edges do not cross each other. This rule was called Rule + by Pach and

[25] As was realized early on, e.g. in [466, 655].

[26] cr^* should not be confused with the simple crossing number which is based on a stronger requirement: each edge is allowed to cross at most one other edge.

Tóth [612], the drawing style itself has also been called *semisimple* [87], and *star-simple* [317]); the similar-looking Rule − is not a drawing rule but affects the counting of crossings: crossings of adjacent edges are allowed, but they do not count. For the standard crossing number, $\mathrm{cr} = \mathrm{cr}_+$, but no similar results are known for other crossing numbers. The only separations we are aware of are for the monotone odd crossing number, mon-ocr, here $\text{mon-ocr}(G) < \text{mon-ocr}_+(G)$ for some graph G [338], and the local crossing number, where $\mathrm{lcr}(G) < \mathrm{lcr}^*(G)$ is possible. The odd crossing number is sensitive to the effects of Rule −: $\mathrm{iocr}(G) < \mathrm{ocr}(G)$ for some graph G [338].

Finally, there is one more restriction which is often made:

At most two edges cross in any point. Depending on how we count, this requirement is not strictly speaking necessary: a crossing is a common interior point of two edges. If k edges cross in the same point, then there are $\binom{k}{2}$ crossings by definition of crossing. To make this point clear, the literature often refers to *pairwise crossing* in the definition of crossing number.[27] A crossing shared by k (distinct) edges can be replaced by k (double-) crossings by perturbing the edges.[28] This assumes that we do not allow touching points, that is, every two edges actually cross at the crossing point (otherwise perturbations may introduce more than k crossings which may, or may not, be reducible based on other drawing conventions). Crossing numbers which allow multiple crossings include degenerate and genus crossing number.

2.2.2 Style of Drawing

Once we get beyond the basics of what constitutes a drawing there are various choices to be made that influence the appearance of the drawing, vertices and edges, as a whole; we are calling this the *style of the drawing*, an admittedly vague term. There seems to have been very little systematic work on this with the exception of Bertin's "Semiology of Graphics" (originally published in 1967). Bertin's book contains a valuable section on networks [116, Part II] which could form the basis of a modern treatment from the perspective of graph drawing. Bertin identifies, among others, linear drawings (book drawings in two pages), circular (that is, convex) drawings, hierarchical drawings, and perspective drawings. For example, about convex drawings he writes "By arranging the elements [...] on a a circle, any relationship can be transcribed by a straight line. This is the construction which produces the least confusing images, whatever the number of intersections stemming from the raw data." [116, p. 271]. This seems like good common sense, and sociologists had used this technique for years [148, 574, 575], but there has been little experimental work on this. Purchase [638, 639] has started investigating metrics based on common aesthetic

[27] While this clarified the method of counting, assuming the reader understood that that was the intention, it may have been a small step in the confusion of the crossing number with the pair crossing number.

[28] Tait [729] in 1877 describes this as follows: "By infinitesimal changes of position of the branches intersecting in it, a triple point is decomposable into 3 double points, a quadruple point into 6, and generally an x-ple point into $\frac{x(x-1)}{1\cdot 2}$ double points". Tait is taking about closed plane curves.

criteria (including crossing minimization, bend minimization, and angle resolution), and there has also been work on angle resolution in particular [446, 449], and how different drawing aesthetics combine [445, 447].

If we look at what drawings researchers have used in practice, two dominant styles emerge, both focussed on edges. Edges are either drawn as curves (or polygonal arcs for computational purposes) or as straight-line segments (or geodesics in metric surfaces).[29] Not surprisingly, the traditional crossing number, cr, and the rectilinear crossing number, $\overline{\text{cr}}$, have remained the main crossing number variants, and many other crossing numbers are wedged between cr and $\overline{\text{cr}}$ since they are obtained by restricting cr or relaxing $\overline{\text{cr}}$. Some variants have been based on restricting common parameters for these drawings; e.g. the t-polygonal crossing number allows at most $t-1$ *bends* in each edge. One could imagine restricting the number of available *slopes* (t-polygonal, k-slope crossing number) or the set of available slopes (e.g. orthogonal drawings, in which all edge segments are axis-parallel), but, as far as we know, this has only been studied for embeddings, not drawings; the crossing minimization problem for port diagrams, which often employ orthogonal drawings, has been studied [490, 686, 756], but no crossing number notion has been explicitly defined. Finally, one can control the angles at which edges meet; the *angular resolution* of a drawing is the smallest angle between any two edges at a common endpoint; more recently, the *crossing resolution* of a drawing has been introduced as the smallest angle between any two edges at a crossing [246]; in *RAC (right-angle crossing) drawings* all crossings have to be at right-angles [250]. Recent progress on the rectilinear crossing number has been based on relaxing the rectilinear drawing requirement to pseudolinear drawings, leading to the pseudolinear crossing number, $\widetilde{\text{cr}}$. It seems to capture both the combinatorial and geometric nature of the rectilinear crossing number well enough to have led to the conjecture that $\widetilde{\text{cr}}(K_n) = \overline{\text{cr}}(K_n)$ [89], but so far this crossing number has not been investigated for other graphs (with the exception of [417]). Further relaxing pseudolinearity to x-monotonicity leads to a whole group of crossing numbers (monotone crossing numbers). Analogously to relaxing rectilinear to pseudolinear drawings, one could relax spherical to pseudospherical drawings to study the spherical crossing number; this has only been done for complete graphs so far [67], without formally introducing a *pseudospherical crossing number.*

A couple of other drawing styles have been added to the graph drawing toolbox recently; there are Lombardi drawings [268], partially drawn lines [102, 150], drawings with fat edges [269], and bold drawings [516], though we are not aware of any crossing number variants based on them. However, reviewing the compendium of crossing number variants suggests that style decisions are typically not made for purely aesthetic reasons, but to reflect some structural characteristics of the graph. For example, the vertices of the graph may be ordered, in x or y-direction (or both) and a drawing has to represent this ordering (or both orderings), or the graph may be bipartite or k-partite, suggesting drawings in which vertices in the same group are visualized close to each other. There is not always a need to create a new name or symbol for a crossing number that is created in this way;

[29] Eppstein [288] has given us a detailed summary and history of various curve drawing styles. Many of those have not been explored in the context of crossing minimization.

for example, if we weight the edges of the graph, it is quite natural to interpret $\mathrm{cr}(e, f)$ as $w(e) \cdot w(f)$ and we can continue to write $\mathrm{cr}(G)$ for the weighted crossing number of G, or $\mathrm{cr}(G, w)$ is we want to emphasize that G is equipped with a special structure. The following list collects style choices made based on structural features of the graph.

Orderings of the vertices. If the vertices of the graph are equipped with a total or partial order, it seems natural to arrange the vertices along a line (or a circle), but then additional restrictions on drawing the edges are necessary to get new variants. For the line, this is done by the fixed linear (total order) and the anchored (partial order) crossing numbers. If one interprets the ordering as ordering the x-coordinates of the vertices and one requires edges to be drawn as straight-line segments (or x-monotone curves), one gets variants of the monotone or leveled crossing numbers. If one interprets the ordering as ordering the vertices by distance from the origin, one gets the radial crossing number. If one interprets the ordering as an angular ordering around the origin, one gets the cyclic level crossing number.

One could also imagine vertices being ordered with respect to both x- and y-coordinates (corresponding to directions NW, NE, SE, SW). Eades, Lai, Misue, and Sugiyama [278, 558] called this an *orthogonal ordering* and studied it as a way to preserve the mental map of a graph in a redrawing. In crossing number terms, this suggests the (so far) uninvestigated bi-monotone crossing number.

Partite Graphs. For bipartite or k-partite graphs it is natural to require that all vertices in a particular group are somehow drawn together; for example, they may lie on a common straight line. For $k = 2$ this gives the bipartite crossing number. For larger k there is the convex k-partite crossing number which requires the vertices to lie on the boundary of a disk so that vertices in the same group are consecutive. Partitions can also be placed on concentric circles (radial crossing number), or parallel lines. If the partitions are ordered (and the vertices are assigned to fixed partitions), we are back in the "Orderings of vertices case" with radial and leveled crossing number. So far, there hasn't been an attempt at a free radial or a free leveled crossing number.

For bipartite graphs there also is the notion of *(line)-separated drawings*, in which there is a line that crosses every edge exactly once, thereby separating the vertices on opposite sides of the bipartition; this notion has been studied in passing, but not systematically, for example, it appears in [90, 194].

Ordering of edges at vertices. If we prescribe, at each vertex, the cyclic ordering of the ends of edges at that vertex, the *rotation*, we are looking at crossing numbers with rotation system. There may also be restrictions on the rotation system based on other structural properties. For example, in a directed graph we may want all the incoming and all the outgoing edges to be consecutive, giving us the bimodal crossing number. Another way in which the rotation at a vertex can be constrained is by identifying its neighbors with leaves of a tree and restricting the ordering of the leaves to an ordering corresponding to an embedding of the tree. This is related to the idea of tanglegrams in computational biology, and has been studied for the bipartite crossing number, and the k-layer crossing number.

Directed edges. A directed acyclic graph can be understood as a graph with a partial ordering of the vertices, leading to hierarchical drawings (upward crossing number), recurrent hierarchical drawings (the uninvestigated clockwise crossing number) or, less restrictive, bimodal drawings (bimodal crossing number).

Disconnected graph. There is not much to say about disconnected graphs in the plane, components are typically moved apart and drawn separately. Interesting problems start appearing when a disconnected graph is drawn on a higher-genus surface.

Pairs of Graphs. Pairs (or tuples) of graphs are no different from disconnected graphs, unless there is some type of interaction between the graphs, for example, a shared vertex set. At that point, there are drawing styles to model different types of interaction, e.g. simultaneous crossing number (shared vertices and edges), red/blue crossing number and joint crossing numbers (shared canvas).

Edge-coloring. If a graph has multiple edges, we can think of the graph as a union of multiple graphs on the same vertex set and apply ideas from "Pairs of Graphs". We could also assign different weights to crossings depending on the colors of the edges that cross (weighted crossing number); one particular example would be to only count crossings between edges of the same color (simultaneous crossing number) or different color (red/blue crossing number). On the other hand, some visualizations, such as metro-line drawings, are naturally done using edge colorings.

Edge-weights. Simple edge weights can be modeled using the weighted crossing number.

Labelings. There are various algorithms and heuristics for labeling graphs, see [474] for a survey. Labels can be drawn within the object to which they apply, leading to styles in which edges and vertices are thickened up as in [269, 516] or the medieval drawings mentioned in Remark 7. We are not aware of any crossing number variants taking the presence of labels into account.

Vertex-coloring. If the vertex coloring is proper, we are back in the case of partite graphs. If it is not, different colors may denote different types of vertices. E.g. the color of a vertex may encode which boundary component (of a surface with holes) a vertex lies on.

Partially embedded graphs. One may want to minimize the number of crossings in the drawing of a graph G which has been partially embedded, this leads to the constrained crossing number. Interesting, but as far as we know, uninvestigated, special cases occur if the locations of some (or all) of the vertices are fixed and the number of bends along each edge is restricted.[30] One may also consider the variant that instead of an embedding one is given a simple drawing, and wants to extend to a crossing-minimal simple drawing [342, 384].

Clusters. There has been much research on clustered drawings in which vertices are grouped into hierarchically nested regions. There are various types of crossings (edge-edge, edge-region, region-region). Typically, all of these crossings are prohibited, and there is significant research on c-planarity (clustered planarity) whose complexity it still open. Recently a first step was taken into allowing some of these types of crossings [50], but a formal notion of a *clustered crossing number* has not yet been introduced. In the visualization of large data sets, one can imagine vertices being located in given geometric clusters, for example the tiles of a 2-dimensional grid, and counting the crossings between edges and tile boundaries [174].

Symmetry. If a graph is symmetric, that is, has some non-trivial automorphism π, one can ask whether there are drawings of the graph which *show* π. For example, one can ask for π to be induced by an isometry of the plane, and minimize crossings under that constraint [153].

2.2.3 Drawing Surface

It's natural to think of a crossing as happening in the plane, so it's hardly surprising that crossing numbers are typically defined for the plane or for locally planar manifolds: surfaces, in other words.[31]

We need to decide on which surface we draw the graph; typically, this is the plane or the two-dimensional sphere S^2 (which can make a difference if metric conditions are in place, as in the geodesic crossing number). Crossing numbers on other surfaces, orientable, S_g, and non-orientable, N_g, were investigated in the earliest papers, including the toroidal crossing number [382] and crossing numbers on the Klein bottle [506]. Often special notations were introduced for surface drawings; we'll follow the convention to write the surface in the index; so cr_{N_1} is the projective plane crossing number and pcr_{S_1} is the toroidal pair crossing number (which has not been investigated as far as we know).

The surface may have holes, in which case some vertices may be forced to lie in certain boundary components (for two holes: radial crossing number with two levels), maybe with their order specified (map crossing number, anchored crossing number). We may also allow disconnected surfaces, for example multiple planes (as in the k-planar and the geometric k-planar crossing numbers).

[30] A potential application is described in [126]; the paper studies the number of crossings in electric transmission networks; vertex locations are fixed, and there are several graphs (corresponding to different voltages) connecting the vertices; the paper considers straight-line realizations, so we get what could be called the *simultaneous geometric crossing number with fixed vertex locations*.

[31] Graph embeddings and graph genus are already defined in Sainte-Laguë's 1926 "Les Reseaux" [670, p.6], sometimes called the zeroth book of graph theory, see [359] for a translation. A definition of planar graphs—called *spherical*—occurs in Sainte-Laguë's earlier thesis [669], which formed the basis of the book.

If we drop the restriction that a manifold be locally planar, we can explore pinched surfaces (such as the spindle) or branched surfaces. Neither of these choices is well-investigated, with the exception of books. Book crossing numbers are typically defined by disallowing edges to cross the spine, so crossings cannot occur on the spine (where the manifold is not locally planar). On the other hand, one may decide to allow edges crossing the spine and try to minimize the number of spine crossings (spine crossing number). For pinched surfaces it is not immediately clear what constitutes a proper drawing (are vertices allowed to lie in pinches, how many edges can pass through a pinched point, may an edge pass through a pinched point without crossing to the other part of the surface, how do we count the crossings, what if we have triple pinches, etc.).

Finally, we can consider drawing the graph in other manifolds, 3-dimensional space, for example. There is the grid crossing number, in which graphs are drawn on d-dimensional grids of limited size, and the space crossing number, which has the flavor of a stabbing number.[32]

2.3 Methods of Counting

In German a crossing of curves is called a "Doppelpunkt" [241, 717], a double point. This term stems from the algebraic tradition and survives in knot theory, but even in graph drawing pairwise counting of crossings is the preferred method, that is, k edges passing through the same point count for $\binom{k}{2}$ crossings. One can imagine counting a k-wise crossing just once (degenerate crossing number, genus crossing number) or k times.[33] As we saw in the short historical section, the algebraic way of counting crossings may precede this way of counting crossings; edges are oriented, and for an ordered pair (e, f) of edges we can assign a crossing a $+1$ or -1 depending on whether f crosses e from left to right or from right to left. For weighted graphs, it is natural to assign weights to crossings, typically using the product of the weights of the edges involved (as far as we know, real weights or weights from other algebraic structures have not been studied). Continuing the philosophy of pairwise counting, the weighted crossing number allows one to assign weights to pairs of edges; typically, these weights do not depend on the drawing, but they could: in the *pairwise minimum local crossing number* the weight assigned to the pair e, f is the smaller of the number of crossings e and f are involved in.

When computing the number of crossings between two edges, $\psi(e, f)$, most crossing numbers ψ add up the counts of the pairwise crossings of e and f. There are some exceptions: the pair crossing number takes the maximum (so each pair contributes at most once, namely if it crosses), the odd crossing number adds up crossings modulo 2, and the algebraic crossing number takes the absolute value of the sum.

To calculate the crossing number of a drawing, most crossing numbers simply add up

[32] There also is a notion of crossing number for geometric hypergraphs, in which hyperedges are represented as simplices, see [54, 55].

[33] The later variant seems not to have been studied; some subtleties immediately arise (as they do for the degenerate crossing number): do we allow an edge to pass through the same point multiple times? Do edges have to cross when passing through the point or may they touch? Do we count every crossing, or do we just count the number of edges involved?

the pairwise crossings. As we saw earlier, the local crossing number takes the maximum per edge: $\max_{e\in E}\sum_{f\in E}\mathrm{cr}(e,f)$. Independent crossing number variants do not include pairs of adjacent edges in the count (independent crossing number, independent odd crossing number, etc.).

Finally, to determine the crossing number of a graph we typically minimize the crossing number over all drawings, although there is the family of maximum crossing numbers (maximum crossing number, maximum rectilinear crossing number, maximum orchard crossing number).

Some crossing numbers count crossings other than edge crossings, e.g. the spine, orchard, edge and space crossing numbers. One could imagine a fan crossing number, based on Kaufmann and Ueckerdt's notion of fan-planarity [484]: instead of counting how many edges a given edge crosses, we count how many fans (stars) it crosses.[34]

Remark 8 (Crossing vs Crossings). Most crossing number variants count the number of crossings in some type of drawing, but there are variants that do not: The independent crossing number ignores some of the crossings (adjacent crossings), and the local crossing number counts crossings along edges, not all crossings. In such cases, we can study the crossing number of drawings that are restricted by that second crossing number. We could ask, for example, what is the smallest number of crossings in a graph with independent crossing number at most k? This type of question has only been approached very occasionally, but there have been some exceptions, including the study of the number of crossings in drawings with bounded local crossing number and various other beyond-planar drawing styles [117, 200], bounded monotone local crossing number [357], bounded rectilinear local crossing number [596, Corollary 2], bounded odd crossing number [620], bounded pair crossing number [681]. In the reverse direction, we can ask whether a bounded crossing number implies that we can bound some more constrained crossing number variant. Famously, the answer is no for rectilinear drawings [121], but there is a polynomial bound if we allow a single bend along each edge [121]. ♦

2.4 Modes of Representation

This leaves us with modes of representation of graphs; there is not much to be said here; the standard mode of representation where a curve between two points is taken to represent the edge connecting the vertices corresponding to the points is predominant. The only alternative model we have seen in the context of crossing numbers is that of confluent drawings introduced by Dickerson, Eppstein, Goodrich, and Meng [248]. A graph is drawn like a train track (with branches and switches), vertices correspond to stations, and an edge to a legal train route (trains cannot make sharp turns at switches).[35] If we allow bridges, points at which one track crosses over another track, then the confluent crossing number is the smallest number of bridges necessary to realize the train track. Using the confluent drawing style (rather than its semantics) as an inspiration, we could allow edges

[34]To make this precise, one would probably count the crossings of an edge as the size of the largest matching it crosses.

[35]Roger Penrose uses a similar idea in his, or his father's, railway mazes [242].

in a drawing to run in parallel temporarily and then separate again (without changing order), just like in a confluent drawing but without the connotation for connectivity. Now let us say we count the crossing of two such bundles of edges as a single crossing (as opposed to weighing it by the number of edges in the bundle), do we get an interesting notion of crossing number? Should we require that every bundle contains each edge at most once? These questions, suggested in an earlier version of this survey, led to the introduced of the bundled crossing number. In an actual drawing we may decide to keep the edges in a bundle slightly separate, maybe by using color for the intervening spaces. This idea has been studied in the context of the Metro-line crossing number under the name “block crossing” [326].

There is one other model of representation that has not been explored yet in the context of crossing numbers: representing graphs as intersection graphs. String graphs will serve as an example. We know that every planar graph is the intersection graph of strings (curves), indeed at this point we know that we can assume that each pair of strings crosses at most once [176], and that the strings are straight-line segments [175] (we do not yet know whether they can be chosen in at most 4 directions, this would imply the 4-color theorem). So in the string representation every vertex becomes a curve (or straight-line segment) and an edge corresponds to a (single) crossing of the curves. One could imagine extending this model by distinguishing two types of crossings: crossings representing edges and crossings that count towards a *string crossing number*. In a drawing the later crossings could be represented by overpasses (as for knots). We are not aware that this approach has been investigated. (The existing string crossing number realizes a slightly different idea.)

3 A Compendium of Crossing Numbers

For the compendium (and indeed for the rest of the paper), I have always tried to go back to the sources; any result reported at second hand is identified as such. (This does not mean that I guarantee the correctness of all results.) I also made heavy use of other tools such as Vrťo’s online bibliography of crossing numbers [755], MathSciNet, zbMATH Open (formerly Zentralblatt), and Google Scholar. In turn, the work on this survey led to my writing the book “Crossing Numbers of Graphs” [674], which presents some of the major results in the area. There is an emerging area of graph drawing called “beyond planar graphs” which, while less focussed on crossing numbers, is all about non-planar graphs, see, for example [251, 442].

I have tried to be exhaustive, but decided to exclude certain areas altogether rather than covering them badly; this includes crossing numbers for objects other than graphs, most notably knots, braids, hypergraphs [192, 244], permutations [118], tropical curves [170, 171], and tanglegrams [46, 231, 320].[36]

For some crossing numbers we had to introduce new notation to avoid conflicts—of which there are many. As the table in Section 3.1 shows, nearly every crossing number

[36] There are also some stabbing number variants called crossing numbers, but the spirit is different; we do not document these variants here.

variant with a parameter k has been called ν_k or cr_k at some point; we tried to minimize the proliferation of notation. E.g. instead of creating new symbols for the toroidal crossing number or the Klein bottle crossing number, we simply modify the notation for the standard crossing number to include the surface: cr_Σ denotes the crossing number on surface Σ. Similarly, if the underlying graph has structure (rotation, ordering, layering) we don't create a new crossing number notation. For example the fixed linear crossing number is simply the book crossing number, bkcr_k restricted to drawings which respect the linear ordering of the vertices, so we use bkcr_k for both variants, writing $\text{bkcr}_k(G, \pi)$ to distinguish the fixed linear crossing number from the book crossing number if necessary. This approach leads to some overloading of notation, but hopefully no confusion.

Many crossing numbers exist under multiple names reflecting various acts of rediscovery; in these cases I've generally decided to go with the older or more established name. In every case, I have tried to document all variant names and symbolism I have encountered, typically in the "comments" section.

As a computer scientist I am particularly interested in questions of computational complexity, so most crossing numbers contain a "complexity" entry detailing what we know about the complexity of each problem. There is an excellent recent survey by Zehavi [789] on the parameterized complexity of crossing minimization problems.

For each crossing number there is an entry for "relationships"; this entry is restricted to relationships between crossing number variants and only the most basic parameters: $n = |V|$ and $m = |E|$ (so, in particular, we list all crossing lemmas we are aware of in this rubric). We make no attempt to try capturing relationships with other graph parameters such as the girth, bisection width, cut width, etc. or the emerging links between crossing number and chromatic number in the study of Albertson's conjecture [35]. A comprehensive survey on some of these results is by Shahrokhi, Sýkora, Székely, and Vrťo [693].

Finally, we include exact (and some asymptotic) crossing number results for major graph families such as the complete, K_n, and complete bipartite graphs, $K_{m,n}$, under the rubric "values"; for lesser-known crossing number variants we tend to include more detail; we use the usual symbols for graph families, such as P_n for the path on n vertices, C_n for cycles of length n, $Q_n = \Box_{i=1}^n K_2$ for the n-dimensional hypercube graph, where $\Box$ is the Cartesian product of two graphs (sometimes written as $\times$), W_n for the wheel graph (on $n+1$ vertices), K_n^r for the complete balanced r-partite graph on n vertices, and $\text{GP}(n, k)$) for the generalized Petersen graph (on $2n$ vertices).

Remark 9 (Parameters and Derived Notions). For a crossing number γ_k parameterized by some parameter k, we can define a new parameter $\mu_\gamma(G)$ as the smallest k for which $\gamma_k(G) = 0$ if such a k exists. For the (surface) crossing numbers, this gives us (Euler, non-orientable, orientable) genus, for the book (or k-page) crossing number, this gives us the notion of pagenumber (or book thickness), for the k-planar crossing number, the thickness of a graph, and for the geometric k-planar crossing number, its geometric thickness; for the (surface) independent odd crossing number we get a homological notion of genus [335–337, 680]. The grid crossing number has two parameters (dimension and volume) which could be used to define area/volume of a graph. We will mention some of these derived parameters below, but without attempting to survey results concerning them. ♦

3.1 Notation for Crossing Numbers

The following table lists the crossing numbers with the symbol we use in the current paper (if any) and other notations found in the literature with references; the alternative notations are listed chronologically (at least with respect to the first occurrences we found). The crossing numbers are listed alphabetically by name. There are several crossing number variants for which symbols have never been introduced, including annulus, bimodal, confluent, map, Metro-line, radial, red/blue and spine crossing numbers, these (and some others) are not listed below.

Table 1: Crossing number variants with symbols used in the text and in the literature.

Name (alternative names)	Symbol	Symbol (literature)
abstract topological graph	$\mathrm{cr}(G, R)$	cr_{at} [513]
algebraic	acr	acr [622], ACR [741], ALG-CR [742]
algebraic +	acr_+	acr_+ [338]
anchored	$\mathrm{bkcr}_k(G, A, \pi)$	acr [163]
average	*no symbol*	acr [635]
bipartite	bcr	ν_2 [393], ν^* [547], $\underline{\nu}_2$ [546], *bcr* [650, 697]
bipartite cylindrical	$\mathrm{cr}_{②}$	$\mathrm{cr}_{\circledcirc}$ [16]
bipartite cylindrical local	$\mathrm{lcr}_{②}$	$\mathrm{lcr}_{②}$ [488]
book (k-page)	bkcr_k	ν_k [695, 771]
book edge (k-page edge)	*no symbol*	cre_k [95]
bundled	bc	bc [33]
centrally symmetric	cr_{cs}	*no symbol*
centrally symmetric rectilinear	$\overline{\mathrm{cr}}_{cs}$	$\overline{\mathrm{cr}}_{cs}$ [6]
constrained (partially predrawn)	*no symbol*	pd-cr$(G, \mathcal{H})$ [33]
convex bundled	bc°	bc° [33]
convex (outerplanar, circular, 1-page)	bkcr_1	ν_1 [695], cr^* [691], χ [100], μ_+ [151]
convex maximum rectilinear	max-$\overline{\mathrm{cr}}^\circ$	obf° [754], CR_c [158]

Name (alternative names)	Symbol	Symbol (literature)
convex k-partite (circular k-partite)	*no symbol*	cpr_k [662]
(minimum, minimal, planar, graph, edge, topological)	cr	c [389], C [380], c_0^+ [506], ν [374], ν_* [370], $\underline{\nu}$ [546], $\theta(G)$ [792], κ [244], CR [612], $cr_{\mathbb{R}^2}$ [344], CR [724], $\nu_{\mathbb{R}^2}$ [771], $N_{\mathbb{R}^2}$ [283], crn [343, Section 4.5]
(joint)	$\mathrm{cr}(G_1, G_2)$	$\mathrm{cr}(G_1, G_2)$ [585], $cr(G_1, G_2)$ [57], $C_r(G_1, G_2)$ [778]
cylindrical	$\mathrm{cr}_{\circledcirc}$	$\mathrm{cr}_{2\circ}$ [271]
degenerate	dcr	$\underline{\mathrm{CR}}$ [608]
diagonal	cr_Δ	cr_Δ [585]
edge	ecr	*no symbol*
fixed convex bundled	$\mathrm{bc}^\circ(G, \pi)$	$\mathrm{bc}^\circ(G, \pi)$ [33]
fixed linear	$\mathrm{bkcr}_k(G, \pi)$	ν_π [542] (for $k = 2$), ν_L [215] (for $k = 2$), $\nu_{L,k}$ [217], μ [787] (for $k = 1$)
genus	gcr	GCR [567]
genus g (surface)	cr_{S_g}	c_g^+ [506], cr_g [463], cr_g^* [505], cr_2 [663]
genus g local (local g)	lcr_{S_g}	λ_g [471]
(d-dimensional volume N) grid	$\overline{\mathrm{cr}}_\#(G, N, d)$	cr [263]
geometric k-planar (rectilinear k-colored)	$\overline{\overline{\mathrm{cr}}}_k$	GCR_k [604], $\overline{\mathrm{cr}}_k$ [28]
independent	cr_-	CR_- [612]
independent algebraic	iacr	s [747], IACR [741], IALG-CR [742], acr_- [338]
independent odd	iocr	ODD-CR_- [612], *CR-IODD* [724], $\nu^{(i)}$ [771], iocr [623], *cr-iodd* [580]

Name (alternative names)	Symbol	Symbol (literature)
independent pair	pcr_	PAIR-CR_ [612], pcr_ [338], ipcr [769]
k-layer	*no symbol*	K [763]
k-planar	cr_k	Cr_k [597], CR_k [727], $\nu_k^{(B)}$ [771], cr_k [696], CR_k [604]
k-planar	*no symbol*	$\mathrm{cr}_{k\text{-}pl}$ [200]
Klein bottle	cr_{N_2}	cr_2 [506], $\overline{\overline{cr_2}}$ [659], $cr_{\mathbb{K}}$ [344]
leveled	$\text{mon-cr}_{\preceq}(G)$	$\mathrm{mon} - \mathrm{cr}(G, \ell)$ [338]
linear (2-page)	bkcr_2	μ [151]
local (crossing parameter)	lcr	λ_0 [471], lcn [216], crs [364], c [713], ξ [365], φ [762]
local convex (local outerplanar)	*no symbol*	$locr(G)$ [470]
local k-page (local book)	*no symbol*	lcr_k [711]
local k-planar	lcr_k	LCR_k [71]
local odd	locr	*no symbol*
local pair	lpcr	lpcr [20]
local toroidal	lcr_{S_1}	ℓ_1 [383], λ_1 [471]
major (major-monotone)	Mcr	Mcr [137]
maximum (maximal)	max-cr	ν_* [370], ν_M [652], cr^{M} [630], CR [411], cr_M [56], CR [158]
maximum bipartite	max-bcr	*no symbol*
maximum edge	max-ecr	*no symbol*
maximum local	max-lcr	E [395]
maximum orchard	*no symbol*	MOCN [311]
maximum rectilinear (maximal rectilinear, obfuscation complexity)	max-$\overline{\mathrm{cr}}$	$\overline{\nu}_*$ [370], $\underline{M}$ [339], $\overline{\nu}^+$ [386], ν'_M [652], $\overline{\mathrm{CR}}$ [43], obf [754], $\overline{CR}$ [158]
maximum rectilinear edge	max-$\overline{\mathrm{ecr}}$	*no symbol*
minor (minor-monotone)	mcr	mcr [137]
monotone	mon-cr	mon-cr [338], MON-CR [610]
monotone independent odd	mon-iocr	mon-iocr [338], mon-ocr_ [87]

Name (alternative names)	Symbol	Symbol (literature)
monotone odd	mon-ocr	mon-ocr [338]
monotone odd + (monotone semisimple odd)	mon-ocr_+	mon-ocr_+ [87]
monotone odd ± (monotone weakly semisimple odd)	$\text{mon-ocr}_\pm$	$\text{mon-ocr}_\pm$ [87]
monotone pair	mon-pcr	$\text{pair-cr}^{\text{mon}}$ [752]
nodal	ncr	*no symbol*
nodal toroidal	ncr_{S_1}	n_1 [383]
non-orientable genus g	cr_{N_g}	c_g [506], $\widetilde{cr}_g$ [467], cr_g [505]
odd	ocr	ODD-CR [613], cr_{odd} [421], *CR-ODD* [724], $\nu^{(o)}$ [771], ocr [623], *cr-odd* [580]
odd +	ocr_+	ODD-CR_+ [612]
odd ± (weakly semisimple odd)	$\text{ocr}_\pm$	$\text{ocr}_\pm$ [338], OCR_* [478]
orchard	orchard-cr	OCN [311]
oriented (joint)	$\overrightarrow{\text{cr}}$	cr_+ [585]
pair (pairwise)	pcr	PAIR-CR [613], cr_{pair} [421], pcr [502], pair-cr [752], *CR-PAIR* [724], $\nu^{(p)}$ [771], *cr-pair* [580]
pair +	pcr_+	PAIR-CR_+ [612], pcr_+ [338]
projective plane	cr_{N_1}	cr_1 [506], $cr_{\mathbb{P}}$ [344], cr_p [536], N_P [283]
pseudolinear	$\widetilde{\text{cr}}$	$\widetilde{cr}$ [89]
quasi	quasi-cr	*no symbol*
rectilinear (straight-line, linear, geometric)	$\overline{\text{cr}}$	$\bar{c}$ [389], $\overline{cr}$ [459], $\overline{\nu}$ [374], $\overline{\nu}_*$ [370], $\overline{\text{cr}}$, R [339], ν' [652], cr_1 [122], $\overline{\kappa}$ [770], LIN-CR [710], *CR-LIN* [724], rcr [456], $\overline{\text{cr}}_1$ [151], *cr-lin* [580]

Name (alternative names)	Symbol	Symbol (literature)
rectilinear bipartite cylindrical	$\overline{\text{cr}}_{\textcircled{2}}$	*no symbol*
rectilinear edge	$\overline{\text{ecr}}$	*no symbol*
rectilinear k-planar	$\overline{\text{cr}}_k$	$\overline{\text{cr}}_k$ [696], RCR_k [604]
rectilinear local	$\overline{\text{lcr}}$	$\overline{cr}_1$ [522], $\overline{\text{lcr}}$ [11]
rectilinear space	space-$\overline{\text{cr}}$	lin-cr_4 [156]
sequential	*no symbol*	SCN [343]
simple	$\text{cr}^{\times}$	scr [192], crs [152]
simple degenerate	dcr^*	CR^* [608], cr^* [19]
simple local	lcr^*	*no symbol*
simple quasi	quasi-cr^*	cr_3 [636]
simultaneous	scr	scr [199], simcr [192]
simultaneous geometric	$\overline{\text{scr}}$	*no symbol*
simultaneously planar	*no symbol*	cr_{sp} [354]
skewness (removal, slimming)	sk	r [552], s [378], μ_0 [463], μ [464], $\zeta(G)$ [792], κ [321], sk [185], *skew* [201]
space	space-cr	cr_4 [156]
spherical (spherical geodesic)	$\overline{\text{cr}}_{S^2}$	$\check{\text{cr}}$ [757], cr_{S^2} [91], N_S [283]
stable	*no symbol*	cr^k [467], $\text{cr}_{\gamma(G)-k}$ [472]
string	str-cr	scr [136]
t-circle	$\text{cr}_{t\circ}$	$\text{cr}_{t\circ}$ [271]
t-partite circle	$\text{cr}_{\textcircled{t}}$	$\text{cr}_{\textcircled{t}}$ [167]
t-polygonal	$\overline{\text{cr}}_t$	cr_t [120]
tile	tile-cr	tcr [635]
toroidal (torus)	cr_{S_1}	cr_1 [382], N_T [283]
triple	triple-cr	tcr [733]
uncrossed	ucr	ucr [423]
upward	mon-$\text{cr}_{\preceq}(G)$	*no symbol*
weighted	$\text{cr}(G, w)$	cr_w [678], wcr [563]
weighted	$\overline{\text{cr}}(G, w)$	*no symbol*
x-monotone	mon-$\text{cr}_{\preceq}(G)$	mon-cr [338]

3.2 Crossing Numbers

1-page crossing number. See convex crossing number, book crossing number.
2-page crossing number. See book crossing number.
Abstract topological graph crossing number. See crossing number of abstract topological graph.

Algebraic crossing number

Definition: Order and orient all edges of G and assign a crossing between edges $e < f$ a $+1$ or -1 depending on whether f crosses e from right to left or from left to right at that point. We let $\text{acr}(e, f)$ be the sum of the values of all crossings of f with e (which can be negative). For a given drawing D (and a given orientation) of G we let $\text{acr}(D) = \sum_{e<f\in E(G)} |\,\text{acr}(e, f)|$, where $<$ is the ordering of $E(G)$.[37] The *algebraic crossing number* of G, $\text{acr}(G)$, is the minimum algebraic crossing number of any drawing of G. The Rule + variant of acr is $\text{acr}_+(G)$, the smallest algebraic crossing number of any drawing of G in which adjacent edges are forbidden to cross. One can define an intermediate variant in which we require $\text{acr}(e, f) = 0$ for every pair of adjacent edges e and f; denote this variant by $\text{acr}_\pm$

Reference: Pelsmajer, Schaefer, Štefankovič [622], also Tutte [747], Winterbach [771].

Comments: One could argue that this crossing number is implicit in Tutte [747]; certainly, the idea of counting crossings algebraically is; however, Tutte insists on not counting adjacent crossings by setting $\text{acr}(e, f) = 0$ for adjacent edges e and f; he writes: "We are taking the view that crossings of adjacent edges are trivial, and easily got rid of." If we read this as a claim that $\text{acr}(G) = \text{iacr}(G)$, then we now know that this claim is wrong. So Tutte did define iacr, but acr seems to have first been isolated as a separate notion in [622].[38] There it was asked whether $\text{acr}(G) = \text{cr}(G)$, a question answered by Tóth in the negative [742].

Complexity: **NP**-complete.[39]

Relationships: $\text{iacr}(G) \leqslant \text{acr}(G) \leqslant \text{acr}_\pm \leqslant \text{acr}_+(G)$ for all G (from definition). There are graphs G for which $\text{iacr}(G) < \text{acr}(G)$ [338]. Tóth showed that there are graphs G with $\text{acr}(G) \leqslant 0.855\,\text{pcr}(G) = \text{cr}(G)$ answering the question from [622].

Open Questions: What is the relationship between acr and pcr?

Also see: Odd crossing number, independent algebraic crossing number, monotone crossing number (for monotone variants).

Anchored crossing number. See fixed linear crossing number.
Annulus crossing number. See map crossing number.

[37]The value of $\text{acr}(D)$ does not depend on the order or orientation of the edges, so $\text{acr}(D)$ is well-defined.

[38]Winterbach [771] defines the *Tutte crossing number*; unlike Tutte, he does not set $\text{acr}(e, f) = 0$ for adjacent edges, but he does order edges by endpoints (to avoid counting both $\text{acr}(e, f)$ and $\text{acr}(f, e)$. As a result he counts some adjacent crossings, e.g. v_1v_2 with v_2v_3 but not others, e.g. v_1v_2 with v_1v_3.

[39]**NP**-hardness is obtained as in Pach and Tóth's proof that ocr is **NP**-hard. The question lies in **NP**, since it can be phrased as an integer linear program (this is one way of looking at Tutte's characterization of planarity [747]).

Bimodal crossing number

Definition: The *bimodal crossing number* of a directed graph G, is the smallest number of crossings in any bimodal drawing of G. A drawing is *bimodal* if at every vertex all in-coming edges (and thus, all out-going edges) are consecutive.

Reference: Buchheim, Jünger, Menze, Percan [154], also Wright, Appa, Jarrett [776].

Comments: Buchheim, Jünger, Menze, and Percan [154] introduce bimodal drawings as a relaxation of hierarchical drawings with the goal of reducing the number of crossings. Wright, Appa, and Jarrett [776] introduce an equivalent model using *duplex vertices* in the context of traffic routing and route crossing minimization.

Complexity: **NP**-complete [154]. The embeddability problem is in **P** (easy reduction to planarity).

Relationships: The upward crossing number is an upper bound on the bimodal crossing number (and they differ, because the upward crossing number is infinite for directed cycles).

Values: The bimodal crossing number of the complete biorientation of $K_{n,n}$ is at most $Z(n) = X(n)X(n)$, where $X(n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$ [777].[40]

Open Questions: Is it true that the bimodal crossing number of a complete biorientation of a bipartite graph G is at most the crossing number of G?

Also see: Upward crossing number.

Bipartite confluent crossing number. See confluent crossing number.

Bipartite crossing number

Definition: A 2-*layer (or bipartite)* drawing of a bipartite graph G is a straight-line drawing in which the vertices of G lie on two parallel lines with the vertices in the same group lying on the same line. The *bipartite crossing number*, $\mathrm{bcr}(G)$, of a bipartite graph G is the smallest number of crossings in a 2-layer drawing of G. The *maximum bipartite crossing number*, $\text{max-}\mathrm{bcr}(G)$, of a bipartite graph G is the largest number of crossings in a 2-layer drawing of G. The 2-*level skewness* of a bipartite graph G is the smallest number of edges whose removal leaves a graph which has a crossing-free 2-layer drawing.

Reference: Harary [385]; Watkins [765]; Harary, Schwenk [392, 393]. Also [208]. The maximum bipartite crossing number is implicit in Chimani, Felsner, Kobourov, Ueckerdt, Valtr, Wolff [194].

Comments: Harary develops this crossing number notion without naming it. Watkins called it the *special crossing number*; Harary and Schwenk coined *bipartite crossing number* and wrote $\nu_2(G)$, May and Mennecke [545, 547], in two papers on circuit layout, call it the *inner crossing number* ν^*. None of these names seem to have stuck; the corresponding optimization problem is now known as the 2-sided (or 2-layer)

[40] This paper describes a construction of a crossing-minimal drawing for $K_{m,n}$ from a paper by Holroyd and Miller from 1966 which I have not been able to track down. The paper shows that the construction is equivalent to Blazek and Koman's Construction A from 1964 [127].

crossing minimization problem (e.g. [791]). In the 1-sided crossing minimization problem the order of vertices on one of the two lines is fixed.[41] If the ordering in both layers is determined, the crossing number can be determined in quadratic time [270]. Hotz [444, Section 3.6.3] discusses an application to circuit layout in which the permutations on either side are restricted by the nature of the circuit. As an extremal question, the bipartite crossing number is even older. In a textbook on algebra from 1889, Chrystal [208, p.34] asks to verify the bipartite crossing number of $K_{m,n}$ (his value is off by a factor of 2). Also, see Singmaster [706, 5.Q.1]. Kircher, in his 1669 "Ars Magna Sciendi" includes several convex, straight-line drawings of a $K_{9,9}$ [489, p.18, 196], and, unbelievably, a $K_{18,18}$ [489, p.170]. The name bipartite crossing number has also been used for $\text{cr}(K_{m,n})$, Zarankiewicz's problem. Arguably, crossing minimization of storyline visualizations [511] could be considered a variant of the bipartite crossing number (in which edges are relaxed to be monotone, but there are conditions on edges having to touch or cross). Reducing the bipartite crossing number using vertex splits is studied in [24]. May and Szkatuła [546] define a generalization, the *p-partite crossing number*, for p-partite graphs, see the comments on the k-layer crossing number. The 2-level skewness problem is explicitly introduced in [169], but it had been studied earlier.

COMPLEXITY: **NP**-complete.[42] Can be approximated in polynomial time to within a factor of $O(\log^2 n)$ [697]. It's trivial for bipartite permutation graphs [146]. The embedding problem is easy, Harary and Schwenk [392] give a complete characterization of graphs with $\text{bcr}(G) = 0$. The 1-sided crossing minimization problem is **NP**-complete [274, 279, 577], but fixed-parameter tractable [266, 500]. The 2-level skewness problem is **NP**-complete [274, 732].

RELATIONSHIPS: $\text{bcr}(G) \geqslant \overline{\text{cr}}(G)$ for all bipartite graphs G, and the inequality can be strict (e.g. $K_{2,2}$). $\text{bcr}(G) \geqslant m - n + 1$ [547]. If G is a 2-connected, bipartite graph, then $\text{bcr}(G) \geqslant (m-1)/3$, where $m = |E(G)|$ [499]. There is a crossing lemma [51, Theorem 4.1], and a lower bound on the local crossing number [51, Theorem 4.2].[43] $\text{bcr}(G) + \text{max-bcr}(G) = \theta(G)$ [194, Lemma 2], where $\theta(G) = \left(m(m+1) - \sum_{v \in V} \deg^2(v)\right)/2$, with $m = |E|$.

VALUES: $\text{bcr}(C_{2n}) = n - 1$ [392]. $\text{bcr}(K_{m,n}) = \binom{m}{2}\binom{n}{2}$ [208, 765]. $\text{bcr}(M_{2,n}) = n - 1$, $\text{bcr}(M_{3,n}) = 5n - 6$, $\text{bcr}(M_{m,n}) = \Theta(m^2 n)$ where $M_{m,n} = P_m \square P_n$ is the $m \times n$

[41] The crossing minimization problem for tanglegrams [46, 320, 775] has a similar flavor; in a tanglegram, the ordering of the vertices in each layer is constrained by a rooted tree.

[42] Shahrokhi and Vrťo [700] write "the **NP**-hardness of the problem was proved for multigraphs, but it is widely assumed that it is also **NP**-hard for simple graphs". The multigraph proof is due to Garey and Johnson [348]. The problem remains **NP**-complete for simple graphs as well (thanks to Daniel Štefankovič for help with this proof): by a result of Even and Shiloah [299] the optimum linear arrangement problem is **NP**-hard for bipartite graphs; take a bipartite graph G and make each of its vertices the center of a sufficiently large star; in a crossing-minimal bipartite drawing of the resulting graph, the leaves of the star can be assumed to be consecutive; this bipartite drawing encodes a solution to the optimum linear arrangement problem of the original graph G, just as in the original proof by Garey and Johnson.

[43] The authors phrase the result slightly differently, and they do not introduce a *local bipartite crossing number*.

mesh, and $\mathrm{bcr}(Q_n) = \Theta(4^n)$ [694].

Open Questions: Is $\mathrm{bcr}(G) + \text{max-}\overline{\mathrm{cr}}(G) = \theta(G)$ for bipartite graphs G [194]?

Also see: Radial crossing number, cylindrical crossing number, tile crossing number, bipartite confluent crossing number (under confluent crossing number), upward crossing number. Generalizations include convex k-partite crossing number and leveled crossing number (under monotone crossing numbers).

Bipartite cylindrical crossing number. See cylindrical crossing number.
Biplanar convex crossing number. See 2-page crossing number (under book crossing number), convex crossing number.
Biplanar crossing number. See k-planar crossing number.

Book crossing number

Definition: A *book* with k pages is a branched surface consisting of k half-planes whose boundary lines have been identified (forming the *spine*). The *book crossing number* for a book with k pages, or *k-page crossing number*, $\mathrm{bkcr}_k(G)$, of a graph G, is the smallest number of crossings in a drawing of G in a book with k pages so that all vertices lie on the spine of the book and every edge lies in a single page. The smallest k for which $\mathrm{bkcr}_k(G) = 0$ is the *pagenumber* of G. The *local k-page crossing number* is the smallest local crossing number of any k-page drawing of the graph.

Reference: Blažek, Koman [127] (for $\mathrm{bkcr}_k(K_n)$); Saaty, Holroyd [668] (for $\mathrm{bkcr}_2(K_n)$); Nicholson [588]; Leclerc and Monjardet [525] (for bkcr_2). Shahrokhi, Sýkora, Székely, Vrťo [695] (for bkcr_k). Sripimonwan [711] for the local k-page crossing number.

Comments: The book crossing number for a single page is the same as the convex crossing number. There are two types of book drawings, *combinatorial*, in which edges are not allowed to cross the spine, and *topological* in which edges are allowed to cross the spine [771, p. 3.1.3.1]. The book crossing number is restricted to combinatorial drawings, and there is good reason for that, since a topological book crossing number would not add anything new: for a single page, the spine cannot be crossed, so we again get the convex crossing number and for two pages, $k = 2$, we would get the standard crossing number as was observed (and proved) by Nicholson [588, Appendix].[44] Even before Nicholson, Blažek and Koman [127], in their paper showing that $\mathrm{cr}(K_n) \leqslant Z(n)$, using 2-page (combinatorial) drawings, asked for the value of $\mathrm{bkcr}_k(K_n)$, and gave an upper bound for $k = 3$. Every graph can be embedded in 3 pages if we allow a topological embedding.[45] The spine crossing number is a variant that does allow topological drawings (but counts crossings differently).

[44] One has to keep in mind that Nicholson proved this result very early in the history of the crossing number; his primary goal is an aesthetic layout (he restricts edge segments on each page to be drawn like semicircles) which minimizes the number of crossings via a heuristic that modifies the permutation along the spine. There was a brief follow-up [111].

[45] This result is due to Atneosen [74]. White [767, page 59] gives a very simple proof he attributes to Babai in 1974 (essentially the same proof found later by Bernhart and Kainen [115]).

Combinatorial drawings in two pages have been called circular [771] or cycle [402] drawings, so the name circular or cycle crossing number for the crossing number bkcr_2 would not be surprising. More typically, though, bkcr_2 is known as the *2-page crossing number* or sometimes the *(free) linear crossing number*, e.g. [542], or the *biplanar convex crossing number* [147, pg. 393].

There are two degrees of freedom in finding a combinatorial book-drawing: finding the best order of vertices along the spine and determining which page each edge is drawn in. We get interesting variants, if we restrict either of these. If one fixes the order of the vertices along the spine, one obtains the *fixed linear crossing number*, discussed in a separate entry. If one assigns each edge to a specific page, one gets what could be called the *partitioned book crossing number*; we treat it as a special case of the convex simultaneous crossing number (see entry for simultaneous crossing number).

If instead of counting crossings, we count edges involved in crossings, we get the book edge crossing number introduced by Bannister, Eppstein, and Simons [95], see the entry on edge crossing number. The local crossing number of book drawings has been investigated for one page, see the local convex crossing number (under convex crossing number), for two pages [125], and even for k pages [533]. The *local book crossing number*, or more precisely, the *local k-page crossing number* was first formally introduced by Sripimonwan [711] using the name k-page book local crossing number.

COMPLEXITY: The problem is interesting even for the special case of embeddings, that is, $\mathrm{bkcr}_k(G) = 0$. Graphs of pagenumber 1 are the outerplanar graphs which can be recognized in linear time. Graphs of pagenumber 2 are the planar subgraphs of Hamiltonian graphs which implies that testing $\mathrm{bkcr}_2(G) = 0$ is **NP**-complete [209].[46] Testing $\mathrm{bkcr}_k(G) = 0$ for fixed $k \geqslant 4$ is also **NP**-complete, since it is for a given ordering of the vertices on the spine (one can easily construct a gadget that forces a given ordering in a book-embedding); see the entry on the fixed linear crossing number, which is the variant of the book crossing number in which the order of the vertices is given (for an alternative proof, see [271]). As far as we know, the complexity of testing $\mathrm{bkcr}_3(G) = 0$ is open. The only general complexity result about the crossing number version we are aware of is the special case of the convex crossing number, $k = 1$: testing $\mathrm{bkcr}_1(G) \leqslant m$ is **NP**-complete [541], but fixed-parameter tractable in m [94]. The computation of $\mathrm{bkcr}_2(G)$ is fixed-parameter tractable (with the sum of bkcr_2 and the treewidth of G as the parameter) [94].

RELATIONSHIPS: $\mathrm{bkcr}_k(G) \leqslant \mathrm{bkcr}_{k-1}(G)$ (by definition). $\mathrm{bkcr}_k(G) \leqslant \mathrm{bkcr}_1(G)/k$ [695], $\text{mon-cr}(G) \leqslant \mathrm{bkcr}_2(G)$ (from definition) and so $\mathrm{bkcr}_{2k}(G) \geqslant \mathrm{cr}_k(G)$ (see k-planar crossing number), also $\mathrm{bkcr}_1(G) \geqslant \overline{\mathrm{cr}}(G)$ (obvious, since bkcr_1 is the convex crossing number). $\mathrm{bkcr}_4(G) = 0$ for all planar graphs G, and the up-

[46]The characterization of pagenumber 2 graphs is due to Bernhart and Kainen [115], but also see [160] on the pre-history of that observation.

per bound is sharp [110, 783, 784, 786].[47] If G is planar with maximum degree four, then $\text{bkcr}_2(G) = 0$ [109].[48] A crossing lemma is known: $\text{bkcr}_k(G) \geqslant m^3/(37k^2n^2) - 27kn/37$ for $n = |V|$, $m = |E|$ [699].

VALUES: For bkcr_1, see the entry on convex crossing number $\text{bkcr}_2(K_n) = Z(n)$ [2, 3][49,50] (for earlier results, see [155, 237]) and $\text{bkcr}_2(K_{m,n}) \leqslant Z(m,n)$ [237], with $Z(n) = X(n)X(n-2)/4$ and $Z(m,n) = X(m)X(n)$, where $X(n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$. Buchheim and Zheng [155] calculate bkcr_2 for several small graphs. Asymptotic results include $\lim_{n\to\infty} \text{bkcr}_2(K_{m,n})/Z(m,n) = 1$ for $7 \leqslant m \leqslant 8$ [237]. Faria, de Figueiredo, Richter and Vrťo [305] give upper bounds on $\text{bkcr}_2(Q_n)$ (improving work by Madej [537]). Satsangi, Srivastava, Srivastava [671] show (computationally) that $\text{bkcr}_2(K_{1,4,n}) = n(n-2)$ for $2 \leqslant n \leqslant 15$. For values of $\text{bkcr}_k(K_n)$ for $k \geqslant 3$ and small values of n as well as asymptotic bounds, see [7, 238]. If $2 < n/k \leqslant 3$, then $\text{bkcr}_k(K_n) = 1/2(n-3)(n-2k)$ [7]. For values of $\text{bkcr}_k(K_{k+1,n})$ for $3 \leqslant k \leqslant 6$, asymptotic results for $\text{bkcr}_k(K_{k+1,n})$, and upper bounds on $\text{bkcr}_k(K_{m,n})$ see [493]. See [711, Table 4.1, 4.3] for values and bounds for the local 2-page crossing numbers of some small complete bipartite graphs.[51]

OPEN QUESTIONS: De Klerk, Pasechnik, and Salazar [238] introduce a function $Z_k(n)$ for which they show that $\text{bkcr}_k(K_n) \leqslant Z_k(n)$; they conjecture that equality holds (as we saw, the case $k = 2$ is known to be true [2, 3]). De Klerk and Pasechnik [237] conjecture $\text{bkcr}_2(K_{m,n}) = Z(m,n)$. ▼ De Klerk, Pasechnik, and Salazar [493] ask whether $\gamma(k) := \lim_{m,n\to\infty} \text{cr}_{2k}(K_{m,n})/\ \text{bkcr}_k(K_{m,n})$ goes to 1 as k goes to infinity? ▼ Faria, de Figueiredo, Richter and Vrťo [305] ask whether $\text{bkcr}_2(Q_n) \leqslant \overline{\text{cr}}(Q_n)$; this is not true for all graphs: as they point out, a non-Hamiltonian planar triangulation G satisfies $\text{bkcr}_2(G) > 0 = \overline{\text{cr}}(G)$. ▼ Satsangi, Srivastava, Srivastava [671] conjecture that $\text{bkcr}_2(K_{1,4,n}) = n(n-2)$ for all n; they also make some conjectures on the pagenumber of certain graph families, based on computational evidence. ▼ Shahrokhi asked whether $\text{bkcr}_2(G) = O(\text{cr}(G) + \sum_{v\in V(G)} \deg(v)^2)$ [147, Problem 9.4.9]. ▼ He, Sălăgean, Mäkinen, and Vrťo [415] show that $\text{bkcr}_2(C_m \,\square\, C_n) \leqslant (m-2)n$ for $n \geqslant m \geqslant 3$, as is true for the standard crossing number, and supply computational evidence that equality may hold; this would be implied by the stronger conjecture by Harary, Kainen and Schwenk [391] that $\text{cr}(C_m \,\square\, C_n) = (m-2)n$ for $n \geqslant m \geqslant 3$.

ALSO SEE: Convex crossing number, fixed linear crossing number, convex simultaneous

[47] The question whether $\text{bkcr}_3(G) = 0$ may be true for all planar graph was a long-standing open question, mentioned, for example, by Kainen [468]. Yannakakis [783, 784] proved that every planar graph has pagenumber at most 4, but his example of a planar graph that needs 4 pages announced in [784] was not included in [783], but was finally published in [786], roughly at the same time that [110]—by a different set of authors—was published.

[48] For a survey on graphs with pagenumber 2, see [267].

[49] Independently rediscovered in [544]. There are also results for $\text{bkcr}_2(K_n)$ if the number of edges on each page is restricted [14].

[50] An early thesis [634] effectively shows how to express the calculation of $\text{bkcr}_2(K_n)$ as a linear integer program.

[51] Comment on Table 4.1: The local 2-page crossing number of $K_{4,5}$ is 2, since $\text{lcr}(K_{4,5}) = 2$ [232]. The first open case appears to be the local 2-page crossing number of $K_{3,6}$.

crossing number (under simultaneous crossing number), spine crossing number, anchored crossing number, book edge crossing number (under edge crossing number).

Book edge crossing number. See edge crossing number.

Bundled crossing number

Definition: A *bundled crossing* in a drawing of a graph is a pseudodisk in which every edge in some edge-set E_1 crosses every edge in another edge-set E_2, and so that there are no other crossings inside the pseudodisk. The *bundled crossing number*, $\mathrm{bc}(D)$ of a drawing of G is the smallest number of disjoint bundled crossings that cover all crossings of D. The *bundled crossing number*, $\mathrm{bc}(G)$, of G is the smallest bundled crossing number of any intersection-simple drawing of G. Let $\mathrm{bc}'(G)$ denote the variant of $\mathrm{bc}(G)$ in which we allow self-crossings and multiple crossings of edges. If we require the drawing to be convex, that is, all vertices lie on the outer face, we get the *convex (circular, outerplanar) bundled crossing number*, $\mathrm{bc}^\circ(G)$; we write $\mathrm{bc}^\circ(G, \pi)$ for the *fixed convex bundled crossing number*, for which the order of vertices along the outer face is determined by permutation π.

Reference: Fink, Hershberger, Suri, Verbeek [324]; Alam, Fink, Pupyrev [33].

Comments: Bundling edges was introduced in [440]. An earlier version of this survey suggested studying the crossing number of drawings with bundled crossings based on the related notion of confluent drawings and crossings in confluent drawings. The bundled crossing number of a drawing was introduced by Fink, Hershberger, Suri, and Verbeek [324]. Alam, Fink, Pupyrev [33] defined the bundled crossing number of a graph.

Complexity: Determining the bundled crossing number of a given drawing is **NP**-complete [324], as is computing $\mathrm{bc}(G)$ for a given G [178]. For sufficiently dense graphs, $\mathrm{bc}(G)$ can be approximated in polynomial time [33]. $\mathrm{bc}^\circ(G, \pi)$ can be approximated to within a factor of 16 in polynomial time [33].[52] Both bc° and $\mathrm{bc}^{\circ\prime}$ are fixed-parameter tractable [178].

Relationships: $\mathrm{cr}(G) \leqslant \mathrm{bc}(G)$ (every crossing can be viewed as a bundled crossing). $\mathrm{bc}(G) \geqslant \mathrm{bc}'(G) = \gamma(G)$, where $\gamma(G)$ is the (orientable) genus of G, and the inequality can be strict [33], and $\mathrm{bc}(G) \leqslant 6\,\mathrm{bc}'(G)$ [682]. $\mathrm{bc}(G) \geqslant (m - (3n - 6))/6$, and $\mathrm{bc}^\circ(G) \geqslant (m - (2n - 3))/6$.

Open Questions: What are $\mathrm{bc}(K_n)$, $\mathrm{bc}(K_{m,n})$, and $\mathrm{bc}(Q_n)$?

Also see: Degenerate crossing number, confluent crossing number, Metro-line crossing number.

[52] If the drawing is fixed, an 8-approximation is known [64].

Centrally symmetric crossing number

Definition: A drawing of a graph is *centrally symmetric* if it invariant under a rotation around the origin by 180 degrees. The *centrally symmetric crossing number* $\mathrm{cr}_{cs}(G)$ is the smallest number of crossings in a centrally symmetric drawing of G. The *centrally symmetric rectilinear (or geometric) crossing number*, $\overline{\mathrm{cr}}_{cs}(G)$ is the smallest number of crossings in a centrally symmetric, rectilinear drawing of G. For $\overline{\mathrm{cr}}_{cs}$ we allow the drawings to be degenerate (more than two edges may cross in a point, each pair of edges crossing in the point counts separately). Both cr_{cs} and $\overline{\mathrm{cr}}_{cs}$ may be infinite.

Reference: Based on Ábrego, Dandurand, Fernández-Merchant [6].

Comments: Ábrego, Dandurand, and Fernández-Merchant [6] determine $\overline{\mathrm{cr}}_{cs}(K_{2n})$ without naming the crossing number, but introducing the notation. For rectilinear drawings multiple crossings have to be allowed (otherwise very few graphs would have finite $\overline{\mathrm{cr}}_{cs}$). Non-bipartite graphs must have one vertex in the origin in a centrally symmetric drawing; this vertex blocks straight-line edges between symmetric vertices, so it may be of interest to consider at geodesic drawings on the sphere. Perlstein and Pinchasi [627] proved that a graph has a centrally symmetric embedding on the sphere if and only if it is a generalized thrackle (also see [675, Theorem 3.11].

Complexity: **NP**-complete for cr_{cs} and $\exists\mathbb{R}$-complete for $\overline{\mathrm{cr}}_{cs}$.[53] Complexity of the embedding problem is open.

Relationships: $\mathrm{cr}(G) \leqslant \mathrm{cr}_{cs}(G)$, $\overline{\mathrm{cr}}(G) \leqslant \overline{\mathrm{cr}}_{cs}(G)$ (by definition). $\mathrm{cr}_{cs}(G) \leqslant \overline{\mathrm{cr}}_{cs}(G)$.[54] Both cr_{cs} and $\overline{\mathrm{cr}}_{cs}$ violate what we called the embedding axiom: $\mathrm{cr}_{cs}(K_3) = \overline{\mathrm{cr}}_{cs}(K_3) = \infty$. $\mathrm{cr}_{cs}(G) = 0$ implies $\overline{\mathrm{cr}}_{cs}(G) = 0$.[55]

Values: $\overline{\mathrm{cr}}_{cs}(K_{2n}) = 2\binom{n}{4} + \binom{n}{2}^2$ [6].

Open Questions: Which graphs G satisfy $\mathrm{cr}_{cs}(G) = 0$ or $\overline{\mathrm{cr}}_{cs}(G) = 0$? ▼ Which graphs have finite centrally symmetric (rectilinear) crossing number? ▼ What are $\mathrm{cr}_{cs}(K_{m,n})$ and $\overline{\mathrm{cr}}_{cs}(K_{m,n})$? ▼ $\overline{\mathrm{cr}}_{cs}(M_{4n}) = 1$, where M_k is the Möbius ladder on k vertices. What about $\overline{\mathrm{cr}}_{cs}(M_{4n+2})$? ▼ Can $\overline{\mathrm{cr}}_{cs}(G)$ be bounded in $\mathrm{cr}_{cs}(G)$?

Also see: Monotone crossing numbers.

Centrally symmetric rectilinear crossing number. See centrally symmetric crossing number.
Circular bundled crossing number. See bundled crossing number.
Circular crossing number. See convex crossing number.
Circular *k*-partite crossing number. See convex crossing number.
Clockwise crossing number. See cyclic level crossing number.

[53]Using $\mathrm{cr}_{cs}(G+G) = 2\,\mathrm{cr}(G)$ and $\overline{\mathrm{cr}}_{cs}(G+G) = 2\,\overline{\mathrm{cr}}(G)$.
[54]Multiple crossings in the rectilinear drawing can be removed by perturbing edges.
[55]The Tutte embedding [745] of a plane graph is symmetric if one starts with a symmetric arrangements of vertices on the outer face.

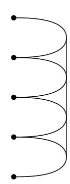
Confluent crossing number

Definition: A *confluent drawing* (sometimes known as a train track) consists of branches (simple curves with two connection points) and switches (homeomorphs of the symbol $\prec$, so three connection points), and nodes. Each of the three connection points of a switch is incident to a node, or to the connection point of exactly one branch or one switch. Each connection point of a branch is incident to a connection point of a switch or a node. The drawing is smooth at connection points and the only crossings allowed are crossings between branches. A confluent drawing represents a graph $G = (V, E)$ as follows: V is the set of nodes of the drawing, and an edge in E corresponds to a smooth curve connecting its endpoints (such a curve cannot make a sharp turn between the upward and the downward branch of the $\prec$) without turning around. Note that a single branch or switch can carry many edges. The *confluent crossing number* of a graph G is the smallest number of crossings required in a confluent drawing of G.

Reference: Based on Eppstein, Goodrich, Meng [292], also Newberry [587].

Comments: Confluent drawings were introduced by Dickerson, Eppstein, Goodrich, and Meng [248] to reduce the number of crossings (which they do dramatically) while emphasizing the connectivity structure visually. A confluent drawing looks like a train track and track crossing number would be a good alternative name. Eppstein, Goodrich, and Meng [292] define this crossing number implicitly as a crossing minimization problem. They restrict themselves to the special case of two-layered drawings where G is bipartite (each group being a layer) and distinguish between various levels of depth. So, in effect, they consider a *bipartite confluent crossing number*. One could consider variants in which switches are also counted as crossings (see Metro-line crossing number). Newberry [587] earlier introduced the technique of edge clustering for layered drawings of directed graphs with the same goal of reducing the total number of crossings. Edges that share the same sources and targets can be bundled (or concentrated) into *edge concentration nodes* (which require new levels).

Complexity: Open, even the special case of testing whether a graph has a confluent embedding (no crossings) is not known to be **NP**-hard (although it is known to lie in **NP** [451]).



Values: Complete and complete bipartite graphs have confluent crossing number 0, see the crossing-free confluent drawing of K_5 in the margin. The Chvátal graph has confluent crossing number 1 [290].

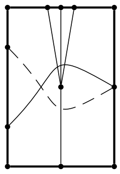
Also see: Metro-line crossing number.

Constrained crossing number

Definition: A *partially drawn graph* is a graph $G = (V, E)$ with a subgraph $H \subseteq G$ and a drawing $\mathcal{H}$ of H in the plane. The *constrained crossing number* of G given $\mathcal{H}$ is the smallest number of crossings in any drawing of G that contains $\mathcal{H}$ minus the number of existing crossings in $\mathcal{H}$.

Reference: Mutzel, Ziegler [581, 582] and Hamm, Hliněný [384].

Comments: Mutzel, Ziegler defined a more restricted variant: they required H to be a connected graph with vertex set V and $\mathcal{H}$ to be an embedding. In that case, $\mathcal{H}$ can be described completely by its rotation system. Hamm and Hliněný allowed partial drawings and named the resulting variant *partially predrawn crossing number*, written pd-cr$(G, \mathcal{H})$.[56] This version had earlier been introduced as a crossing minimization problem [342]. If $\mathcal{H}$ is not an embedding, then the drawing of G may not be simple, as the Figure in the margin shows, where $E(G) - E(H)$ is the dashed edge, and the outer face is empty. The notion of a *simple constrained crossing number* is implicit in [384].[57] It is known to be **NP**-complete to test whether there is a simple drawing of G extending a plane $\mathcal{H}$, even if $E(G) - E(H) = e$ [65]. Hamm and Hliněný also introduce the *partially predrawn c-planar crossing number* in which the drawing of G must have local crossing number at most c.



Complexity: **NP**-complete (since crossing number is a special case); the restricted case defined by Mutzel and Ziegler is also **NP**-complete since fixed linear crossing number is a special case. Testing whether there is an embedding of G containing $\mathcal{H}$ is in linear time [52]. The constrained crossing number is fixed parameter tractable [384, Theorem 1.1], and this remains true if the resulting drawing of G has to have bounded local crossing number [384, Theorem 1.3].

Open Questions: Is the constrained crossing number fixed-parameter tractable for parameter $k = |E(G)| - |E(H)|$?

Also see: Fixed linear crossing number, crossing number of graphs with rotation, map crossing number, wire crossing number.

Convex crossing number

Definition: The *convex crossing number* of a graph G, $\text{bkcr}_1(G)$, is the smallest number of crossings in a drawing of G in which all vertices of G lie on the boundary of a convex set and edges have to lie within the convex set (a *convex drawing* of G). If G is a k-partite graph we can require that all vertices belonging to a particular group occur consecutively on the boundary. Call this variant the *convex k-partite crossing number* of G.

Reference: Melihov, Kureĭčik, Seljankin, Tiščenko [553], Mäkinen [538], Kainen [470], Riskin [662].

Comments: The convex crossing number is the same as bkcr_1, the 1-*page book crossing number*; other names include *outerplanar crossing number* [695] and *circular crossing number* [707]. Extremal problems that, in effect, ask for the calculation of the convex crossing number for certain graphs are even older: In 1839, Gräfe asks for $\text{bkcr}_1(K_n)$,

[56]Earlier versions of the survey restricted the constrained crossing number to partial embeddings, but partial drawings are a natural generalization.

[57]If simplicity of G does not matter, then one can planarize $\mathcal{H}$ by introducing dummy vertices for crossings.

and derives a rather complicated (but correct) formula to compute it [362, p.200][58]; a brief note in the "Archiv für Mathematik und Physik" by a high-school student proves $\mathrm{bkcr}_1(K_n) = \binom{n}{4}$ inductively (Englert [285]); a follow-up communication by a better-known mathematician includes another counting argument, as well as the now common argument counting K_4-subgraphs (Saalschütz [665])[59]; in 1889 the problem appears as an exercise in an algebra textbook (Chrystal [208, p.34]). A convex straight-line drawing of a K_9 can be found in an early, illustrated edition of Ramon Llull's "Ars Magna" [534] from 1517. Athanasius Kircher includes the same drawing in his 1669 "Ars Magna Sciendi" [489, p.8]. See Singmaster [706, 5.Q.1] for related puzzles. The variant $\mathrm{bkcr}_1(G, \pi)$ in which the order of the vertices along the boundary is prescribed is a special case of the *fixed linear crossing number*. According to the reviews on zbMATH and MathSciNet, the paper by Melihov, Kureĭčik, Seljankin, Tiščenko [553] studies the convex crossing number for fixed and changing orderings of the vertices. Mäkinen [538] mentions the possibility of minimizing edge crossings in convex drawings, but immediately dismisses it, preferring circular dilation to optimize drawings. Kainen [470] introduced the *local outerplanar crossing number*, which he abbreviated as *locr*(G), and which we would call the *local convex crossing number*), in which we try to minimize the largest number of crossings along any edge; drawings with local convex crossing number at most 1 have been called *outer* 1-*planar* [75, 281], also see [125].[60,61] The local k-page crossing number (see book crossing number) generalizes this notion to more than one page. Riskin [662] introduced the convex k-partite crossing number as the circular k-partite crossing number. There really is no reason to restrict this crossing number to k-partite graphs, it also makes sense if we allow crossings within each group.[62] In the case that the order of the groups around the boundary is fixed, this variant has been studied under the name *grouped circular layouts* in [360]. For $k = 2$, the circular k-partite crossing number equals the bipartite crossing number, for $k = |V|$, it is simply the convex crossing number. For a version maximizing the number of crossings, see the convex maximum rectilinear crossing number (under maximum rectilinear crossing number). One could also imagine allowing multiple nested layers of points in convex position; for the special case of rectilinear drawings of the complete graph, this has been studied in [535]; that approach could also be viewed as a refinement of the

[58]The second edition of the book, which is available online, does not contain the general formula, it only computes the values for K_6 and K_9 [363, p.193].

[59]Gräfe, Englert and Saalschütz phrase the problem as counting the number of (inner) crossings of diagonals in a convex n-gon; Gräfe intriguingly suggests that the number does not just depend on n, e.g. when multiple crossings are counted as one (the way the degenerate crossing number counts), as well as when the drawing is not convex; he follows up a bit on the first suggestion, but not the second. Saalschütz asks the reader to determine the number of regions in the convex drawing of K_n.

[60]Eggleton [281] introduced a degenerate version of outer 1-planarity, see the discussion under the entry for local crossing number.

[61]We avoid this term, since there is a conflicting notion of outer k-planarity.

[62]Arguably, this is exactly the crossing number variant discussed by Bronfenbrenner [148] in a 1945 sociology paper unearthed by David Eppstein [289].

rectilinear crossing number. Allowing multiple, superimposed, layers, we can define the *biplanar convex crossing number* as the smallest number of crossings between edges of the same color in any two-coloring of the edges of G in a convex drawing of G. This is the same as the 2-page crossing number (see under book crossing number). Kainen [465] introduced the *average outerplanar crossing number* $\hat{\nu}_1(G)$, as the average of $\text{bkcr}_1(G, \pi)$ over all π, but strictly speaking we would not consider it a crossing number.

Complexity: **NP**-complete [541]. $\text{bkcr}_1(G, \pi)$ can be computed in time $O(n^2)$ [270]. Testing whether the local convex crossing number is at most 1 is in linear time [75].

Relationships: $\text{bkcr}_1(G) \geqslant \overline{\text{cr}}(G)$ for all graphs G (from definition). There is a crossing lemma, $\text{bkcr}_1(G) \geqslant m^3/(27n^2)$ [692], an improved lower bound has been announced in [9]. $\text{bkcr}_1(G) = O((\text{cr}(G) + \sum_{v \in V(G)} \deg(v)^2) \log n)$ [691]. $\text{bkcr}_1(G) \leqslant (m+1)^3/(3(n-2)^2$ [9]. Graphs with local convex crossing number at most k have minimum degree at most $(4k+1)^{1/2} + 1$ [180].

Values: Obviously, $\text{bkcr}_1(K_n) = \binom{n}{4}$ [285, 665]. $\text{bkcr}_1(K_{m,n}) = \binom{n}{2}\binom{m}{2} - (1/12)(n^2m^2 - n^2 - m^2 + (\gcd(m,n))^2$ [10].[63] For results on the convex k-partite crossing number of $K_{m,n}$ see [661], for results on $K_{n,n,\ldots,n}$, see [334]. Let $M_{m,n} = P_m \square P_n$ denote the $m \times n$ mesh. $\text{bkcr}_1(M_{3,n}) = 2n - 3$ if n even and $2n - 4$ otherwise, $n \geqslant 3$ [334], $\text{bkcr}_1(M_{4,n}) = 4(n-2)$ for $n \geqslant 2$ [414]. Asymptotically, $\text{bkcr}_1(M_{n,n}) = \Omega(n^2 \log n)$ [692]. For Halin graphs, see [334], for circulant graphs see [414], for the cone graph $C_n * \overline{K}_2$ see [468], and for random labeled trees, see [58]. For randomly, convexly embedded graphs, there is a central limit theorem [59].

Open Questions: Ábrego and Fernández-Merchant conjecture that the *convex midrange crossing constant* exists and equals 1/3; that is, if we define $\text{bkcr}_1(n, m)$ as the minimum of $\text{bkcr}_1(G)$ over all graphs G with n vertices and m edges, they conjecture that the limit of $\text{bkcr}_1(n,m)n^2/m^3$ as n goes to infinity and $n \ll m \ll n^2$ equals 1/3. ▼ What is $\text{bkcr}_1(C_m \square C_n)$? Kainen [465] showed that $\text{bkcr}_1(C_m \square C_n) \leqslant (m^3 + 3n^2)/2$ for even m and n.

Also see: Fixed linear crossing number, bipartite crossing number, tile crossing number, disk crossing number (under map crossing numbers), convex simultaneous crossing number, biplanar crossing number, book crossing number.

Convex k-partite crossing number. See convex crossing number.
Convex maximum rectilinear crossing number. See maximum rectilinear crossing number.
Convex simultaneous crossing number. See simultaneous crossing number.
Cross index. See local crossing number.
Crossing edge number. See edge crossing number.

[63] This generalizes Riskin's result that $\text{bkcr}_1(K_{m,n}) = 12n(m-1)(2mn - 3m - n)$ if $m|n$ [660]. Ábrego and Fernández-Merchant [10] also show that $\text{bkcr}_1(K_{m,n}) = \text{cr}_{②}(K_{m,n})$ by giving a correspondence between convex and cylindrical drawings of $K_{m,n}$.

Crossing number

Definition: The *crossing number* of G, $\mathrm{cr}(G)$, is the smallest number of crossings in any drawing of G. We write $\mathrm{cr}_\Sigma(G)$ for the crossing number of G on surface Σ; cr_{S_g} is also known as the *genus g crossing number*, cr_{S_1} is the *toroidal crossing number*, cr_{N_1} is the *projective plane crossing number* and cr_{N_2} is the *Klein bottle crossing number.* If the graph is equipped with a rotation (embedding) scheme ρ, we write $\mathrm{cr}_\Sigma(G,\rho)$ for the crossing number of the graph with the prescribed rotation (embedding) scheme ρ.

Reference: Turán [744], Harary and Hill [389], also Harary [387, 388].

Comments: For a detailed account of the early history of the crossing number, see Beineke and Wilson's "The Early History of the Brick Factory problem" [103], but also see Remark 4. Influenced by Turan's problem [744], research during the initial phase (1950s) focussed on the crossing number of the complete bipartite graph (Zarankiewicz [788], Urbanik [750]) and in the 1960s expanded to include investigation of complete graphs (e.g. Guy [373], who credits Anthony Hill and C.A. Rogers, and writes that Erdős claimed to have been thinking about the problem for 20 years; also Saaty [667], Goodman [361]). As far as we can tell, the first paper defining the crossing number for arbitrary graphs is due to Harary and Hill in 1963 [389], and one of the first papers in which the crossing number of an infinite family of graphs was determined is by Guy and Harary [380] showing that Möbius ladders have crossing number 1. The toroidal crossing number was introduced in [382, 506], and the Klein bottle crossing number together with general surface crossing numbers in [506] (also [463]).

Complexity: **NP**-complete [348], remains **NP**-complete for almost planar graphs [163]—even if there are only a small number of high-degree vertices [420, 426], for cubic graphs [424] and if the drawing of the graph is restricted by a given rotation (embedding) system ρ [621]. There is a constant $c > 0$ so that approximating the crossing number to within a factor of c (even for cubic graphs) is **NP**-complete [161, 637], but it can be approximated to within a polynomial bound for graphs of bounded degree [181, 210, 212], and there is a subpolynomial approximation algorithm [213]. An algorithm achieving an additive error of $o(|G|^4)$ in randomized polynomial time has also been announced [628]. The embedding problem $\mathrm{cr}_\Sigma(G) = 0$ can be solved in linear time for any (compact orientable or non-orientable) surface Σ [562]. The surface crossing number problem, $\mathrm{cr}_\Sigma(G)$, remains **NP**-complete for all surfaces Σ (via an easy reduction from the planar case). Testing $\mathrm{cr}(G) \leqslant k$ can be decided in time $O(f(k)n)$, that is, the problem is fixed-parameter tractable [368, 487].

Relationships: $\mathrm{cr}(G) \geqslant 1/29\ m^3/n^2$ for $m > 7n$, a result known as the *crossing lemma* [17].[64] The lower bound can be refined for graphs with an imbalanced degree sequence [332, 601], and it can be improved for sufficiently dense graphs [571].

[64] The original versions of the crossing lemma (with smaller constants), but $m > 4n$, are due to Ajtai,Chvátal, Newborn, Szemerédi [32] and Leighton [528]. The previous best bound was $\mathrm{cr}(G) \geqslant 1024/31827\ m^3/n^2$ for $m > 103/16\ n$ [600].

There are crossing lemmas for multigraphs, $\mathrm{cr}(G) = \Omega(m^3/(kn^2))$, where k is an upper bound on the edge multiplicity, and $m > cnk$ for some constant c [22, 606, 726]. This result is asymptotically tight in general, but the dependence on the multiplicity can be removed if additional assumptions are made [22, 330, 483, 605, 606]. The limit of $\mathrm{cr}(G)n^2/m^3$ as n goes to infinity, and $n \ll m \ll n^2$, exists and is denoted as γ, and known as the *midrange crossing constant* [227, 228, 295, 600, 602, 609]. For $\Sigma \in \{S_g, N_g\}$ we have $\mathrm{cr}_\Sigma(G) = \Omega(m^3/n^2)$ if $0 \leqslant g < n^2/m$ and $\mathrm{cr}_\Sigma(G) = \Omega(m^2/g)$ if $n^2/m \leqslant g \leqslant m/64$ [698]. Asymptotically, $\mathrm{cr}(G) = O(g(\mathrm{cr}_{S_g}(G) + n))$ for graphs of bounded degree as long as $g = o(n)$ [259]. If $\mathrm{cr}_\Sigma(G) = 0$, then $\mathrm{cr}(G) \leqslant c_\Sigma \Delta n$, where Δ is the maximum degree of G [140], for an algorithmic view of this result, see [198]. The behavior of the sequence $\mathrm{cr}_{S_0}(G), \mathrm{cr}_{S_1}(G), \mathrm{cr}_{S_2}(G), \ldots$ (and similarly for non-orientable surfaces) has been studied by Širáň and others, see [548] for a recent survey and results.

VALUES: See [218] for a comprehensive survey of bounds and values of the crossing number.[65] The **planar** crossing number of the **complete** graph K_n is at most $Z(n) = X(n)X(n-2)/4$, where $X(n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$ [127, 373].[66] Guy's, or Harary and Hill's, or Hill's conjecture states that $\mathrm{cr}(K_n) = Z(n)$ [103, 389]; the conjecture is known to be true for $n \leqslant 12$ [616], and $\mathrm{cr}(K_{13}) \in \{219, 221, 223, 225\}$ [550]. (For a computer-free proof that $\mathrm{cr}(K_9) = 36$, see [551].) For a strengthened version of the conjecture, see [87]. It is known that $\mathrm{cr}(K_n) > 0.985\ Z(n)$ [91].[67] The crossing number of the **complete 2-partite** graph $K_{m,n}$ is conjectured to be given by Zarankiewicz's function $Z(m,n) = X(m)X(n)$, which counts the number of crossings in Zarankiewicz's drawing of $K_{m,n}$. This is now known as Zarankiewicz's conjecture.[68] As in the case for complete graphs, the upper bound $\mathrm{cr}(K_{m,n}) \leqslant Z(m,n)$ is easy, but the lower bound is hard. The conjecture is known to be true for $n \leqslant 6$ [492] and $n \leqslant 8, m \leqslant 10$ [773]. $\mathrm{cr}(K_{7,n}) \geqslant 2.203n^2 - 4.5n > 0.979\ Z(7,n)$ [261], building on [236]. Using semi-definite programming, it can be shown that $\mathrm{cr}(K_{m,n}) > \alpha Z(m,n)$ for sufficiently large m and n [149, 494]; currently we have $\alpha = 0.8878$ [149] (the paper also contains more specific bounds for $10 \leqslant m \leqslant 13$). For every m there is an $N(m)$ so that if $\mathrm{cr}(K_{m,n}) = Z(m,n)$ for all $n \leqslant N(m)$, then $\mathrm{cr}(K_{m,n}) = Z(m,n)$ for all n [206].[69] If $\lfloor m/2 \rfloor \lfloor n/2 \rfloor$ divides $\mathrm{cr}(K_{m,n})$ for all m, n, then Zarankiewicz's conjecture is true (and a similar result holds for Hill's conjecture) [327]. For the balanced bipartite graph, the best current lower asymp-

[65] The official journal version is [219], but the authors promise to keep updating the arXiv version [218].

[66] It should be pointed out that verifying the upper bound is a tedious exercise in counting. Mohar [565, 566] discovered a geodesic embedding of K_n for which the bound can be verified much more easily.

[67] This improves a lower bound of $\mathrm{cr}(K_n) > 0.8594\ Z(n)$ for sufficiently large n which follows by combining a lower bound on $\mathrm{cr}(K_{m,n})$ from [494] with the main theorem from [650] discussed below.

[68] Zarankiewicz [788] claimed equality, but his proof (like Urbanik's [750]), contained a subtle error which was later found by Kainen and Ringel (as described by Guy [376]), and by Blažek, as mentioned in [503].

[69] For a partial extension to arbitrary surfaces, see [647].

totic lower bound is $\text{cr}(K_{n,n}) \geqslant 0.9118Z(n,n)$ [90].[70] The conjectures for complete and complete bipartite graphs are related: the truth of Zarankiewicz's conjecture implies that $\lim_{n\to\infty} \text{cr}(K_n)/Z(n) = 1$ [466]; in fact, $\lim_{n\to\infty} \text{cr}(K_n)/Z(n) \geqslant \lim_{n\to\infty} \text{cr}(K_{n,n})/Z(n,n)$ [650], so asymptotic improvements on $\text{cr}(K_{n,n})$ lead to corresponding improvements on $\text{cr}(K_n)$.[71] For **complete 3-partite** graphs we know that $\text{cr}(K_{1,3,n}) = Z(4,n) + \lfloor n/2 \rfloor$ and $\text{cr}(K_{2,3,n}) = Z(5,n) + n$ [68], $\text{cr}(K_{1,4,n}) = n(n-1)$ [432, 448]. It is known that $\text{cr}(K_{1,m,n}) = Z(m+1, n+1) - \lfloor m/2 \rfloor \lfloor n/2 \rfloor$ if Zarankiewicz's conjecture is true [432, 780, 781]. $\text{cr}(K_{2,4,n}) = Z(6,n) + 2n$ [434]. $\text{cr}(K_{3,3,n}) \geqslant Z(6,n) + n + 1$ [356],[72] and $\text{cr}(K_{3,3,n}) = Z(6,n) + 2n + 2\lfloor n/2 \rfloor + 1$ if Zarankiewicz's conjecture is true for $m = 7$, and the cases up to $n = 2-$ are true [431]. For **complete 4-partite** graphs we have $\text{cr}(K_{1,1,1,n}) = X(n)$ [404], $\text{cr}(K_{1,1,4,n}) = Z(6,n) + 2n + 2\lfloor n/2 \rfloor$ [719], $\text{cr}(K_{1,2,2,n}) = Z(5,n) + \lfloor \frac{3n}{2} \rfloor$ [429], $\text{cr}(K_{2,2,2,n}) = Z(6,n) + 3n$ [433]. For **complete 5-partite** graphs $\text{cr}(K_{1,1,1,1,n}) = Z(4,n) + n$ [771], also [429], and $\text{cr}(K_{1,1,1,2,n}) = Z(5,n) + 2n$ [429]. For **complete *k*-partite** graphs Harborth [404] found a function $Z(n_1, \ldots, n_k)$ for which $\text{cr}(K_{n_1,\ldots,n_k}) \leqslant Z(n_1, \ldots, Z_{n_k})$, and he conjectures this upper bound to be the correct value (*Harborth's conjecture*).[73] It is known that $0.666\ Z(n_1, n_2, n_3) \leqslant \text{cr}(K_{n_1,n_2,n_3}) \leqslant Z(n_1, n_2, n_3)$ [355].[74,75] For the **projective plane**, $\text{cr}_{N_1}(K_n)$ is known up to $n \leqslant 10$ and there are asymptotic bounds: $(41/273)\binom{n}{4} \leqslant \text{cr}_{N_1}(K_n) \leqslant c_{N_1}\binom{n}{4}$ for sufficiently large n and $c_{N_1} < 3/\pi^2$ [66, 283, 506].[76] It is known that $\text{cr}_{N_1}(K_{4,n}) = \lceil n/3 \rceil (2n - 3(1 + \lceil n/3 \rceil))$ [435]. Also, $\text{cr}_{N_1}(C_3 \,\square\, C_n) = n - 1$ for $n \geqslant 5$ and $\text{cr}_{N_1}(C_3 \,\square\, C_4) = 2$ [664]. For other graphs in the projective plane, see [184, 436, 536, 759, 760]. For the **torus**, $\text{cr}_{S_1}(K_n)$ is known for $n \leqslant 10$ and $\text{cr}_{S_1}(K_{m,n})$ for $m, n \leqslant 6$ [383]. Asymptotically, $(23/210)\binom{n}{4} \leqslant \text{cr}_{S_1}(K_n) \leqslant (22/81)\binom{n}{4}$ [283, 383][77] and $1/15\binom{m}{2}\binom{n}{2} \leqslant \text{cr}_{S_1}(K_{m,n}) \leqslant 1/6\binom{m-1}{2}\binom{n-1}{2}$ [382]. Also, $\text{cr}_{S_1}(K_{3,n}) =$

[70] Using Razborov's method of flag algebras which was first applied to crossing numbers by Norin and Zwols in unpublished work, see [90].

[71] Apparently Székely phrases this as "If Zarankiewicz's conjecture is asymptotically X% true, then the Harary-Hill conjecture is also asymptotically X% true", thanks to one of the referees for supplying that quote. Székely's survey [728] contains more details on the current status of the Zarankiewicz conjecture.

[72] The paper also derives an upper bound which agrees with the general upper bound found by Harborth [404].

[73] Harborth calls his function S. He mentions a paper by Blažek and Kolman [128] which contains a similar expression, without proof; a proof may be contained in the hard-to-locate [129].

[74] The authors of [355] use a different expression $A(n_1, n_2, n_3)$ which equals $Z(n_1, n_2, n_3)$ in values. Their drawings differ from Harborth's [404] in that they are rectilinear, leading them to conjecture that $\overline{\text{cr}}(K_{n_1,n_2,n_3}) = \text{cr}(K_{n_1,n_2,n_3}) = Z(n_1, n_2, n_3)$.

[75] There are many further results for (planar) crossing numbers of complete k-partite graphs, hypercubes, Cartesian (and Kronecker) products of cycles, paths, and stars and other families of graphs; for a survey on these results, see [218].

[76] Koman's upper bound of $(13/16)Z(n) \approx 39/128\binom{n}{4}$ stood for nearly 50 years, until Elkies [283] showed that a randomized construction a la Moon [573] gives a better bound of $3/\pi^2\binom{n}{4}$. Arroyo, McQuillan, Richter, Salazar, and Sullivan [66] then showed that Elkies construction is not asymptotically optimal.

[77] Elkies [283] bound of 22/81 improves a much older result by Guy, Jenkyns and Schaer [383] using a randomized construction.

$\lceil (n-3)^2/12 \rceil$ [382] and $\mathrm{cr}_{S_1}(K_{4,n}) = \lfloor n/4 \rfloor (2n - 4(1 + \lfloor n/4 \rfloor))$ [430, 437]. There are asymptotic bounds for $\mathrm{cr}_{S_1}(K_{m,n})$ [382]. $\mathrm{cr}_{S_1}(C_n^3) = 0$ for $n \geqslant 7$, and $\mathrm{cr}_{S_1}(C_n^4) = n$ for $n \geqslant 9$ [390], where G^k is the k-th power of G.[78] For the crossed toroidal grid graph $X_{m,n}$, which is embeddable on the Klein bottle, it is known that $\mathrm{cr}_{S_1}(X_{3,n}) = 1$ and $\mathrm{cr}_{S_1}(X_{4,n}) = 2$ for $n \geqslant 4$ and there is an upper bound for $\mathrm{cr}_{S_1}(X_{m,n})$ conjectured to be tight [658]. For the **Klein bottle**, $\mathrm{cr}_{N_2}(K_n)$ is known for $n \leqslant 9$ [506] and there are asymptotic bounds: $(1/14)\binom{n}{4} \leqslant \mathrm{cr}_{N_2}(K_n) < (59/216)\binom{n-1}{4}$ for $n \geqslant 16$ [505]. $\mathrm{cr}_{N_2}(K_{m,n})$ is known for $3 \leqslant m \leqslant 6$ and $n \leqslant N(m)$ with $N(3) = 12$, $N(4) = 8$, $N(5) = N(6) = 6$; for these ranges $\mathrm{cr}_{N_2}(K_{m,n}) = \mathrm{cr}_{S_1}(K_{m,n})$ [504]. $\mathrm{cr}_{N_2}(C_m \square C_n)$ is known for $m \leqslant 6$ [659] and for sufficiently large m and n [460].[79] For the **double torus**, $\mathrm{cr}_{S_2}(K_9) = 4$ [663].[80] For the **triple and quadruple torus** it has been announced that $\mathrm{cr}_{S_3}(K_{10}) = 3$, and $\mathrm{cr}_{S_4}(K_{11}) = 4$ [527].

Exact values of $\mathrm{cr}_\Sigma(K_{3,n})$ are known for all surfaces Σ [428, 648]. Lower and upper bounds on $\mathrm{cr}_\Sigma(K_n)$, $\mathrm{cr}_\Sigma(K_{m,n})$, and $\mathrm{cr}_\Sigma(Q_n)$ are surveyed in [690, 698]. Gross [369] showed that $\mathrm{cr}_{S_g}(O_p) = p(p-1)/2$, where $p \equiv 1 \bmod 4$ is a prime power, $g = (p-1)(p-4)/4$, and $O_p = K_{2p} - pK_2$, the *octahedral graph*. Lower bounds on a generalized notion of periodic graphs on surfaces have been announced in [779].

Open Questions: There is a well-known conjecture by Harary, Kainen and Schwenk [391] that $\mathrm{cr}(C_m \square C_n) = n(m-2)$ for $n \geqslant m \geqslant 3$; the conjecture is known to be true for $3 \leqslant m \leqslant 7$ [21, 106, 496, 645, 653], and for $n \geqslant m(m+1)$, $m \geqslant 3$ [358]; for surveys predating the more recent developments ($6 \leqslant m \leqslant 7$, and $n \geqslant m(m+1)$), see [651, 693]. It is also known that for every m there is a $c_m \geqslant 0$ so that $\mathrm{cr}(C_m \square C_n) = n(m-2) - c_m$ for $n \geqslant 3$ [635]. A weaker version of the conjecture, suggested by computational evidence in [415], would be that $\mathrm{bkcr}_2(C_m \square C_n) = n(m-2)$. ▼ Erdős and Guy [295] conjectured a value for $\mathrm{cr}(Q_n)$ which was disproved after nearly fifty years by Yang, Wang, Wang and Zhou [782].[81] The first open value is $\mathrm{cr}(Q_7) \leqslant 1744$. ▼ Chia and Lee [186] conjecture that $\mathrm{cr}(K_n - e) = Z(n) - \binom{\lfloor (n-1)/2 \rfloor}{2}$ (true for $n \leqslant 12$), and $\mathrm{cr}(K_{m,n} - e) = Z(m,n) - \lfloor (m-1)/2 \rfloor \lfloor (n-1)/2 \rfloor$ (true for $m \in \{3,4\}$). ▼ Czabarka, Singgih, Székely, and Wang [228] ask whether the midrange crossing constant (defined above) $\gamma = 8/(9\pi^2)$ (in that case, it would equal the rectilinear midrange crossing constant). ▼ DeVos, Mohar, and Šámal asked whether it is true that in any cr-minimal drawing of the disjoint union of two graphs G_1 and G_2 on a surface Σ, the drawings of G_1 and G_2 are disjoint? Trivially true for plane, and also known for projective plane [243] and the Klein bottle [101], also see [164]. ▼ Böröczky, Pach, and Tóth [140] ask whether $\mathrm{cr}(G) = O(g\Delta n)$, where g is the genus of G, and Δ its maximum degree (this is known to be true of the torus [607]). ▼ Shahrokhi, Székely, and Sýkora [698] conjecture that $\mathrm{cr}_\Sigma(K_n) = O(n^4/g)$, where $\Sigma \in \{S_g, N_g\}$. ▼ Richter asked whether $\mathrm{cr}(G) \geqslant \mathrm{cr}(K_n)$ implies

[78] G^k, the *k-th power* of G, is a graph on $V(G)$ with edge uv if G contains a path of length at most k between u and v.

[79] See Riskin's MathSciNet review MR1974148 of that paper.

[80] This refuted Conjecture 3.3 in [463].

[81] Also see discussion in [218, Section 6.1.1].

that $\mathrm{cr}(G+v) \geqslant \mathrm{cr}(K_{n+1})$, where + denotes the join of two graphs; the answer turns out to be no, but it is open how small the gap between $\mathrm{cr}(G+v)$ and $\mathrm{cr}(G)$ can be. For multigraphs, the gap is $\mathrm{cr}(G)^{1/2}$ [38]; the exact gap is also known for graphs with $\mathrm{cr}(G) \leqslant 7$ [38, 256]. ▼ How hard it is to decide whether G is 4-colorable for graphs G with $\mathrm{cr}(G) \leqslant 1$ [350].[82] ▼ Does every graph with crossing number at least 2 contain a subgraph with crossing number 2? [130, 642, 643].[83] ▼ Mohar [566] shows that for $K_n - M$, where M is a (not necessarily perfect) matching, we have $\mathrm{cr}(K_n - M) \leqslant Z(n) - |M|/2\ (\lfloor n/2 \rfloor - 1)(\lfloor n/2 \rfloor - 2)$ and conjectures that equality holds. ▼ Ho [437] conjectures that the crossing number of $K_{4,n}$ on a surface Σ of Euler genus $\mathrm{eg} = \mathrm{eg}(\Sigma)$ is $\lfloor \frac{n}{\mathrm{eg}+2} \rfloor (2n - (\mathrm{eg}+2)(1 + \lfloor \frac{n}{\mathrm{eg}+2} \rfloor))$ (which is known to be an upper bound).[84] ▼ Sequence A110507 in OEIS [452] is defined as $a(n)$, the smallest order of a cubic graph with crossing number n. The first open value is $a(12)$ [220, 766]. ▼ For two conjectures by Eric W. Weisstein on $\mathrm{cr}(K_{n,n} - M)$, where M is a perfect matching in $K_{n,n}$, and [453] on $\mathrm{cr}(\overline{nP_2})$, see [453]. ▼ Is it true that the number of good drawings of a 3-connected graph G is $O(f(\mathrm{cr}(G))n^2)$ [753]?[85] ▼ If G is a 4-connected graph with $\mathrm{cr}(G) \leqslant 3$, is G Hamiltonian? This is true for $\mathrm{cr}(G) \leqslant 2$ and false for $\mathrm{cr}(G) \leqslant 6$, see [598]. ▼ Staš and Valiska [715] conjecture that every K_n, $n \geqslant 8$ has a crossing-minimal drawing such that removing all edges with crossings disconnects the graph. The show that the conjecture is true for $8 \leqslant n \leqslant 12$, and true for $n \geqslant 13$ assuming the Harary-Hill conjecture.[86] ▼ In [473, Open problem 3.1] the authors ask whether computing the crossing number of G joined with a vertex, an apex graph, is **NP**-complete particularly if G is a partial hypercube; for general G this can be shown to follow from Cabello's hardness proof [161], but the partial hypercube variant is open.

Also see: Stable crossing number.

Crossing number of abstract topological graph

Definition: A graph G with a symmetric relation R over $E(G)$ is called an *abstract topological graph* or *AT-graph*. A drawing D is a *weak realization* of (G, R) if every pair of edges (e, f) that cross in D belongs to R. The *crossing number* of (G, R), $\mathrm{cr}(G, R)$, is the smallest number of crossings in a weak realization of (G, R). If there is no weak realization of (G, R) we let $\mathrm{cr}(G, R) = \infty$.

[82] Oporowski and Zhao [592] showed that such graphs are always 5-colorable, and 3-colorability is **NP**-complete, since it is for planar graphs [346].

[83] This was claimed to be true in [130], a paper on crossing numbers in linguistics (keyword: eodermdromes); Richter established the conjecture for several special cases of graphs, including cubic graphs [642]. The conjecture does not extend to crossing number 3, since $K_{3,5}$ has crossing number 4, but all its subgraphs have crossing number at most 2.

[84] It appears from [430] that an earlier version of [437] contained an attempted proof of this result for the Klein bottle, i.e. $\mathrm{eg} = 2$, but there were missing cases.

[85] This could be a first step towards understanding whether graph isomorphism is fixed-parameter tractable for graphs with bounded crossing number.

[86] Mengersen [557] calls what remains after removing the crossing edges of a good (not necessary optimal) drawing a *skeleton*, and studies skeletons of complete n-partite graphs.

REFERENCE: Kratochvíl [512].

COMMENTS: Kratochvíl introduced the crossing number $\mathrm{cr}_{at}(G, R)$ of an abstract topological graph (G, R) in his study of string graphs. Intersection graph theory studies graphs (G, R) which have weak realizations for restricted R. Trivially, if R contains no edges, then G has a linear number of edges (since it is planar). Graphs G which are weakly realizable with an R excluding the complete graph K_k are known as *k-quasi-planar*. Linear bounds on $|E(G)|$ are also known if R excludes a complete bipartite [599] or tripartite [736] graph. The study of twisted graphs [13, 405] falls into this category. This crossing number can be viewed as a special case of the weighted crossing number (weights being restricted to 1 and ∞).

COMPLEXITY: **NP**-complete [678].

RELATIONSHIPS: $\mathrm{cr}(G) \leqslant \mathrm{cr}(G, R)$ (by definition). There are abstract topological graphs (G, R) for which $\mathrm{cr}(G, R) \geqslant 2^{cn}$ for some $c > 0$ [512, 515], where $n = |V(G)|$. If $\mathrm{cr}(G, R) < \infty$, then $\mathrm{cr}(G, R) \leqslant m2^n$ [678], where $m = |E(G)|$ and $n = |V(G)|$.

OPEN QUESTIONS: Kratochvíl [512] conjectured that in any crossing minimal weak realization of (G, R) any edge which is involved in crossings is crossed by some edge exactly once.

ALSO SEE: Weighted crossing number, quasi crossing number.

Crossing parameter. See local crossing number.

CYCLIC LEVEL CROSSING NUMBER

DEFINITION: A *cyclic k-level graph* $G = (V, E, \ell)$ is a directed graph (V, E) with a *leveling* ℓ, a mapping from V to $\{1, \dots, k\}$ which assigns a level $\ell(u)$ to each vertex u. Fix k rays, all starting at the origin, and number them 1 through k in clockwise order. A *cyclic* drawing of a cyclic k-level graph is a drawing in which a vertex u is placed on ray $\ell(u)$, and a directed edge (u, v) is drawn in the clockwise wedge between rays $\ell(u)$ and $\ell(v)$ so that the edge crosses all rays starting at the origin (not just the k rays we chose) at most once. The *cyclic level crossing number* of a cyclic k-level graph is the smallest number of crossings in a cyclic drawing of the graph.

REFERENCE: Based on Bachmaier, Brandenburg, Brunner, Hübner [81].

COMMENTS: The idea of realizing a leveled graph in a cyclic drawing can be found in a paper by Sugiyama, Tagawa and Toda [721], where cyclic k-level graphs are introduced in an appendix under the name *recurrent hierarchies*. The crossing minimization problem for cyclic k-level graphs is studied by Bachmaier, Brandenburg, Brunner, Hübner [81], without introducing a name for the corresponding crossing number. The authors also refer to a 2009 master's thesis by Hübner, which is entitled "A global approach on crossing minimization in hierarchical and cyclic layouts of leveled graphs". A cyclic layout could be visualized in a non-cyclic way by repeating one of the layers at the beginning and end; this is what Bertin [116, Figure 4, p.109] does in his visualization of a tripartite perfect matching in which the order of vertices is fixed in each group; he uses the number of crossings between two layers as a measure of similarity: "The nearer the order between the columns, the less numerous are the intersections.".

One could also consider a *clockwise crossing number*, in which a directed graph $G = (V, E)$ is given, and the problem is to find a leveling ℓ that minimizes the cyclic level crossing number of (V, E, ℓ). This clockwise crossing number is to the cyclic level crossing number what the upward crossing number is to the leveled crossing number.

COMPLEXITY: **NP**-complete, since the bipartite crossing number is a special case. The embedding problem can be solved in quadratic time [80].

CYLINDRICAL CROSSING NUMBER

DEFINITION: A *cylindrical drawing* of a graph G is a drawing in which all vertices of G lie on two concentric circles, and no edge crosses a circle. The *cylindrical crossing number* of G, $\text{cr}_{\circledcirc}(G)$, is the smallest number of crossings in a cylindrical drawing of G. For a bipartite G a *bipartite cylindrical drawing* is a drawing in which the vertices in the same group of the partition lie on the same circle, and the inner face of the small circle, and the outer face of the large circle are empty. For a bipartite graph G, the *bipartite cylindrical crossing number*, $\text{cr}_{②}(G)$, is the smallest number of crossings in a bipartite cylindrical drawing of G. The local version of that is known as the *bipartite cylindrical local crossing number*, $\text{lcr}_{②}(G)$. One can also consider the *rectilinear bipartite cylindrical crossing number*, $\overline{\text{cr}}_{②}(G)$, in which edges are allowed to cross circles, but have to be straight-line; in this variant one also minimizes over the radii of the circles.

REFERENCE: Ábrego, Aichholzer, Fernández-Merchant, Ramos, and Salazar [2], based on earlier suggestion by Richter and Thomassen [650]. The bipartite cylindrical crossing number was introduced by Ábrego, Fernández-Merchant, and Sparks [16], there written as $\text{cr}_{\circledcirc}$. Nafar [583] introduces the rectilinear version. The local version of the bipartite cylindrical crossing number was introduced by Khachatryan [488].

COMMENTS: Bipartite cylindrical drawings were introduced in Richter and Thomassen [650] as a stepping stone to constructing cylindrical drawings of K_n, which is a class of drawings realizing the conjectured minimal crossing number $Z(n)$ of K_n, where $Z(n) = X(n)X(n-2)/4$, and $X(n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$.[87] If in addition to requiring the inner and outer face to be empty, we fix the cyclic order of the vertices on the concentric circles, we obtain the annulus crossing number. The cylindrical crossing number for general (non-bipartite) graphs was introduced in [2]. One way to generalize the cylindrical crossing number is to allow t circles (arbitrarily located, but disjoint), on which all vertices have to lie; this leads to the *t-circle crossing number* and the *t-partite circle crossing number* introduced in [271]. Nafar [583] studies, but does not name, the rectilinear bipartite cylindrical crossing number.

[87] Can one obtain constructions of crossing-minimal drawings of K_n if one starts with three, instead of two circles? Section 4 in [167] suggests that the answer is no.

Complexity: Testing whether $\mathrm{cr}_{\circledcirc}(G) = 0$ is **NP**-complete [271]. Testing whether $\mathrm{cr}_{\textcircled{2}}(G) \leqslant k$ is **NP**-complete for bipartite graphs G, an easy reduction from the bipartite crossing number, while $\mathrm{cr}_{\textcircled{2}}(G) = 0$ can be decided in linear time (it is the same as radial level planarity for two levels).

Relationships: $\mathrm{cr}_{\textcircled{2}}(K_{m,n}) = \mathrm{bkcr}_1(K_{m,n})$ [10].

Values: $\mathrm{cr}_{\circledcirc}(K_n) = Z(n)$ [2]. $\mathrm{cr}_{\textcircled{2}}(K_{n,n}) = n\binom{n}{3}$ and $\mathrm{cr}_{\textcircled{2}}(K_{m,n}) = \binom{n}{2}\binom{m}{2} - (1/12)(n^2m^2 - n^2 - m^2 + (\gcd(m,n))^2$ [10, 16, 650].[88] $\overline{\mathrm{cr}}_{\textcircled{2}}(K_{m,n}) = \binom{m}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$, for $m \geqslant n$ [583]. $\mathrm{lcr}_{\textcircled{2}}(K_{m,n})$ is known for $2 \leqslant m \leqslant 4$ and in case m divides n (there are also upper and lower bounds for the general case) [488].

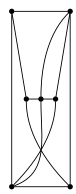
Also see: Radial crossing number, annulus crossing number (under map crossing number), t-circle crossing number.

Degenerate crossing number

Definition: The *degenerate crossing number* of a drawing D of a graph G is the number of points in which edges cross each other (that is, we count each point at which crossings occur only once, not $\binom{k}{2}$ times for k edges passing through it); recall that edges are not allowed to touch, and may not cross themselves. The *degenerate crossing number* of a graph G, $\mathrm{dcr}(G)$, is the smallest number of crossing points in a drawing of G. If we minimize over simple drawings only (each pair of edges crosses at most once), we obtain the *simple degenerate crossing number*, $\mathrm{dcr}^*(G)$.

Reference: Pach, Tóth [608]. Also see Harborth [394, 398].

Comments: Harborth [394, 398] studies multiple crossings in drawings of the complete graph, but does not consider the problem of minimizing the number of multiple crossings.[89] Pach and Tóth [608] credit Günter Rote and M. Sharir with asking "what happens if multiple crossings are counted only *once*". If we allow self-crossings we get the genus crossing number. Some papers use the term degenerate crossing number for dcr^* [19]. The definition of dcr^* is ambiguous. It is not clear whether the definition by Pach and Tóth [608] is aiming for crossing-simple or intersection-simple. There is a difference between the two, for example the graph shown in the margin has crossing-simple degenerate crossing number 1, but it requires at least two crossings, if adjacent edges are not allowed to cross.



Complexity: The degenerate crossing number is **NP**-complete even for cubic graphs [682].

Relationships: $\mathrm{gcr}(G) \leqslant \mathrm{dcr}(G) \leqslant \mathrm{dcr}^*(G) \leqslant \mathrm{cr}(G)$ by definition. There are examples with $\mathrm{dcr}(G) < \mathrm{dcr}^*(G)$ [608]. $\mathrm{dcr}(G) \leqslant 3\,\mathrm{gcr}(G)$, and $\mathrm{gcr}(G) = \mathrm{dcr}(G)$ for $\mathrm{dcr}(G) \leqslant 3$ [682]. There is an asymptotically optimal crossing lemma for the simple version,

[88] The case $m = n$ was solved in [650], the general case in [16]; the formula given in [16] is simplified in [10].

[89] Harborth's goal is the opposite: he tries to maximise the number of multiple crossings of the largest number of edges; in particular, he shows that K_{2m} can be drawn with two m-fold crossings. He conjectures that there cannot be drawings with three m-fold crossings (and verifies that for $m = 3$ if crossings between adjacent edges are not allowed).

$\text{dcr}^*(G) \geqslant c \cdot m^3/n^2$ for $m \geqslant 4n$ [19], while, on the other hand, $\text{dcr}(G) < m$, where $m = |E(G)|$, $n = |V(G)|$ [608].

Values: Pach and Tóth [608] claim that $\text{dcr}(K_{5,5}) \leqslant 15$, comparing it to $\text{cr}(K_{5,5}) = 16$.

Also see: Genus crossing number, bundled crossing number, triple crossing number.

Degenerate local crossing number. See local crossing number.
Diagonal crossing number. See joint crossing numbers.
Directed crossing number. See upward crossing number.
Disk crossing number. See map crossing number.

Edge crossing number

Definition: The *edge crossing number* of a drawing D of a graph G is the number of edges involved in crossings in D. The *edge crossing number* of G, $\text{ecr}(G)$, is the smallest edge crossing number of any drawing of G. The *rectilinear edge crossing number* of G, $\overline{\text{ecr}}(G)$, is the smallest edge crossing number of any rectilinear drawing of G. We can also define maximum variants (requiring drawings to be simple). The *book edge crossing number* of G is the smallest edge crossing number of any k-page book drawing of G.

Reference: Based on Ringel [655], Harborth and Mengersen [406, 407], Harborth and Thürmann [410], Ishiguro [455], Gange, Stuckey, Marriott [341], Bannister, Eppstein, Simons [95].

Comments: *Crossing edge number* may be a better name to avoid confusion with the standard crossing number (which is sometimes called edge crossing number). However, the term crossing edge number has also been used for skewness [353] with which ecr is easily confused. The skewness of G, $\text{sk}(G)$, is the smallest number of edges whose removal make a graph planar, while $\text{ecr}(G)$ minimizes the number of edges involved in crossings. By definition, $\text{sk}(G) \leqslant \text{ecr}(G)$ and it is easy to construct graphs G for which $\text{sk}(G) = 1$ and $\text{ecr}(G)$ is arbitrarily large.[90] Ringel [655] showed that every drawing of a K_n has at most $2n-2$ crossing-free edges, in other words, he studied $|E(G)| - \text{ecr}(G)$ for $G = K_n$.[91] Harborth, Mengersen and Thührmann [396, 406, 407, 409, 556] studied parameters h_s and H_s, the minimum and maximum number of edges with at most s crossings in an intersection-simple drawing of complete multipartite graphs. Extending their notation to arbitrary graphs, one could write $\text{ecr}(G) + H_0(G) = |E(G)|$, and $\text{max-}\text{ecr}(G) + h_0(G) = |E(G)|$. The smallest s for which $h_s(K_n) > 0$, that is, the smallest s for which every intersection-simple drawing of a complete graph contains an edge with at most s crossings, has been

[90] Albertson [34] defined the *crossing cover number*, $\rho(G)$, which is the smallest number of vertices so that in some drawing of G every crossing lies on an edge incident to one of the vertices. Analogously we could define the *edge crossing cover number*, $\rho'(G)$, to be the smallest number k of edges for which there is a drawing of G, called a *skewness-k drawing* in [251], in which every crossing lies on one of the k edges. Then $\rho'(G) = \text{sk}(G)$.

[91] There is one subtlety here: Ringel, and later Harborth, require drawings to be (intersection)-simple, but it is immediate that an ecr-minimal drawing is simple, so this does not lead to an inconsistency in this case.

studied starting with [409], and we know that $\Omega(n^{3/2}) \leqslant s \leqslant O(n^{7/4})$, see [519, 722]. Harborth and Thürmann [410] introduce the parameters $r_s(n)$ and $R_s(n)$ which they define as the minimum and maximum number of edges with at most s crossings in a straight-line drawing of K_n. If again we extend this notation to arbitrary graphs, we have $\overline{\mathrm{ecr}}(G) + R_0(G) = |E(G)|$ and max-$\overline{\mathrm{ecr}}(G) + r_0(G) = |E(G)|$. Gange, Stuckey, Marriott [341], in passing, mention the possibility of minimizing the number of edges involved in crossings. Ishiguro [455] defines a notion he calls *minimum non-crossing edge number*, $\mathrm{nce}(G)$, which, in our terminology, is $|E(G)| -$ max-$\overline{\mathrm{ecr}}(G)$, or $r_0(G)$ in the notation of Harborth and Thürmann [410]. Bannister, Eppstein, and Simons [95] define edge crossing numbers for 1-page and 2-page embeddings, denoting them as $\mathrm{cre}_1(G)$ and $\mathrm{cre}_2(G)$. The edge crossing number, unlike the skewness of a graph, can be made to fit our general notion of crossing number: $\sum_{e \in E} \max_{f \in E} \mathrm{pcr}(e, f)$, where $\mathrm{pcr}(e, f) = 1$ if and only if e and f cross at least once. Eggleton [281] uses "edge crossing number" to denote what we would call the simple degenerate local crossing number (see entry for local crossing number).

COMPLEXITY: Open. Bannister, Eppstein, and Simons [95] show that the 1-page and 2-page variants are fixed-parameter tractable for k-almost trees (with k being the parameter).

RELATIONSHIPS: $\mathrm{ecr}(G) \leqslant \overline{\mathrm{ecr}}(G)$ (by definition). $\mathrm{ecr}(G) \leqslant 2\,\mathrm{cr}(G)$, $\overline{\mathrm{ecr}}(G) \leqslant 2\,\overline{\mathrm{cr}}(G)$ and inequality can be strict (since $\mathrm{ecr}(G)$ and $\overline{\mathrm{ecr}}(G)$ are bounded by $|E|$).

VALUES: $\mathrm{ecr}(K_n) = \binom{n}{2} - (2n-2)$ [655]. $\mathrm{ecr}(K_{n_1,\dots,n_k})$ is known [556]. $\overline{\mathrm{ecr}}(K_n) = \binom{n}{2} - (2n-2)$ [410]. max-$\mathrm{ecr}(K_n) = \binom{n}{2}$ for $n \geqslant 8$, and values of max-$\mathrm{ecr}(K_n)$ are known for $n < 8$ [407]. max-$\overline{\mathrm{ecr}}(K_n) = \binom{n}{2} - 5$ for $n \geqslant 8$, and values for $n < 8$ are known [410].[92]

Faithful crossing number. See string crossing number.
Fixed convex bundled crossing number. See bundled crossing number.

FIXED LINEAR CROSSING NUMBER

DEFINITION: The *fixed linear crossing number*, $\mathrm{bkcr}_k(G, \pi)$ of an ordered graph (G, π) in a book with k pages, is the smallest number of crossings in a drawing of G in a book with k pages so that all vertices lie on the spine of the book in the order prescribed by π and each edge lies on a single page. If π orders only a subset $A \subseteq V(G)$ of the vertices (the *anchors*) and the remaining vertices are not required to lie on the spine, we obtain the *anchored crossing number*, $\mathrm{bkcr}_k(G, A, \pi)$.

REFERENCE: Melihov, Kureĭčik, Seljankin, Tiščenko [553] for $\mathrm{bkcr}_1(G, \pi)$ and $\mathrm{bkcr}_2(G, \pi)$. Masuda, Nakajima, Kashiwabara, Fujisawa [542] for $\mathrm{bkcr}_2(G, \pi)$. Cabello, Mohar [163] for $\mathrm{bkcr}_1(G, A, \pi)$.

COMMENTS: According to the reviews on zbMATH and MathSciNet, Melihov, Kureĭčik, Seljankin, Tiščenko [553] study the fixed linear crossing number for one and two

[92] This result is also announced in the later [455], without proof. The author also claims that $\mathrm{nce}(G) \leqslant \min\{\chi(G), 5\}$ unless G is $K_{1,7}$ or K_7.

pages. A close variant of the book crossing number, it could also be called the fixed book crossing number; $\text{bkcr}_1(G,\pi)$ has been called the *chordal crossing number* [787]. Cabello and Mohar defined the special case of anchors lying on the boundary of a disk and the drawing lying within the disk, which is equivalent to $\text{bkcr}_1(G,A,\pi)$.

COMPLEXITY: $\text{bkcr}_1(G,\pi)$ can be computed in $O(n^2)$ [270]. $\text{bkcr}_2(G,\pi)$ is **NP**-complete [542] (even if each connected component is a single edge). This implies that $\text{bkcr}_k(G,\pi)$ is **NP**-complete for $k \geqslant 2$.[93] As in the case of the book crossing number, the embedding problem is of special interest here. The problem of deciding whether $\text{bkcr}_k(G,\pi) = 0$ on input (G,π) and k was shown **NP**-complete by Garey, Johnson, Miller, and Papadimitriou [345], but they left open the question of what happens for fixed k. Unger claims that $\text{bkcr}_3(G,\pi) = 0$ can be tested in time $O(n \log n)$ [749], while testing $\text{bkcr}_k(G,\pi) = 0$ is **NP**-complete for any fixed $k \geqslant 4$ [748].[94,95] Cimikowski [215] has studied various heuristics for computing $\text{bkcr}_2(G,\pi)$. For the anchored version, Cabello and Mohar [163] showed that $\text{bkcr}_1(G,A,\pi)$ is **NP**-complete even if G consists of two vertex disjoint planar graphs[96]; Hliněný [420] shows **NP**-completeness for $|A| \leqslant 6$ (more precisely, if each of the two H_i has at most three anchors).

RELATIONSHIPS: $\text{bkcr}_2(G,\pi) \leqslant \text{bkcr}_1(G,\pi)/2 - (\text{bkcr}_1(G,\pi)/8 + 1/64)^{1/2} + 1/8$ [38, Corollary 6], $\text{mon-cr}(G,\pi) \leqslant \text{bkcr}_2(G,\pi)$ for ordered graphs (G,π) (from definition).

ALSO SEE: Book crossing number, convex crossing number.

Fixed monotone crossing number. See monotone crossing numbers.
Fractional crossing number. See weighted crossing number.

GENUS CROSSING NUMBER

DEFINITION: The *genus crossing number* of a drawing D of a graph G is the number of points in which edges cross each other (that is, we count this point only once, not $\binom{k}{2}$ times for k edges passing through it); we do not allow edges to touch in the shared point, but we do allow self-crossings of an edge (so an edge can pass through the same crossing point multiple times at no additional cost). The genus crossing number of a graph G, $\text{gcr}(G)$, is the smallest number of crossing points in a drawing of G.

[93] To add a page, surround each vertex by many nested edges. Then all these added edges have to lie in a separate page. This simple construction fails, of course, if the ordering cannot be specified.

[94] Unger expresses the embedding results for colorings of circle graphs, but the reduction is easy: given a graph G with an ordering π, add a Hamiltonian cycle to G extending that ordering, yielding G'. Then every non-cycle edge is a chord of the graph, and the endpoints of two chords alternate along the cycle if and only if the chords have to go into different pages in a book embedding of G. Let G'' be the circle (chord intersection) graph of G. Then k-colorability of G'' is equivalent to G being embeddable in k pages with the given ordering. This is sufficient to show that testing $\text{bkcr}_k(G,\pi)$ is **NP**-complete for $k \geqslant 4$: Given a circle graph one can use Spinrad's algorithm to construct a circle model G' for it, from which one can get a graph G with an ordering of vertices π, so that the circle graph is k-colorable, if and only if (G,π) has a k-page embedding respecting π, that is $\text{bkcr}_k(G,\pi) = 0$.

[95] Both papers have been criticized for lack of details, see [267, Footnotes 104-105] and [291].

[96] This was the main intermediate step in their proof that computing the crossing number of an almost planar graph is **NP**-complete.

Reference: Mohar [567].

Comments: Mohar proves that the genus crossing number equals the non-orientable genus of a graph. He conjectures that $\mathrm{gcr}(G) = \mathrm{dcr}(G)$ [567].

Complexity: **NP**-complete [567] (since Carsten Thomassen showed that determining the non-orientable genus of a graph is **NP**-complete [569]).

Relationships: $\mathrm{gcr}(G) \leqslant \mathrm{mcr}(G)$ since gcr is minor-monotone. There are graphs for which $\mathrm{gcr}(G) < \mathrm{mcr}(G)$ [567]. Also, $\mathrm{gcr}(G) \leqslant \mathrm{dcr}(G)$ by definition.

Values: Exact results for the non-orientable genus of K_m and $K_{m,n}$ were given by Ringel, see [284] for a discussion.

Also see: Degenerate crossing number.

Geodesic crossing number

Definition: The *geodesic crossing number*, $\overline{\mathrm{cr}}_\Sigma(G)$, on a metric surface Σ, is the smallest number of crossings in a drawing of G on Σ where each edge is represented by a geodesic (with respect to the metric) in Σ.[97] Special cases include the rectilinear crossing number, where Σ is the plane with the Euclidean metric (in which case we write $\overline{\mathrm{cr}}$), the *spherical (geodesic) crossing number* [523, 573, 757], where Σ is the unit ball S^2 in three-dimensional Euclidean space, and the *toroidal geodesic crossing number*, where Σ is a (geometric) torus in three-dimensional Euclidean space.

Reference: Guy, Jenkyns, Schaer [383], also Harary, Hill [389].

Comments: The spherical geodesic crossing number of complete graphs is discussed by Harary and Hill [389]. Moon [572, 573], also see [37, 565, 566], studies the number of crossings in a random geodesic drawing of K_n on the sphere (vertices are picked at random, edges are shortest arcs). Both spherical and toroidal geodesic crossing numbers are introduced and studied explicitly in [383]. It is not clear from the paper whether the authors believe that the toroidal geodesic crossing number is independent of the actual geometric shape of the torus; they concentrate on a single model (the unit square with opposite sides identified). They explicitly equate the rectilinear crossing number with the geodesic crossing number, even though Harary and Hill [389] had earlier realized that K_8 has a geodesic drawing on the sphere with at most 18 crossings, whereas $\mathrm{cr}(K_8) = 19$ was unproven, but expected to be true at the time. Guy [374, 375] later realized that the spherical crossing number of K_n is at most $Z(n) = X(n)X(n-2)/4$, where $X(n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$; this again shows that the spherical crossing number of K_8 is at most 18. Since he could also show that $\overline{\mathrm{cr}}(K_8) = 19$ (also Barton [97] and Singer [705]), this separates rectilinear and spherical crossing number. It is not clear whether all papers discussing geodesic crossing numbers distinguish between shortest arcs and geodesics (exceptions are [573, 757] which explicitly define the geodesic crossing

[97] Intuitively, geodesics are locally shortest arcs. A geodesic is not necessarily a shortest arc between two points on a surface, and it need not be unique, as the example of antipodal points on the sphere shows.

number in terms of shortest arcs rather than geodesics). This question is studied in [450], which uses the term *shortest path crossing number*. Elkies [283] extends Moon's work by studying random geodesic drawings on the projective plane and the torus. For a connection between the spherical geodesic crossing number and counting regular triangulations of higher-dimensional pointsets, see [319].

COMPLEXITY: Open, but likely to be $\exists\mathbb{R}$-hard (and in $\exists\mathbb{R}$ assuming the metric is natural), see [673] for $\exists\mathbb{R}$.

RELATIONSHIPS: $\overline{\text{cr}}_{S^2}(G) \leqslant \overline{\text{cr}}(G)$ (a sufficiently small drawing of G will realize this).

VALUES: $\overline{\text{cr}}_{S^2}(K_n) \leqslant Z(n)$, where $Z(n) = X(n)X(n-2)/4$, with $X(n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$, is Zarankiewicz's function, the conjectured upper bound on $\text{cr}(K_n)$ [374, 566, 650, 757].[98] It is known that for the unit sphere, $\overline{\text{cr}}_{S^2}(K_n) > 0.996\ Z(n)$ [91]. Extending Moon's work on randomized geodesic constructions, Elkies showed that $\overline{\text{cr}}_{N_1}(K_n) \leqslant (3/\pi^2)\binom{n}{4}$ for $n \geqslant 15$, and $\overline{\text{cr}}_{S_1}(K_n) \leqslant (22/81)\binom{n}{4}$ [283] for natural geometric models of N_1 and S_1. Let $s(r, n)$ be the expected number of crossings in a random geodesic drawing of a complete, balanced r-partite graph K_n^r. Then $\lim_{n\to\infty} s(r, n)/\,\text{max-cr}(K_n^r) = \zeta(r)$, where $\zeta(r) := \frac{3(r^2-r)}{8(r^2+r-3)}$, see [355].

OPEN QUESTIONS: Is there a Fary theorem for metric surfaces? That is, is it true that $\text{cr}_\Sigma(G) = 0$ implies that $\overline{\text{cr}}_\Sigma(G) = 0$ for a surface Σ equipped with a "natural" metric?[99] There are Fary-theorems for metrics of non-positive curvature [224, 450]. ▼ Does it matter whether the geodesic crossing number is defined in terms of geodesics or shortest arcs?[100] Shortest arcs can cross more than once (without overlapping) in some surfaces; are there examples of graphs for which every optimal geodesic (or shortest arc) drawing requires some edges to cross more than once?.[101]

ALSO SEE: Rectilinear crossing number.

Geometric *k*-planar crossing number. See k-planar crossing number.

[98] This result is claimed by Guy in [374] without any details. One can use the cylindrical drawings of Richter and Thomassen [650] to see that the inequality is true. Wagner [757] obtains this result as an application of Gale duality. Mohar [565, 566] gives a novel, simple construction, which sheds new light on Moon's original paper.

[99] Thomassen [737] points out that it is likely that one can construct metrics for which this fails, but what about standard metrics?

[100] It is known that there are graphs for which the geodesic crossing number differs from the *shortest arc crossing number* on the Klein bottle [450], but the situation on orientable surfaces with a Euclidean metric seems to be open. By [450, Theorem 1] there is a metric for the torus (with zero curvature) for which all geodesic embeddings are shortest-arc embeddings.

[101] The answer is yes for pseudosurfaces: take a sphere and two tori and attach each torus to the sphere at a single point (using two distinct points). Take two copies of a graph whose planar crossing number is large but which can be embedded on the torus. Connect the two graphs by two edges whose endpoints are adjacent in the toroidal graphs. Then the graph has a geodesic drawing in which only the two edges cross, namely in the points of attachment. In particular, the geodesic pair crossing number differs from the geodesic crossing number for this pseudosurface.

Grid crossing number

Definition: A *d-dimensional grid drawing* of a graph G is a geometric (straight-line) embedding of G into $\mathbb{N}^d$, that is, vertices are assigned to points in $\mathbb{N}^d$, edges are straight-line segments between their endpoints, and we require that no vertex lies on an edge, unless it is an endpoint of that edge. The *volume* of a d-dimensional grid drawing of G is the volume of a smallest axis-parallel box containing all points of the grid drawing. The *d-dimensional volume N grid crossing number* of G, $\overline{\text{cr}}_{\#}(G, N, d)$ is the smallest number of crossings in a d-dimensional grid drawing of G of volume at most N.

Reference: Based on Dean [240], Dujmović, Morin, Sheffer [263], Swamy [723, Q5] for name.

Comments: Dean [240] asked whether it is hard to tell whether a given graph has a crossing minimal-drawing on an $n \times n$-grid, where n is part of the input. Dujmović, Morin, Sheffer [263] introduce the crossing number of a grid graph (what we called a grid drawing), which they write $\text{cr}(G)$, G being a grid graph/drawing, and then study the crossing number of that, in particular, the parameter $\text{cr}_d(N, m) = \min\{\text{cr}(G) : G$ is a d-dimensional grid drawing of a graph with m edges and volume at most $N\}$, which is quite natural, since their main goal is a crossing lemma result for grid graphs. They point to several previous papers that have studied grid embeddings, that is, grid drawings without crossings (also called non-crossing grid graphs in the literature), but theirs seems to be the first paper to study the crossing number notion. The 2-dimensional grid crossing number is a refinement of the rectilinear crossing number. It is well-known that $\overline{\text{cr}}(G)$ can be realized on a grid of double exponential size and that grids of that size are necessary for some graphs (Bienstock [120]). It is in this context that Swamy [723] coined the term grid crossing number.

Complexity: **NP**-complete for $d = 2$.[102]

Relationships: $\overline{\text{cr}}(G) \leqslant \overline{\text{cr}}_{\#}(G, N, 2)$ (by definition), and $\overline{\text{cr}}(G) = \overline{\text{cr}}_{\#}(G, N, 2)$ for $N = 2^{2^{cn}}$ for some $c > 0$[103] and there are graphs for which $\overline{\text{cr}}(G) < \overline{\text{cr}}_{\#}(G, N, 2)$ if $N = 2^{2^{dn}}$ for some $0 < d$ [120]. $\overline{\text{cr}}_{\#}(G, N, 2) = \Theta(m^3/N^2)$ for $m \geqslant 4N$ (follows from [32] as observed in [263]), $\overline{\text{cr}}_{\#}(G, N, 3) = \Omega(m^2/N \log\log(m/N))$ for $m \geqslant 2(2^d - 1)N$, $\overline{\text{cr}}_{\#}(G, N, 3) = \Omega(m^2/N \log(m/N))$, and $\overline{\text{cr}}_{\#}(G, N, d) = \Omega(m^2/N)$ [263].

Values: $\text{cr}(G, (n-2)^2, 2) = 0$ for planar graphs G [685]. $\text{cr}(G, O(n), 3) = 0$ for planar graphs [761]. For complete graphs, it is known that $\text{cr}(K_n, 4n^3, 3) = 0$, and

[102] Bienstock [120] showed that for every G there is a G' with $\text{cr}(G) = \overline{\text{cr}}(G')$, where G' is obtained from G by subdividing each edge at most cn^2 times (for some fixed $c > 0$). We claim that $\text{cr}(G) = \overline{\text{cr}}_{\#}(G', cn^2, 2)$ which implies that computing $\overline{\text{cr}}_{\#}(G, N, 2)$ is **NP**-hard. To see that $\overline{\text{cr}}(G') = \overline{\text{cr}}_{\#}(G', cn^2, 2)$, take a $\overline{\text{cr}}$-optimal drawing of G'. Replace each crossing with a (very small) C_4 close to that crossing, so that the corners of C_4 become the endpoints of the four half-edges meeting at the crossing. Triangulate the drawing, keeping the C_4-faces empty; the resulting graph is 3-connected, so by a result from [207], it has an embedding on the $(n-2) \times (n-2)$ grid in which all faces are convex. In particular, we can replace each C_4 by two diagonal edges, and remove all triangulation edges to obtain a grid drawing of G'.

[103] Folklore result; true, because $\overline{\text{cr}}(G) \leqslant k$ can be expressed in the existential theory of the reals, see [673], for example.

$\text{cr}(K_n, o(n^3), 3) > 0$ [223].

OPEN QUESTIONS: What is the complexity of computing $\text{cr}(G, N, d)$ for dimensions $d > 2$?

ALSO SEE: Space crossing number, rectilinear crossing number.

INDEPENDENT ALGEBRAIC CROSSING NUMBER

DEFINITION: The *independent algebraic crossing number* of G, $\text{iacr}(G)$, is defined like $\text{acr}(G)$ except that we do not count $\text{acr}(e, f)$ for adjacent edges e and f.

REFERENCE: Tutte [747].

COMMENTS: Tutte's paper "Toward a Theory of Crossing Numbers" is often cited claiming it (implicitly) contains all kinds of crossing number definitions. A look at the text shows that Tutte defines two crossing numbers: the standard crossing number (which he calls $c(G)$) and what we now call the independent algebraic crossing number; his crossing chains count crossings algebraically, that is, over $\mathbb{Z}$, not modulo 2 as the odd crossing numbers do; moreover, he sets the coefficients of pairs of adjacent edges to 0 so they don't count. The crossing number he defines based on that, $s(G)$, is $\text{iacr}(G)$. Tutte writes: "It is clear that $c(G) \geqslant s(G)$. Does equality always hold?" This question was answered in the negative by Tóth [742] who constructed a graph G with $\text{iacr}(G) = \text{acr}(G) < \text{cr}(G)$.

COMPLEXITY: In **NP** (similar to algebraic crossing number). It is possible that **NP**-hardness can be achieved along similar lines as in [621].

RELATIONSHIPS: $\text{iacr}(G) \leqslant \text{acr}(G)$ and $\text{iocr}(G) \leqslant \text{iacr}(G)$ (by definition). It follows from results in [622] that there are graphs G for which $\text{iocr}(G) < \text{iacr}(G)$.

OPEN QUESTIONS: Fulek and Kynčl [336] ask whether $\text{iacr}_\Sigma = 0$ implies that $\text{cr}_\Sigma = 0$ for all graphs G and surfaces Σ. Since $\text{iacr}_\Sigma(G) \leqslant \text{iocr}_\Sigma(G)$, this $\mathbb{Z}$-version of the Hanani-Tutte theorem is true if the traditional Hanani-Tutte theorem holds for Σ. The traditional Hanani-Tutte theorem fails for surfaces Σ of orientable genus 4 and higher [336].

ALSO SEE: Algebraic crossing number, independent odd crossing number.

INDEPENDENT CROSSING NUMBER

DEFINITION: The *independent crossing number* of G, $\text{cr}_-(G)$, is the smallest number of crossings between pairs of independent edges in any drawing of G.

REFERENCE: Pach, Tóth [612].

COMMENTS: The first explicit definition of the independent crossing number seems to be in Pach, Tóth [612]. Not counting crossings between adjacent edges is implicit in many early papers, and, for straight-line or geodesic drawings, entirely justified [573].

COMPLEXITY: **NP**-complete.

Relationships: $\mathrm{pcr}_-(G) \leqslant \mathrm{cr}_-(G) \leqslant \mathrm{cr}(G)$ (from definition). The spectrum of cr_- has been studied for K_5, $K_{3,3}$, and K_6 in [165].[104]

Open Questions: It is not known whether $\mathrm{cr}_-(G) < \mathrm{cr}(G)$ is possible. This would follow from a separation of the corresponding monotone crossing numbers [338].

Independent odd crossing number

Definition: The *independent odd crossing number* of G, $\mathrm{iocr}(G)$, is the smallest number of independent pairs of edges crossing an odd number of times in any drawing of G.

Reference: Székely [724].

Comments: This variant seems to have been introduced and named by Székely. He attributes it to Tutte [747], but Tutte really defined the independent *algebraic* crossing number.[105]

Complexity: **NP**-complete [621] even if restricted to cubic graphs.

Relationships: $\mathrm{iocr}(G) \leqslant \mathrm{ocr}(G)$ for all graphs G (by definition). $\mathrm{iocr}(G) = \mathrm{ocr}(G) = \mathrm{cr}(G)$ for $\mathrm{iocr}(G) \leqslant 2$ [625], generalizing the Hanani-Tutte theorem (Footnote 7). There are graphs G for which $\mathrm{iocr}(G) < \mathrm{ocr}(G)$ [338]. $\mathrm{cr}(G) \leqslant \binom{2\,\mathrm{iocr}(G)}{2}$ [625]; this implies that ocr, acr, pcr, cr and all their $+$ and $-$ variants are within a square of each other. There is a crossing lemma: $\mathrm{iocr}(G) \geqslant 1/64\ m^3/n^2$.[106] There are algebraic sufficiency criteria for $\mathrm{iocr}(G) = \mathrm{cr}(G)$ [725]. $\mathrm{iocr}(G) \geqslant \mathrm{sk}(G)$.[107] For surfaces other than the sphere, the only known result is that $\mathrm{iocr}_{N_1}(G) = 0$ implies $\mathrm{cr}_{N_1}(G) = 0$ [618]. A smallest counterexample to $\mathrm{iocr}_\Sigma(G) = 0$ implying $\mathrm{cr}_\Sigma(G) = 0$ must be 2-connected [680]. There is a graph G with $\mathrm{iocr}_\Sigma(G) = 0$ and $\mathrm{cr}_\Sigma(G) > 0$ for any surface Σ of genus at least four [336]. See Remark 3 for a discussion of crossing lemmas for iocr_Σ.

Values: $\mathrm{iocr}(\mathrm{GP}(12,4)) = 4$, where $\mathrm{GP}(12,4)$ is the generalized Petersen graph [328].[108]

Open Questions: Are there interesting graphs G for which $\mathrm{cr}(G) = \mathrm{sk}(G)$? For each such G we have $\mathrm{iocr}(G) = \mathrm{cr}(G)$ (settling all intermediate crossing numbers, such as pcr and cr_- as well). ▼ Using the notion of edge distance, we can express $\mathrm{iocr}(G) > 0$ as saying that drawings of G always contain two edges of distance at least one that cross oddly. In [454] it is shown that the Petersen graph always contains two edges at distance one that cross oddly, and the Heawood graph contains two edges at distance two that cross oddly. Are there general patterns on how crossings distribute between different edge distances?

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[104] Here, the spectrum of a graph G is the set of all values $\mathrm{cr}_-(D)$ of drawings D of G in which pairs of independent edges cross at most once.

[105] Parity is only mentioned in one short passage in Tutte's paper [747], and that occurs when he observes that for two edges e and f, $\mathrm{acr}(e,f) \equiv \mathrm{cr}(e,f) \bmod 2$.

[106] For a proof, see the section on crossing lemma variants in Section 1.

[107] Not claimed anywhere, but easy: If $\mathrm{iocr}(G) = k$, we can remove at most k edges, so that $\mathrm{iocr}(G) = 0$, implying $\mathrm{cr}(G) = 0$, and the graph is planar.

[108] As so far the only non-trivial result for iocr, this deserves some comment. The article [328] actually shows (though it doesn't claim so) that $\mathrm{sk}(\mathrm{GP}(12,4)) \geqslant 4$. Since iocr is lower-bounded by sk, and there is a drawing of $\mathrm{GP}(12,4)$ with four crossings, the result follows. The same paper determines $\mathrm{cr}(\mathrm{GP}(3k,k))$ for all k, but the inductive step seems to use cr in an essential way.

Also see: Odd crossing number, independent algebraic crossing number (under algebraic crossing number), monotone crossing number (for monotone version).

Independent pair crossing number. See pair crossing number.
Independent string crossing number. See string crossing number.
Inner crossing number. See bipartite crossing number.

Joint crossing numbers

Definition: Suppose G_1 and G_2 are graphs embedded in the same surface Σ; a *joint embedding* of G_1 and G_2 is a simultaneous embedding of homeomorphic copies of G_1 and G_2 in which the only shared points between G_1 and G_2 are (transversal) crossings of an edge of G_1 with an edge of G_2; if we restrict the homeomorphisms to be orientation-preserving, we speak of a *joint orientation-preserving embedding.* If we restrict the homeomorphisms so that all vertices of G_1 lie in a face of G_2 and vice versa, we call the joint embedding *single-faced.* The *(joint) (homeomorphic) crossing number* of G_1 and G_2, $\text{cr}(G_1, G_2)$, is the smallest number of crossings in any joint embedding of G_1 and G_2 in Σ, the *oriented crossing number*, $\overrightarrow{\text{cr}}(G_1, G_2)$ of G_1 and G_2, is the smallest number of crossings in any joint orientation-preserving embedding of G_1 and G_2. The *single-faced crossing number*, $\text{cr}_{sf}(G_1, G_2)$, is the smallest number of crossings in any single-faced joint embedding of G_1 and G_2. Similarly, $\overrightarrow{\text{cr}}_{sf}(G_1, G_2)$, is the *single-faced oriented crossing number.* We can relax the notion of joint embedding to a *diagonal embedding* by allowing vertices of G_1 to coincide with vertices of G_2 and edges of G_1 to coincide with edges of G_2. The smallest number of crossings in a diagonal embedding is the *diagonal crossing number*, $\text{cr}_\Delta(G_1, G_2)$. If we want to emphasize the underlying surface, we write $\text{cr}(G_1, G_2; \Sigma)$, for example. If instead of embedded graphs G_1, G_2 we have abstract topological graphs that are embeddable in Σ, we can still define the *(joint) crossing number* and the *diagonal crossing number* of G_1 and G_2 by additionally minimizing over all embeddings of G_1 and G_2. Richter and Salazar [646] suggest the notation $\text{cr}(\phi_1(G_1), \phi_2(G_2))$ for the embedded graph variant ($\phi_i(G_i)$ is a class of homeomorphic embeddings of G_i), Hliněný and Salazar [426] suggested the name *joint homeomorphic crossing number* for this case to distinguish it from the topological case; we will rely on context.

Reference: Negami [585, 586]. Also, Archdeacon, Bonnington [57], and Richter, Salazar [646].

Comments: Joint crossing numbers, that is crossings numbers of pairs of (embedded) graphs were first introduced by Negami [585, 586]. Archdeacon and Bonnington [57] restrict joint embeddings to orientation-preserving homeomorphisms, so their joint crossing number is what Negami called the oriented crossing number. Negami simply uses crossing number for the joint crossing number. Richter and Salazar [646] explicitly define the single-faced crossing number which is implicit in Archdeacon, Bonnington [57]. As examples for values of joint crossing numbers, Negami gives $\text{cr}(K_5, K_{3,3}; S_1) = 2$ and $\text{cr}_\Delta(K_5, K_{3,3}; S_1) = 0$. Since G_1 and G_2 are both required to be embeddable on Σ, the crossing number of pairs is always 0 for the plane.

Complexity: Joint crossing number (both homeomorphic and topological version), and joint oriented crossing number are **NP**-complete, for any orientable surface of genus at least 3, even for simple, 3-connected graphs [420, 426]. As [426] point out, an earlier result by Archdeacon and Bonnington [57, Theorem 2.2] implies that the joint homeomorphic crossing number of two graphs on the projective plane can be solved in polynomial time.

Relationships: $\mathrm{cr}_\Delta(G_1, G_2) \leqslant \mathrm{cr}(G_1, G_2)$ (from definition). If $\gamma(\Sigma)$ is the (orientable or non-orientable) genus of Σ, then $\overrightarrow{\mathrm{cr}}(G_1, G_2; \Sigma) \leqslant 4\gamma(\Sigma)|E(G_1)| \cdot |E(G_2)|$, and $\overrightarrow{\mathrm{cr}}(G_1, G_2; S_1) \leqslant 2/3|E(G_1)| \cdot |E(G_2)|$ [57, 333, 585].[109]

Values: $\mathrm{cr}(G_1, G_2; S_n) = 2n$ if both G_1 and G_2 are 2-cell embedded on S_n so that each embedding has a single face [778].

Open Questions: Negami [585] conjectures that $\mathrm{cr}(G_1, G_2) \leqslant c|E(G_1)| \cdot |E(G_2)|$ for some constant c independent of Σ; Archdeacon and Bonnington [57] believe this conjecture to be false. They conjectured that $\overrightarrow{\mathrm{cr}}(G_1, G_2) \leqslant c_\Sigma \cdot \overrightarrow{\mathrm{cr}}_{sf}(G_1, G_2)$ for embedded graphs G_1 and G_2 which was shown to be false by Richter and Salazar [646] (who suggest a revised conjecture).

Also see: Simultaneous crossing number. Red/blue crossing number.

k-layer crossing number

Definition: A *leveling* of a graph $G = (V, E)$ is a mapping from V to $\{1, \ldots, k\}$, assigning each vertex a level. The leveling is *proper* if all edges of G are between vertices at adjacent levels. A *layered* drawing of a properly leveled (layered) graph is a drawing in which the vertices are placed on k parallel lines, with vertices in layer i assigned to the ith line, and edges are drawn as straight-line segments. The *k-layer crossing number* of a layered graph is the smallest number of crossings in a k-layer drawing of the graph.

Reference: Warfield [763], Sugiyama, Tagawa, Toda [721], Shahrokhi, Vrťo [700].

Comments: Shahrokhi and Vrťo [700] introduced (and named) the 3-layer crossing number, but as a crossing minimization problem the k-layer crossing number is already present in papers by Warfield [763] and Sugiyama, Tagawa, and Toda [721]; these earlier papers write $K(M)$ for the layered crossing number of a leveled graph represented by a matrix M. The 2-layer crossing number is just the bipartite crossing number. May and Szkatuła [546] defined the *p-partite crossing number*, $\underline{\nu}_p$ for p-partite graphs: the vertices of each part are drawn on one of p parallel lines, and a subdivision vertex is added whenever an edge crosses a line, so this corresponds to the k-layer crossing number of a properly leveled graph. Di Bartolomeo, Lang, and Dunne [245] implement an algorithm for maximizing the number of crossings in a k-layer drawing.[110] Layered crossing numbers are similar to leveled crossing

[109] The argument for Theorem 1 in [585] contains a gap which is fixed in [333].

[110] They do so tongue-in-cheek, but one could study the *maximum k-layer crossing number* extending the maximum bipartite crossing number.

numbers, except that for the layered crossing numbers edges have to be realized as straight-line segments (rather than just being monotone); if the leveling is proper, the leveled and layered crossing numbers coincide. Leveling a graph imposes a linear structure on the graph. One could also imagine allowing other structures, for example trees [632], or cycles as in the cyclic level crossing number. Wotzlaw, Speckenmeyer and Porschen [775] consider the case in which the ordering of the vertices in each layer is restricted by a tree (a generalization of the tanglegram problem, also see the comment in the entry on the bipartite crossing number).

COMPLEXITY: **NP**-complete [348], even for trees [412].[111] Can be approximated to within a factor of $O(\log n)$ in polynomial time [700]. The embeddability problem can be decided in polynomial time and this remains true if the ordering of vertices in each layer is constrained by trees [775].

RELATIONSHIPS: The k-layer crossing number of G is at least $\overline{\text{cr}}(G)$ and it can be strictly greater than $\overline{\text{cr}}(G)$. The leveled crossing number is a lower bound on the k-layer crossing number.

VALUES: There is an upper bound on the k-layer crossing number of L_n^k, the complete balanced proper layered graph on k layers [301].

OPEN QUESTIONS: If a graph has leveled crossing number zero, that is, if it has a monotone leveled embedding, it has an embedding in which all edges are straight-line segments [276, 611], though the area of the graph may increase exponentially [531]. Are there leveled graphs for which the k-layer crossing number is strictly larger than the leveled crossing number?

ALSO SEE: Bipartite crossing number, leveled crossing number (under monotone crossing number), cyclic level crossing number.

Klein bottle crossing number. See crossing number.
***k*-page crossing number.** See book crossing number.

k-PLANAR CROSSING NUMBER

DEFINITION: The *k-planar crossing number*, $\text{cr}_k(G)$, of $G = (V, E)$ is the minimum of $\sum_{i=1}^k \text{cr}(G_i)$, where the minimum is taken over all $G_i = (V, E_i)$ with $\bigcup_{i=1}^k E_i = E$. The special case cr_2 is also known as the *biplanar crossing number*. If we restrict the drawings to be rectilinear, we get $\overline{\text{cr}}_k$, the *rectilinear k-planar crossing number*. Given a rectilinear drawing D of G, the *geometric k-planar crossing number*, $\overline{\overline{\text{cr}}}_k(D)$, is the minimum of $\sum_{i=1}^k \text{cr}(D_i)$, where the minimum is taken over all $G_i = (V, E_i)$ with $\bigcup_{i=1}^k E_i = E$, and D_i is D restricted to G_i. The *geometric k-planar crossing number*, $\overline{\overline{\text{cr}}}_k(G)$, is the minimum of $\overline{\overline{\text{cr}}}_k(D)$ over all rectilinear drawings D of G.[112] The *thickness*, $\Theta(G)$, is the smallest k such that $\text{cr}_k(G) = 0$; similarly, the *geometric thickness*, $\overline{\Theta}(G)$, is the smallest k such that $\overline{\overline{\text{cr}}}_k(G) = 0$. The *local k-planar crossing*

[111] The reduction by Garey and Johnson [348] is to bipartite multigraphs. The middle layer can be used to replace multiple edges by parallel paths.

[112] Equivalently, the geometric k-planar crossing number is the smallest number of crossings between edges of the same color in any k-coloring of the edges of G in any rectilinear drawing of G.

number, $\operatorname{lcr}_k(G)$, is the minimum of $\max_{i=1}^k \operatorname{lcr}(G_i)$, where the minimum is taken over all $G_i = (V, E_i)$ with $\bigcup_{i=1}^k E_i = E$.

REFERENCE: Kodres [501], Owens [597], Shahrokhi, Sýkora, Székely, Vrťo [696], Pach, Székely, Tóth, Tóth [604]. The local, k-planar crossing number was introduced by Asplund, Do, Hamm, Jain [71].

COMMENTS: Kodres [501] overlooked paper (see Remark 5) discusses (and implements) crossing minimization for biplanar and k-planar drawings of graphs (based on electronic circuits); the paper shows that $K_{7,7}$ is not biplanar and conjectures that $\operatorname{cr}_2(K_{7,7}) = 4$ (the correct value is 1 [229]). Owens [597] introduced the k-planar crossing number for arbitrary k, but focussed on the biplanar case, Shahrokhi, Sýkora, Székely, Vrťo introduced the rectilinear version. The k-planar crossing numbers have also been called the *multiplanar crossing numbers* [493]. The k-planar crossing number should not be confused with the crossing number of a k-planar drawing which has only been studied for $k = 1$, where it is called the simple crossing number. The geometric variant was introduced by Pach, Székely, Tóth, Csaba, Tóth [604], refining Kainen's notion of real thickness [471]. In [28] the geometric variant is called the *rectilinear k-colored crossing number*. If the geometric drawings were restricted to be convex, then one would get the k-page crossing number. The study of the k-planar crossing number is often motivated by linking it to questions of VLSI design. Interestingly, there is a book from 1896 on electrical wiring which includes two diagrams illustrating how to draw connections without any crossings in two layers ("die Verbindungen [können] in zwei übereinander liegenden Ebenen ohne Kreuzung gelegt werden") [63, Figures 51, 52]. In this context it may also be interesting to look at a grid (lattice)-version of these crossing numbers; the notion of *lattice thickness* was studied in [36].

COMPLEXITY: The k-planar crossing number is **NP**-complete, since the embedding problem $\operatorname{cr}_k(G) = 0$ is equivalent to the thickness of G being at most k and even for $k = 2$ this problem is **NP**-complete [539]. The rectilinear and geometric k-planar crossing numbers are $\exists\mathbb{R}$-complete, since they coincide with $\overline{\operatorname{cr}}$ for $k = 1$, but the case $k \geqslant 2$ is open, though likely to be $\exists\mathbb{R}$-complete as well.

RELATIONSHIPS: $\operatorname{cr}_k \leqslant \overline{\operatorname{cr}}_k \leqslant \overline{\overline{\operatorname{cr}}}_k \leqslant \operatorname{bkcr}_k$ (by definition). $\operatorname{cr}_1 = \operatorname{cr}$ and $\overline{\overline{\operatorname{cr}}}_1 = \overline{\operatorname{cr}}_1 = \overline{\operatorname{cr}}$ (by definition). $\operatorname{cr}_2(G) \leqslant (3/8)\operatorname{cr}(G)$ [230]. $\operatorname{cr}_k(G) \leqslant c\operatorname{cr}(G)$ and $\overline{\operatorname{cr}}_k(G) \leqslant c\,\overline{\operatorname{cr}}(G)$, where $c = (2/k^2 - 1/k^3)$ (and $c \geqslant 1/k^2$ for some graphs G) [604]. It has been announced that the upper bound is $c = 1/k^2(1 + o(1))$ [70]. $\overline{\overline{\operatorname{cr}}}_k(G) \leqslant c\,\overline{\operatorname{cr}}(G)$ for $c = 1/k$ (and $c \geqslant 1/k^2$ for some graphs G) [604], and $\overline{\overline{\operatorname{cr}}}_k(G) \leqslant (1/k)\,\overline{\operatorname{cr}}(G) - c$, $k \geqslant 2$, for sufficiently dense graphs [28, 300]. $\operatorname{cr}_k(G) \leqslant \operatorname{bkcr}_{2k}(G)$.[113] There is a crossing lemma,
$\operatorname{cr}_k(G) \geqslant 1/64\ m^3/(n^2k^2)$, where $n = |V(G)|$ and $m = |E(G)|$ [696]. On the other hand, $\operatorname{cr}_k(G) \leqslant 1/(12k^2)(1 - 1/(4k))m^2 + O(m^2/(kn))$ [696]. $\operatorname{lcr}_k(G) \leqslant$

[113] Observed by Winterbach [771], follows from $\operatorname{cr}(G) \leqslant \text{mon-cr}(G) \leqslant \operatorname{bkcr}_2(G)$. Winterbach [771, Question 8.2.5] asks whether there are graphs G for which $\operatorname{cr}_k(G) < \operatorname{bkcr}_{2k}(G)$. De Klerk, Pasechnik, and Salazar give a positive answer in [493] for $G = K_{2k+1,k^2+2000k^{7/4}}$ by showing that $\operatorname{bkcr}_{2k}(G) > 0$, while $\operatorname{cr}_k(G) = 0$ by a result of Beineke's.

$(1/k+\varepsilon)\,\mathrm{lcr}(G)$ for graphs G with sufficiently large $\mathrm{lcr}(G)$ [71], and the bound can be lowered to $O(1/k^2)$ under additional assumptions; this bound would be tight, as witnessed by the complete graph. It has been announced that 1-planar graphs have geometric thickness at most 2, that is, $\mathrm{lcr}(G) \leqslant 1$ implies that $\overline{\overline{\mathrm{cr}}}_2(G) = 0$ [144].

VALUES: See [229] for a comprehensive survey of biplanar crossing numbers of complete graphs, complete bipartite graphs and some other graph families, also [521, 661]. For values of k-planar crossing numbers of complete and complete bipartite graphs, see [272, 559, 696, 701, 702]. For the biplanar crossing number, exact values are known up to $k = 10$: $\mathrm{cr}_2(K_9) = 1$[114] and $\mathrm{cr}_2(K_{10}) = 2$ [272]; we also know that $4 \leqslant \mathrm{cr}_2(K_{11}) \leqslant 6$ [272] and $6 \leqslant \mathrm{cr}_2(K_{12}) \leqslant 12$ [701]. For an improved upper bound on cr_2 of the hypercube Q_8, see [221]. For random graphs, see [72, 709]. $\overline{\overline{\mathrm{cr}}}_2(K_n) = 0$ for $n \leqslant 8$ and $\overline{\overline{\mathrm{cr}}}_2(K_9) = 2 > 1 = \mathrm{cr}_2(K_9)$ [28], Upper and lower bounds on $\overline{\overline{\mathrm{cr}}}_2(K_n)$ can be found in [28]. It has been announced that $\overline{\overline{\mathrm{cr}}}_4(G) = 0$ for all 2-degenerate graphs G [458].

OPEN QUESTIONS: Czabarka, Sýkora, Székely, and Vrťo [229] ask for the smallest c with $\mathrm{cr}_2(G) \leqslant c\,\mathrm{cr}(G)$ for all G. They show that $8/119 \leqslant c \leqslant 3/8$, where the lower bound is witnessed by K_n. ▼ Shavali and Zarrabi-Zadeh ask for the largest k for which $\mathrm{cr}(G) \leqslant k$ implies that $\mathrm{cr}_2(G) = 0$; they can show that $10 \leqslant k \leqslant 35$ [701].[115]

ALSO SEE: Simultaneous crossing number, red/blue crossing number, biplanar convex crossing number (under convex crossing number).

Leveled crossing number. See monotone crossing numbers.
Linear crossing number. See book crossing number. Very rarely used as synonym for rectilinear crossing number.
Local book crossing number. See book crossing number.
Local convex crossing number. See convex crossing number.

LOCAL CROSSING NUMBER

DEFINITION: The *local crossing number* of a drawing D of a graph G, $\mathrm{lcr}(D)$, is the largest number of crossings on any edge of G. The *local crossing number* of G, $\mathrm{lcr}(G)$, is the minimum of $\mathrm{lcr}(D)$ over all drawings of G. Define the *simple local crossing number* $\mathrm{lcr}^*(G)$ as the minimum of $\mathrm{lcr}(D)$ over all *intersection-simple* drawings D of G (every two edges intersect at most once). For the local crossing number on a surface Σ, we write lcr_Σ. If we count multiple crossings only once, we get the (simple) degenerate local crossing number. If we maximize $\mathrm{lcr}(D)$ over all *intersection-simple* drawings D of G, we obtain max-$\mathrm{lcr}(G)$, the *maximum local crossing number*. If we restrict drawings to be straight-line, we get the *rectilinear local crossing number*, $\overline{\mathrm{lcr}}(G)$.

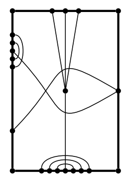
REFERENCE: Kainen [471]. Also, Ringel [656], Guy, Jenkyns, Schaer [383]. For the simple local crossing number, see Schumacher [688] and Pach, Tóth [609]. The simple

[114] The fact that K_9 is not biplanar was first shown in [98, 746]. A much simper proof is now available [123].

[115] The authors ask a more general question, which is also of interest: find bounds for a function $k(c)$ so that $\mathrm{cr}(G) \leqslant k(c)$ implies that $\mathrm{cr}_2(G) \leqslant c$.

degenerate local crossing number was introduced by Eggleton [281]. The maximum local crossing number is based on a paper by Harborth [395]. The rectilinear local crossing number seems to have first been mentioned in an earlier version of this survey.

COMMENTS: The local crossing number was first introduced by Gerhard Ringel in lectures and conversations in the 1960s [377, 462]. Guy, Jenkyns, and Schaer [383] define the *local toroidal crossing number*, the local crossing number on a torus, lcr_{S_1}. Kainen [471] introduces the local crossing number on arbitrary surfaces, and also credits Ringel [656]. Ringel's paper shows that a graph with at most one crossing per edge can be 7-colored,[116] but Ringel doesn't name the local crossing number explicitly in this paper. Graphs that can be drawn with at most one crossing per edge were later called 1-*embeddable* (Ringel [654]), 1-*planar*[117] (Schumacher [689]) and even *simple*, on occasion [152]; the drawn graph has been called 1-*immersed* [510]. Kainen [469] considered the local crossing number on arbitrary surfaces, he shows that $\Theta_\Sigma(G) \leqslant 1 + \mathrm{lcr}_\Sigma(G)$, with $\Theta_\Sigma(G)$ being the thickness of G on surface Σ. Cimikowski [216] in his definition of local crossing number restricts drawings to be cr-minimal. It is easy to see that this leads to a different notion of local crossing number. Harary, Kainen, and Schwenk [391] gave as an example $W_5 \square K_2$ which has crossing number 2 and local crossing number 1, but any drawing of $W_5 \square K_2$ realizing crossing number 2 has local crossing number at least 2. They conjecture that their example is the smallest possible. Eggleton [281] introduces a degenerate version of the local crossing number, that is, he counts multiple crossings as a single crossing (he also restricts drawings to be intersection-simple); he calls this variant the "edge crossing number", not to be confused with the notion of edge crossing number we introduce. Eggleton shows that every outerplanar drawing in which each edge has at most one degenerate crossing is rectifiable (realizable by straight-line segments and maintaining topological equivalence). Thomassen [738] calls $\mathrm{lcr}(D)$ the *cross-index* of D and studies conditions under which drawings D with $\mathrm{lcr}(D) \leqslant 1$ are rectifiable (realizable by straight-line segments, maintaining topological equivalence); this suggests the notion of *geometric/straight-line* 1-*planarity* [249, 441, 676], or, more generally, a *rectilinear local crossing number*, $\overline{\mathrm{lcr}}$, called *crossing index* in [522]. Schumacher [688] uses the term n-*embeddable* for graphs G with $\mathrm{lcr}(G) \leqslant n$, and claims that if we take a drawing D of G with $\mathrm{lcr}(D) \leqslant n$ and a minimal number of crossings, "none of G's edges is crossing itself; two different edges with one vertex in common do not cross either, and two different edges without a vertex in common cross once at the most." The claim about self-crossings is obviously true, but the remaining two claims are false. See the graph in the margin for an example showing that adjacent edges can be forced to cross.[118] A slight modification of this example shows



[116]Borodin [141] shows that they can even be 6-colored, which is sharp, because K_6 is 1-planar. Eppstein and Huynh [293] point out that the chromatic number of graphs with local crossing number k is of order $\Theta(\sqrt{k})$.

[117]Not to be confused with the notion of k-planarity in the multi-planar crossing number.

[118]This was also observed, without detailed proof, in [600, Figure 1]. Some explanation of our example:

that two edges can be forced to cross an arbitrary number of times in an lcr-optimal drawing. One could ask for an upper bound on the minimum number of crossings in a drawing D of a graph with $\mathrm{lcr}(D) \leqslant n$. For $n = 1$ this yields the simple crossing number. Pach and Tóth [609] study the parameter we called the simple local crossing number without naming it. Bodlaender and Grigoriev [364] rediscovered the local crossing number, calling it *crossing parameter*. In a later paper, Grigoriev, Koutsonas, and Thilikios [365] use the term *ξ-nearly planar* for graphs with local crossing number at most ξ, and give an equivalent structural characterization of these graphs. For a convex (our outerplanar) version see the local convex crossing number (under convex crossing number). Feng, Ye, and Xu [318] suggest studying the minimal number of crossings along longest paths in a network (to model optical router networks); this has a similar flavor to the local crossing number, but is not strictly speaking a crossing number in our sense. With a similar motivation, Stallman and Gupta [712, 713] consider heuristics for the local crossing number of layered graphs, which they call the *bottleneck crossing number*[119]; to be precise, they really define what amounts to the local pair crossing number in which we minimize the largest number of edges crossing each edge (not the actual crossings), see the entry for pair crossing number.[120] Harborth [395] studies the largest number of crossings along an edge in (intersection)-simple drawings of complete (multipartite) graphs. Also see [442, Chapters 4-7]. In [124] a drawing is called *min-k-planar* if for every pair of crossing edges one of the edges is involved in at most k crossings; one could define a crossing number based on this, maybe called the *pairwise minimum local crossing number*, but the authors do not do so; they do show a crossing lemma lower bound of the form $\Omega(m^2/n^2)$ for this notion. Hliněný [425] points out that the definition of min-k-planarity is sensitive to whether drawings have to be simple or not.

COMPLEXITY: Deciding whether $\mathrm{lcr}(G) \leqslant 1$ is **NP**-complete, even if the graph is 3-

consider a drawing of the graph with $\mathrm{lcr}(D) \leqslant 4$ in which the outer face is empty, in particular, the edges of the outer cycle are free of crossings. Then it is easy to argue that the two adjacent left/right edges have to cross in D. Here is how we enforce that the outer face is empty: add a new vertex and connect it to all vertices on the outer cycle. The vertices of this newly added star and the outer cycle form the *outer frame*. For each edge uv in the outer frame, add $4|V(G)| + 1 = 89$ parallel paths P_3 between u and v; let the new graph be G' and fix a drawing D' with $\mathrm{lcr}(D') \leqslant 4$ and minimizing $\mathrm{cr}(D')$. We can assume that no two adjacent edges cross in D' (otherwise we're done). Let uv be an edge of the outer frame, and xy be another edge. Then uv and xy cannot cross oddly: pick a cycle C containing xy, but not uv (if xy also belongs to the outer frame, then the cycle can be completed with a P_3). The cycle has length at most $|V(G)| = 22$. Each of the 89 cycles of the form $uv + P_3$ crosses C evenly, so if uv crosses xy oddly, then each of the P_3 must cross C oddly, so some edge in C has at least $89/22 > 4$ crossings, contradicting $\mathrm{lcr}(D') \leqslant 4$. So uv crosses every edge evenly, so it crosses either one, or two edges. One can reduce the number of crossings in all cases, so uv and thus all edges of the outer frame are free of crossings.

[119] More recent heuristics papers have called this the *min-max edge crossing problem*, see [626].

[120] The local pair crossing number differs from the local crossing number, using examples similar to the ones presented above to separate local and simple local crossing numbers. The distinction was probably not intended by the authors of [712, 713], since they also define the crossing number as pcr. For layered drawings there is no difference between counting all local crossings or only counting local pair crossings.

connected, and a rotation system is known [76, 162, 364, 510], or a plane embedding of $G - P$, where P is a path, is fixed [481]. There are results on the parameterized complexity of lcr [93]. Testing whether $\mathrm{lcr}(G) \leqslant k$ is **NP**-complete, and remains **NP**-hard to approximate to within a factor of $2 - \varepsilon$ [751]. Maximal graphs with $\mathrm{lcr}(G) \leqslant 1$ ("optimal 1-planar graphs") can be recognized in linear time [143]. Known results imply that testing $\overline{\mathrm{lcr}} \leqslant 1$ is **NP**-complete [676], while testing $\overline{\mathrm{lcr}}(G) \leqslant k$ is $\exists\mathbb{R}$-complete, even for a fixed k [672].

RELATIONSHIPS: $\mathrm{lcr}(G) \leqslant \mathrm{lcr}^*(G) \leqslant \min\{\mathrm{cr}(G), E(G) - 1\}$ and $\mathrm{lcr}^*(G) \leqslant \overline{\mathrm{lcr}}(G) \leqslant \min\{\overline{\mathrm{cr}}(G), E(G) - 1\}$ by definition. $\mathrm{lcr}(G) = \mathrm{lcr}^*(G)$ for $\mathrm{lcr}(G) \leqslant 3$,[121] and there are graphs G with $4 = \mathrm{lcr}(G) < \mathrm{lcr}^*(G)$ (Footnote 118). $\mathrm{lcr}^*(G) \leqslant f(\mathrm{lcr}(G))$ for an exponential function f, with $f(4) = 8$ [439].[122] For every $\ell > k \geqslant 4$ there is a graph G such that $\mathrm{lcr}(G) = k$ and $\overline{\mathrm{lcr}}(G) \geqslant \ell$ [641, Chapter 5]. $\mathrm{lcr}(G) \geqslant \mathrm{cr}(G)/|E(G)|$ by definition. There is a crossing lemma for the local crossing number of bipartite drawings of bipartite graphs [51, Theorem 4.2]. For graphs with fixed $\mathrm{lcr}(G)$, $\sum_{v \in V(G)} \deg(v)^k \leqslant 2(n-1)^k + o(n)$ [790]. $\sum_{e \in E} (\mathrm{cr}_D(e) + 1)^{-1} \leqslant 3n - 6$, a result called the harmonic crossing lemma in [22]; here $\mathrm{cr}_D(e)$ is the number of crossings along e in the drawing D. For every surface Σ and every k there is a graph so that $\mathrm{lcr}_\Sigma(G) = 1$ and $\mathrm{cr}_\Sigma(G) \geqslant k$ [391]. There is a graph G with $\mathrm{cr}(G) = 2$ for which any drawing D with $\mathrm{lcr}(D) \leqslant 1$ fulfills $\mathrm{cr}(D) \geqslant 3$ [152, 391]. For infinitely many n there is a graph G with $\mathrm{cr}(G) = 2$ for which any drawing D with $\mathrm{lcr}(D) \leqslant 1$ fulfills $\mathrm{cr}(D) \geqslant n - 2$ (and there are such drawings); the result is tight [200].[123] Since $\mathrm{cr}(G) \leqslant m\,\mathrm{lcr}^*(G)/2$, edge-bounds for graphs with bounded lcr^* imply crossing number bounds. The parameter $r(G) = \min_D \mathrm{lcr}(D)\,\mathrm{cr}(D)/(\mathrm{lcr}(G)\,\mathrm{cr}(G))$ measures how well cr and lcr can be minimized simultaneously; it is known that $cn^{1/2} \leqslant \max r(G) \leqslant c'n$, where the maximum is taken over all non-planar graphs of order n [751]. Let $m = |E(G)|$ and $n = |V(G)|$. Schumacher [687, 688] showed that $m \leqslant (\mathrm{lcr}^*_\Sigma(G) + 3)(n - \chi)$, where χ is the Euler characteristic of the surface Σ as long as $\mathrm{lcr}^*_\Sigma(G) \leqslant 2$, and that these bounds are tight.[124] Pach and Tóth showed that $m \leqslant (\mathrm{lcr}^*(G) + 3)(n - 2)$ as long as $\mathrm{lcr}^*(G) \leqslant 4$, and that these bounds are tight for $\mathrm{lcr}^*(G) \leqslant 2$ [609]. As it turns out, this is where the obvious pattern stops: $m \leqslant 5.5(n - 2)$ for $\mathrm{lcr}^*(G) \leqslant 3$ [600], and $m \leqslant 6(n - 2)$ for $\mathrm{lcr}^*(G) \leqslant 4$ [17] and both results are tight up to additive constants.[125] There are also edge bounds for multi-partite graphs G with $\mathrm{lcr}^*_\Sigma(G) = 1$ [479, 704]. For unbounded $\mathrm{lcr}^*(G)$, the best current result is $m \leqslant 3.81\,\mathrm{lcr}^*(G)n$ [17], improving an earlier bound by [609]. Every

[121] The fact that $\mathrm{lcr}^*(G)$ is finite if $\mathrm{lcr}(G) = 1$ was observed by Ringel [656]; for $\mathrm{lcr}(G) \leqslant 3$, see [600, Lemma 1.1].

[122] Answering an open question from a previous version of the survey.

[123] Chimani, Kindermann, Montechhiani and Valtr [200] introduce the *k-planar crossing number* as the smallest number of crossings in a drawing of G with local crossing number at most k, and they initiate the study of $\mathrm{cr}_{k\text{-}pl}(G)/\mathrm{cr}(G)$; the naming of the crossing number clashes with that of the traditional k-planar crossing number.

[124] The special case, $m \leqslant 4n - 8$ for graphs with $\mathrm{lcr}^*(G) \leqslant 1$ on the sphere seems to go back to [134].

[125] Ackerman [17] uses his result to derive an improved constant for the crossing lemma for cr, following the same approach as [600].

planar graph has a non-planar drawing D with $\operatorname{lcr}^*(D) \leqslant 3$ [509] and that bound is tight [507, 508]. For the rectilinear local crossing number Didimo [249] showed that $\overline{\operatorname{lcr}}(G) \leqslant 1$ implies $m \leqslant 4n-9$ (and this bound is tight for infinitely many n). $\operatorname{lcr}_{S_g}(G) = O(\frac{m\log^2 g}{g})$ [264], improving an earlier bound [364]. If $\operatorname{lcr}_\Sigma(G) \leqslant k$, then $\operatorname{tw}(G) = O(\sqrt{(\operatorname{eg}+1)(k+1)n})$, where $\operatorname{tw}(G)$ is the treewidth of G, and $\operatorname{eg} = \operatorname{eg}(\Sigma)$ is the Euler genus of Σ [264]. It has been announced that $\operatorname{lcr}(G) \leqslant 1$ implies $\overline{\overline{\operatorname{cr}}}_2(G) = 0$, in other words, the geometric thickness of G is at most 2 [144].

Values: $\operatorname{lcr}(K_n)$ and $\operatorname{lcr}^*(K_n)$ are known for $n \leqslant 9$; $\operatorname{lcr}(K_{m,n})$ and $\operatorname{lcr}^*(K_{m,n})$ are known for various values of m, n (various sources, see [49, Table 1][126]). The local crossing number of several families of generalized Petersen graphs $\operatorname{GP}(n,k)$) are known [96].[127] $\operatorname{lcr}_{S_1}(K_n)$ is known for $n \leqslant 9$, and there are asymptotic results for $\operatorname{lcr}_{S_1}(K_n)$ [383]. It is known which complete multi-partite graphs G satisfy $\operatorname{lcr}_\Sigma(G) = 1$, where Σ is an orientable surfaces [232] or the projective plane [703]. $\text{max-}\operatorname{lcr}(K_n) = \binom{n-2}{2}$, and $\text{max-}\operatorname{lcr}(K_{n_1,n_2}) = (n_1-1)(n_2-1)$, and, more generally, $\text{max-}\operatorname{lcr}(K_{n_1,\dots,n_k}) = n_1+n_2+\binom{\ell}{2}-2\ell-\sum_{i=1}^k \binom{n_i}{2}$, where $\ell = \sum_{i=1}^k n_i$, and $n_1 \geqslant n_2 \cdots \geqslant n_k$ [395]. $\overline{\operatorname{lcr}}(K_n)$ is known for all n [11]. For complete bipartite graphs, $\overline{\operatorname{lcr}}(K_{3,n}) = \lceil (n-2)/4 \rceil$, $\overline{\operatorname{lcr}}(K_{4,n}) = \lceil (n-2)/2 \rceil$ and there are asymptotic upper and lower bounds [8]. For Cartesian products of cycles with small graphs, as well as paths and cycles with stars, see [578].

Open Questions: Is it true that $m \leqslant (\operatorname{lcr}^*_\Sigma(G)+3)(n-\chi)$, where χ is the Euler characteristic of Σ, even just for Σ being the sphere? ▼ Has $\operatorname{lcr}(K_n)$ been studied? ▼ Is there a relationship between $\operatorname{lcr}(G)$ and the pagenumber of G, that is, the smallest k for which $\operatorname{bkcr}_k(G) = 0$? (It is known that 1-planar graphs have pagenumber at most 39 [107, 108].) ▼ Dujmović, Eppstein, and Wood mention the conjecture that $\operatorname{lcr}_{S_g}(G) = O(\frac{m}{g+1})$ [264]. ▼ Brandenburg asks for an upper bound on the geometric thickness of k-planar graphs [144]. ▼ For various open questions and conjectures on the local crossing number of infinite graphs, see [298].

Also see: Local convex crossing number (under convex crossing number), local book crossing number (under book crossing number), nodal crossing number, simple crossing number, local pair crossing number (under pair crossing number), local odd crossing number (under odd crossing number), local k-planar crossing number (under k-planar crossing number), local simultaneous planar crossing number and local pair simultaneous planar crossing number (under simultaneous crossing number), local spine crossing number (under spine crossing number).

Local k-page crossing number. See book crossing number.

[126]The paper works with k-planar drawings which are, by definition, (intersection)-simple (why? good question). So the results as stated in the table are for lcr*, but, since lcr and lcr* are the same up to value 3, and upper bounds carry over, this also yields results for lcr. The paper implements an algorithm to efficiently generate all (intersection)-simple drawings of small complete and small complete bipartite graphs. This could probably be used to determine other crossing number as well.

[127]Table 5.1 in [96] summarizes the values of $\operatorname{lcr}(\operatorname{GP}(n,k))$ for various small values of n and k. The smallest open case is $\operatorname{lcr}(\operatorname{GP}(16,4))$, which is either 1 or 2.

Local odd crossing number. See odd crossing number.
Local outerplanar crossing number. See convex crossing number.
Local pair crossing number. See pair crossing number.
Local toroidal crossing number. See local crossing number.
Major Crossing number. See minor crossing number.

Map crossing number

Definition: A *map* is a graph $G = (V, E)$ and a surface Σ with boundary $\partial\Sigma$ so that $V \subseteq \partial\Sigma$. In a drawing of G each edge is realized by a properly embedded arc (a connected curve that intersects $\partial\Sigma$ in its endpoints only). The *crossing number* of the map is the smallest number of crossings in a drawing of the map. Similarly, one can define odd, algebraic and pair crossing number for maps. We can introduce special names based on the number of boundary components of Σ: *disk crossing number* (one hole), *annulus crossing number* (two holes), *pair of pants crossing number* (three holes), and so on.

Reference: Pelsmajer, Schaefer, Štefankovič [622].

Comments: The map crossing numbers were introduced in [622] to separate ocr from cr. One can turn every boundary component into a single vertex with rotation; as long as one is considering a crossing number variant in which adjacent crossings count the same as independent crossings, the crossing number notion does not change, so one can alternatively look at map crossing numbers as crossing numbers of graphs with rotation system; map crossing numbers can also be considered a special case of the constrained crossing number. If we allow vertices to arbitrarily move on their boundary component, the disk crossing number becomes the convex crossing number, and the annulus crossing number turns into the radial crossing number on two levels. (The general case does not seem to have been considered so far.)

Complexity: The disk crossing number can be computed in time $\Theta(m \log m)$, where $m = |E|$; the annulus (algebraic) crossing number can be computed in polynomial time [621].[128] The complexity of computing the pair-of-paints crossing number is open. The general problem is **NP**-complete, since computation of the crossing number of a graph with a given rotation is **NP**-complete [621].

Relationships: $\mathrm{ocr}(M) \leqslant \mathrm{pcr}(M) \leqslant \mathrm{acr}(M) = \mathrm{cr}(M)$ for any map M; there is a map M for which $13 = \mathrm{ocr}(M) < \mathrm{pcr}(M) = 15$; if Σ has n boundary components, then $\mathrm{cr}(M) \leqslant \mathrm{ocr}(M)\binom{n+4}{4}/5$ [622].

Also see: Radial crossing number (on two levels), crossing number (with rotation system), constrained crossing number, convex crossing number, cylindrical crossing number, joint crossing numbers, wire crossing number.

Maximum bipartite crossing number. See bipartite crossing number.

[128]Results in that paper are phrased for graphs with rotation systems.

Maximum crossing number

Definition: The *maximum crossing number* of a graph G, max-cr(G), is the largest number of crossings in any drawing of G in which every pair of edges has at most one point in common (including a shared endpoint; touching points are forbidden).[129] The set of possible values $\{\text{cr}(D) : D \text{ is an intersection-simple drawing of } G\}$, is the *spectrum* of G for cr.

Reference: Ringel [655], Grünbaum [370].

Comments: In a 1972 paper, Grünbaum [370] expresses surprise that max-cr(K_n) and max-cr$(K_{m,n})$ have not been studied; he mentions max-cr$(K_4) = 1$ and Saaty's claim that max-cr$(K_n) = \binom{n}{4}$ [666] which he calls "probably true but unsubstantiated". Ringel had already settled this problem earlier [655]. This crossing number has also been called *maximal crossing number* [370]. One can try to relax the simplicity condition without allowing an infinite number of crossings. One model allows independent edges to cross an arbitrary number of times, as long as they do not form empty lenses (bigons consisting of two subarcs of the edges that do not enclose a vertex). In this drawing model K_n has at most $n!$ many crossings (and an exponential number of crossings is possible) [317].[130]

Complexity: **NP**-complete [194].

Relationships: max-$\overline{\text{cr}}(G) \leqslant$ max-cr(G) for all graphs G. max-cr$(G) \leqslant \theta(G)$, where $\theta(G) = \left(m(m+1) - \sum_{v\in V} \deg^2(v)\right)/2$, with $m = |E|$, the *thrackle bound* [630], and max-cr$(G) \leqslant \theta'(G) := \theta(G) - c_4 + k_4$, the *sub-thrackle bound* [652], where c_4 is the number of C_4-subgraphs of G, and k_4 the number of K_4-subgraphs of G.

Values: max-cr$(K_n) = \binom{n}{4}$ [655]. max-cr$(K_{n_1,\dots,n_k}) = \binom{n}{4} - \sum_{i=1}^{k}(\binom{n_i}{4} + (n-n_i)\binom{n_i}{3}))$, where $n = \sum_{i=1}^{k} n_i$ and $k \geqslant 2$ [401]. For trees T, max-cr$(T) = \theta(T)$, with $\theta(T)$ as defined above [630]. max-cr$(C_4) = 1$, and max-cr$(C_n) = n(n-3)/2$, for $n \neq 4$ [371, 399, 774], see [158] for the spectrum of C_n. max-cr$(Q_3) = 34$, where Q_3 is the 3-dimensional hypercube graph [400]. Asymptotically, max-cr(W_n) is $5n^2/4$ [411]. Also, max-cr is known for all graphs on up to 6 vertices [411]. max-cr$(\text{GP}(2,5)) = 68$ [403], where GP$(2,5)$ is the generalized Petersen graph; see [159] for the spectrum of GP$(2,5)$). max-cr$(C_3 \,\square\, C_3) = 78$ [413].

Open Questions: Ringeisen, Stueckle, and Piazza [652] introduced the *Subgraph Problem*: is it true that max-cr$(H) \leqslant$ max-cr(G) if H is a subgraph of G? Archdeacon [56] conjectures that it is. The conjecture is unsettled even for induced subgraphs H of G. For the maximum rectilinear crossing number, it is easy to see that max-$\overline{\text{cr}}(H) \leqslant$ max-$\overline{\text{cr}}(G)$ if H is a subgraph of G [652]. The same authors also conjecture that max-cr$(G) = \theta(G)$ if and only if G contains at most one cycle and that cycle is not C_4, where $\theta(G)$ is as defined above. This conjecture is equivalent to Conway's thrackle conjecture, according to which every graph for which max-cr$(G) = \theta(G)$ satisfies $|E(G)| \leqslant |V(G)|$ [774]. ▼ Harborth [397] mentions some

[129] In other words: an intersection-simple drawing.

[130] There are graphs, however, that can be drawn with arbitrarily many crossings in this model, which may explain why the authors did not introduce a new crossing number notion.

unpublished partial results on the spectrum of K_n for cr, can these be proved and extended?

ALSO SEE: Maximum rectilinear crossing number, maximum bipartite crossing number (under bipartite crossing number).

Maximum edge crossing number. See edge crossing number.
Maximum local crossing number. See local crossing number.
Maximum orchard crossing number. See orchard crossing number.

MAXIMUM RECTILINEAR CROSSING NUMBER

DEFINITION: The *maximum rectilinear crossing number* of a graph G, max-$\overline{\text{cr}}(G)$, is the largest number of crossings in any simple straight-line drawing of G (by requiring the graph to be simple we avoid edge overlap). If we restrict drawings to be convex (all vertices on the boundary of a circle), we get the *convex maximum rectilinear crossing number*, here denoted by max-$\overline{\text{cr}}^\circ(G)$. The set of possible values $\{\text{cr}(D) : D$ is a simple straight-line drawing of $G\}$, is the *spectrum* of G for $\overline{\text{cr}}$.

REFERENCE: Grünbaum [370]. Also, Furry, Kleitman [339].

COMMENTS: Originally defined by Grünbaum who mentions several results, including the calculation of max-$\overline{\text{cr}}(C_n)$ due to Steinitz [717].[131] Other names for this crossing number include *maximal rectilinear crossing number* [370] and *obfuscation complexity* [754]. Verbitsky writes obf(G) for max-$\overline{\text{cr}}$ and obf$^\circ$ for max-$\overline{\text{cr}}^\circ$. Thürmann [739] considers a variant max-$\overline{\text{cr}}^h$ of max-$\overline{\text{cr}}$ parameterized by the number h of vertices that lie on the boundary of the convex hull of all vertices (but only for complete graphs).

COMPLEXITY: The maximum rectilinear crossing number is **NP**-hard [85], but not known to lie in **NP**. Can be approximated efficiently to within a factor of 1/3 [85, 754]. For triangulations this bound can be improved to 56/39 [475]. The convex maximum rectilinear crossing number is **NP**-complete [85].

RELATIONSHIPS: max-$\overline{\text{cr}}(G) < 3|V(G)|^2$ [754]. max-$\overline{\text{cr}}(G) \leqslant$ max-cr(G) (by definition) and the inequality can be strict (e.g. compare Steinitz's result on max-$\overline{\text{cr}}(C_n)$ to max-cr(C_n) when n is even). max-$\overline{\text{cr}}(G) \leqslant \theta'(G)$, the sub-thrackle bound (see maximum crossing number for $\theta'(G)$), and there is a characterization of which graphs achieve max-$\overline{\text{cr}}(G) = \theta'(G)$ [718]. max-$\overline{\text{cr}}^\circ(G) \leqslant$ max-$\overline{\text{cr}}(G) \leqslant 3\,$max-$\overline{\text{cr}}^\circ(G)$ [86, 754]. For extremal values of max-$\overline{\text{cr}}$ given order and size of G, see [132]; for given order and degree, see [43, 84, 131]. max-$\overline{\text{cr}}^\circ(G) =$ max-$\overline{\text{cr}}^h(G)$ for $h = |V(G)|$ (by definition).

VALUES: max-$\overline{\text{cr}}(K_{n_1,\dots,n_k}) = \binom{n}{4} - \sum_{i=1}^{k}(\binom{n_i}{4} + (n-n_i)\binom{n_i}{3})$, where $n = \sum_{i=1}^{k} n_i$ and $k \geqslant 2$ (follows from [401], also see [45, 340]). max-$\overline{\text{cr}}(tK_4) = 20\binom{t}{2} + t$ [42]; max-$\overline{\text{cr}}(2K_5) = 60$, max-$\overline{\text{cr}}(2K_3 + 4K_1) = 136$, max-$\overline{\text{cr}}(2K_3 + 2K_3) = 357$, max-$\overline{\text{cr}}(2K_5 + 5K_1) =$

[131] Steinitz's result from 1923 was preceded by several incorrect or incomplete results, including a note by Baltzer [92] who seems to have originated the problem in 1885; in turn, it was rediscovered multiple times, e.g. in [339], for a (partial) survey see [372].

442 [84], where + is the join of two graphs. max-$\overline{\text{cr}}(C_n) = n(n-3)/2$ if n is odd and max-$\overline{\text{cr}}(C_n) = n(n-4)/2+1$ if n is even [717].[132] The spectrum of C_n for $\overline{\text{cr}}$ was determined in [158, 339, 372, 717]. max-$\overline{\text{cr}}^\circ(C_n) = n(n-3)/2$ if n is odd, and $n(n-4)/2+1$ if n is even, and the spectrum of C_n is known [158]. There is a closed formula for max-$\overline{\text{cr}}$ of 2-regular graphs (disjoint union of cycles) [131], and its complement [84]. The value of max-$\overline{\text{cr}}^h(K_n)$ is known [408]. max-$\overline{\text{cr}}(W_n) = (2n^2-5n-1)/2$ if n is odd and n^2-3n+1 if n is even [308]; for generalized wheel graphs $W_{m,n}$ see [45]. max-$\overline{\text{cr}}(Q_3) = 28$, where Q_3 is the 3-dimensional hypercube graph [44]. max-$\overline{\text{cr}}(\text{GP}(2,5)) = 49$ [313], where $\text{GP}(2,5)$ is the Petersen graph. max-$\overline{\text{cr}}(M(3,3)) = 35$ [314], and max-$\overline{\text{cr}}(M(2,n)) = (9n^2-11n+4)/2$ [316], where $M(m,n) = P_m \,\square\, P_n$ is the $m \times n$ mesh. For more results on meshes and other graphs based on tessellations and polyominos, see [312, 314–316]. For spiders, see [113, 303], for trees of diameter at most 4, see [113]. Calculating max-$\overline{\text{cr}}(nP_2)$, the largest number of crossings of n line segments, is an old puzzle, as in Sam Loyd Jr's "When Drummers Meet", see [706, 5.Q.1], in educational writing [252, 253],[133] textbooks [362, p.70, 497, p.5, 3rd part], and, with variations, in [716].

Open Questions: Alpert, Feder and Harborth [43] asked if max-$\overline{\text{cr}}(G) =$ max-$\overline{\text{cr}}^\circ(G)$ for every graph G; it is now known that this is not the case [194], but it is still possible that equality holds for bipartite graphs. (Also, see [133].) ▼ It is not known whether max-$\overline{\text{cr}}$ lies in **NP**, the best known upper bound is $\exists\mathbb{R}$. ▼ Alpert, Feder, Harborth and Klein [44] show that max-$\overline{\text{cr}}(Q_n) \geqslant 2^{n-2}[2^{n-1}(n^2-2n+3)-n^2-1]$ and conjecture that this lower bound is tight. ▼ What is max-$\overline{\text{cr}}(C_k \cup C_\ell)$? This question may be hard, since we only know the maximum number of intersections between two polygons if at least one of them has an even number of sides [254]; if both polygons have an odd number of sides we only know the value up to a constant [18, 172].

Also see: Maximum crossing number, maximum (rectilinear) edge crossing number (under edge crossing number), convex crossing number.

Maximum rectilinear edge crossing number. See edge crossing number.

Metro-line crossing number

Definition: Let G be a graph embedded in the plane, and $\mathcal{L}$ a set of paths (without repeated vertices) in G called *lines*. A *routing* of the lines orders all lines passing through an edge *at each end* of the edge. An *edge crossing* of two lines occurs if the ordering of the two lines at the two ends of some edge have switched. A vertex (*station*) is represented as a (convex) polygon with one side for each incident edge. The routing determines the order at each side of the station. If the entry and exit

[132] For more recent proofs in English, see [43, 339].

[133] Diesterweg's books are in the (long) tradition of Pestalozzi's "Formenlehre", an educational approach to shapes and figures in preparation for Euclid's geometry; it discusses many questions we would now classify as basic coincidence or combinatorial geometry. The first book [253] contains the exercises, the second book [252] the solutions (instructions for teachers). Problem 13 in chapter 10 is the relevant problem here, though there are variants as well (e.g. what happens if some lines are parallel).

points of two lines alternate along the boundary of a station, a *station crossing* occurs; that is, the two lines have to cross within the station. The *Metro-line crossing number* of a particular routing of $\mathcal{L}$ in the embedding of G is the number of edge and station crossings of lines in edges. The *Metro-line crossing number* of $\mathcal{L}$ is the smallest Metro-line crossing number of any routing of L.[134]

REFERENCE: Based on Benkert, Nöllenburg, Uno, Wolff [112], Argyriou, Bekos, Kaufmann, Symvonis [61].

COMMENTS: The concept of metro-line crossing minimization was introduced in Benkert, Nöllenburg, Uno, Wolff [112], a more general model was suggested by Argyriou, Bekos, Kaufmann, Symvonis [61]. Both these papers consider the problem a crossing minimization problem and study it in various variants (e.g. stations have to be 2-sided or 4-sided or the end of lines may be forced to be in particular positions), so the metro-line crossing number defined above is just one possible variant.

COMPLEXITY: Optimizing the Metro-line crossing number of a single edge in G can be done in polynomial time [112] and there are **NP**-hard variants even if the underlying graph is a path [61] or a caterpillar [325]. There are polynomial-time and fixed-parameter tractable cases for some variants [590].

ALSO SEE: Confluent crossing number, wire crossing number.

Minimum non-crossing edge number. See edge crossing number.

MINOR CROSSING NUMBER

DEFINITION: The *minor crossing number*, $\text{mcr}(G)$, of a graph G is the smallest crossing number of any graph having G as a minor. The *major crossing number*, $\text{Mcr}(G)$, of a graph G is the largest crossing number of any minor of G. We write mcr_Σ for the minor crossing number on surface Σ.

REFERENCE: Bokal, Fijavž, Mohar [137].

COMMENTS: The definition of the minor crossing number was motivated by an attempt to find a crossing number that works well with minors, indeed it is minor-monotone by definition (the genus crossing number also addresses this issue), and is sometimes called the *minor monotone crossing number*. Robertson and Seymour identified the 41 forbidden minors of the set $\{G : \text{mcr}(G) \leqslant 1\}$ [137]. Chimani and Gutwenger [195] introduce a variant $\text{mcr}_W(G)$, for $W \subseteq V(G)$, in which only vertices in W are allowed to be expanded in the minor relationship; this allows them to draw connections to a hypergraph crossing number variant.

[134] One can distinguish between avoidable and unavoidable station crossings: two lines entering a station through the same edge need not cross within the station, such a crossing can always be turned into an edge crossing without increasing the Metro-line crossing number of the drawing. Since the unavoidable station crossings can be computed in polynomial time, several papers restrict themselves to drawings without avoidable station crossings, and then only count edge crossings. This also gives a more interesting variant if one studies fixed-parameter tractability.

COMPLEXITY: **NP**-complete [424, 621].[135] Testing $\text{mcr}(G) \leqslant k$ is in polynomial time for any fixed k, since the property is closed under minors. However, only for $k = 1$ is the set of forbidden minors known [137].

RELATIONSHIPS: $\text{mcr}_\Sigma(H) \leqslant \text{mcr}_\Sigma(G)$ if H is a minor of G (from definition), $\text{mcr}_\Sigma(G) \leqslant \text{cr}_\Sigma(G) \leqslant \text{Mcr}_\Sigma(G)$ (from definition). $\text{cr}_\Sigma(G) \leqslant \lfloor \Delta/2 \rfloor^2 \text{mcr}_\Sigma(G)$ [137], where Δ is the maximum degree of G. $\text{mcr}_\Sigma(G) \geqslant (m - (3(n + \text{eg}(\Sigma)) + 6))/2$, where $\text{eg}(\Sigma)$ is the Euler genus of Σ and $n = |V(G)|, m = |E(G)|$ [137]. There is a constant $c(H)$ for every graph H so that $\text{mcr}(G) \leqslant c(H)|V(G)|$ for every G that does not contain H as a minor [138].

VALUES: $\text{mcr}(K_n)$ is known for $n \leqslant 8$ [137]. There are asymptotic bounds for complete graphs, complete bipartite graphs and hypercubes [136, 137].

ALSO SEE: Genus crossing number.

Minor-monotone crossing number. Alternative name for minor crossing number.
Monotone crossing number. See monotone crossing numbers.

MONOTONE CROSSING NUMBERS

DEFINITION: A drawing is *monotone* if every vertical line in the plane intersects each edge at most once. The *monotone crossing number* of G, mon-cr(G), is the smallest number of crossings in a monotone drawing of G. If G is equipped with a preorder $\preceq$ (reflexive and transitive) of its vertices we restrict the drawings of G to drawings which *respect* the preorder $\preceq$ in the sense that the total preorder created by the x-coordinates of the vertices extends $\preceq$. We write mon-cr$_\preceq(G)$ for the resulting (fixed) monotone crossing number. If there is no danger of confusion, we will drop $\preceq$ in the notation. If $\preceq$ is the trivial preorder, then mon-cr$_\preceq$ is simply the monotone crossing number mon-cr; if $\preceq$ is a total preorder we get the *leveled crossing number*[136] of which the bipartite crossing number and the k-layer crossing number are special cases. If $\preceq$ is a total order (at most one vertex per level, by anti-symmetry), we get the *x-monotone crossing number*. For a directed acyclic graph G with its induced preorder $\preceq$ we get the *upward crossing number* as mon-cr$_\preceq(G)$.

For any crossing number notion ψ one can introduce the corresponding monotone version mon-ψ as above (with or without a given preorder), for example, one can talk about the *monotone pair crossing number*, mon-pcr or the *monotone odd crossing number*, mon-ocr.

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[135] Neither of those sources shows that the problem lies in **NP**. For that one needs to observe that for every G there is a graph H so that $\text{mcr}(G) = \text{cr}(H)$ and G can be obtained from H using a polynomial (in size of G) number of contractions and deletions.

[136] More typically called the multi-level crossing minimization problem. A *level* is a set of vertices that are equivalent in the sense that $u \preceq v$ and $v \preceq u$. Levels realized as parallel lines in a drawing are often called *layers*. In crossing minimization problems the first step typically consists in assigning vertices to layers and then ordering the vertices within each layer. One can consider crossing number variants in which orderings of some layers are already specified. E.g. in the well-known *one-sided crossing minimization problem* the bipartite graph is drawn on two layers and the ordering of one layer is pre-specified.

Reference: Valtr [752], Fulek, Pelsmajer, Schaefer, Štefankovič [338].

Comments: The monotone crossing number was introduced by Valtr [752] who also mentions monotone pair crossing number and monotone odd crossing number. The preorder versions are introduced in [338], but many of these problems are implicit in the crossing minimization problems studied in leveled (layered) graph drawing. The preorder version mon-$\text{cr}_{\preceq}$ suggested here is a general tool to unify many of these notions. One could imagine a bi-monotone crossing number in which orderings are prescribed both for the x and the y direction. Balko, Fulek, and Kynčl [87] introduce the *monotone odd + crossing number*, mon-ocr_+ (under the name *monotone semisimple odd crossing number*), and the *monotone odd ± crossing number*, $\text{ocr}_\pm$ (using the name *monotone weakly semisimple odd crossing number*).

Complexity: mon-cr(G) is **NP**-complete.[137] With two levels, crossing minimization is **NP**-complete (see bipartite crossing number for a discussion), even if the ordering of one level is given (one-sided crossing minimization) [274, 279]. Testing whether a directed graph has upward crossing number 0 is **NP**-complete [282].

Relationships: $\text{cr}(G) \leqslant \text{mon-cr}(G) \leqslant \overline{\text{cr}}(G)$ (definition). $\text{mon-cr}(G) \leqslant 4\,\text{mon-pcr}(G)^{4/3}$ for all G [752]. $\text{mon-iocr}(G) \leqslant \text{mon-ocr}(G) \leqslant \text{mon-ocr}_\pm(G) \leqslant \text{mon-ocr}_+(G) \leqslant \text{mon-cr}(G)$ (definition). $\text{mon-cr}(G) \leqslant \binom{2\,\text{cr}(G)}{2}$, and there are graphs G for which $\text{mon-cr}(G) \geqslant 7/6\,\text{cr}(G) - 6$ [610]. If there is a graph G with a linear order $\preceq$ of its vertices so that $\text{mon-}\psi_{\preceq}(G) < \text{mon-}\phi_{\preceq}(G)$ for $\psi, \phi \in \{\text{ocr}, \text{iocr}, \text{acr}, \text{iacr}, \text{pcr}_+, \text{pcr}, \text{pcr}_-, \text{cr}, \text{cr}_-\}$, then there is a graph G' for which $\psi(G') < \phi(G')$; there is a graph G with a linear order $\preceq$ of its vertices, so that $\text{mon-iocr}_{\preceq}(G) < \text{mon-ocr}_{\preceq}(G)$ and consequently, there is a graph G' so that $\text{iocr}(G) < \text{ocr}(G)$ [338].

Values: $\text{mon-cr}(K_n) = Z(n)$ [1, 2], where $Z(n) = X(n)X(n-2)/4$ is Zarankiewicz's function, with $X(n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$.[138] The same result was also found by [87] who prove the stronger result $\text{mon-ocr}_\pm(K_n) = \text{mon-ocr}_+(K_n) = Z(n)$.

Open Questions: Is $\text{mon-iocr}(K_n) = Z(n)$ [87]? ▼ We can define a maximum version of the monotone crossing number (each pair of edges may cross at most once). What is the maximum monotone crossing number of C_n? The interesting case here are cycles of even length, for which Pach and Sterling showed that there can be at most $n(n-3)/2-1$ crossings [603, Lemma], compared to a lower bound of $n(n-4)/2+1$ via max-$\overline{\text{cr}}(C_n)$.

Also see: Bipartite crossing number, radial crossing number, upward crossing number, pseudolinear crossing number, local crossing number (bottleneck crossing minimization).

Multiplanar crossing number. See k-planar crossing number.

[137] **NP**-hardness follows from the hardness of crossing number [348], simply subdivide each edge sufficiently often so each part can be drawn as a monotone edge. The problem lies in **NP**: guess an ordering of the vertices and the ordering in which edges pass above and below each vertex. That is sufficient to calculate the crossing number of the drawing.

[138] The result remains true if the edges in the drawing are only *x-bounded*, that is, each edge lies (horizontally) entirely between its endpoints.

Nodal crossing number

Definition: Let $\text{cr}_D(e)$ be the number of crossings involving e in a drawing D. Let $\text{cr}_D(v)$ be the sum of $\text{cr}_D(e)$ over all e incident to v (in the literature, $\text{cr}_D(v)$ is known as the *responsibility of v in D*). The *nodal crossing number* of a drawing D of a graph G, $\text{ncr}(D)$, is the largest $\text{cr}_D(v)$ over all vertices of G. The *nodal crossing number* of G, $\text{ncr}(G)$, is the minimum of $\text{ncr}(D)$ over all drawings of G. For the nodal crossing number on a surface Σ, we write ncr_Σ.

Reference: Guy, Jenkyns, Schaer [383].

Comments: The nodal toroidal crossing number, ncr_{S_1} was introduced by Guy, Jenkyns, Schaer [383]; Guy [296, p.364] also referred to it as the *vertical crossing number*. Radermacher and Rutter consider a related parameter they call the *co-crossing number* of a vertex [640]. One can imagine a related notion of "vertex-skewness" in which we ask for the smallest number of vertices that can be removed to make a graph planar. This notion was first mentioned by Harary [387], and called k-apex graphs in [117], also see Footnote 157. In [714] the non-increasing sequence of vertex responsibilities is called a crossing sequence.

Complexity: Open.

Relationships: $\text{lcr}(G) \leqslant \text{ncr}(G) \leqslant \text{cr}(G)$ (by definition).

Values: $\text{ncr}_{S_1}(K_n)$ is known for $n \leqslant 9$, and there are asymptotic results for $\text{ncr}_{S_1}(K_n)$ [383].

Also see: Local crossing number, simple crossing number, vertex-skewness (under skewness).

Non-crossing edge number. See edge crossing number.

Odd crossing number

Definition: The *odd crossing number* of G, $\text{ocr}(G)$, is the smallest number of pairs of edges crossing an odd number of times in any drawing of G. The Rule + variant of ocr is $\text{ocr}_+(G)$, the smallest number of pairs of edges crossing an odd number of times in any drawing of G in which adjacent edges are forbidden to cross (called *semisimple* in [87], and *star-simple* in [317]). One can define an intermediate variant in which adjacent edges have to cross evenly (such drawings are called *weakly semisimple* in [87]); denote this variant by $\text{ocr}_\pm$.[139] Karl and Tóth [478] call a graph *k-odd planar* if it can be drawn so that every edge is crossed by at most k other edges an odd number of times (edges crossing an even number of times do not count). Based on that one can introduce the *local odd crossing number* of a graph G as the smallest k for which the graph is k-odd planar.

Reference: Pach, Tóth [613], also Levow [530].

[139] The + rule for crossing numbers looks rather straightforward: we prohibit drawings in which adjacent edges cross. One may ask, however, in what sense of the word cross? The standard interpretation is that $\text{cr}(e, f) = 0$ for all pairs of adjacent edges e and f. But why not require that $\psi(e, f) = 0$ if we are considering the crossing number ψ? For cr and pcr (and $\overline{\text{cr}}$, of course), this makes no difference, but for ocr and acr we get a new variant which we denote by $\psi_\pm$ [338]. By definition, $\psi \leqslant \psi_\pm \leqslant \psi_+$.

Comments: First explicitly defined (and named) by Pach and Tóth [613], although Levow [530] deserves some credit; he realized that Tutte's algebraic theory of crossing number could be developed over binary fields (Wu developed a theory parallel to Tutte's over binary fields, but he didn't touch on the subject of crossing numbers); Levow defines a parameter that could be algebraic or odd crossing number (or, indeed, an independent version). His definition is not precise enough to decide.

Complexity: **NP**-complete [613] and remains **NP**-complete if the graph is cubic or rotation system is given [621]. The problem is fixed-parameter tractable [619].

Relationships: There is a crossing lemma, $\mathrm{ocr}(G) \geqslant 1/64\ m^3/n^2$ for $m > 4n$ [613].[140] For $\mathrm{ocr}_{\pm}$, a stronger bound is known: $\mathrm{ocr}_{\pm}(G) \geqslant 1/54\ m^3/n^2$ [478]. $\mathrm{iocr}(G) \leqslant \mathrm{ocr}(G) \leqslant \mathrm{ocr}_{\pm} \leqslant \mathrm{ocr}_{+}(G)$ for all graphs G (by definition). $\mathrm{ocr}(G) \leqslant \mathrm{acr}(G) \leqslant \mathrm{cr}(G)$ (by definition). $\mathrm{ocr}(G) = \mathrm{cr}(G)$ if $\mathrm{ocr}(G) \leqslant 3$ [623]. There are graphs for which $\mathrm{ocr}(G) < (\sqrt{3}/2 + o(1))\,\mathrm{acr}(G) = \mathrm{pcr}(G) = \mathrm{cr}(G))$ [622]. $\mathrm{ocr}_\Sigma(G) \leqslant \binom{2\,\mathrm{cr}_\Sigma(G)}{2}$ for all surfaces Σ, and $\mathrm{ocr}_\Sigma(G) = \mathrm{cr}_\Sigma(G)$ if $\mathrm{ocr}_\Sigma(G) \leqslant 2$ for all surfaces Σ [624]. $\mathrm{locr}(G) \leqslant \mathrm{lpcr}(G)$, so there are graphs for which $\mathrm{locr}(G) < lcr(G)$ (since there are graphs separating lpcr and lcr). If $\mathrm{locr}(G) \leqslant k$, then $m = |E(G)| \leqslant \sqrt{32kn}$ [478].

Also see: Independent odd crossing number, algebraic crossing number, monotone crossing number (for monotone version).

Orchard crossing number

Definition: An *orchard drawing* of G is a straight-line drawing of G with vertices in general position to which are added straight (infinite) lines through every pair of vertices. The *orchard crossing number*, orchard-cr(D), of an orchard drawing D of G is the total number of crossings between edges and lines (not counting the line an edge lies on). The *orchard crossing number* of G, orchard-cr(G), is the smallest orchard crossing number of any orchard drawing of G. The *maximum orchard crossing number* of G is the largest orchard crossing number of any orchard drawing of G.

Reference: Feder, Garber [311].

Comments: One can also imagine a pseudoline version of the orchard crossing number. Replacing lines with line segments in the definition of the orchard crossing number leads to the *airport crossing number* [297]. For the airport crossing number, a non-rectilinear version may be of interest as well.

Complexity: Open.

Relationships: $\overline{\mathrm{cr}}(G) \leqslant$ orchard-cr$(G)/2$ [311] (since every edge crossing counts twice). The drawing maximizing the orchard crossing number of K_n realizes $\overline{\mathrm{cr}}(K_n)$ [311].

Values: orchard-cr$(K_n) = 2\binom{n}{4}$, and the values of orchard-cr$(K_{1,n})$ and orchard-cr(W_n) are known [311]; orchard-cr$(K_{n,n}) = 4n\binom{n}{3}$ [310]. Further results are in [309]. The maximum orchard crossing number of $K_{m,n}$ is known [311].

Also see: Rectilinear crossing number

[140] See the section on crossing lemma variants in Section 1.

Oriented crossing number. See joint crossing numbers.
Outerplanar crossing number. See convex crossing number.

Pair crossing number

Definition: The *pair crossing number* of G, $\mathrm{pcr}(G)$, is the smallest number of pairs of edges crossing in any drawing of G. The *independent pair crossing number* of G, $\mathrm{pcr}_-(G)$, is the smallest number of pairs of independent edges crossing in any drawing of G. The Rule + variant of pcr is $\mathrm{pcr}_+(G)$, the smallest number of pairs of edges crossing in any drawing of G in which adjacent edges are forbidden to cross. The *local pair crossing number* of G, $\mathrm{lpcr}(G)$, is the smallest k so that G has a drawing in which every edge crosses at most k other edges (possibly multiple times).

Reference: Mohar (attributed in [502]), Pach, Tóth [612, 613], Ackerman, Schaefer [20] for lpcr.

Comments: According to Kolman and Matoušek [502], the pair crossing number was first explicitly introduced by Mohar who asked whether pcr = cr at an AMS Conference on topological graph theory in 1995. The first mention in print seems to be by Pach and Tóth [613] (as the *pairwise crossing number*), who pointed out that crossing number is often defined as pair crossing number (whether intentionally or not), see Section 1 for a discussion. The independent pair crossing number was also defined by Pach and Tóth [612]; Alon [41] and Tao and Vu [735] discuss the crossing lemma for the independent pair crossing number. The local pair crossing number was explicitly introduced by Ackerman, Schaefer [20], though there had been earlier implicit definitions of this notion [712, 713].

Complexity: The pair crossing number is **NP**-complete [613, 679] and remains **NP**-complete if the graph is cubic or rotation system is given [621]. The independent pair crossing number is also **NP**-complete. The pair crossing number is fixed-parameter tractable [619].

Relationships: There is a crossing lemma for the independent pair crossing number, $\mathrm{pcr}_-(G) \geqslant 1/64\ m^3/n^2$ for $m > 4n$ [41].[141] For pcr_+ a stronger lower bound is known, $\mathrm{pcr}_+(G) \geqslant 1/32.4 m^3/n^2$ for $m \geqslant 6.75n$. $\mathrm{ocr}(G) \leqslant \mathrm{pcr}(G) \leqslant \mathrm{cr}(G)$, $\mathrm{pcr}_-(G) \leqslant \mathrm{pcr}(G) \leqslant \mathrm{pcr}_+(G)$ for all G. If $\mathrm{pcr}_-(G) = \mathrm{pcr}(G)$, then $\mathrm{pcr}(G) = \mathrm{pcr}_+(G)$.[142] There are graphs G for which $\mathrm{ocr}(G) < \mathrm{pcr}(G)$ [622], indeed $\mathrm{ocr}(G) = \mathrm{acr}(G) \leqslant 0.855\,\mathrm{pcr}(G)$ is possible [742]. It has been announced that $\mathrm{cr}(G) = O(\mathrm{pcr}(G)^{3/2})$ [629]. The best previous bound was $\mathrm{cr}(G) = O(\mathrm{pcr}(G)^{3/2} \log \mathrm{pcr}(G))$ which follows from combining a result by Matoušek [543] (based on a proof by Tóth [740]), who showed that $\mathrm{cr}(G) = O(\mathrm{pcr}(G)^{3/2} \log^2 \mathrm{pcr}(G))$ with a stronger string graph separator [526], see [674, Corollary 9.23] and, independently, [477]. Earlier bounds on cr in terms

[141] See the section on crossing lemma variants in Section 1.

[142] Consider a pcr-minimal drawing of G. Since $\mathrm{pcr}_-(G) = \mathrm{pcr}(G)$, it does not have any crossings between adjacent edges (otherwise it would witness $\mathrm{pcr}_-(G) < \mathrm{pcr}(G)$). So the drawing shows that $\mathrm{pcr}_+(G) = \mathrm{pcr}(G)$.

of pcr (using different techniques) are due to Valtr and Tóth [742, 752]. $\mathrm{lpcr}(G) \leqslant \mathrm{lcr}(G)$ (by definition), and there are graphs G for which $\mathrm{lpcr}(G) < \mathrm{lcr}(G)$ (see Footnote 120), however, $\mathrm{lpcr}(G) = \mathrm{lcr}(G)$ as long as $\mathrm{lcr}(G) \leqslant 2$ [20],[143] or G is sufficiently connected [480]. For a discussion of crossing lemmas for pcr and pcr_ on surfaces, see Remark 3.

Pair-of-pants crossing number. See map crossing number.
Pair string crossing number. See string crossing number.
Pairwise crossing number. See pair crossing number.
Projective plane crossing number. See crossing number.

Pseudolinear crossing number

Definition: A *pseudoline* is a simple closed curve in the projective plane that is non-separating. A *pseudoline arrangement* is a set of pseudolines so that each pair of pseudolines has exactly one point in common. A *pseudolinear drawing* of G is a drawing of G in the projective plane so that each edge lies on a pseudoline in a pseudoline arrangement. Edges are then called *pseudosegments*. The *pseudolinear crossing number* of G, $\widetilde{\mathrm{cr}}(G)$, is the smallest number of crossings between pseudosegments in a pseudolinear drawing of G.

Reference: Balogh, Leaños, Pan, Richter, and Salazar [89, 615].

Comments: The pseudolinear crossing number was introduced in Pan's thesis [615]. De Avila-Martínez, Leaños, Medina [78] introduce the *m-symmetric pseudolinear crossing number*, which restricts the drawings to pointsets closed with respect to rotations by $2\pi/m$. This crossing number is only defined for complete graphs.

Complexity: **NP**-complete [417]. It is $\exists\mathbb{R}$-complete to test whether $\widetilde{\mathrm{cr}}(G) = \overline{\mathrm{cr}}(G)$ [417].

Relationships: $\text{mon-cr}(G) \leqslant \widetilde{\mathrm{cr}}(G) \leqslant \overline{\mathrm{cr}}(G)$ (since pseudolines can be realized as x-monotone curves and because every rectilinear drawing can be extended to a pseudoline drawing). The pseudolinear crossing number differs from the standard crossing number, even for complete graphs: $18 = \mathrm{cr}(K_8) < \widetilde{\mathrm{cr}}(K_8) = \overline{\mathrm{cr}}(K_8) = 19$. Hernández-Vélez, Leaños, Jesús and Salazar [417] show that the graphs G_m introduced by Bienstock and Dean [121] separate cr from the pseudolinear crossing number, since $\mathrm{cr}(G_m) = 4$ and $\widetilde{\mathrm{cr}}(G_m) = m$. This also separates mon-cr from $\widetilde{\mathrm{cr}}$ since $\text{mon-cr} \leqslant \binom{2\,\mathrm{cr}}{2}$ [610]. For every m there is an H_m so that $\widetilde{\mathrm{cr}}(H_m) \leqslant \overline{\mathrm{cr}}(H_m) - m$ [417].

Values: $\widetilde{\mathrm{cr}}(K_n) = \overline{\mathrm{cr}}(K_n)$ for $n \leqslant 27$ [5]. $\widetilde{\mathrm{cr}}(K_n) \leqslant 0.380448\binom{n}{4} + O(n^3)$ [88]. $\widetilde{\mathrm{cr}}(K_n) \geqslant 0.379972\binom{n}{4} - O(n^3)$ [5]. Some of the best asymptotic lower bounds for $\overline{\mathrm{cr}}(K_n)$ are achieved via $\widetilde{\mathrm{cr}}(K_n)$. Since $\widetilde{\mathrm{cr}}(K_n)/\binom{n}{4}$ is nondecreasing and bounded, the limit $\tilde{\rho} = \lim_{n\to\infty} \widetilde{\mathrm{cr}}(K_n)/\binom{n}{4}$ exists and is known as the *pseudolinear crossing constant*; for bounds on $\tilde{\rho}$, see [27]. The value of $\widetilde{\mathrm{cr}}(K_{2n})$ is known for centrally symmetric drawings [235].

Open Questions: Balogh, Leaños, Pan, Richter, and Salazar [89] conjecture that $\widetilde{\mathrm{cr}}(K_n) = \overline{\mathrm{cr}}(K_n)$. Supporting this conjecture is the fact that the convex hull of

[143] A fact used in the proof of the crossing lemma for pcr_+.

both a $\widetilde{\text{cr}}$-optimal and a $\overline{\text{cr}}$-optimal drawing of K_n is a triangle [31, 89]; it is open whether this is true for the second convex hull as well (an earlier paper on this topic has been withdrawn [524]). ▼ Aichholzer, Duque, Fabila-Monroy, García-Quintero, and Hidalgo-Toscano [27] mention the question whether $\tilde{\rho} < \rho$, where ρ is the rectilinear crossing constant, as a "challenging open problem". ▼ Extending a question by Pegg [617], we can ask whether $\text{cr}(G) = \widetilde{\text{cr}}(G)$ for cubic graphs G.

ALSO SEE: Rectilinear crossing number, monotone crossing number.

QUASI CROSSING NUMBER

DEFINITION: A drawing of a graph is *quasi-plane* if it does not contain three pairwise-crossing edges. The *quasi(-plane) crossing number*, quasi-cr(G), of G is the smallest number of triples of edges crossing pairwise in any drawing of the graph. If we restrict the drawings to be intersection-simple, we get the *simple quasi(-plane) crossing number*, quasi-cr$^*(G)$.

REFERENCE: Pitchanathan, Shannigrahi [636].

COMMENTS: Pitchanathan and Shannigrahi [636] introduce the simple quasi crossing number, and use the notation cr_3 (which we use for the 3-planar crossing number); we also distinguish the variant without the simplicity requirement. It's natural to extend both notions to the k-quasi(-plane) setting, yielding the k-quasi(-plane) crossing number and its simple variant.

COMPLEXITY: Open.

RELATIONSHIPS: There is a crossing lemma, quasi-cr$^*(G) \geqslant c\ m^5/n^4$. If $\text{lcr}^*(G) \leqslant k$, then quasi-cr$^*_{k+1}(G) = 0$ [48].

VALUES: quasi-cr$^*(K_{10}) = 0$ [145]. quasi-cr$^*(K_{11}) = 4$ [636].

OPEN QUESTIONS: Is there a non-trivial upper bound on quasi-cr$^*(K_n)$ [636]. ▼ Is quasi-cr(G) = quasi-cr$^*(G)$, or can quasi-cr$^*(G)$ be bounded in quasi-cr(G)? ▼ Is there a crossing lemma for quasi-cr? ▼ One could consider quasi-$\overline{\text{cr}}$, the *quasi rectilinear crossing number.* Brandenburg [145] mentions in passing that quasi-$\overline{\text{cr}}(K_{10}) > 0$, attributing it to [30]. What is the exact value, and how does quasi-$\overline{\text{cr}}$ relate to quasi-cr*? ▼ It is a long-standing open question (e.g. [147, Problem 1, Section 9.6]), whether quasi-cr$^*_k(G) = 0$ implies that $|E(G)|$ is linear, this is only known for $k \leqslant 4$ [331].

ALSO SEE: Crossing number of abstract topological graph.

RADIAL CROSSING NUMBER

DEFINITION: A *leveling* of a graph $G = (V, E)$ is a mapping from V to $\{1, \ldots, k\}$, assigning each vertex a *level.* A *radial drawing* of G is a drawing in which vertices of level i are placed on the ith circle of k concentric circles; edges are required to be monotone in the sense that they cross every circle that is concentric with the level circles at most once. The *radial crossing number* of G is the smallest number of crossings in a radial drawing of G.

Reference: Bachmaier [79]. Richter, Thomassen [650]. Also, Northway [589].

Comments: Bachmaier [79] introduced the general concept of radial crossing number. If G is bipartite one can assign the vertices of each group to one of two circles, resulting in the bipartite cylindrical drawings introduced by Richter and Thomassen [650] to study the crossing number of K_n via bipartite cylindrical drawings of $K_{n,n}$; there also is a concept of cylindrical crossing number for non-bipartite graphs. In a paper from 1940, Northway [589], suggested radial layouts and used the number of crossing lines as an aesthetic criterion.

Complexity: Radial level planarity can be tested in linear time [82]. For two levels, the radial crossing number is **NP**-complete (this easily follows from **NP**-hardness of the bipartite crossing number), as is the one-sided version (in which the ordering of the vertices on one level is fixed) [79, 274, 279]. If orderings of vertices on both sides are fixed, the problem is in polynomial time [621].[144]

Relationships: The leveled crossing number of G is an upper bound on its radial crossing number. In particular, the bipartite crossing number, bcr, is an upper bound on radial crossing number with two levels (the upper bound may be strict, e.g. for $K_{2,2}$).

Values: The radial crossing number of $K_{n,n}$ on two levels is $n\binom{n}{3}$ [650] (with each group on a separate level). More recently, Sparks [708] showed that under the same restrictions the radial crossing number of $K_{m,n}$ can be calculated.

Also see: Bipartite crossing number, leveled crossing number (under monotone crossing numbers), annulus crossing number (under map crossing number), cylindrical crossing number.

Rectilinear crossing number

Definition: The *rectilinear crossing number* of G, $\overline{\text{cr}}(G)$, is the smallest number of crossings in a straight-line drawing of G.

Reference: Harary, Hill [389].

Comments: The rectilinear crossing number for arbitrary graphs was introduced by Harary and Hill [389]. It is sometimes claimed that the rectilinear crossing number is also known as the *linear* or *geometric(al)* crossing number, but published evidence for that is slim.[145] De Avila-Martínez, Leaños, Medina [78], based on [4, 5, 173], introduce the *m-symmetric rectilinear crossing number*, which restricts the drawings to pointsets closed with respect to rotations by $2\pi/m$. This crossing number is only defined for complete graphs.

[144] In this case, the radial crossing number turns into the annulus crossing number.

[145] If it is used at all, the term "linear crossing number" typically refers to the linear crossing number introduced by Nicholson [588], the only exceptions I found are [56, 104]. The use of "geometric drawing" for straight-line drawing is quite common, but there only seem to be a small number of papers using the term geometric crossing number [6, 15, 56].

COMPLEXITY: $\exists\mathbb{R}$-complete [120, 239], see [673] for $\exists\mathbb{R}$. Can be approximated to within an additive error of $o(|G|^4)$ in polynomial time [329]. The crossing number of a straight-line drawing of a graph can be computed in time $O(n^2 \log n)$ [270].

RELATIONSHIPS: $\mathrm{cr}(G) \leqslant \overline{\mathrm{cr}}(G)$ for all graphs G, and inequality can be strict, e.g. $18 = \mathrm{cr}(K_8) < \overline{\mathrm{cr}}(K_8) = 19$ [97, 705].[146] $\mathrm{cr}(G) = \overline{\mathrm{cr}}(G)$ if $\mathrm{cr}(G) \leqslant 3$, but for every k there is a G such that $\mathrm{cr}(G) = 4$ and $\overline{\mathrm{cr}}(G) \geqslant k$ [121].[147] $\mathrm{cr}(G) = \overline{\mathrm{cr}}(G)$ for maximal graphs of pathwidth 3 [119]. Also, $\overline{\mathrm{cr}}(G) = O(\Delta\,\mathrm{cr}^2(G))$, where Δ is the maximum degree of G [122]; this was improved to $\overline{\mathrm{cr}}(G) = O(\Delta\,\mathrm{cr}(G) \log \mathrm{cr}(G))$ if $|E| \geqslant 4|V|$ [695]. Wilf [770] points out that $\overline{\mathrm{cr}}(G) \leqslant \rho M/3$, where M is the number of times $2K_2$ occurs as a subgraph in G, and $\rho \approx 0.38$ is the rectilinear crossing constant (definition under values).[148] The *rectilinear midrange crossing constant* is the limit of $\overline{\mathrm{cr}}(G)n^2/m^3$ as n goes to infinity and $n \ll m \ll n^2$; this limit exists [602, 604], and is at least as large as the midrange crossing constant.

VALUES: The values of $\overline{\mathrm{cr}}(K_n)$ are now known up to $n = 27$ and for $n = 30$ (see [15] for a recent survey, also [25]). $\overline{\mathrm{cr}}(K_n) > \mathrm{cr}(K_n)$ for $n = 8$ and $n \geqslant 10$. $277/729\binom{n}{4} \leqslant \overline{\mathrm{cr}}(K_n) \leqslant 9363184/24609375\binom{n}{4} + \Theta(n^3)$ (lower bound: [5], upper bound:
[570]; current techniques are described in [15]). Since $\overline{\mathrm{cr}}(K_n)/\binom{n}{4}$ is nondecreasing and bounded, $\rho = \lim_{n\to\infty} \overline{\mathrm{cr}}(K_n)/\binom{n}{4}$—the *rectilinear crossing constant* [323]—exists, and is, surprisingly, related to Sylvester's Four Point Problem [684]; for bounds on ρ, see [27]. The value of $\overline{\mathrm{cr}}(K_{2n})$ is known for centrally symmetric drawings [235]. For complete bipartite $\overline{\mathrm{cr}}(K_{m,n}) \leqslant Z(m,n)$, where $Z(m,n) = X(m)X(n)$ and $X(n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$ [788]. It has been conjectured that $\overline{\mathrm{cr}}(K_{m,n}) = \mathrm{cr}(K_{m,n})$ [56]. This conjecture is implied by Zarankiewicz's conjecture as Guy observed [379]. For the balanced case $\overline{\mathrm{cr}}(K_{n,n}) \geqslant 0.987n^4/16 + o(n^4)$ [90]. For tripartite complete graphs K_{n_1,n_2,n_3} there is a function $A(n_1, n_2, n_3)$ introduced in [355] for which the authors conjecture $\overline{\mathrm{cr}}(K_{n_1,n_2,n_3} = \mathrm{cr}(K_{n_1,n_2,n_3}) = A(n_1, n_2, n_3)$;[149] they can show that

$$0.973A(n_1, n_2, n_3) \leqslant \overline{\mathrm{cr}}(K_{n_1,n_2,n_3}) \leqslant A(n_1, n_2, n_3)$$

(and a slighter weaker lower bound for cr). $\overline{\mathrm{cr}}(K_8^4) = 8$ and $\overline{\mathrm{cr}}(K_9^4) = 15$, where K_n^r is the complete balanced r-partite graph, and there is an upper bound for complete balanced multipartite graphs [301]; the same paper also contained bounds for layered

[146] Barton's thesis [97] and Singer's unpublished manuscript [705] also contain early upper bounds on $\overline{\mathrm{cr}}(K_n)$, Barton obtains $\overline{\mathrm{cr}}(K_n) \leqslant 11/648\; n^4 + O(n^3)$ and Singer shows $\overline{\mathrm{cr}}(K_n) \leqslant 5/312n^4 + O(n^3)$; see the section on values for current best bounds.

[147] Some more light is thrown on these separating examples in [417]

[148] The paper doesn't supply an argument, but one imagines Wilf would have argued as follows: fix a $\overline{\mathrm{cr}}$-optimal drawing of K_n, where $n = |V(G)|$. Randomly assign vertices in $V(G)$ to vertices in the drawing of K_n. Then the probability that four vertices of $V(G)$ are in convex position, is ρ by definition of ρ. The probability that two edges of G are mapped to the four endpoints so that the two edges cross, is $1/3$; hence, the expected number of crossings of G is at most $\rho M/3$.

[149] The function $A(n_1, n_2, n_3)$ is a special case of Harborth's function (see crossing number of k-partite graphs).

graphs. $\overline{\text{cr}}(C_3 \square C_n) = n$ [653], $\overline{\text{cr}}(C_4 \square C_n) = 2n$ [106]. For complements of cycles, see [381]. Faria, de Figueiredo, Richter and Vrťo [305] give upper bounds on $\overline{\text{cr}}(Q_n)$.

OPEN QUESTIONS: Harary, Kainen, and Schwenk conjectured that $\text{cr}(C_m \square C_n) = n(m-2)$ for $n \geqslant m \geqslant 3$; since there are straight-line drawings of $C_m \square C_n$ with $n(m-2)$ crossings, a weaker conjecture would be: $\overline{\text{cr}}(C_m \square C_n) = n(m-2)$ for $n \geqslant m \geqslant 3$; the conjecture is known to be true for the same cases as the original conjecture which is discussed in the entry on the crossing number. ▼ The separation of cr and $\overline{\text{cr}}$ by Bienstock and Dean [121] implies that $\overline{\text{cr}}$ cannot be bounded in cr; however, Hernández-Vélez, Leaños, and Salazar [417] conjecture that this can be done, that is, $\overline{\text{cr}}(G) \leqslant f(\text{cr}(G))$ for some function f, as long as G is 3-connected. ▼ What is the complexity of $\overline{\text{cr}}(G) \leqslant 4$ (in comparison: cr is fixed-parameter tractable; pseudo-linear crossing number is also open)? ▼ Pegg [617] asks whether $\overline{\text{cr}}(G) = \text{cr}(G)$ for cubic graphs G.[150] ▼ Dean [239] asks how hard it is to test whether $\overline{\text{cr}}(G) = \text{cr}(G)$. ▼ Fabila-Monroy, Paul, Viafara-Chanchi, and Weinberger [301] conjecture that $\text{cr}(K_n^4) < \overline{\text{cr}}(K_n^4)$ for almost all n; it is in this context that they observe that $\overline{\text{cr}}(K_8^4) = 8$, while $\text{cr}(K_8^4) = 6$ and $\overline{\text{cr}}(K_9^4) = 15 = \text{cr}(K_9^4)$. ▼ It had been conjectured [29] that for every n there is a $\overline{\text{cr}}$-minimal drawing of K_n which contains a $\overline{\text{cr}}$-minimal drawing of K_{n-1}. This conjecture was proven wrong by [12] which shows that K_{18} has a unique $\overline{\text{cr}}$-minimal drawing which does not contain a $\overline{\text{cr}}$-minimal drawing of K_{17}. The authors of [12] ask whether K_{6n} always has a unique $\overline{\text{cr}}$-minimal drawing (this is known to be true for $n \in \{1, 2, 3\}$. ▼ So far undefined is the *circular-arc crossing number* in which edges have to be drawn as circular arcs; it differs from both crossing number and rectilinear crossing number. Can the rectilinear crossing number be polynomially bounded in the circular-arc crossing number?

ALSO SEE: t-polygonal crossing number, pseudolinear crossing number, maximum rectilinear crossing number, simultaneous geometric crossing number (under simultaneous crossing number), grid crossing number, rectilinear local crossing number (under local crossing number), rectilinear weighted crossing number (under weighted crossing number), centrally symmetric rectilinear crossing number (under centrally symmetric crossing number).

Rectilinear bipartite cylindrical crossing number. See cylindrical crossing number.
Rectilinear edge crossing number. See edge crossing number.
Rectilinear *k*-planar crossing number. See k-planar crossing number.
Rectilinear local crossing number. See local crossing number.
Rectilinear space crossing number. See space crossing number.
Rectilinear weighted crossing number. See weighted crossing number.

[150] By a result of Bienstock and Dean [122], $\overline{\text{cr}}(G) = O(\text{cr}^2(G))$ in this case, so no unbounded separation is possible in this case. Is K_8 the graph of smallest degree for which we know that $\overline{\text{cr}}(G) > \text{cr}(G)$?

Red/blue crossing number

Definition: Given graphs $G_i = (V_i, E_i)$, and point-sets P_i in the Euclidean plane with $|P_i| = |V_i|$, $i \in \{1, 2\}$, a *red/blue* drawing consists of straight-line embeddings of G_i on vertex set P_i, $i \in \{1, 2\}$ (each graph by itself is free of crossings). The *red/blue crossing number* is the smallest number of crossings in a red/blue drawing (necessarily between edges of G_1, the red edges, and G_2, the blue edges; in other words, we count red/blue crossings). It is possible that the G_i have no red/blue drawing on the P_i, in which case we say that the red/blue crossing number is infinite.

Reference: Based on Bereg, Jiang, Yang, Zhu [114].

Comments: Bereg, Jiang, Yang, Zhu [114] are interested in the smallest number of crossings between any two crossing-free, geometric spanning trees on P_1 and P_2. However, they do go on to study the special case where the G_i are paths.

Complexity: Testing whether the red/blue crossing number of two paths is 0 is **NP**-complete [114]. (Finding red/blue spanning trees with the minimum number of crossings can be solved in time $O(n \log n)$.)

Also see: Simultaneous crossing number, joint crossing numbers, geometric k-planar crossing number.

Right-angle crossing number

Definition: The *right-angle crossing number* of G is the smallest number of crossings in a straight-line drawing of G in which all pairs of crossing edges have to be orthogonal. If no such drawing exists, the right-angle crossing number is infinite.

Reference: Based on Didimo, Eades, and Liotta [250].

Comments: Didimo, Eades, and Liotta [250] introduced the notion of RAC (Right Angle Crossing) drawing based on the aesthetic heuristic that drawings are easier to read if angles at crossings are large [449]. One can imagine a t-polygonal right-angle crossing number, in which each edge is allowed to consist of t line segments. Didimo, Eades, and Liotta [250] showed that every graph has finite 4-polygonal right-angle crossing number. A more relaxed version may only require angles to be at least some large $\alpha \leqslant 90$, see [246, 265], or edges to be drawn as circular arcs [179].

Complexity: It is $\exists\mathbb{R}$-complete [677] to decide whether the right-angle crossing number is finite (see [60] for earlier **NP**-hardness result). The problem remains hard even if there are at most eleven crossings per edge and in the fixed embedding setting.

Relationships: The right-angle crossing number of G is at least $\overline{\mathrm{cr}}(G)$. If G has finite right-angle crossing number, then $m \leqslant 4n - 10$ assuming that $n \geqslant 4$ [250].[151]

Open Questions: What is the complexity of deciding whether the right-angle crossing number is at most k for small values of k such as 1, 2, or 3 [677]? ▼ Does the right-angle crossing number remain $\exists\mathbb{R}$-hard for graphs of bounded degree [677]? ▼ What is the complexity of the right-angle crossing number if one (or two) bends

[151] Linear edge bounds are also known for graphs with finite t-polygonal right-angle crossing number, where $t \in \{2, 3\}$ [62, 482].

per edge are allowed, so called RAC_1 (RAC_2) drawings [677]? Does the right-angle crossing number problem remain $\exists\mathbb{R}$-hard if we restrict it to drawings with local crossing number 1, that is, at most one crossing per edge?

Rotational crossing number. Crossing number of graph with rotation (or embedding) system. See entry for crossing number.

Sequential crossing number

DEFINITION: A *sequential drawing* of a family of graphs $\mathcal{G} = (G_i)_{i=1}^k$, with $G_i = (V, E_i)$, is a family of drawings $\mathcal{D} = (D_i)_{i=1}^k$, so that D_i is a drawing of G_i, for all $1 \leqslant i \leqslant k$ and the drawing of $G_i \cap G_{i+1}$ is the same in D_i and D_{i+1}, for all $1 \leqslant i < k$. The *sequential crossing number* of $\mathcal{D}$ is $\sum_{1 \leqslant i \leqslant k} \text{cr}(D_i)$. The *sequential crossing number* of $\mathcal{G}$ is the smallest sequential crossing number of a sequential drawing of $\mathcal{G}$.

REFERENCE: Hamm [343, Sections 4.5, 5.4].

COMPLEXITY: **NP**-complete, since crossing number is the special case $k = 1$, remains **NP**-complete even if $|E_i| \leqslant 5$ for all $1 \leqslant i \leqslant k$ [343, Section 5.4], using [234].

ALSO SEE: Simultaneous crossing number.

Simple crossing number

DEFINITION: The *simple crossing number* of G, $\text{cr}^\times(G)$, is the smallest number of crossings in any drawing of G in which every edge has at most one crossing.[152] If there is no such drawing, we let $\text{cr}^\times(G) = \infty$; the name "simple crossing number" conflicts with the usual notion of a simple drawing (which only requires that every two edges cross at most once).[153] Kainen [469] called a drawing in which every edge has at most one crossing *nearly planar*, Ringel [654] called it a 1-*embedding*; the graphs with $\text{cr}^\times(G) \leqslant 1$ are called 1-*planar* [689].

REFERENCE: Buchheim, Ebner, Jünger, Klau, Mutzel, Weiskircher [152].

COMMENTS: Buchheim, Ebner, Jünger, Klau, Mutzel, Weiskircher [152] introduce this variant to simplify their integer linear program for crossing minimization; the usefulness of the simple crossing number lies in the fact that every graph G has a subdivision G' for which $\text{cr}(G) = \text{cr}^\times(G')$. 1-planar graphs can be 6-colored [141, 656] (in principle) and 7-colored in linear time [183].

COMPLEXITY: Deciding whether $\text{cr}^\times(G) < \infty$ is **NP**-complete [364]. Deciding $\text{cr}^\times(G) \leqslant k$ for 1-planar graphs is also **NP**-complete, even if the graph is 3-connected, and a rotation system is given [76].

[152] Ringel [656] already observed that crossings between two adjacent edges can always be removed in such a drawing.

[153] Another possible name, the 1-planar crossing number, clashes with the k-planar crossing number introduced by Owens. This is doubly unfortunate, since that name suggests a nice generalization beyond 1-planarity.

Relationships: $\text{cr}^\times(G) < \infty$ is equivalent to $\text{lcr}(G) \leqslant 1$. If $\text{cr}^\times(G) < \infty$, then $m \leqslant 4n - 8$ and $\text{cr}^\times(G) \leqslant n - 2$ [134].[154] All sufficiently large 3-connected, 2-crossing critical graphs have simple crossing number 2 [135].

Also see: Local crossing number.

Simple degenerate crossing number. See degenerate crossing number. **Simple degenerate local crossing number.** See local crossing number.
Simple local crossing number. See local crossing number.
Simple quasi crossing number. See quasi crossing number.

Simultaneous crossing number

Definition: A *simultaneous drawing* of a family of graphs $\mathcal{G} = (G_i)_{i=1}^k$, with $G_i = (V_i, E_i)$, is a drawing of $G = (V, E)$ with $V = \bigcup_{i=1}^k V_i$ and $E = \bigcup_{i=1}^k$. In other words, vertices or edges that belong to more than one graph are drawn only once. There are two different types of crossings in the drawing of G: a *proper crossing* is a crossing between two edges e and f that belong to the same graph G_i for some i, otherwise the crossing is a *phantom crossing*. The *simultaneous crossing number* of $\mathcal{G}$, $\text{scr}(\mathcal{G})$, of a family of graphs $\mathcal{G} = (G_i)_{i=1}^k$ is the smallest number of proper crossings in any simultaneous drawing of G as defined above. A proper crossing of two edges e and f counts once for each graph G_i in which it occurs. A family of graphs is *simultaneous planar* if $\text{scr}(\mathcal{G}) = 0$. If we restrict the drawings to be straight-line drawings, we get the *simultaneous geometric crossing number* of G, $\overline{\text{scr}}$. If we restrict the drawings to be convex (all vertices on the boundary of a disk, all edges inside the disk), we get the *convex simultaneous crossing number*.

Reference: Chimani, Jünger, Schulz [199], He, Sălăgean,and Mäkinen [414].

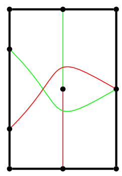
Comments: The crossing number $\text{scr}(\mathcal{G})$ was introduced in Chimani, Jünger, Schulz along with several minimization problems, including the minimization of phantom crossings in an scr-minimal drawing. Geißer [354] studies the number of phantom crossings in a simultaneous embedding of $\mathcal{G}$ (so no proper crossings are allowed). This could be called the *simultaneously planar crossing number*. Chimani, Jünger, Schulz also consider a weighted variant of $\text{scr}(\mathcal{G})$ which is still restricted to counting only proper crossings. One could consider a more general variant in which phantom crossings are assigned weights. The restriction to drawings in which edges belonging to more than one graph are drawn only once is typically known as the *simultaneous embedding with fixed edges (SEFE)* style (an unfortunate name). When defining the crossing number version, the *fixed edges* epithet was dropped. One could consider defining a *free* version in which edges belonging to multiple graphs may be drawn differently for each graph. Families of graphs on the same vertex set are known as multiplex networks in information visualization, and there is research on layout algorithms in that area [307]. The convex simultaneous crossing number is based on an observation by He, Sălăgean,and Mäkinen [414] which implies that it corresponds

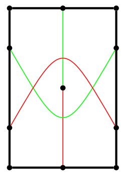
[154] The result that any 1-planar drawing of a graph G on n vertices has at most $n - 2$ crossings is implicit in several papers, e.g. [134, 302], an explicit statement can be found in [233].

to a book drawing in which edges belonging to the same G_i are assigned to the same page. It extends the partitioned book crossing number; it is more powerful, since in an edge in a simultaneous drawing can belong to multiple graphs. In the SEFE model, Grilli [366] introduced what amounts to a local and local pair version of the simultaneous planar crossing number.

COMPLEXITY: **NP**-complete [199].[155] Testing simultaneous planarity is **NP**-complete for three graphs (the complexity of testing simultaneous planarity of two graphs is open) [351]. The simultaneously planar crossing number is **NP**-complete [163, 177, 354], as is the local simultaneous planar crossing number [366]. The convex simultaneous crossing number generalizes the convex crossing number and therefore is **NP**-complete. Testing convex simultaneous planarity is **NP**-complete if the number of graphs k is not bounded [443]; it is open whether the problem remains **NP**-complete for fixed k.

RELATIONSHIPS: $\operatorname{scr}(\mathcal{G}) \leqslant k \operatorname{cr}(G)$, where $G = (V, E)$ with $V = \bigcup_{i=1}^k V_i$ and $E = \bigcup_{i=1}^k E_i$ [199]. The number of phantom crossings in an scr-minimal drawing can be forced to be exponential [199], though it is not clear whether this is true for fixed k; the case $k = 2$ would be particularly interesting. The top picture in the margin shows that for $k = 2$ adjacent edges may have to cross in an embedding; a simple modification shown just below shows that two independent edges may have to cross at least twice.[156]





ALSO SEE: Red/blue crossing number, joint crossing numbers, sequential crossing number.

Simultaneous geometric crossing number. See simultaneous crossing number.
Single-faced crossing number. See joint crossing numbers.

SKEWNESS

DEFINITION: The *skewness* $\operatorname{sk}(G)$ of a graph G is the smallest number of edges whose removal from G leaves a planar graph. We write $\operatorname{sk}_\Sigma(G)$ for the smallest number of edges whose removal leaves a graph embeddable in the surface Σ.

REFERENCE: Guy [378], also Harary [387].

COMMENTS: As discussed in Example 1, skewness does not fit our definition of crossing number, but it is included because of its close relationship with both the crossing and the edge crossing number. Harary [387], in a 1965 paper, asks "how can one find a set of $c(G)$ edges whose removal results in a planar graph", where $c(G)$ is the crossing number of G. In 1972, Guy [378] introduces skewness under a different name,

[155] **NP**-hardness follows since for $k = 1$ scr is the same as cr. **NP**-membership is non-trivial for $k > 1$ [678].

[156] In both examples, there are two graphs: green and red, and the black edges belong to both the green and the red graph; the outer face is forced to be empty. These examples also show that not allowing adjacent or multiple phantom crossings can increase the simultaneous crossing number. The obvious generalizations of these examples, e.g. showing that two edges may be made to cross an arbitrary number of times, are incorrect.

writing "J. Ch. Boland suggested, and Mrs. Sheehan named the idea of the *slimming number*", citing a 1967 Oberwolfach meeting for the planar version, and a 1969 Oberwolfach meeting for the surface version, which he calls the *generic slimming number* in the orientable case, and the *characteristic slimming number* in the non-orientable case. The use of "skew" for non-planar graphs probably traces back to Kuratowski's paper [105]. One also occasionally finds the name *removal number* for skewness [495, 552]. Guy [378] used a bound that was first explicitly stated and proved by Kainen [463], namely that $\mathrm{sk}_\Sigma(G) \geqslant m - g/(g-2)\ (n-2+2\gamma)$, where g is the girth of G and $\gamma = \gamma(\Sigma)$ the genus of the surface Σ. This lower bound, and its special form for the plane, have been rediscovered several times [187, 633]. A graph with skewness at most k is sometimes called *k-skew*, though often the term is applied to a specific drawing of the graph. 1-skew graphs are also often called *almost planar*. Chia and Sim [191] call a graph π-skew if $\mathrm{sk}(G) = \pi(G) := \lfloor m - g/(g-2)\ (n-2)\rfloor$, where g is the girth of G.

Skewness is often discussed in its equivalent form of finding a maximum planar subgraph of a graph [166, 768]. Finding a maximum *induced* planar subgraph corresponds to removing the smallest number of vertices from a graph making it planar.[157]

One could introduce a version of skewness which asks for the fewest number of edges that need to be removed to reduce the crossing number below a certain bound. The corresponding maximum-bounded-crossing subgraph problem was introduced and studied in [211].

Skewness can be based on crossing numbers other than the standard crossing number (planarity) and the surface crossing number (surface embeddability). We are aware of three such variants, based on the convex crossing number, the bipartite crossing numbers, and a local version of the fixed linear crossing number: Kainen [465] introduces the *outerplanar skewness* of a graph, which he writes as $\mu_1(G)$, to give a lower bound on $\mathrm{bkcr}_1(C_m \,\square\, C_n)$; the same concept was called *convex skewness* in [39]. Starting with the bipartite crossing number, one obtains the notion of 2-level skewness (discussed under bipartite crossing number). The authors of [343, Section 5.1] study how many vertices in a fixed 1-page drawing need to be removed to ensure that all edges have at most a given number of crossings; the version for k-page drawings is studied in [23].

Complexity: **NP**-complete [532],[158] and remains **NP**-hard to approximate within some constant factor, even for cubic graphs [306].[159] Testing $\mathrm{sk}(G) \leqslant k$ can be done

[157] In the entry on nodal crossing number, we suggest the name *vertex-skewness* for this notion. The idea is old, dating back to at least Harary [387], but there seems to be no standard name, though *apex number* [486] and *(non-planar) vertex-deletion number* [554] have been used, and there is a notion of *generalized vertex-skewness* [498].

[158] There also is a proof-sketch in [785], and a proof of a more general result in [764]; for a simple proof, see [197].

[159] Cabello's **NP**-completeness proof for cr [161] also works for skewness, yielding another proof of the non-approximability result for cubic graphs.

in linear time for fixed k [487], though no practical algorithm seems to be known. Convex skewness is fixed-parameter tractable [343, Section 5.1].

RELATIONSHIPS: $\mathrm{sk}(G) \leqslant \mathrm{cr}(G)$, and for every k there are 1-skew graphs with crossing number k [216]; there is a bound $\mathrm{cr}(G) \leqslant (3\,\mathrm{sk}(G)^2 + (4n-17)\,\mathrm{sk}(G))/6$ which is tight for infinitely many 1-skew graphs [255, 257]. It is known that $\mathrm{sk}(G) = \mathrm{cr}(G)$ if and only if $\mathrm{lcr}(D) \leqslant 1$ for every crossing-minimal drawing D of G [255]. For surfaces we have $\mathrm{sk}_\Sigma(G) \leqslant \mathrm{cr}_\Sigma(G)$. $\mathrm{sk}(G) \geqslant \gamma(G)$, where $\gamma(G)$ is the orientable genus of G (which equals the bundled crossing number $\mathrm{bc}'(G)$), and the toroidal grid shows that there are toroidal graphs with arbitrary large skewness. $\mathrm{sk}(G) \leqslant \mathrm{ecr}(G)$ (by definition), and a planar grid with an additional edge shows that there are 1-skew graphs with arbitrarily large ecr. $\mathrm{sk}(G) \geqslant m - g/(g-2)\ (n-2)$, where g is the girth of G, and, more generally, $\mathrm{sk}_\Sigma(G) \geqslant m - g/(g-2)\ (n-2+2\gamma(\Sigma))$ [378, 463]. $\mathrm{sk}(G) = O((\Delta\gamma(G)n)^{1/2})$ [258, 260]. If G is 1-planar, then $\mathrm{sk}(G) \leqslant n-2$ [233] (also see Footnote 154). If G has a drawing with c crossings and skewness k, then $\overline{\mathrm{cr}}_{2k}(G) \leqslant c$ [247]. Kainen [464] showed that $\chi(G) \geqslant r$ implies that $\mathrm{sk}(G) \geqslant \mathrm{sk}(K_r)$, an early precursor of Albertson's conjecture; for generalizations of this result, see [225, 226, 614].

VALUES: $\mathrm{sk}(K_n) = \binom{n-3}{2}$, for $n \geqslant 3$ [378, 464]. $\mathrm{sk}(K_{m,n}) = mn - 2(m+n-2)$, for $m, n \geqslant 2$ [378]. The skewness of complete 3-partite and 4-partite graphs is known [187, 191]. The complete k-partite graph $K_{2,\dots,2}$ is π-skew, that is $\mathrm{sk}(K_{2,\dots,2}) = 2(k-1)(k-3)$ [187]. For the skewness of some other complete k-partite graphs, see [191]. $\mathrm{sk}(Q_n) = 2^{n-1}n = 2(2^n-2)$, for $n \geqslant 3$ [216, 378]. Grötsch's graph has skewness 3 [633]. $\mathrm{sk}(C_m \,\square\, C_n) = 2$ for $m = 3$ and $3 \leqslant n \leqslant 4$ and m otherwise, assuming $3 \leqslant m \leqslant n$ [555]. For the skewness of a triangulated $C_m \square C_n$ and its dual, see [554]. If G is triangle-free and contains a Hamiltonian path, then $C_4 \,\square\, G$ is π-skew [633]. $\mathrm{sk}(\overline{C}_n) = (n^2 - 9n + 12)/2$ for $n \geqslant 8$, and $\mathrm{sk}(\overline{C}_n) = 1$ for $n = 7$, and $\mathrm{sk}(\overline{C}_n) = 0$ for $3 \leqslant n \leqslant 6$ [594]. $\mathrm{sk}(C_m + P_n) = (m-2)(n-2) + 1$, where $+$ is the join of two graphs; $\mathrm{sk}(K_n \,\square\, T) = (m-1)\binom{n-2}{2} + \binom{n-3}{2}$, where $m = |V(T)|$, and T is a tree with max-degree at most $2n-4$; $\mathrm{sk}(K_{1,m} \,\square\, P_n) = (m-2)\lfloor (n-1)/2 \rfloor$, for $m \geqslant 2$, $n \geqslant 1$; $\mathrm{sk}(W_m \,\square\, P_n) = (m-2)\lfloor (n-1)/2 \rfloor + \lfloor n/2 \rfloor$ [595]. $\mathrm{sk}(K_{1,m} \,\square\, C_3) = (m-2)$, and $(C_m \,\square\, P_2) \,\square\, P_n$ and $(C_m \,\square\, P_2) \,\square\, C_n$ are π-skew for $m \geqslant 4, n \geqslant 2$, see [190], which contains further results on Cartesian and join products with paths and cycles. For the skewness of generalized Petersen graphs, see [83, 185, 187–189, 552], for a family of graphs generalizing the Heawood graph, see [731]. Upper bounds for some network topologies can be found in [214]. For the convex skewness of circulant graphs, see [40]. For a **surface** Σ with Euler characteristic χ, we have $\mathrm{sk}_\Sigma(K_n) = n(n-7)/2 + 3\chi$ for sufficiently large n.[160] For an orientable surface Σ with genus $\gamma = \gamma(\Sigma)$, we have $\mathrm{sk}_\Sigma(K_{m,n}) = (m-2)(n-2) - 4(2-2\gamma)$ if $m \equiv n \equiv 0 \bmod 2$ for sufficiently large m and n, and $\mathrm{sk}_\Sigma(Q_n) = (n-4)2^{n-1} + 4(2\gamma - 1))$ for sufficiently

[160] The lower bound is an application of Euler's formula, as observed in [378]. The upper bound is harder, it follows from a result by Jungerman and Ringel [461, Theorem 1.2] who show that for sufficiently large n there is a triangulation of Σ with $\binom{n}{2} - ((n-3)(n-4)/2 - 6\gamma)$ edges, where $\gamma = 1 - \chi/2$ is the orientable genus of Σ. For non-orientable surfaces the lower bound follows from [657].

large n [378].

Open Questions: What is $\text{sk}_\Sigma(K_{m,n})$ for an orientable (or non-orientable) surface Σ? For orientable Σ, Guy [378] conjectures $\text{sk}_\Sigma(K_{m,n}) = (m-2)(n-2)-4(2-2\gamma)$ for all sufficiently large m and n. ▼ Chia and Lee [188] conjectured that $\text{sk}(\text{GP}(4k,k)) = k+2$ for odd $k \geqslant 3$, where $\text{GP}(n,k))$ is the generalized Petersen graph. The conjecture was mostly settled in [189], but cases $k=5$ and $k=7$ remain open. ▼ Orthaber [593] conjectures that every crossing-minimal drawing of K_n contains a set of $\text{sk}(K_n)$ edges whose removal makes the drawing crossing-free (leaving the maximal possible number $3n-6$ of edges). ▼ Chia and Sim [190] ask whether $\text{sk}(K_{1,m} \,\square\, C_n) = (m-2)\left(\lfloor\frac{n-1}{2}\rfloor + 1\right)$?

Also see: Crossing number, edge crossing number, uncrossed crossing number, bipartite crossing number (2-level skewness).

Space crossing number

Definition: A *spatial drawing* of a graph G is a continuous embedding of G in $\mathbb{R}^3$, it is *rectilinear* if edges are line segments. A *spatial crossing* is any (straight) line that crosses four[161] vertex-disjoint edges. The *space crossing number* of G, $\text{space-cr}(G)$, is the smallest number of spatial crossings in any spatial drawing of G. The *rectilinear space crossing number*, $\text{space-}\overline{\text{cr}}(G)$, is the smallest number of spatial crossings in any rectilinear spatial drawing of G.

Reference: Bukh, Hubard [156].

Comments: For a notion of crossing number for geometric hypergraphs, see [54, 55].

Complexity: Open.

Relationships: $\text{space-cr}(G) \leqslant \binom{\text{cr}(G)}{2}$; for every k there is a graph G with $\text{space-cr}(G) = 0$ and $\text{cr}(G) \geqslant k$ [156]. There is a crossing lemma, $\text{space-cr}(G) \geqslant m^6/(cn^4 \log^2 n)$ for $c = 4^{179}$, and $n = |V|$, $m = |E|$ as long as $m \geqslant 4^{41}n$ [156].

Open Questions: Bukh and Hubbard ask whether graphs with $\text{space-cr}(G) = 0$ are minor-closed and whether $\text{space-cr}(G) = 0$ is equivalent to $\text{space-}\overline{\text{cr}}(G) = 0$. They conjecture negative answers in both cases.

Also see: Grid crossing number.

Spherical crossing number. See geodesic crossing number.

Spine crossing number

Definition: The *spine crossing number*[162] of G in a book of k pages is the smallest number of crossings between edges and the spine in a k-page topological book embedding of G. In a *topological book embedding* edges are allowed to cross the spine.

[161] Bukh and Hubbard also, in passing, mention the possibility of counting lines that cross *three* edges.

[162] This crossing parameter has never been named, the closest is the occasional use of the phrase *crossings over the spine*. It has also been studied as a minimization problem for upward planar drawings [549].

Reference: Based on Miyauchi [561].

Comments: Miyauchi gives an upper bound on the number of spine crossings for K_n in a 3-page book (also see discussion in the entry on book crossing number).

Complexity: Open.

Relationships: Any graph $G = (V, E)$ has a $k+1$-page topological book embedding in which each edge crosses the spine at most $\lceil \log_k |V| \rceil$ times, so the spine crossing number of a graph $G = (V, E)$ in a $(k+1)$-page book is at most $|E| \lceil \log_k |V| \rceil$ [286, 560], and this bound is tight [287]. If G is planar, then the spine crossing number in a 2-page book is at most $|E|$, by a result of Kaufmann and Wiese [485].[163]

Values: Graphs with spine crossing number zero in 1-page book are exactly the subgraphs of outerplanar graphs, and in 2-page books, the subgraphs of planar Hamiltonian graphs.

Also see: Book crossing number

Stable crossing number

Definition: The *stable crossing number* of G with parameter k is $\mathrm{cr}_\Sigma(G)$ where $\Sigma = S_{\gamma(G)-k}$ and $\gamma(G)$ is the (orientable) genus of G.

Reference: Kainen [467].

Comments: Kainen's motivation in introducing the stable crossing number seems to have been to investigate infinite families of graphs in surfaces in which they are nearly embeddable and show that this can lead to small constant (stable) crossing numbers [467, Abstract].

Complexity: **NP**-complete even for $k = 1$, since determining the planar crossing number of a toroidal graph is **NP**-complete, e.g. by the result of Cabello, Mohar [163].

Values: $4k \leqslant \mathrm{cr}_\Sigma(Q_n) \leqslant 8k$ for $\Sigma = S_{\gamma(Q_n)-k}$ and $0 \leqslant k \leqslant \gamma(Q_n)$ [467]. $\mathrm{cr}_\Sigma(Q_n \square K_{4,4}) = 4k$, where $0 \leqslant k \leqslant 2^n$, $\Sigma = S_{\gamma(Q_n \square K_{4,4})-k}$ [472]; for a generalization of this result, see [631, Theorem 9].

Open Questions: Kainen [467] conjectured $\mathrm{cr}_\Sigma(Q_n) = 8k$ for $\Sigma = S_{\gamma(Q_n)-k}$.

String crossing number

Definition: The *string crossing number* of G, str-cr(G), is the smallest number of crossings in any string drawing of G minus $|E(G)|$. A *string drawing* of G is a set of curves $(c_v)_{v \in V(G)}$ so that c_u and c_v cross for every edge $uv \in E(G)$.[164]

Reference: Bokal, Czabarka, Székely, Vrťo [136].

[163] Kaufmann and Wiese show that every planar graph has a 2-page drawing in which every edge crosses the spine at most once. This suggests the notion of a *local spine crossing number*.

[164] Crossings between c_u and c_v are allowed even if there is no edge uv; so a string drawing is not a string representation in the strict sense in which a string graph is the intersection graph of a set of curves in the plane.

COMMENTS: Bokal, Czabarka, Székely, Vrťo [136] also suggest the *independent string crossing number* (they call it the faithful crossing number) and the *pair string crossing number*; string graphs correspond to graphs of pair string crossing number 0. Richter, Thomassen [649] study a similar notion for closed curves in their proof that $\mathrm{cr}(C_5 \,\square\, C_5) = 15$. One could also imagine a *local string crossing number*, which counts crossings along the strings. For (strict) string representations, this has been studied [204, 514].

COMPLEXITY: Open.

RELATIONSHIPS: $\text{str-cr}_\Sigma(G) \leqslant 4\,\mathrm{mcr}_\Sigma(G)$ [136].

Surface crossing number. See crossing number.

t-CIRCLE CROSSING NUMBER

DEFINITION: A *t-circle drawing* of a graph G is a drawing in which the vertices of G lie on t disjoint circles which are empty; that is, the face bounded by each circle contains no part of G or any other circle. The *t-circle crossing number*, $\mathrm{cr}_{t\circ}(G)$, of a graph G is the smallest number of crossings in a t-circle drawing of G. For a t-partite graph G with a fixed partition, a *t-partite circle drawing* is a t-circle drawing of G in which the vertices of each part of G lie on the same, distinct circle. The *t-partite circle crossing number*, $\mathrm{cr}_{\textcircled{t}}(G)$, is the smallest number of crossings in a t-partite circle drawing of G (for a given partition). The 2-partite circle crossing number is also known as the *bipartite cylindrical crossing number*.

REFERENCE: Duque, González-Aguilar, Hernández-Vélez, Leaños, Medina [271]. The bipartite cylindrical crossing number was introduced by Ábrego, Fernández-Merchant, and Sparks [16], there written as $\mathrm{cr}_{\circledcirc}$; the tripartite circle crossing number was introduced in [167].

COMMENTS: The 1-circle crossing number is the same as the 2-page crossing number. 2-partite circle drawings are more commonly known as *bipartite cylindrical drawings* [650], and this is where this family of crossing numbers originated. From there the cylindrical crossing number, and then the t-circle crossing number developed. The notation $\mathrm{cr}_{\textcircled{t}}$ for the t-partite circle crossing number was introduced in [167]. If we fix the cyclic order of the vertices on the circles, we obtain the map crossing number. A practical crossing minimization studied in [720] can be viewed as a variant of the 2-circle crossing number problem with constraints on lengths of edges.

COMPLEXITY: Testing whether $\mathrm{cr}_{t\circ}(G) = 0$ is **NP**-complete for any fixed $t \geqslant 2$, $t \neq 3$ [271].[165] Testing $\mathrm{cr}_{\textcircled{t}}(G) \leqslant k$ is **NP**-complete for every fixed $t \geqslant 2$.[166]

RELATIONSHIPS: $\mathrm{cr}_{1\circ}(G) = \mathrm{bkcr}_2(G)$, and $\mathrm{cr}_{2\circ}(G) = \mathrm{cr}_{\circledcirc}(G)$ (by definition).

VALUES: $\mathrm{cr}_{\circledcirc}(K_n) = Z(n)$ [2]. $\mathrm{cr}_{\textcircled{2}}(K_{n,n}) = n\binom{n}{3}$ [650], and $\mathrm{cr}_{\textcircled{2}}(K_{m,n})$ is known [16]. $\mathrm{cr}_{\textcircled{3}}(K_{2,2,n}) = 3\lceil n^2/2\rceil - n - 3$, and upper and lower bounds on $\mathrm{cr}_{\textcircled{3}}(K_{m,n,p})$ are known [167, 168]

[165] The reduction is from $\mathrm{bkcr}_t(G) = 0$, which is not known to be hard for $t = 3$. The problem remains hard if $= 0$ is replaced with $\leqslant k$ for any fixed k.

[166] Since the special case of $t = 2$, the bipartite cylindrical crossing number, is **NP**-complete.

Open Questions: Can $\text{cr}_{ⓣ}(G) = 0$ be decided in polynomial time, for fixed, or unbounded t?

Also see: Cylindrical crossing number, radial crossing number, map crossing number.

t-polygonal crossing number

Definition: The *t-polygonal crossing number* of G, $\overline{\text{cr}}_t(G)$, is the smallest number of crossings in a straight-line drawing of G in which every edge is allowed to consist of up to t line segments.

Reference: Bienstock [120].

Comments: Introduced by Bienstock [120] to bridge the gap between cr and $\overline{\text{cr}}$. In the area of graph drawing, t-polygonal drawings would also be called $(t-1)$-bend drawings (each edge having at most $t-1$ bends).

Complexity: $\exists\mathbb{R}$-complete [120] for $t = 1$, see [673] for $\exists\mathbb{R}$. Open for $t > 1$.

Relationships: $\overline{\text{cr}}_1(G) = \overline{\text{cr}}(G)$ (by definition), $\overline{\text{cr}}_2(G) \leqslant 2\,\text{cr}(G)^2$ [121]. Let $t(k)$ be the smallest t so that $\overline{\text{cr}}_t(G) = \text{cr}(G)$ for all G with $\text{cr}(G) \leqslant k$. Then $t(k) = \Theta(k^{1/2})$ [120].

Also see: Rectilinear crossing number.

Tile crossing number

Definition: A *tile* T is a graph $G = (V, E)$ together with two disjoint sequences $L = \{u_1, \ldots, u_k\}$ and $R = \{v_1, \ldots, v_k\}$ of vertices in V. A *tile drawing* of T is a drawing of T in the unit square with all vertices of L on the left boundary of the square in order, that is, u_i above u_{i+1}, and all vertices of R on the right boundary with v_i above v_{i+1}. The *tile crossing number* of T is the smallest number of crossings in a tile drawing of T. T^2 is the tile obtained from T by placing two copies of T next to each other and identifying v_i of the left copy with u_i of the right copy, for $1 \leqslant i \leqslant k$. This defines tiles T^n for arbitrary integer powers n. The *average crossing number* of T is the limit of the tile crossing number of T^n divided by n as n goes to infinity.

Reference: Pinontoan, Richter [635].

Comments: Pinontoan and Richter [635] do not require that $|L| = |R|$, but they mostly study tiles they call self-compatible for which this is the case, since for those tiles the average crossing number is defined. They can show that the average crossing number of a tile always exists. The tile crossing number is rather specific to constructions of crossing-critical graphs. It bears similarity to bipartite and convex crossing number, but differs from them by allowing additional vertices within the square. In that respect, it resembles the anchored crossing number most closely. For an extension of the tile crossing number, the panel crossing number, see [473].

Complexity: The tile crossing number is **NP**-complete,[167] and remains **NP**-complete for twisted planar tiles (tiles which become planar after twisting one of the boundaries) [422]. If $L \cup R = V$, then the problem is in polynomial time. The complexity

[167] The regular crossing number is a special case for $k = 0$.

of the average crossing number is open, but Dvořák and Mohar [273] show that it can be approximated in exponential time in the absolute error.

RELATIONSHIPS: tile-cr(T^n) $\leqslant n$ tile-cr(T) [635]. Let $o(T^n)$ be the graph constructed from T^n by identifying L and R of the tile T^n (in order). Then the average crossing number of T equals $\lim_{n\to\infty} \mathrm{cr}(o(T^n))/n$ [635].

OPEN QUESTIONS: Pinotoan and Richter [635] conjecture that if the average crossing number of T equals tile-cr(T), then there is an N so that $\mathrm{cr}(o(T^n))/n =$ tile-cr(T) for all $n \geqslant N$. ▼ Dvořák and Mohar [273] conjecture that the average crossing number of a tile is always a rational number.

ALSO SEE: Anchored crossing number (under fixed linear crossing number), bipartite crossing number, convex crossing number.

Toroidal crossing number. See crossing number.
Toroidal geodesic crossing number. See geodesic crossing number.

TRIPLE CROSSING NUMBER

DEFINITION: The *triple crossing number* of G, triple-cr(G), is the smallest number of triple crossings (a point in which three edges cross) in a drawing in which there are only triple crossings. We assume that there are no self-crossings, no crossings between adjacent edges, and that independent edges cross at most once and do not touch. The triple crossing number may be infinite.

REFERENCE: Tanaka, Teragaito [733].

COMMENTS: As the definition shows, Tanaka, Teragaito [733] introduce a very restrictive version of a triple crossing number (which more accurately could be called the simple triple crossing number). In this version, triple-cr(K_5) $= \infty$, since crossings have to occur between independent edges (forcing at least 6 endpoints in a non-planar graph). However, it it easy to give a drawing of K_5 with two triple crossings if crossings between adjacent edges are allowed. Another condition that could be relaxed is that independent edges cross at most once. Tanaka and Teragaito in passing also introduce the *n-fold crossing number*. Harborth [394, 398] studied multiple crossings (see Footnote 89).

COMPLEXITY: Open.

RELATIONSHIPS: $\mathrm{cr}(G) \leqslant 3$ triple-cr(G) (perturb triple crossings). The triple crossing number is not monotone (for example, triple-cr($K_{4,4}$) $= \infty$, while triple-cr($K_{6,4}$) $=$ 4 [733].

VALUES: Tanaka and Teragaito [733] determine the triple crossing number for all complete k-partite graphs; in particular, they show that triple-cr(K_n) $= \infty$ for $n \geqslant$ 5, and triple-cr($K_{m,n}$) $= \infty$ for non-planar $K_{m,n}$ with the following exceptions: triple-cr($K_{3,3}$) $=$ triple-cr($K_{3,4}$) $= 1$, triple-cr($K_{3,6}$) $= 2$, and triple-cr($K_{4,6}$) $= 4$.

ALSO SEE: Degenerate crossing number.

Tutte crossing number. See algebraic crossing number.

Uncrossed crossing number

Definition: A collection of drawings of G is *uncrossed* if every edge of G is crossing-free in at least one of the drawings. The *uncrossed crossing number*, $\mathrm{ucr}(G)$, of G is the smallest $\sum_{D\in\mathcal{D}} \mathrm{cr}(D)$ over all uncrossed collections $\mathcal{D}$ of drawings of G. If $\mathcal{D}$ is restricted to consists of at most k drawings, one gets the *uncrossed crossing number in at most k drawings*, denoted $\mathrm{ucr}_k(G)$.

Reference: Hliněny and Masařík [423].

Comments: Hliněny and Masařík [423] also introduce the new parameter $\mathrm{ounc}(G)$, the *crossing-optimal uncrossed number*, the smallest k for which $\mathrm{ucr}(G) = \mathrm{ucr}_k(G)$.

Complexity: Testing whether $\mathrm{ucr}(G) \leqslant m$ and $\mathrm{ucr}_2(G) \leqslant m$ are both **NP**-complete and testing $\mathrm{ucr}_k(G) \leqslant m$ is fixed-parameter tractable in parameter m [423].

Relationships: For every m there is a graph G with $\mathrm{cr}(G) = 1$ and $\mathrm{ucr}(G) \geqslant m$ [423]. For every m there is a graph G with $\mathrm{ucr}(G) = 2$ and $\mathrm{ounc}(G) \geqslant m$ [423]. There is a crossing lemma: $\mathrm{ucr}(G) \geqslant m^4/(87n^3)$ [423].

Values: $\mathrm{ucr}(K_7) = 3$, and $\mathrm{ucr}(K_n) = \Theta(n^5)$ [423].

Also see: Non-crossing edge number (under edge crossing number).

Upward crossing number

Definition: A drawing is *monotone* if every vertical line in the plane intersects each edge at most once. The *upward crossing number* of a directed acyclic graph G is the smallest number of crossings in a monotone drawing of G in which all edges point in the same direction. We write $\text{mon-cr}_{\preceq}(G)$, where $\preceq$ is the partial ordering induced by the orientation of G. For mixed graphs, containing both directed and undirected edges, the *mixed upward crossing number* is the smallest number of crossings in a monotone drawing of G in which all *directed* edges point in the same direction.

Reference: Based on Eiglsperger, Kaufmann [282], also Chimani, Zeranski [202].

Comments: One of the monotone crossing numbers. The upward crossing number corresponds to the layer-free upward crossing minimization problem [196]. Eiglsperger and Kaufmann define the notion of a crossing number for a (mixed) upward planarization, calling it the *(mixed) upward crossing minimal problem.* Chimani and Zeranski [202] then use term *upward crossing number.* The upward crossing number could also be called the *directed crossing number* or the *hierarchical crossing number*; the latter term has been used in the context of leveled graphs [579]. Generalizing to recurrent hierarchies, one could define a clockwise crossing number (see cyclic level crossing number).

Complexity: Even testing whether a graph is *upward planar*, that is, has upward crossing number 0, is **NP**-complete [349]. See [203] for a survey on upward planarity testing, and [202] for a survey on exact upward crossing minimization.

Relationships: $\text{mon-cr}(G) \leqslant \text{mon-cr}_{\preceq}(G)$, where $\preceq$ is the partial ordering induced by the orientation of G. The bimodal crossing number is a lower bound on $\text{mon-cr}_{\preceq}(G)$.

Open Questions: Computing the upward crossing number remains **NP**-complete even if we restrict the number of levels at which vertices can be placed: for two levels, the **NP**-complete bipartite crossing number is a special case. Is upward planarity fixed-parameter tractable if the parameter is the number of levels?

Also see: Monotone crossing numbers, bimodal crossing number, bipartite crossing number, clockwise crossing number (under cyclic level crossing number).

Weighted crossing number

Definition: The *weighted crossing number*, $\mathrm{cr}(D, w)$ of a drawing D of a graph $G = (V, E)$ with *weights* $w : E^2 \to \mathbb{R}_{\geqslant 0}$, is defined as $\sum_{e,f \in E} w(e, f) \cdot\ i_D(e, f)$, where $i_D(e, f)$ is the number of crossings between e and f in D. The *weighted crossing number*, $\mathrm{cr}(G, w)$ is the minimum of $\mathrm{cr}(D, w)$ over all drawings of G. The *weighted rectilinear crossing number*, $\overline{\mathrm{cr}}(G, w)$ is the minimum of $\mathrm{cr}(D, w)$ over all straight-line drawings D of G.

Reference: Jackson, Ringel [457], Mohar [563], Schaefer, Sedgwick, Štefankovič [678]. Biedl, Chimani, Derka, Mutzel [119] for the rectilinear variant.

Comments: Assigning weights to edges (as opposed to edge pairs) is an old idea. Integer weights are typically interpreted as parallel copies of simple edges; for many crossing number variants, it is easy to show that k parallel edges correspond to a single edge of weight k. This argument may have first occurred in a paper by Kainen [463] in which he shows that $\mathrm{cr}_\Sigma(G) \leqslant k^2 \,\mathrm{cr}_\Sigma(G')$ where G is a graph with at most k parallel edges between every pair of vertices, and G' is the underlying simple graph of G. If G has exactly k parallel edges between every pair of vertices, then equality holds. This shows, as Scheinerman and Ullman [683, Theorem 7.1.4] observed, that the *fractional crossing number* equals the crossing number and thus is of no independent interest. Jackson and Ringel explicitly introduce the problem for determining the weighted crossing number of complete bipartite graphs. Some crossing number variants, like independent crossing number and the crossing number of abstract topological graphs, can be considered special cases of the weighted crossing number. Mohar and Stephen [568] study the expected value of randomly weighted graphs and derives a crossing lemma for this case. A special case based on partitioning edges into three (but it could be more) classes is introduced in [47] as *hierarchical partial planarity*.

Complexity: **NP**-complete [678].[168] The problem remains **NP**-complete even if the underlying graph is a $K_{3,n}$ [119]. The weighted rectilinear crossing number problem is $\exists\mathbb{R}$-complete (since $\overline{\mathrm{cr}}$ is $\exists\mathbb{R}$-complete).

Also see: Crossing number of abstract topological graph.

[168] This assumes w is considered part of the input (so weights can be large). **NP**-hardness follows from Garey, Johnson [348] since the regular crossing number is a special case. **NP**-membership is harder.

Wire crossing number

Definition: A *layout* is a partition of a rectangle (the chip area) into two types of smaller rectangles: *modules*, where wires end, and *regions*, through which wires are routed. Vertices are located on the boundary of modules. An edge between two vertices has associated with it the *netlist*, the list of regions it passes through (in the given order) to connect its endpoints. The *wire crossing number* is the smallest number of crossings with which all the netlists can be realized.

Reference: Based on Groenveld [367]. Also, Chen and Lee [182].

Comments: The study of crossings numbers for VLSI layouts traces back to the work of Leighton [528],[169] but after a while more specialized models developed.[170] The one described above is closest in spirit to Groenveld's description [367] and Chen and Lee's later version [182]. The name *wire crossing number* was not used in those papers, but first appears, as far as we know, in [476], a paper that describes a slightly different model, and introduces the notion of *hypercrossings*, crossings of hyperedges (Groenveld [367] also considers hyperedges, multi-terminal nets in his terminology, but deals with them differently). The wire crossing number as defined above is not particularly interesting as a graph crossing number, because the topology of the edges does not change (with respect to the modules). Any two edges cross at most once, and their isotopy class determines whether they have to cross or not. We decided to include the wire crossing number, since it contains aspects of several other crossing numbers: it is really a special case of the map crossing number or the constrained crossing number in which the isotopy type of each edge is fixed. The idea of routing along given tracks (the netlists) is also similar to the Metro-line crossing number. Marek-Sadowska and Sarrafzadeh [540] also consider what Chen and Lee [182] call the *unconstrained* crossing minimization problem in which the isotopy type of the edges is not fixed. Both papers claim a polynomial time algorithm for the problem in this case, which is unlikely, since the unconstrained version of the problem is equivalent to computing a map crossing number, which is **NP**-complete [621].[171]

Complexity: Polynomial time [367].

Relationships: Map crossing number, constrained crossing number, Metro-line crossing number.

***x*-monotone crossing number.** See monotone crossing numbers.

[169] See Remark 5 on an early forerunner.

[170] We should mention that Hotz [444, Section 3.6] develops a notion of (hyperedge) crossing number for circuit layout and poses at least one interesting special problem for the bipartite crossing number. Unfortunately, he works over an abstract notion of circuits introduced using category theory, which makes his text unnecessarily hard to read. His notation for the crossing number of a circuit computing a Boolean function f is $L_V(f)$.

[171] The two papers really show that one can efficiently find a drawing in which every pair of edges crosses at most once. Such a drawing need not be crossing-minimal, of course.

4 Some New Questions on Crossing Numbers

Several open questions have already been embedded in the text above, we don't want to repeat these here. The following questions, as far as we know, are new.

Several authors have studied the parity of crossing numbers of complete graphs, Guy [375], Kleitman [491, 492],[172] Archdeacon, Richter, and others, but how hard is it to compute?

Question 2. What is the complexity of determining $\mathrm{cr}(G) \bmod 2$?

Hliněny and Thomassen [427] show that the problem is **NP**-complete under Turing (Cook) reductions; it remains open whether the problems is **NP**-complete under many-one (Karp) reductions.

It's common knowledge that adjacent crossings don't matter, so the following should be easy:

Question 3. Is $\mathrm{cr}(K_n) = \mathrm{cr}_-(K_n)$?

In reality, we do not even know whether there is a good bound on the total number of crossings in a cr_--minimal drawing of K_n. There are many similar open questions for other crossing numbers, for example, $\mathrm{pcr}(K_n) = \mathrm{cr}(K_n)$ and $\mathrm{ocr}_+(K_n) = \mathrm{ocr}(K_n) = \mathrm{iocr}(K_n)$. For monotone crossing numbers some progress has been made [87].

We know that the $\overline{\mathrm{cr}}$ problem is $\exists\mathbb{R}$-complete so, as Bienstock realized, optimal drawings can require exponential precision in the coordinates. What happens if we only have polynomial precision available?

Question 4. Is there a function f so that G has a straight-line grid drawing on a $O(n) \times O(n)$ grid (that is, vertices are grid points) with at most $f(\overline{\mathrm{cr}}(G))$ crossings?

We can broaden the question by using the grid crossing number: is there a function f so that $\overline{\mathrm{cr}}_{\#}(G, n^k, 2) \leqslant f(\overline{\mathrm{cr}}(G))$ for some k?

One can also consider games as the source of crossing number definitions; here is a pen and paper crossing game based on an idea from [576]:

Question 5. Suppose we arrange $2n$ points on the boundary of a disk; players alternate connecting pairs of points; crossing your own edge costs two points, crossing your opponent's edge costs one point. Who wins?

A recent computer game [99] suggests a concrete notion of a game crossing number:

Question 6. Two players alternate placing vertices of a graph (a C_n in the original game) for a straight-line drawing of the graph in the plane. A vertex once placed cannot be moved. The first player attempts to minimize the number of crossings, the second player tries to maximize them. What is the largest number of crossings the second player can force in the final drawing?

[172] Kleitman's parity argument was anticipated by Guy and Harary [380] who observed that the parity of the crossing number of a drawing does not change if all vertices have even degree (they credit this observation to Zeeman).

By Fary's theorem, $\mathrm{cr}(G) = 0$ implies that $\overline{\mathrm{cr}}(G) = 0$. Does Fary's theorem generalize to other crossing numbers? For most, it is either an immediate consequence (pair crossing number, local crossing number) or irrelevant (bipartite and book crossing number, for example). The answer is "no" for the simultaneous crossing number, since $\mathrm{scr}(T, P) = 0$ for any tree T and path P, and there are trees and paths for which $\overline{\mathrm{scr}}(T, P) > 0$ [53]. What about metric surfaces other than the plane? To take the easiest open example:

Question 7. If a graph can be embedded in a torus, does it always have a geodesic embedding in the torus?

We assume the torus is a standard geometric torus with the natural distance metric inherited from 3-dimensional space. There are related results by de Verdière [224], Mohar [564] and Hubard, Kaluža, de Mesmay and Tancer [450] for other metrics.

While it's been conjectured that $\widetilde{\mathrm{cr}}(K_n) = \overline{\mathrm{cr}}(K_n)$, we do not even know whether the rectilinear crossing number can be bounded in the pseudolinear crossing number.

Question 8. Is there a function f so that $\overline{\mathrm{cr}}(G) \leqslant f(\widetilde{\mathrm{cr}}(G))$ for all graphs G?

Remark 10 (More Open Questions). Lists of open questions can also be found in the book by Brass, Moser, and Pach [147, Chapter 9] and articles by Pach and Tóth [612], Brandenburg, Eppstein, Goodrich, Kobourov, Liotta, and Mutzel [142, Section 6.3], Richter and Salazar [644], and Archdeacon [56]. For biplanar crossing numbers, see [229]. *Warning:* Some of these questions are no longer open. ⧫

Acknowledgments

Thanks to Markus Chimani, Jan Kynčl, Gelasio Salazar and Paul Kainen for corrections and comments, Heiko Harborth for supplying me with many sources which I had missed, and the anonymous referees for many helpful suggestions. Finally, thanks are due to the DePaul librarians who have tracked down many papers for me which are not available online.

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