Distance-regular graphs

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Abstract

This is a survey of distance-regular graphs. We present an introduction to distance-regular graphs for the reader who is unfamiliar with the subject, and then give an overview of some developments in the area of distance-regular graphs since the monograph 'BCN' [Brouwer, A.E., Cohen, A.M., Neumaier, A., *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989] was written.

Contents

1 Introduction

Distance-regular graphs are graphs with a lot of combinatorial symmetry, in the sense that given an arbitrary ordered pair of vertices at distance h, the number of vertices that are at distance i from the first vertex and distance j from the second is a constant (i.e., does not depend on the chosen pair) that only depends on h, i, and j. Biggs introduced distance-regular graphs, by observing that several combinatorial and linear algebraic properties of distance-transitive graphs were holding for this wider class of graphs, see Biggs' monograph [48] from 1974. Well-known examples are the Hamming graphs and the Johnson graphs, as these graphs link the subject of distance-regular graphs to coding theory and design theory, respectively. But there are many more interesting links to other subjects, such as finite group theory (and distance-transitive graphs), representation theory, finite geometry, association schemes, and orthogonal polynomials. Moreover, distance-regular graphs are frequently used as test instances for problems on general graphs and other combinatorial structures, such as problems related to random walks and from combinatorial optimization. An example is Hoffman's (unpublished; see [85, Thm. 3.5.2]) coclique bound, which was first proved by Delsarte [189, p. 31] for distance-regular graphs with diameter two (also known as strongly regular graphs), as an example of his linear programming method. Distance-regular graphs have applications in several fields besides the already mentioned classical coding and design theory, such as (quantum) information theory, diffusion models, (parallel) networks, and even finance.

In this survey of distance-regular graphs, we give an overview of some developments in the area of distance-regular graphs since the monograph 'BCN' by Brouwer, Cohen, and Neumaier [78] from 1989 was written. This influential monograph, which is almost like an encyclopedia of distance-regular graphs, inspired many researchers to work on distance-regular graphs, such as the authors of this survey. Since then, many papers have been written, many more than the ones we will discuss in this overview. We intend to discuss the most relevant developments of the past twenty-seven years, realizing that 'most relevant' is quite subjective. Perhaps we should say that we give our personal view on the past twenty-seven years. The same is true when we discuss the major open problems in the area. A recent major breakthrough is the proof of one of the Bannai-Ito conjectures made in the influential monograph by Bannai and Ito [38] from 1984, i.e., the one that states that there are finitely many distance-regular graphs with given valency (at least three). Just as important is the theorem stating that there are finitely many non-geometric distance-regular graphs with both valency and diameter at least three and smallest eigenvalue at least a given number; a generalization of a well-known result about strongly regular graphs. The classification of tridiagonal pairs is an example of an important recent breakthrough in algebraic combinatorics that is completely inspired by the major (still) open problem of classifying the Q-polynomial distance-regular graphs. The construction of the twisted Grassmann graphs, that is, of this family of strange examples that were not expected to be in the picture, gave a better perspective on how difficult this classification problem really is. It seems to suggest that the problem cannot be solved just by algebraic methods. In addition, we need to better understand geometric distance-regular graphs.

This survey is organized as follows. After this brief introduction, we present an introduction to distance-regular graphs for the reader that is unfamiliar with the subject. We then present the classical examples of distance-regular graphs, and an overview of the most important constructions since 'BCN' [78]. In Section 4, we give more necessary and advanced background for the remaining part of the paper. We then treat several subjects in Sections 5-14, for example Q-polynomial distance-regular graphs, the Terwilliger algebra, the Bannai-Ito conjecture, geometric distance-regular graphs, and spectral characterizations. In Section 15, we discuss important applications of distance-regular graphs, namely in combinatorial optimization and in the area of random (classical and quantum) walks (which model diffusion models, dynamic stock portfolios, and the abelian sandpile, for example). In Section 16, we then discuss some miscellaneous topics, and in Section 17 we report progress on the 'feasibility' and 'uniqueness' of the intersection arrays that were listed in the tables of parameter sets of distance-regular graphs in 'BCN' [78]. We conclude with a section on open problems and some directions for future research.

Note that we will focus our attention on distance-regular graphs with diameter at least three. We do not completely exclude strongly regular graphs (the diameter two case), but we are of the opinion that they form a subject of their own. A separate survey of strongly regular graphs would therefore be warmly welcomed. For some information we refer to the recent book by Brouwer and Haemers on spectra of graphs [85, Ch. 9] and the paper by Cohen and Pasechnik [132]. Also bipartite distance-regular graphs with diameter three

form a separate subject. These graphs are equivalent to symmetric designs, for which we refer to the monograph by Ionin and Shrikhande [346].

2   An introduction to distance-regular graphs

In this section we intend to introduce some basics about distance-regular graphs to the reader that is unfamiliar with the topic. This includes some basic proofs and questions to give some (first) flavors of the area of distance-regular graphs.

2.1   Definition

Let Γ denote a simple, undirected, connected graph, with vertex set $V = V_\Gamma$ of size $v = |V|$. Whenever there is an edge between two vertices x and y, we say that x is *adjacent* to y, or that x and y are *neighbors*, use the notation $x \sim y$, and denote the edge by xy. The *distance* in the graph between two vertices x and y is denoted by $d(x, y) = d_\Gamma(x, y)$, and is given by the length of the shortest path between x and y. The *diameter* of the graph is $D = D_\Gamma = \max_{x,y \in V} d(x, y)$. The set of vertices at distance i from a given vertex $z \in V$ is denoted by $\Gamma_i(z)$, for $i = 0, 1, \ldots, D$. The *distance-i graph* Γ_i is the graph with vertex set V, where two vertices x and y are adjacent if and only if $d_\Gamma(x, y) = i$. A graph is called *bipartite* if the vertex set can be partitioned into two parts such that every edge has one end (vertex) in each part.

A connected graph Γ with diameter D is called *distance-regular* if there are constants c_i, a_i, b_i — the so-called *intersection numbers* — such that for all $i = 0, 1, \ldots, D$, and all vertices x and y at distance $i = d(x, y)$, among the neighbors of y, there are c_i at distance $i - 1$ from x, a_i at distance i, and b_i at distance $i + 1$. It follows that Γ is a regular graph with valency $k = b_0$, and that $c_i + a_i + b_i = k$ for all $i = 0, 1, \ldots, D$. By these equations, the intersection numbers a_i can be expressed in terms of the others, and it is standard to put these others in the so-called *intersection array*

$$\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}.$$

Note that $b_D = 0$ and $c_0 = 0$ are not included in this array, whereas $c_1 = 1$ is included (note that all numbers in the intersection array are positive integers). Also the number of vertices can be obtained from the intersection array. In fact, every vertex has a constant number of vertices k_i at given distance i, that is, $k_i = |\Gamma_i(z)|$ for all $z \in V$. Indeed, this follows by induction and counting the number of edges between $\Gamma_i(z)$ and $\Gamma_{i+1}(z)$ in two ways. In particular, it follows that $k_0 = 1$ and $k_{i+1} = b_i k_i / c_{i+1}$ for all $i = 0, 1, \ldots, D - 1$. The number of vertices now follows as $v = k_0 + k_1 + \cdots + k_D$. In combinatorial arguments such as the above, it helps to draw pictures; in particular, of the so-called *distance-distribution diagram*, as depicted in Figure 1.
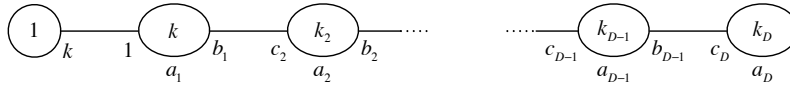
Figure 1: Distance-distribution diagram

2.2 A few examples

2.2.1 The complete graph

The complete graphs K_v (i.e., the graphs where all vertices are adjacent to each other) are the distance-regular graphs with diameter 1, and have intersection array $\{v-1;1\}$ if $v > 1$.

2.2.2 The polygons

The polygons (cycles) C_v are the distance-regular graphs with valency 2, and have intersection array $\{2, 1, \ldots, 1; 1, 1, \ldots, 1\}$ if v is odd, and $\{2, 1, \ldots, 1; 1, \ldots, 1, 2\}$ if v is even.

2.2.3 The Petersen graph and other Odd graphs

The well-known Petersen graph is a distance-regular graph with diameter 2, and has intersection array $\{3, 2; 1, 1\}$. The distance-regular graphs with diameter 2 are very special, and form a subject of their own. They are exactly the connected strongly regular graphs (for more on such graphs, see [85, Ch. 9]).

The Petersen graph is the same as the Odd graph O_3. For an integer $k \geqslant 2$, the vertices of the Odd graph O_k are the $(k-1)$-subsets of a set of size $2k-1$, and two vertices are adjacent if the corresponding subsets are disjoint. The Odd graph O_k is distance-regular with diameter $k-1$. For odd $k = 2l-1$, its intersection array is $\{k, k-1, k-1, \ldots, l+1, l+1, l; 1, 1, 2, 2, \ldots, l-1, l-1\}$. For even $k = 2l$, the intersection array is $\{k, k-1, k-1, \ldots, l+1, l+1; 1, 1, 2, 2, \ldots, l-1, l-1, l\}$. Consequently, the numbers a_i are zero for all $i = 0, 1, \ldots, D-1$, but $a_D = l > 0$.

2.3 Which graphs are determined by their intersection array?

All graphs in the above examples have the property that they are the only ones that are distance-regular with the given intersection array. In other words, given the particular intersection array, it is possible to reconstruct the graph uniquely (up to isomorphism). A typical combinatorial argument can be used to show this for the Petersen graph.

Proposition 2.1. *The Petersen graph is determined as distance-regular graph by its intersection array.*

Proof. Consider a distance-regular graph with intersection array $\{3, 2; 1, 1\}$. Take a vertex z; it has $b_0 = 3$ neighbors, each of which has $b_1 = 2$ neighbors at distance 2 from z (and hence there are no triangles in the graph; $a_1 = 0$). Each of the vertices at distance 2 from

z has precisely $c_2 = 1$ common neighbors with z (hence there are no 4-cycles in the graph either). This already determines the $10 = 1 + 3 + 6$ vertices and all edges except those having both ends in $\Gamma_2(z)$. The graph induced on $\Gamma_2(z)$ is regular with valency $a_2 = 2$, and because the graph has no triangles, this must be a 6-cycle. Now there is (up to isomorphism) only one way to make this 6-cycle if one takes into account that the entire graph has no triangles and 4-cycles; we obtain the Petersen graph as the only graph with intersection array $\{3, 2; 1, 1\}$; see Figure 2. $\qquad\square$
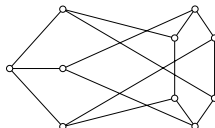
Figure 2: The Petersen graph

This is clearly a very interesting property; however it does not hold for all intersection arrays. The smallest intersection array (smallest in terms of the number of vertices) that corresponds to more than one graph is $\{6, 3; 1, 2\}$; it corresponds to the Hamming graph $H(2, 4)$ (also known as the lattice graph $L_2(4)$) and the Shrikhande graph.

One of the problems in the field of distance-regular graphs is therefore to determine which graphs are determined by their intersection array, and more generally, to determine all graphs that have the same intersection array as a given graph. While for many graphs this problem is still open, for the Odd graphs the problem was settled already long ago by Moon [498]. Her result was later generalized by Koolen [406] as follows.

Proposition 2.2. *Let Γ be a non-bipartite distance-regular graph with diameter $D \geqslant 4$, and intersection numbers $a_1 = a_2 = a_3 = 0$, $c_2 = 1$, and $c_3 = c_4 = 2$. Then Γ is an Odd graph.*

This result shows that we do not always need all intersection numbers to determine a graph. This is very typical in the characterization results that we know. We will see more examples of this later on, for example in the characterizations in Section 9.1. Note that the condition that the graph is non-bipartite is also a condition on the intersection numbers; it is not hard to see that a distance-regular graph is bipartite (i.e., has no odd cycles) if and only if $a_i = 0$ for all $i = 0, 1, \ldots, D$. To obtain their results, both Moon and Koolen used the correspondence to a certain Johnson graph; and Moon characterized this Johnson graph by just a few intersection numbers. Hiraki [315] also strengthened the result by Moon; he showed — among others — that $a_1 = a_2$, $a_4 = 0$, $c_2 = 1$, and $c_3 = c_4 = 2$ suffices to determine the Odd graphs among the non-bipartite distance-regular graphs with diameter $D \geqslant 5$. These results also 'eliminate' intersection arrays that match the intersection arrays of the Odd graphs partially.

This brings us to the following question: which intersection arrays should we look at? Do we need a distance-regular graph first, before we consider its intersection array? Perhaps there are beautiful distance-regular graphs that we do not know of yet. How can we find these? One way is to first try to classify possible intersection arrays.

In order to find putative intersection arrays of distance-regular graphs, we should in principle find as many conditions on such arrays as possible. In this introduction, we will however only mention some elementary standard conditions. There are many other — more technical — conditions known (some of which we will see in later sections) that eliminate certain intersection arrays, but these are beyond the scope of this introduction. We begin with some combinatorial conditions, and then bring linear algebra into the game to obtain algebraic conditions.

2.4 Some combinatorial conditions for the intersection array

The first trivial conditions that should hold for the intersection array $\{b_0, b_1, \ldots, b_{D-1};$ $c_1, c_2, \ldots, c_D\}$ of a distance-regular graph is that the intersection numbers listed are positive integers. Moreover, the intersection number $a_i = b_0 - b_i - c_i$ is a nonnegative integer. But we also have some divisibility conditions as follows.

Proposition 2.3. *With notation as above, the following conditions hold:*

(i) $k_{i+1} = \frac{b_0 b_1 \cdots b_i}{c_1 c_2 \cdots c_{i+1}}$ *is an integer for $i = 0, 1, \ldots, D - 1$,*

(ii) $v k_i$ *is even for $i = 1, 2, \ldots, D$,*

(iii) $k_i a_i$ *is even for $i = 1, 2, \ldots, D$,*

(iv) $v k a_1$ *is divisible by 6.*

Proof. (i) Earlier on, in Section 2.1, we obtained the recurrence $k_{i+1} = b_i k_i / c_{i+1}$ for all $i = 0, 1, \ldots, D - 1$, and this implies that

$$k_{i+1} = \frac{b_0 b_1 \cdots b_i}{c_1 c_2 \cdots c_{i+1}}$$

for $i = 0, 1, \ldots, D - 1$. These numbers are clearly positive integers.

(ii) By doubly counting all pairs (z, e), where z is an end vertex of edge e in Γ_i, it follows that the number of edges in Γ_i equals $v k_i / 2$, which should be an integer.

(iii) Similarly, there are $k_i a_i / 2$ edges of Γ within $\Gamma_i(z)$ for a fixed vertex z, and this should be an integer.

(iv) Finally, the number of triangles in Γ equals $v k a_1 / 6$. □

There is also a nice order in the intersection numbers, and consequently the k_i are unimodal, as we shall see next.

Proposition 2.4. *With notation as above, the following conditions hold:*

(i) $1 = c_1 \leqslant c_2 \leqslant \cdots \leqslant c_D$,

(ii) $k = b_0 \geqslant b_1 \geqslant \cdots \geqslant b_{D-1}$,

(iii) *If $i + j \leqslant D$, then $c_i \leqslant b_j$,*

(iv) *There is an i such that $k_0 \leqslant k_1 \leqslant \ldots \leqslant k_i$ and $k_{i+1} \geqslant k_{i+2} \geqslant \ldots \geqslant k_D$.*

Proof. (i) and (ii) Let $i = 1, 2, \ldots, D$. Consider two vertices x and y at distance i, and a vertex z that is adjacent to x and at distance $i - 1$ from y. Now the c_{i-1} neighbors of y that are at distance $i - 2$ from z are all at distance $i - 1$ from x. Therefore $c_i \geqslant c_{i-1}$. Similarly, the b_i neighbors of y that are at distance $i + 1$ from x are at distance i from z, hence $b_{i-1} \geqslant b_i$.

(iii) Consider two vertices x and y at distance $i + j$, and a vertex z at distance i from x and j from y. Then the c_i neighbors of z that are at distance $i - 1$ from x are at distance $j + 1$ from y. Hence $c_i \leqslant b_j$.

(iv) It follows from (i), (ii), and Proposition 2.3 that $k_i^2 \geqslant k_{i-1}k_{i+1}$ for $i = 1, 2, \ldots, D - 1$. This implies that the k_i are unimodal: there is an i such that $k_0 \leqslant k_1 \leqslant \ldots \leqslant k_i$ and $k_{i+1} \geqslant k_{i+2} \geqslant \ldots \geqslant k_D$. $\qquad \square$

Even though these and other combinatorial conditions are important, they are insufficient to obtain most of the advanced results. We need linear algebra.

2.5 The spectrum of eigenvalues and multiplicities

The adjacency matrix A of a (simple, undirected) graph Γ is the $v \times v$ symmetric matrix with entries 0 and 1 whose rows and columns are indexed by the vertices of Γ, and where $A_{xy} = 1$ if and only if $x \sim y$. Because A is real and symmetric, its eigenvalues are real numbers. The spectrum of eigenvalues of a graph (that is, of its adjacency matrix) contains quite some (but in general not all) information about the graph. Spectra of graphs is a very fruitful subject on its own, and it has many more applications to distance-regular graphs than the ones that we shall see here. Good references for spectra of graphs are the classic monograph by Cvetković, Doob, and Sachs [160] and the more recent one by Brouwer and Haemers [85].

The *adjacency algebra* of Γ, denoted by $\mathbb{A} = \mathbb{A}(\Gamma)$, is the matrix subalgebra of $M_{v \times v}(\mathbb{R})$ of polynomials in A, that is, $\mathbb{A} = \mathbb{R}[A]$. This algebra plays an important role for distance-regular graphs, as we shall see later on. We note that the powers of A count walks in the graph, that is, $(A^\ell)_{xy}$ equals the number of walks of length ℓ in the graph from x to y. Using this, we can relate the number of distinct eigenvalues to the diameter of the graph. To do this, assume that Γ is an arbitrary graph with distinct eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$. Because the minimal polynomial of A now has degree $d + 1$, it is clear that $\{I, A, A^2, \ldots, A^d\}$ is a basis of $\mathbb{A}$, and hence that $\dim \mathbb{A} = d + 1$.

Proposition 2.5. *Let Γ be a connected graph with diameter D and distinct eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$. Then $D \leqslant d$.*

Proof. Consider two vertices x and y at distance $i \leqslant D$. Then $(A^\ell)_{xy} = 0$ if $\ell < i$ and $(A^i)_{xy} \neq 0$. This implies that the set of matrices $\{I = A^0, A, \ldots, A^D\}$ is linearly independent in $\mathbb{A}$, and hence that $D + 1 \leqslant \dim \mathbb{A} = d + 1$. $\qquad \square$

For $i = 0, 1, \ldots, d$, we define the matrix $E_i = \prod_{j=0, j \neq i}^{d} \frac{A - \theta_j I}{\theta_i - \theta_j}$. The matrix E_i is the orthogonal projection onto the eigenspace V_i of A corresponding to θ_i. The set $\{E_0, E_1, \ldots, E_d\}$

forms another basis of $\mathbb{A}$. Indeed, let $\mathbf{v}$ be an eigenvector of A with respect to θ_j. Then $E_i\mathbf{v} = \delta_{ij}\mathbf{v}$. This implies that $\{E_0, E_1, \ldots, E_d\}$ forms a linearly independent set of matrices in $\mathbb{A}$, and hence that it is a basis of $\mathbb{A}$. We shall see more of this basis in the next section.

The adjacency matrix A_i of Γ_i is called the *distance-i matrix* of Γ, for $i = 0, 1, \ldots, D$. Let us now consider the case that Γ is distance-regular. In this case, we shall see that also $\{I = A_0, A = A_1, A_2, \ldots, A_D\}$ is a basis of $\mathbb{A}$, and hence that $D = d$. Translating the combinatorial definition of distance-regularity into matrix language, we obtain the equation

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \tag{1}$$

for $i = 0, 1, \ldots, D$. Note that for $i = 0$ and $i = D$, the indices in this equation attain undefined values. Here — and in similar equations that will follow later — we will have the sensible convention that the corresponding summands are zero (so $b_{-1}A_{-1} = c_{D+1}A_{D+1} = 0$). From this recurrence (note that the coefficients c_{i+1} are nonzero for $i = 0, 1, \ldots, D-1$), it follows that there exist polynomials v_i of degree i such that

$$A_i = v_i(A) \tag{2}$$

for $i = 0, 1, \ldots, D$. These polynomials also satisfy a three-term recurrence relation like (1), and hence they form a system of orthogonal polynomials. Because $\sum_{i=0}^{D} A_i = J$ (the all-ones matrix) and $AJ = kJ$ (because Γ is regular with valency k), it follows that $(\sum_{i=0}^{D} v_i(A))(A - kI) = 0$. This shows that $\dim \mathbb{A} \leqslant D + 1$. We may conclude the following.

Proposition 2.6. *Let Γ be a distance-regular graph with diameter D. Then $\dim \mathbb{A} = D + 1$. In particular, Γ has exactly $D + 1$ distinct eigenvalues.*

Remarkably, these $D + 1$ distinct eigenvalues of the distance-regular graph Γ can be computed from the intersection numbers only. To see this, consider the tridiagonal $(D + 1) \times (D + 1)$ matrix *intersection matrix*

$$L = \begin{bmatrix} 0 & b_0 & & & \\ c_1 & a_1 & b_1 & & 0 \\ & c_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & 0 & & \cdot & \cdot & b_{D-1} \\ & & & & c_D & a_D \end{bmatrix}. \tag{3}$$

This matrix is diagonalizable because it is similar to a symmetric matrix. In fact, if Δ is the diagonal matrix with diagonal entries $\Delta_{ii} = k_i$ for $i = 0, 1, \ldots, D$, then by using that $k_{i+1}/k_i = b_i/c_{i+1}$, it can be verified that $\Delta^{1/2}L\Delta^{-1/2}$ is indeed a symmetric tridiagonal matrix.

Let θ be an eigenvalue of L, and let $\mathbf{u} = (u_0, u_1, \ldots, u_D)^\top$ be a corresponding (right) eigenvector, that is, $L\mathbf{u} = \theta\mathbf{u}$, with $u_0 = 1$. Then $u_1 = \theta/k$ and

$$c_iu_{i-1} + a_iu_i + b_iu_{i+1} = \theta u_i \tag{4}$$

for $i = 1, 2, \ldots, D$. The sequence $(u_i)_{i=0}^D$ is called the *standard* (or *cosine*) *sequence* of Γ with respect to θ.

A consequence of the above symmetrization of L is that the row vector $\mathbf{v} = \mathbf{u}^\top \Delta$ is a left eigenvector of L. Thus, the components of $\mathbf{v}$ also satisfy a recurrence involving the intersection numbers. It can be verified that these components can be obtained from the polynomials v_i in (2), that is, $\mathbf{v} = (v_0(\theta), v_1(\theta), \ldots, v_D(\theta))$. This gives an alternative way to obtain the standard sequence.

Proposition 2.7. *Let Γ be a distance-regular graph with diameter D. Then the $D + 1$ distinct eigenvalues of Γ are precisely the eigenvalues of L.*

Proof. Let $\mathbf{u}$ be as above, i.e., an eigenvector of L with respect to eigenvalue θ, and fix a vertex x of Γ. Define the vector $\mathbf{w}$ by $w_y = u_{d(x,y)}$ for $y \in V$. It is not hard (but a bit technical) to check that $A\mathbf{w} = \theta\mathbf{w}$. Indeed, if $\mathbf{a_i}$ denotes column x of A_i, then $\mathbf{w} = \sum_{i=0}^D u_i \mathbf{a_i}$. By (1) and the above equations for the standard sequence, we obtain that

$$A\mathbf{w} = \sum_{i=0}^D u_i (b_{i-1}\mathbf{a_{i-1}} + a_i\mathbf{a_i} + c_{i+1}\mathbf{a_{i+1}})$$
$$= \sum_{i=0}^D (c_i u_{i-1} + a_i u_i + b_i u_{i+1})\mathbf{a_i} = \sum_{i=0}^D \theta u_i \mathbf{a_i} = \theta\mathbf{w}.$$

This shows that all eigenvalues of L are eigenvalues of Γ.

What remains is to show that L has $D + 1$ distinct eigenvalues. We already observed that L is diagonalizable, or in other words, that it has $D + 1$ eigenvalues. Because the intersection numbers $c_1, c_2, \ldots, c_D$ are all nonzero, it follows that the rank of $L - \theta I$ is at least D for all $\theta \in \mathbb{R}$. This shows that all eigenvalues of L are distinct, which finishes the proof. $\qquad\square$

Finally, also the multiplicities of the eigenvalues of Γ follow from the intersection numbers, via the standard sequence. This is known as *Biggs' formula*.

Theorem 2.8. (Biggs' formula) *Let Γ be a distance-regular graph with diameter D and v vertices. Let θ be an eigenvalue of Γ and $(u_i)_{i=0}^D$ be the standard sequence with respect to θ. Then the multiplicity $m(\theta)$ of θ as an eigenvalue of Γ satisfies*

$$m(\theta) = \frac{v}{\sum_{i=0}^D k_i u_i^2}.$$

Proof. Let E be the matrix corresponding to the orthogonal projection onto the eigenspace of Γ with respect to θ (i.e., it is one of the matrices E_i defined before). The idempotent matrix E only has eigenvalues 0 and 1, and the multiplicity $m(\theta)$ of θ as an eigenvalue of Γ is the same as the multiplicity of eigenvalue 1 of E, which implies that $m(\theta) = \operatorname{tr} E$. Because $E \in \mathbb{A}$ and $\{A_0, A_1, \ldots, A_D\}$ is a basis of $\mathbb{A}$, there are real numbers $\nu_i, i = 0, 1, \ldots, D$ such that $E = \sum_{i=0}^D \nu_i A_i$. Note that $AE = \theta E$, which implies that

$c_i\nu_{i-1} + a_i\nu_i + b_i\nu_{i+1} = \theta\nu_i$ for $i = 0, 1, \ldots, D$. From this it follows that $\nu_i = \nu_0 u_i$ for $i = 0, 1, \ldots, D$. By considering the diagonal of the equation $E^2 = E$, we find that $\sum_{i=0}^D k_i\nu_i^2 = \nu_0$, which implies that $\sum_{i=0}^D k_i u_i^2 = 1/\nu_0$. Now it follows that

$$m(\theta) = \operatorname{tr} E = \sum_{x \in V} E_{xx} = v\nu_0 = \frac{v}{\sum_{i=0}^D k_i u_i^2}. \qquad \square$$

Thus, it is relatively easy to compute the spectrum of a distance-regular graph from its intersection array. Remarkably, the fact that multiplicities of eigenvalues are positive integers is a condition that many intersection arrays (that satisfy all earlier conditions) do not satisfy. Note also that algebraically conjugate eigenvalues must have the same multiplicities. The latter plays an important role in the proof of the Bannai-Ito conjecture, see Section 8.1.

Related to the vectors **w** in the proof of Proposition 2.7 and the standard sequence is the representation associated to an eigenvalue θ. Let U be a matrix having as columns an orthonormal basis of the eigenspace of eigenvalue θ. Then $UU^\top$ is the corresponding idempotent matrix E. For every vertex $x \in V$, we denote by $\hat{x}$ the x-th row of U. The map $x \mapsto \hat{x}$ is called a *representation* (associated to θ) of Γ. Given two vertices $x, y \in V$, we have that $\langle \hat{x}, \hat{y} \rangle = E_{xy} = \nu_0 u_{d(x,y)}$, which is why the standard sequence is also called the cosine sequence. The vectors $\hat{x}$ ($x \in V$) all have the same length, $\sqrt{\nu_0}$, hence we call the representation spherical.

2.6 Association schemes

In the previous section we described three different bases for the adjacency algebra $\mathbb{A}$ of a distance-regular graph: $\{I, A, A^2, \ldots, A^D\}$, $\{E_0, E_1, \ldots, E_D\}$, and $\{A_0, A_1, \ldots, A_D\}$. The last one was obtained by explicit use of the property of distance-regularity. A consequence of this is that there are real numbers p_{ij}^h ($h, i, j = 0, 1, \ldots, D$) such that

$$A_i A_j = \sum_{h=0}^D p_{ij}^h A_h. \tag{5}$$

for all $i, j = 0, 1, \ldots, D$. This expression has a combinatorial interpretation: for each two vertices x and y at distance h, there are p_{ij}^h vertices z that are at distance i to x and distance j to y. So the *intersection numbers* p_{ij}^h are nonnegative integers. Note that $p_{1,i-1}^i = c_i, p_{1,i}^i = a_i$, and $p_{1,i+1}^i = b_i$. Also the other intersection numbers p_{ij}^h can be expressed in terms of the intersection array. This gives further conditions on the intersection numbers.

What we have here is a special case of an *association scheme*: an edge decomposition of the complete graph into spanning subgraphs Γ_i ($i = 1, 2, \ldots, D$) whose adjacency matrices A_i ($i = 1, 2, \ldots, D$), together with $A_0 = I$ satisfy (5) for all $i, j = 0, 1, \ldots, D$. Let us look a bit closer at such an association scheme. Clearly also here $\{A_0, A_1, \ldots, A_D\}$ is a basis of an algebra: the *Bose-Mesner algebra*. Because the matrices in this algebra are symmetric, they also commute by (5) (and hence $p_{ij}^h = p_{ji}^h$). This implies that they share a basis of

eigenvectors and there is also a basis of primitive idempotents E_i ($i = 0, 1, \ldots, D$) for $\mathbb{A}$ (so ME_i is a multiple of E_i for all $M \in \mathbb{A}$). These E_i are the projections onto the common eigenspaces, and are the same as before in case of a distance-regular graph. They satisfy the equations $E_i E_j = \delta_{ij} E_i$ for all $i, j = 0, 1, \ldots, D$ and $\sum_{i=0}^{D} E_i = I$.

The coefficients to change from one of the two bases to the other are collected in the so-called *eigenmatix* P and *dual eigenmatrix* Q. That is,

$$A_i = \sum_{i=0}^{D} P_{ji} E_j \quad \text{and} \quad E_i = \frac{1}{v} \sum_{j=0}^{D} Q_{ji} A_j$$

for $i = 0, 1, \ldots, D$. Note that so far we did not order the eigenvalues (or the E_is), so there is some ambiguity in the definition of P and Q. This is not really a problem (as long as we keep some ordering fixed), except that it has become habit that the first row of P contains the valencies of the graphs Γ_i. For this reason, we order the eigenspace of constant vectors first, so that $E_0 = \frac{1}{v} J$, the *trivial* primitive idempotent of $\mathbb{A}$. This is also justified by the fact that dually we could reshuffle the A_i (and Γ_i), except the trivial A_0, and not really get a 'different' association scheme (for a distance-regular graph there is of course an order given!). Note also that column i of P gives the eigenvalues of the corresponding graph Γ_i. The normalization factor $\frac{1}{v}$ for Q is there to make sure that the entries of Q can be seen as 'dual eigenvalues'; for example the multiplicities $m_i = \operatorname{tr} E_i$ of the eigenvalues are in the first row of Q. Just like in the case of distance-regular graphs, the eigenvalues and multiplicities, and more generally, all entries of P and Q can be derived from the intersection numbers p_{ij}^h. In the case of distance-regular graphs, we see now in the proof of Biggs' formula (Theorem 2.8) that a column of Q is a multiple of the corresponding standard sequence.

Observe that the Bose-Mesner (or adjacency) algebra $\mathbb{A}$ is not just closed under ordinary matrix multiplication but also under entrywise (*Hadamard* or *Schur*) matrix multiplication, denoted by $\circ$. The matrices $A_0, A_1, \ldots, A_D$ are the primitive idempotents of $\mathbb{A}$ with respect to $\circ$, i.e., $A_i \circ A_j = \delta_{ij} A_i$, $\sum_{i=0}^{D} A_i = J$. This implies that we may write

$$E_i \circ E_j = \frac{1}{v} \sum_{h=0}^{D} q_{ij}^h E_h \tag{6}$$

for some real numbers q_{ij}^h ($h, i, j = 0, 1, \ldots, D$), known as the *Krein parameters* (or *dual intersection numbers*) of Γ. Because $\frac{1}{v} q_{ij}^h$ is an eigenvalue of $E_i \circ E_j$, which is a principal submatrix of the positive semidefinite matrix $E_i \otimes E_j$, we get the following so-called *Krein conditions*.

Proposition 2.9. *The Krein parameters q_{ij}^h of an association scheme are nonnegative numbers.*

The Krein parameters can be calculated using the dual eigenmatrix as

$$q_{ij}^h = \frac{1}{v m_h} \sum_{l=0}^{D} k_l Q_{lh} Q_{li} Q_{lj}.$$

This follows from working out the sum of entries of the matrix $E_i \circ E_j \circ E_h \circ J$ in different ways. The Krein conditions thus put further constraints on the intersection array of a distance-regular graph. Moreover, if a Krein parameter equals zero, then this has certain consequences. This is perhaps best illustrated in the case of the Q-polynomial distance-regular graphs of Section 5 (see also Section 2.7), where many Krein parameters vanish. See also Section 6.3 for consequences of vanishing Krein parameters.

From the definition of P and Q, it is clear that $PQ = QP = vI$. A different relation between P and Q can be obtained by working out the trace of $A_i E_j$ (which equals the sum of entries of $A_i \circ E_j$) in different ways. This gives the relation $m_j P_{ji} = k_i Q_{ij}$ for all $i, j = 0, 1, \ldots, D$. Together with $PQ = vI$, this gives certain *orthogonality relations* between the columns (and rows) of P. For a distance-regular graph Γ, this relation also follows from the fact that the polynomials v_i ($i = 0, 1, \ldots, D$) form a system of orthogonal polynomials. Here $P_{ji} = v_i(P_{j1})$, which follows from (2), where we remind the reader that P_{j1} ($j = 0, 1, \ldots, D$) are the distinct eigenvalues of Γ.

A final condition that we would like to mention is the *absolute bound*.

Proposition 2.10. *The multiplicities m_i of an association scheme satisfy the following bound:*
$$\sum_{q_{ij}^h \neq 0} m_h \leqslant \begin{cases} m_i m_j & \text{if } i \neq j \\ m_i(m_i + 1)/2 & \text{if } i = j. \end{cases}$$

Proof. The left hand side equals the rank of $E_i \circ E_j$, because of (6) and the fact that the idempotents are mutually orthogonal (and can be diagonalized simultaneously) so that their ranks are additive. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{m_i}$ be a basis of $E_i \mathbb{R}^v$, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m_j}$ be a basis of $E_j \mathbb{R}^v$. Then the column space of $E_i \circ E_j$ is contained in the subspace spanned by the vectors $\mathbf{u}_s \circ \mathbf{v}_t$ ($s = 1, 2, \ldots, m_i$, $t = 1, 2, \ldots, m_j$), thus proving the inequality for $i \neq j$. For $i = j$, note that the latter subspace is spanned by the vectors $\mathbf{u}_s \circ \mathbf{u}_t$ with $s \leqslant t$. $\qquad\square$

For more information on association schemes, we refer to the handbook chapter by Brouwer and Haemers [83] and the recent survey by Martin and Tanaka [470].

2.7 The Q-polynomial property

We already noted that the ordering of graphs and idempotents in an association scheme is not really important. However, in an association scheme that comes from a distance-regular graph, the graphs Γ_i are ordered naturally according to distance in the graph. This ordering is called a P-polynomial ordering. This term comes from the fact that there are polynomials v_i of degree i such that $A_i = v_i(A_1)$, as we have seen. The association scheme is therefore also called *P-polynomial*. An equivalent property of this ordering is that the intersection numbers are such that $p_{ij}^h = 0$ whenever $0 \leqslant h < |i - j|$ or $i + j < h$, and $p_{ij}^{i+j} > 0$ (for $i + j \leqslant D$). Because of this property, we call a P-polynomial association scheme also *metric*. An association scheme can have at most two P-polynomial orderings (that is, there can be at most two distance-regular graphs in it), except for the

association schemes coming from the polygons. For more on association schemes with two P-polynomial orderings, see Section 13.2.

It turns out that many important families of distance-regular graphs, that is, their corresponding association schemes, satisfy the following dual property. We say that an association scheme (and in particular, a distance-regular graph) is Q-*polynomial* if there is an ordering $E_0, E_1, \ldots, E_D$ and there are polynomials q_i of degree i such that $E_i = q_i(E_1)$, where the matrix multiplication is entrywise (so that $(E_i)_{xy} = q_i((E_1)_{xy})$ for all vertices x and y). We also say that the corresponding ordering and the idempotent E_1 are Q-*polynomial*. Also here there is an equivalent property in terms of — in this case — the Krein parameters: an association scheme is called *cometric* (with ordering $E_0, E_1, \ldots, E_D$) if $q_{ij}^h = 0$ whenever $0 \leqslant h < |i - j|$ or $i + j < h$, and $q_{ij}^{i+j} > 0$ (for $i + j \leqslant D$). It is well known though that to check the cometric property, it suffices to check the above conditions for $i = 1$ (just like in the metric case). Dual to the intersection numbers of a distance-regular graph, we here define $c_i^\star = q_{1,i-1}^i, a_i^\star = q_{1,i}^i,\ b_i^\star = q_{1,i+1}^i$, and the Krein array $\{b_0^\star, b_1^\star, \ldots, b_{D-1}^\star; c_1^\star, c_2^\star, \ldots, c_D^\star\}$.

It was conjectured by Bannai and Ito [38, p. 312] that for large enough D, a primitive D-class association scheme is P-polynomial if and only if it is Q-polynomial.

2.8 Delsarte cliques and geometric graphs

Delsarte [189, p. 31] obtained a linear programming bound for cliques in strongly regular graphs. It was observed by Godsil [264, p. 276] that the same *Delsarte bound* holds for distance-regular graphs, as follows.

Proposition 2.11. *Let Γ be a distance-regular graph with valency k and smallest eigenvalue $\theta_{\min}$. Let C be a clique in Γ with c vertices. Then $c \leqslant 1 - \frac{k}{\theta_{\min}}$.*

Proof. Let χ be the characteristic vector of C, and let E be the primitive idempotent corresponding to $\theta_{\min}$. The result follows from working out $\chi^\top E \chi \geqslant 0$. □

A clique C in a distance-regular graph Γ that attains this Delsarte bound is called a *Delsarte clique*. In Section 4.4.2 we will characterize such cliques as certain completely regular codes.

A distance-regular graph Γ is called *geometric* (with respect to $\mathcal{C}$) if it contains a collection $\mathcal{C}$ of Delsarte cliques such that each edge is contained in a unique $C \in \mathcal{C}$. The concept of a geometric distance-regular graph was introduced by Godsil [265] and generalizes the concept of a geometric strongly regular graph as introduced by Bose [66] (and indeed the concepts are the same for diameter two).

Many classical examples of distance-regular graphs (see Section 3.1), such as Johnson graphs, Grassmann graphs, and Hamming graphs are geometric. Bipartite distance-regular graphs are trivially geometric because in this case every edge is a Delsarte clique.

Even though the class of non-bipartite geometric distance-regular graphs is clearly much more restricted than the class of arbitrary distance-regular graphs, Koolen and Bang [410] showed that for fixed smallest eigenvalue there are only finitely many non-geometric distance-regular graphs with both valency and diameter at least three (see

Theorem 9.10). They in fact conjectured that for fixed smallest eigenvalue there are finitely many distance-regular graphs with diameter at least three that are not a cycle, Hamming graph, Johnson graph, Grassmann graph, or bilinear forms graph. This would generalize a result by Neumaier [509] on strongly regular graphs. On the other hand, because geometric distance-regular graphs have more structure than arbitrary distance-regular graphs, it may be possible to classify them, or at least the Q-polynomial ones with large diameter.

2.9 Imprimitivity

A connected graph Γ with diameter D is called imprimitive if not all graphs Γ_i ($i = 1, 2, \ldots, D$) are connected. Bipartite graphs are examples of imprimitive graphs (Γ_2 is disconnected). Among the distance-regular graphs, there are also the antipodal graphs that are imprimitive. These are the graphs for which Γ_D is a disjoint union of complete graphs. In fact, Smith's theorem states that these are all possibilities (see [78, Thm. 4.2.1]), except for the polygons (indeed, for example C_9 has $D = 4$ and only Γ_3 is disconnected in this case).

Theorem 2.12. (Smith's theorem) *An imprimitive distance-regular graph with valency $k > 2$ is bipartite and/or antipodal.*

There is much more to say than this seemingly clear and simple statement. For this we refer to Alfuraidan and Hall [4, Thm. 2.9], who revisited Smith's theorem by working out more precisely all the cases that can occur.

If Γ is a bipartite distance-regular graph, then Γ_2 is a graph with two components. The induced graphs on these components are called the *halved graphs* of Γ.

Proposition 2.13. *The halved graphs of a bipartite distance-regular graph are distance-regular.*

We already noted before that bipartiteness of a distance-regular graph can be seen from its intersection numbers. Clearly this is the case whenever $a_i = 0$ for all $i = 1, 2, \ldots, D$.

A distance-regular graph is antipodal whenever $b_i = c_{D-i}$ for all $i = 0, 1, \ldots, D$, except possibly $i = \lfloor D/2 \rfloor$. If Γ is an antipodal distance-regular graph, then by definition, Γ_D is a disjoint union of cliques. These cliques are called the *fibres* of Γ. We can also construct a smaller distance-regular graph from an antipodal distance-regular graph: its *folded graph* $\overline{\Gamma}$. Its vertices are the fibres of Γ, and two such fibres are adjacent whenever there is an edge (in Γ) between them. We also say the Γ is an *antipodal r-cover* of $\overline{\Gamma}$, where r is the size of the cliques of Γ_D.

Proposition 2.14. *The folded graph of an antipodal distance-regular graph is distance-regular.*

Typically, but certainly not always (see [4]), the halved graphs or folded graphs of an imprimitive distance-regular graph are primitive (that is, not imprimitive). This suggests

that the theory of distance-regular graphs can be boiled down to that of primitive distance-regular graphs. This is not the case however. There is no unique recipe to construct imprimitive distance-regular graphs from the primitive ones, for example. The halving and folding constructions mentioned above cannot be reversed in a generic way, at least not in general. This is best illustrated by the imprimitive distance-regular graphs with diameter three. All of these have as halved or folded graph a complete graph. Sometimes, however, there is an easy way to construct an imprimitive distance-regular graph from a primitive one as follows. The *bipartite double* of a graph Γ with vertex set V is the graph with vertex set $V \times \{0, 1\}$, where two vertices (x, i) and (y, j) are adjacent whenever x is adjacent to y in Γ and $i \neq j$. The *extended bipartite double* of Γ is a variation on this: it has the same vertex set, and besides the edges of the bipartite double, it has additional edges between $(x, 0)$ and $(x, 1)$, $x \in V$.

If Γ is a distance-regular *generalized odd graph* (also called *almost bipartite graph*) with diameter D, that is, if it has intersection numbers $a_i = 0$ for $i < D$ and $a_D > 0$ (like the Odd graphs), then the bipartite double of Γ is distance-regular with diameter $2D+1$. This situation is interesting for several reasons, one of them being that this bipartite double is not just bipartite, but it is also an antipodal 2-cover of Γ. The *Doubled Odd graphs* (for example) are thus showing that bipartiteness and antipodality can occur in the same graph. Note by the way that the folded graph of this Doubled Odd graph is again the Odd graph, but the halved graphs are not (these are isomorphic to Γ_2, a Johnson graph).

More generally, one can see from the intersection array of a distance-regular graph whether the bipartite double or extended bipartite double is distance-regular (see [78, §1.11]).

2.10 Distance-transitive graphs

Distance-regular graphs were 'invented' by Biggs (for an early account, see his monograph [48]) while he was studying so-called distance-transitive graphs. An automorphism of a graph is a bijection from the vertex set to itself that respects adjacencies, i.e., that maps edges to edges. A graph is called *distance-transitive* if it has a group of automorphisms that acts transitively on each of the sets of pairs of vertices at distance i, for $i = 0, 1, \ldots, D$. In other words, for each i and all pairs of vertices (x_1, y_1) and (x_2, y_2) with $d(x_1, y_1) = i = d(x_2, y_2)$, there is an automorphism that maps x_1 to x_2 and y_1 to y_2. This property is easily seen to imply the property of distance-regularity. Many — but not all — classical families of distance-regular graphs, for example the Hamming graphs, are also distance-transitive. The earlier mentioned Shrikhande graph is the smallest distance-regular graph that is not distance-transitive. In fact, it is part of an infinite family of graphs that are distance-regular but not distance-transitive: the so-called Doob graphs. It also indicates that distance-transitivity of a distance-regular graph is not a property that can be recognized from the intersection array.

A distance-transitive graph is clearly also *vertex-transitive*, that is, it has a group of automorphisms such that for all x_1 and x_2, there is an automorphism that maps x_1 to x_2. Although there is no apparent relation between vertex-transitivity and distance-regularity, it was long believed that distance-regular graphs with large enough diameter would have

to be vertex-transitive. This belief was proven wrong by the construction of the twisted Grassmann graphs; see Section 3.2.1.

3 Examples

3.1 The classical families with unbounded diameter

The *Johnson graph* $J(n, D)$ has as vertices the subsets of size D of a set of size n. Two subsets are adjacent if and only if they differ in precisely one element; cf. [78, §9.1]. Note that $J(n, D)$ is isomorphic to $J(n, n-D)$; in the following we therefore restrict to $n \geqslant 2D$. The Johnson graph $J(n, D)$ is characterized as distance-regular graph by its intersection array unless $n = 8$ and $D = 2$, in which case there are also three so-called Chang graphs.

The *Grassmann graph* $J_q(n, D)$ has as vertices the D-dimensional subspaces of a vector space of dimension n over $GF(q)$. Two subspaces are adjacent if and only if they intersect in a $(D-1)$-dimensional subspace; cf. [78, §9.3]. Note that $J_q(n, D)$ is isomorphic to $J_q(n, n - D)$; again we therefore restrict to $n \geqslant 2D$. Metsch [478] showed that the Grassmann graphs are determined by the intersection array if $D \neq 2, \frac{n}{2}$, or $\frac{n-1}{2}$ (for all q) and $(D, q) \neq (\frac{n-2}{2}, 2), (\frac{n-2}{2}, 3)$, or $(\frac{n-3}{2}, 2)$; see also Section 9.1. For $D = 2$, the Grassmann graphs are in general not determined by the intersection array, as the line graph of a 2-$((q^n - 1)/(q - 1), q + 1, 1)$ design has the same array. Van Dam and Koolen [176] constructed the *twisted Grassmann graphs*; these are distance-regular graphs with the same array as the Grassmann graphs for $n = 2D + 1, D \geqslant 2$, see Section 3.2.1.

The *Hamming graph* $H(D, e)$ is defined on vertex set X^D of words of length D from an alphabet X of size e. Two words are adjacent if and only if they differ in precisely one position; cf. [78, §9.2]. The Hamming graph $H(D, e)$ is characterized by its intersection array unless $e = 4$ and $D > 1$, in which case there are also so-called Doob graphs. A *Doob graph* is a cartesian product of cliques of size 4 and Shrikhande graphs. The Hamming graph $H(D, 2)$ is also called a *(hyper)cube* or the *D-cube*. Its halved graph is called a *halved cube* $\frac{1}{2}H(D, 2)$ and is characterized by its intersection array (see [78, §9.2.D]).

The *bilinear forms graph* $Bil(D \times e, q)$ has as vertices all $D \times e$ matrices with entries from the field $GF(q)$, where two matrices are adjacent if and only if their difference has rank 1; cf. [78, §9.5.A] or [191]. We shall assume $D \leqslant e$ in the following, so that D is the diameter. The bilinear forms graph can be considered as the q-analogue of the Hamming graph (view the vertices of the latter as the maps from a set of size D to a set of size e), hence also the notation $H_q(D, e)$ is used in the literature. The bilinear forms graph has an alternative description on the D-dimensional subspaces of a $(D + e)$-dimensional vector space that intersect a fixed e-dimensional subspace trivially, where two such subspaces are adjacent if they intersect in a $(D - 1)$-dimensional subspace; this shows that it is isomorphic to a subgraph of the Grassmann graph $J_q(D + e, D)$. Rifa and Zinoviev [550] showed that the bilinear forms graph is also a quotient (as defined in Section 4.4) of the Hamming graph. Metsch [481] showed that the bilinear forms graph $Bil(D \times e, q)$ is characterized by its intersection array if $q = 2$ and $e \geqslant D + 4$ or $q \geqslant 3$ and $e \geqslant D + 3$; see also Section 9.1. Gavrilyuk and Koolen [242] extended this characterization with the

case $q = 2$ and $e = D$.

The *alternating forms graph* $Alt(n, q)$ has as vertices all $n \times n$ skew-symmetric matrices with zero diagonal and entries from $GF(q)$. Two matrices are adjacent if and only if their difference has rank 2. Note that a skew-symmetric matrix has even rank; cf. [78, §9.5.B] or [192].

The *Hermitian forms graph* $Her(D, q^2)$ has as vertices the $D \times D$ Hermitian matrices with entries in $GF(q^2)$, i.e., matrices H such that $H_{ij} = (H_{ji})^q$ for all i and j. Two matrices are adjacent if and only if their difference has rank 1; cf. [78, §9.5.C]. The Hermitian forms graphs are determined by their intersection arrays for $D \geqslant 3$, see Section 5.2.

The *quadratic forms graph* $Qua(n, q)$ has as vertices the quadratic forms in n variables over $GF(q)$. In the quadratic forms graph two forms are adjacent if and only if the rank of their difference equals 1 or 2; cf. [78, §9.6] or [208]. Under the group of invertible linear transformations of variables, the quadratic forms fall into $2n + 1$ (q odd) or $\lceil \frac{3n+1}{2} \rceil$ (q even) orbits: each form of rank $k \neq 0$ is of one of two types. For even rank there is the well-known distinction between hyperbolic and elliptic forms; in the case of odd rank, a (parabolic) form is equivalent to $x_1 x_2 + \cdots + x_{k-2} x_{k-1} + c x_k^2$, for some c, and the type depends on whether c is a square or not (cf. [514, Ch. IV]). If q is even then each field element is a square, hence there is no distinction for odd rank.

The *dual polar graphs*[1] have as vertices the maximal isotropic (D-dimensional) subspaces of one of the below vector spaces V endowed with a non-degenerate quadratic form. Two subspaces are adjacent if and only if they intersect in a $(D - 1)$-dimensional space; cf. [78, §9.4]. The following dual polar graphs can be distinguished:

$\mathcal{C}_D(q)$ for $V = GF(q)^{2D}$ with a symplectic form; $e = 1$;
$\mathcal{B}_D(q)$ for $V = GF(q)^{2D+1}$ with a quadratic form; $e = 1$;
$\mathcal{D}_D(q)$ for $V = GF(q)^{2D}$ with a quadratic form of Witt index D; $e = 0$;
$^2\mathcal{D}_{D+1}(q)$ for $V = GF(q)^{2D+2}$ with a quadratic form of Witt index D; $e = 2$;
$^2\mathcal{A}_{2D}(\sqrt{q})$ for $V = GF(q)^{2D+1}$ with a Hermitian form; $e = \frac{3}{2}$;
$^2\mathcal{A}_{2D-1}(\sqrt{q})$ for $V = GF(q)^{2D}$ with a Hermitian form; $e = \frac{1}{2}$.

Here the mentioned parameter e is related to the classical parameter β of the next section (see Table 1).

The dual polar graphs $^2\mathcal{A}_{2D-1}(\sqrt{q})$ are determined by their intersection arrays for $D \geqslant 4$, see Section 5.2. The dual polar graphs $\mathcal{B}_D(q)$ and $\mathcal{C}_D(q)$ have the same intersection array but are non-isomorphic unless q is even. The dual polar graph $\mathcal{D}_D(q)$ is the extended bipartite double of $\mathcal{B}_{D-1}(q)$, and its halved graph, called a *half dual polar graph* $\mathcal{D}_{D,D}(q)$, is the distance 1-or-2 graph of $\mathcal{B}_{D-1}(q)$. The extended bipartite double of $\mathcal{C}_{D-1}(q)$ is also distance-regular and is called a *Hemmeter graph* [86]; its halved graph is the distance 1-or-2 graph of $\mathcal{C}_{D-1}(q)$ and is called an *Ustimenko graph* [362].

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[1]These graphs already appear as distance-transitive graphs in disguise in a paper by Hua [338] from 1945.

3.1.1 Classical parameters

The 'classical' distance-regular graphs from the previous section have intersection numbers that can be expressed in terms of four parameters, that is, diameter D and numbers b, α, β, in the following way:

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (i = 0, 1, \ldots, D - 1),$$

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i - 1 \\ 1 \end{bmatrix} \right) \quad (i = 1, 2, \ldots, D), \tag{7}$$

where $\begin{bmatrix} j \\ 1 \end{bmatrix} = 1 + b + b^2 + \cdots + b^{j-1}$ is a Gaussian binomial coefficient. Therefore, a distance-regular graph is said to have classical parameters (D, b, α, β) if its intersection numbers can be expressed as in (7). We note that the parameter b must be an integer not equal to 0 or -1. The classical examples of distance-regular graphs from the previous section have classical parameters as in Table 1 (note that one family of dual polar graphs has intersection numbers that can be expressed in two ways). More basic information on distance-regular graphs with classical parameters can be found in [78, Ch. 6, 9]. Important to mention is that distance-regular graphs with classical parameters must be Q-polynomial. In Section 5, we will therefore include also some results on distance-regular graphs with classical parameters.

3.1.2 Other families with unbounded diameter

One of the ultimate problems in this area is to classify the families of distance-regular graphs with unbounded diameter. Besides the above known families of distance-regular graphs with classical parameters and the *polygons* (see Section 2.2.2), also the below six families are known. All of them are related to the classical ones, but they do not have classical parameters themselves.

The *folded cube* is obtained by folding the hypercube $H(n, 2)$. Unless $n = 6$, it is determined by its intersection array. For $n = 6$, every graph with the relevant intersection array is the incidence graph of a symmetric 2-$(16, 6, 2)$ design. This gives two other distance-regular graphs (see [78, §9.2.D]).

For n even, the folded cube is still bipartite (and the halved cube is still antipodal). Its halved graph is the *folded halved cube* and it is determined by its intersection array for $n \geqslant 12$ (that is, when its diameter is at least 3; see Section 5.3).

The Johnson graph $J(2n, n)$ is antipodal, and its folding is called a *folded Johnson graph*. This folded graph is determined by its intersection array for $n \geqslant 6$ (that is, when its diameter is at least 3; see Section 5.3).

The folded cube, folded halved cube, and folded Johnson graph are so-called *partition graphs* and these are known to be Q-polynomial (see [78, §6.3]).

In Section 2.2.3, we already described the *Odd graphs*, which are determined by their intersection array by Proposition 2.2. The Odd graph is the distance-D graph of the Johnson graph $J(2D + 1, D)$, and it is Q-polynomial.

	D	b	α	β
Johnson graph $J(n,D), n \geqslant 2D$	D	1	1	$n-D$
Grassmann graph $J_q(n,D)$, $n \geqslant 2D$; twisted Grassmann graph $(n = 2D+1)$	D	q	q	$\frac{q^{n-D+1}-1}{q-1}-1$
Hamming graph $H(D,e)$; Doob graph $(e=4)$	D	1	0	$e-1$
Halved Cube $\frac{1}{2}H(n,2)$	$\lfloor \frac{n}{2} \rfloor$	1	2	$2\lceil \frac{n}{2} \rceil - 1$
Bilinear forms graph $Bil(D \times e, q)$, $D \leqslant e$	D	q	$q-1$	$q^e - 1$
Alternating forms graph $Alt(n,q)$, $m = 2\lceil \frac{n}{2} \rceil - 1$	$\lfloor \frac{n}{2} \rfloor$	q^2	q^2-1	$q^m - 1$
Hermitian forms graph $Her(D,q^2)$	D	$-q$	$-q-1$	$-(-q)^D - 1$
Quadratic forms graph $Qua(n,q)$, $m = 2\lfloor \frac{n}{2} \rfloor + 1$	$\lfloor \frac{n+1}{2} \rfloor$	q^2	q^2-1	$q^m - 1$
Dual polar graph; Hemmeter graph $(e=0)$;	D	q	0	q^e
$^2\mathcal{A}_{2D-1}(\sqrt{q})$ also:	D	$-\sqrt{q}$	$\sqrt{q}\frac{1+\sqrt{q}}{1-\sqrt{q}}$	$\sqrt{q}\frac{1+(-\sqrt{q})^D}{1-\sqrt{q}}$
Half dual polar graph $\mathcal{D}_{n,n}(q)$, $m = 2\lceil \frac{n}{2} \rceil - 1$; Ustimenko graph	$\lfloor \frac{n}{2} \rfloor$	q^2	q^2+q	$\frac{q^{m+1}-1}{q-1}-1$

Table 1: Classical parameters of families of distance-regular graphs with unbounded diameter

Also the bipartite double of the Odd graph, the *Doubled Odd graph*, is determined by its intersection array (see [78, §9.1.D]), but it is not Q-polynomial.

The final known family of distance-regular graphs with unbounded diameter is the family of *Doubled Grassmann graphs*. This graph is the bipartite double of the distance-D graph of the Grassmann graph $J_q(2D + 1, D)$. Like the Doubled Odd graph, it is determined by its intersection array (see Section 9.2), and it is not Q-polynomial.

3.2 New constructions

In this section, we mention some relatively new constructions of distance-regular graphs.

3.2.1 The twisted Grassmann graphs

Van Dam and Koolen [176] constructed the first family of non-vertex-transitive distance-regular graph with unbounded diameter. These graphs have the same intersection array as certain Grassmann graphs, and are constructed as follows. Let q be a prime power, and let $D \geqslant 2$ be an integer. Let W be a $(2D + 1)$-dimensional vector space over $GF(q)$, and let H be a hyperplane in W. Vertices are the $(D + 1)$-dimensional subspaces of W that are not contained in H, and the $(D - 1)$-dimensional subspaces of H. Two vertices of the first kind are adjacent if they intersect in a D-dimensional subspace; a vertex of the first kind is adjacent to a vertex of the second kind if the first contains the second; and two vertices of the second kind are adjacent if they intersect in a $(D-2)$-dimensional subspace. This graph is distance-regular with the same intersection array as the Grassmann graph $J_q(2D + 1, D)$. In fact, this Grassmann graph and the twisted Grassmann graph are the point graph and line graph, respectively, of a partial linear space whose points are the D-dimensional subspaces of W, and where a $(D + 1)$-dimensional subspace of W that is not contained in H is incident to the D-dimensional subspaces that it contains, and a $(D - 1)$-dimensional subspace of H is incident to the D-dimensional subspaces of H containing it.

The twisted Grassmann graph is not vertex-transitive (it has two orbits of vertices), and hence it is not isomorphic to the Grassmann graph. Fujisaki, Koolen, and Tagami [235] showed that the automorphism group of the twisted Grassmann graphs is $P\Gamma L(2D + 1, q)_{2D}$, the subgroup of $P\Gamma L(2D + 1, q)$ that fixes H. Bang, Fujisaki, and Koolen [25] determined the spectra of the local graphs, and studied in some detail its Terwilliger algebras (as defined in Section 4.3). Remarkably, these algebras with respect to vertices in distinct orbits are not the same. The twisted Grassmann graphs are also counterexamples to two conjectures by Terwilliger [616, p. 207-210], see [25]. Jungnickel and Tonchev [372] constructed designs that are counterexamples for Hamada's conjecture. Munemasa and Tonchev [506] showed that the twisted Grassmann graphs are isomorphic to the block graphs of these designs. Munemasa [504] showed that the twisted Grassmann graphs can also be obtained from the Grassmann graphs by Godsil-McKay switching (cf. [85, §1.8.3]).

3.2.2 Brouwer-Pasechnik and Kasami graphs

For prime powers q, Pasechnik [93] constructed a distance-regular graph with intersection array $\{q^3, q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1, q^3\}$ as a subgraph of the dual polar graph $\mathcal{D}_4(q)$; in particular, the induced subgraph on the set of vertices at maximal distance from an edge.

Brouwer [93] constructed related distance-regular graphs with intersection array $\{q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1\}$ as follows. Consider the vector space $GF(q)^3$ equipped with a cross product $\times$. The vertex set is $(GF(q)^3)^2$, where a pair (u, v) is adjacent to a distinct pair (u', v') if and only if $u' = u + v \times v'$. The extended bipartite doubles of these graphs are the above mentioned graphs constructed by Pasechnik. In fact, Brouwer's graph is a subgraph of the dual polar graph $\mathcal{B}_3(q)$; in particular, the induced subgraph on the set of vertices at maximal distance from a vertex, see [93].

For even q, the mentioned graphs have the same intersection arrays as certain Kasami graphs, cf. [78, Thm. 11.2.1 (11),(13)]. Pasini and Yoshiara [538] constructed distance-regular graphs with the same intersection array as (bipartite, diameter 4) Kasami graphs using dimensional dual hyperovals. Also the symmetric bilinear forms graphs for q even and $n = 3$ are distance-regular with the same intersection array as (diameter 3) Kasami graphs, cf. [78, p. 285-286] and [74].

Van Dam and Fon-Der-Flaass used almost bent functions to generalize the Kasami graphs, cf. [168], [169, Con. 3]: Let W be an n-dimensional vector space over $GF(2)$, and f be an almost bent function on W with $f(0) = 0$. Then the graph with vertex set W^2, where two distinct vertices (x, a) and (y, b) are adjacent if $a + b = f(x + y)$ is distance-regular with intersection array $\{2^n - 1, 2^n - 2, 2^{n-1} + 1; 1, 2, 2^{n-1} - 1\}$. Recently, a lot of new almost bent functions have been discovered in the guise of quadratic almost perfect nonlinear functions in odd dimensional vector spaces over $GF(2)$, cf. [67, 97, 206].

3.2.3 De Caen, Mathon, and Moorhouse's Preparata graphs and crooked graphs

De Caen, Mathon, and Moorhouse [100] constructed distance-regular antipodal 2^{2t-1}-covers of the complete graph $K_{2^{2t}}$, i.e., with intersection array $\{2^{2t} - 1, 2^{2t} - 2, 1; 1, 2, 2^{2t} - 1\}$. These graphs are defined as follows. Consider the vertex set $V = GF(2^{2t-1}) \times GF(2) \times GF(2^{2t-1})$, and let two vertices (x, i, a) and (y, j, b) be adjacent if

$$a + b = x^2y + xy^2 + (i + j)(x^3 + y^3).$$

The construction is a bit more general, cf. [100], and is related to the Preparata codes. The construction also allows for taking quotients. In this way, distance-regular graphs with intersection arrays $\{2^{2t} - 1, 2^{2t} - 2^h, 1; 1, 2^h, 2^{2t} - 1\}$ for $h = 1, 2, \ldots, 2t$ arise. Prior to this construction, no distance-regular graphs with these intersection arrays were known for $h < t$.

It is noteworthy that the Kasami graphs of the previous section are induced subgraphs of the Preparata graphs. Because of this relation, it is not surprising that variations of the above construction are possible. To obtain these, De Caen and Fon-Der-Flaass [99]

used Latin squares, whereas Bending and Fon-Der-Flaass [43] and Van Dam and Fon-Der-Flaass [169] used highly nonlinear functions such as crooked functions and almost bent functions with accomplices: Let W be an n-dimensional vector space over $GF(2)$, and f be a crooked function on W. Then the (crooked) graph with vertex set $W \times GF(2) \times W$, where two distinct vertices (x, i, a) and (y, j, b) are adjacent if $a + b = f(x + y) + (i + j + 1)(f(x) + f(y))$ is distance-regular with the same intersection array as the Preparata graphs. Godsil and Roy [275] determined that the above equation defines a distance-regular graph precisely when f is crooked. The Gold functions, given by $f(x) = x^{2^e+1}$ on $GF(2^n)$ with $\gcd(e, n) = 1$ and $n = 2t - 1$, give the Preparata graphs.

It follows from the observations in [169, p. 92] that bijective quadratic almost perfect nonlinear functions (that map 0 to 0) are crooked. A new family of such functions was thus constructed by Budaghyan, Carlet, and Leander [96, Prop. 1]. See also [47], but beware that Bierbrauer used a less strict definition of crookedness (compared to the original one) there.

The paper by De Caen and Fon-Der-Flaass [99] initiated the prolific construction by Fon-Der-Flaass [232] of distance-regular n-covers of complete graphs K_{n^2} by using affine planes of order n. Fon-Der-Flaass realized that in general, his method produces many (potentially) non-isomorphic such graphs; at least $2^{\frac{1}{2}n^3 \log n(1+o(1))}$ to be more precise. Computational results by Degraer and Coolsaet [188] confirm this; they verified that at least 80 of the 94 distance-regular antipodal 4-covers of K_{16} can be constructed by Fon-Der-Flaass' prolific construction. Also the (three) distance-regular antipodal 4-covers of K_{10} [188], the (two) distance-regular antipodal 3-covers of K_{14} [187], and the (four) distance-regular antipodal 3-covers of K_{17} [187] were classified by computer by Degraer and Coolsaet. We also remark that Muzychuk [507] extended Fon-Der-Flaass' ideas further.

Godsil and Hensel [269] (see also [100]) described a relation between regular antipodal covers of complete graphs and generalized Hadamard matrices. By constructing skew generalized Hadamard matrices, Klin and Pech [401] thus constructed new infinite families of distance-regular antipodal covers of complete graphs. Their paper contains a good overview of the state of the art concerning such covers, and has many interesting ideas and connections. For more background on antipodal covers of complete graphs, we also refer to Godsil and Hensel [269] and Godsil [266]. For the classification of distance-transitive antipodal covers of complete graphs, we refer to the paper by Godsil, Liebler, and Praeger [271].

3.2.4 Soicher graphs and Meixner graphs

Soicher [564] obtained three distance-regular graphs of diameter four, each being a triple cover of a strongly regular graph. The first has intersection array $\{416, 315, 64, 1; 1, 32, 315, 416\}$, and is a triple cover of the Suzuki graph. The second has intersection array $\{56, 45, 16, 1; 1, 8, 45, 56\}$, and is a triple cover of the second subconstituent of the McLaughlin graph. In an unpublished manuscript, Brouwer [71] (see also [74]) showed that this cover is the only cover of the second subconstituent of the McLaughlin graph, hence it is the only graph with the given intersection array. The third cover constructed by Soicher is the second subconstituent of the second one, it has intersection

array $\{32, 27, 8, 1; 1, 4, 27, 32\}$, and is a triple cover of the Goethals-Seidel graph (the second subconstituent of the second subconstituent of the McLaughlin graph). Soicher [566] also showed that this graph is the only graph with the given intersection array.

Meixner [476] implicitly constructed two distance-transitive antipodal covers with intersection arrays $\{176, 135, 36, 1; 1, 12, 135, 176\}$ and $\{176, 135, 24, 1; 1, 24, 135, 176\}$, as the collinearity graphs of the geometries in [476, Prop. 4.3], see also [74]. Jurišić and Koolen [380] showed that the antipodal Meixner 4-cover is uniquely determined by its intersection array.

Munemasa observed that the Meixner 2-cover is the extended Q-bipartite double of the Moscow-Soicher graph of the next section, cf. [469, Ex. 3.4].

3.2.5 The Koolen-Riebeek graph and the Moscow-Soicher graph

Brouwer, Koolen, and Riebeek [91] gave a construction of a bipartite distance-regular graph with intersection array $\{45, 44, 36, 5; 1, 9, 40, 45\}$ from the ternary Golay code. Each of its halved graphs is the complement of the Berlekamp-van Lint-Seidel graph.

Soicher [565] constructed another distance-regular graph related to one of the Golay codes, in this case the binary. It has intersection array $\{110, 81, 12; 1, 18, 90\}$. Faradžev, Ivanov, Klin, and Muzychuk [213, p. 119] already mentioned the underlying association scheme of this graph without realizing it was metric.

4 More background

4.1 Miscellaneous definitions

A non-complete k-regular graph Γ on v vertices is called *strongly regular* with parameters (v, k, λ, μ) if each two adjacent vertices have λ common neighbors, and each two nonadjacent vertices have μ common neighbors. Thus, the connected strongly regular graphs are precisely the distance-regular graphs with diameter two. The definition of an *amply regular graph* with parameters (v, k, λ, μ) is obtained by replacing the condition on the nonadjacent vertices by the condition that each two vertices at distance 2 have μ common neighbors.

For a graph Γ and $x \in V$, the graph induced on the set $\Gamma_i(x)$ is called an *i*-th *subconstituent* of Γ. The first subconstituent in consideration is also called a *local graph* of Γ, and is denoted by $\Upsilon(x)$. We say that Γ is *locally* Δ if all local graphs are isomorphic to Δ. More generally, we let $\Upsilon(x, y)$ be the induced subgraph on the set of common neighbors of x and y (so it is a local graph of a local graph if x and y are adjacent), etc.. A *Terwilliger graph* is a non-complete graph such that $\Upsilon(x, y)$ is a clique of size μ for each two vertices x and y at distance two, for some μ. Thus, a Terwilliger graph has no induced quadrangles.

Let Γ be a distance-regular graph with valency k and diameter D. Let $\ell(c, a, b) = |\{i = 1, 2, \ldots, D-1 : (c_i, a_i, b_i) = (c, a, b)\}|$. In particular, let $h = h(\Gamma)$ and $t = t(\Gamma)$ be defined by $h(\Gamma) = \ell(c_1, a_1, b_1)$ and $t(\Gamma) = \ell(b_1, a_1, c_1)$. The parameter $h(\Gamma)$ is called the *head* of Γ and $t(\Gamma)$ is called the *tail* of Γ.

The *girth* of Γ is the length of its shortest cycle. The *numerical girth* of Γ is $2h + 3$ if $c_{h+1} = 1$, else it is $2h + 2$. If $a_1 = 0$, then the girth is equal to the numerical girth. If Γ is locally a disjoint union of cliques, then the *geometric girth* of Γ is the minimal length of a cycle for which the induced subgraph on each triple of its vertices is not a triangle; this equals the numerical girth. If Γ has a local graph that is not a disjoint union of cliques, then the geometric girth is defined as 3. For geometric graphs, the geometric girth is half the girth of the incidence graph of the corresponding partial linear space (see Section 4.5). Note that the girth and the numerical girth are determined by the intersection array, but in general the geometric girth is not (for example, the Doob graphs have geometric girth 3, whereas the Hamming graphs (with the same intersection array) have geometric girth 4).

A quadruple (x, y, z, u) of vertices is called a *parallelogram of length i* if $d(x, y) = 1 = d(z, u)$, $d(x, z) = d(y, u) = d(y, z) = i - 1$, and $d(x, u) = i$. The graph Γ is called *m-parallelogram-free* for some $m = 2, 3, \ldots, D$ if Γ does not contain any parallelogram of length at most m. We say Γ is parallelogram-free if it does not contain any parallelogram. Related conditions called $(\text{CR})_m$ and $(\text{SS})_m$ are given by Hiraki [307, 310].

A quadruple (x, y, z, u) of vertices of Γ is called a *kite of length i* if $d(x, y) = d(x, z) = d(y, z) = 1$, $d(x, u) = i$, and $d(y, u) = d(z, u) = i - 1$.

A subgraph Δ of Γ is called *geodetically closed*, or *closed* for short, if $z \in V_\Delta$ for all $x, y \in V_\Delta$ and z on a geodetic between x and y. (A closed subgraph is also called *convex* by some authors.) The subgraph Δ is called *strongly closed* if $z \in V_\Delta$ for all vertices $x, y \in V_\Delta$ and $z \in V_\Gamma$ such that $d_\Gamma(x, z) + d_\Gamma(z, y) \leqslant d_\Gamma(x, y) + 1$. (The term *weak-geodetically closed* is also used for strongly closed.) It is known that if $c_2 > 1$ then all strongly closed subgraphs are regular; cf. [649, Lemma 5.2] or [582]. A distance-regular graph Γ with diameter D is said to be *m-bounded* for some $m = 1, 2, \ldots, D$ if for all $i = 1, 2, \ldots, m$ and all vertices x and y at distance i there exists a strongly-closed subgraph $\Delta(x, y)$ with diameter i, containing x and y as vertices. (Note that Weng [648, 650] also required that $\Delta(x, y)$ is regular.)

4.2 A few comments on the eigenspaces

Consider an association scheme with primitive idempotents $E_0, E_1, \ldots, E_D$. By computing the squared norm, it follows that

$$\sum_{x \in V} E_i \mathbf{e}_x \otimes E_j \mathbf{e}_x \otimes E_h \mathbf{e}_x = 0 \quad \text{if and only if} \quad q_{ij}^h = 0 \quad (h, i, j = 0, 1, \ldots, D), \quad (8)$$

where $\mathbf{e}_x \in \mathbb{R}^v$ denotes the characteristic vector of $\{x\}$. In fact, this computation also gives an alternative proof of the Krein conditions; cf. Proposition 2.9. Recall that the absolute bound (cf. Proposition 2.10) was an immediate consequence of the obvious observation that $(E_i \circ E_j)\mathbb{R}^v \subseteq \text{span}(E_i\mathbb{R}^v \circ E_j\mathbb{R}^v)$. We remark here that these two subspaces indeed coincide:

$$\text{span}(E_i\mathbb{R}^v \circ E_j\mathbb{R}^v) = (E_i \circ E_j)\mathbb{R}^v = \sum_{q_{ij}^h \neq 0} E_h\mathbb{R}^v \quad (i, j = 0, 1, \ldots, D). \quad (9)$$

To see this, note that $\langle \mathbf{u} \circ \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}, \sum_{x \in V} E_i \mathbf{e}_x \otimes E_j \mathbf{e}_x \otimes E_h \mathbf{e}_x \rangle$ for all $\mathbf{u} \in E_i \mathbb{R}^v$, $\mathbf{v} \in E_j \mathbb{R}^v$, and $\mathbf{w} \in E_h \mathbb{R}^v$, where $\langle \, , \, \rangle$ denotes the standard inner product. Therefore, it follows from (8) that $\mathrm{span}(E_i \mathbb{R}^v \circ E_j \mathbb{R}^v)$ is orthogonal to $E_h \mathbb{R}^v$ whenever $q_{ij}^h = 0$. These results are due to Cameron, Goethals, and Seidel [106] (cf. [38, §II.8], [596]), and are quite fundamental in the theory of distance-regular graphs and association schemes; see, e.g., Sections 4.3 and 6.3.1. We note that, in view of (9), the ordering $E_0, E_1, \ldots, E_D$ is Q-polynomial if and only if $\sum_{\ell=0}^{i} (E_1 \mathbb{R}^v)^{\circ \ell} = \sum_{\ell=0}^{i} E_\ell \mathbb{R}^v$ for all $i = 0, 1, \ldots, D$, where $(E_1 \mathbb{R}^v)^{\circ \ell} = E_1 \mathbb{R}^v \circ E_1 \mathbb{R}^v \circ \cdots \circ E_1 \mathbb{R}^v$ (ℓ times).

4.3 The Terwilliger algebra

The *Terwilliger* (or *subconstituent*) *algebra* of an association scheme was introduced in [616]. Though it should be stressed that this algebra also plays an important role in the theory of general distance-regular graphs (cf. Section 6), it is particularly well-suited for Q-polynomial distance-regular graphs. In fact, this algebra has (part of) its roots in the study of *balanced sets* (cf. (14)); see, e.g., [614, p. 93, Note 1].

In the context of the Terwilliger algebra, the Bose-Mesner algebra of an association scheme is always assumed to be *over* $\mathbb{C}$, that is,

$$\mathbb{A} = \mathrm{span}_{\mathbb{C}}\{A_0, A_1, \ldots, A_D\} \subset M_{v \times v}(\mathbb{C}).$$

Fix a 'base vertex' $x \in V$. For each $i = 0, 1, \ldots, D$, let $E_i^\star = E_i^\star(x)$, $A_i^\star = A_i^\star(x)$ be the diagonal matrices[2] in $M_{v \times v}(\mathbb{C})$ with diagonal entries $(E_i^\star)_{yy} = (A_i)_{xy}$, $(A_i^\star)_{yy} = v(E_i)_{xy}$. Note that $E_i^\star E_j^\star = \delta_{ij} E_i^\star$, $\sum_{i=0}^{D} E_i^\star = I$, and moreover $A_i^\star A_j^\star = \sum_{h=0}^{D} q_{ij}^h A_h^\star$. These matrices span the *dual Bose-Mesner algebra* $\mathbb{A}^\star = \mathbb{A}^\star(x)$ *with respect to* x:

$$\mathbb{A}^\star = \mathrm{span}_{\mathbb{C}}\{E_0^\star, E_1^\star, \ldots, E_D^\star\} = \mathrm{span}_{\mathbb{C}}\{A_0^\star, A_1^\star, \ldots, A_D^\star\} \subset M_{v \times v}(\mathbb{C}).$$

Note that if the association scheme is Q-polynomial with respect to the ordering $(E_i)_{i=0}^{D}$ then $A_1^\star$ generates $\mathbb{A}^\star$. The *Terwilliger algebra* $\mathbb{T} = \mathbb{T}(x)$ *with respect to* x is the subalgebra of $M_{v \times v}(\mathbb{C})$ generated by $\mathbb{A}$ and $\mathbb{A}^\star$ [616]. The following are relations in $\mathbb{T}$:

$$E_i^\star A_j E_h^\star = 0 \ \Leftrightarrow \ p_{ij}^h = 0, \quad E_i A_j^\star E_h = 0 \ \Leftrightarrow \ q_{ij}^h = 0 \quad (h, i, j = 0, 1, \ldots, D). \tag{10}$$

We note that the latter is a variation of (8). Because $\mathbb{T}$ is closed under conjugate-transposition, it is semisimple and every two non-isomorphic irreducible $\mathbb{T}$-modules in $\mathbb{C}^v$ are orthogonal. Let G be the full automorphism group of the association scheme. Then $\mathbb{T}$ is a subalgebra of the centralizer algebra[3] of the action of the stabilizer G_x of x on $\mathbb{C}^v$. The two algebras are known to be equal, e.g., for Hamming graphs; cf. [251, Prop. 3]. We also note that the structure of $\mathbb{T}$ may depend on the choice of x if G is not transitive on V; cf. Section 3.2.1.

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[2] We use $\star$-notation instead of the usual $*$-notation in order to avoid confusion with the conjugate transpose.

[3] Dunkl [202, 203, 204, 205] and Stanton [570] studied this latter algebra in detail in the context of addition theorems for orthogonal polynomials associated with some classical families of distance-regular graphs.

Let W be an irreducible $\mathbb{T}$-module. When the association scheme is P-polynomial (resp. Q-polynomial) with respect to the ordering $(A_i)_{i=0}^D$ (resp. $(E_i)_{i=0}^D$), we define the *endpoint* (resp. *dual endpoint*) of W by $\min\{i : E_i^\star W \neq 0\}$ (resp. $\min\{i : E_i W \neq 0\}$). We call W *thin* (resp. *dual thin*) if $\dim E_i^\star W \leqslant 1$ (resp. $\dim E_i W \leqslant 1$) for $i = 0, 1, \ldots, D$. We also define the *diameter* and the *dual diameter* of W by $|\{i : E_i^\star W \neq 0\}| - 1$ and $|\{i : E_i W \neq 0\}| - 1$, respectively. If the association scheme is P-polynomial (resp. Q-polynomial), then thin (resp. dual thin) implies dual thin (resp. thin) [616]. There is a unique irreducible $\mathbb{T}$-module with $E_0^\star W \neq 0$ and $E_0 W \neq 0$, called the *primary* (or *trivial*) $\mathbb{T}$-module; it is thin, dual thin, and given by $\mathrm{span}_{\mathbb{C}}\{A_0 \mathbf{e}_x, A_1 \mathbf{e}_x, \ldots, A_D \mathbf{e}_x\}$, where $\mathbf{e}_x \in \mathbb{C}^v$ denotes the characteristic vector of $\{x\}$. We say the association scheme is *i-thin with respect to* x if every irreducible $\mathbb{T}(x)$-module W with $E_i^\star W \neq 0$ is thin.[4] It is said to be *thin with respect to* x if it is i-thin with respect to x for all $i = 0, 1, \ldots, D$. Finally, we say the association scheme is *thin* (resp. *i-thin*) if it is thin (resp. i-thin) with respect to x for all $x \in V$.

In the study of the Terwilliger algebra, it is often quite important to consider the following three matrices:

$$L = \sum_{i=1}^D E_{i-1}^\star A E_i^\star, \quad F = \sum_{i=0}^D E_i^\star A E_i^\star, \quad R = \sum_{i=0}^{D-1} E_{i+1}^\star A E_i^\star, \tag{11}$$

called the *lowering*, *flat*, and *raising matrices*, respectively. Note that $A = L + F + R$. As an illustrative example, suppose Γ is the D-cube $H(D, 2)$, and let $A^\star = A_1^\star = \sum_{i=0}^D (D - 2i) E_i^\star$ correspond to the Q-polynomial idempotent E_1 associated with the second largest eigenvalue $\theta_1 = D - 2$. Then $F = 0$ because Γ is bipartite, and it follows that L, R, and $A^\star$ generate $\mathbb{T}$. Moreover, we can easily verify that $LR - RL = A^\star$, $RA^\star - A^\star R = 2R$, and $LA^\star - A^\star L = -2L$, so that the Terwilliger algebra $\mathbb{T}$ is a homomorphic image of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Therefore, every irreducible $\mathbb{T}$-module W has the structure of an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module, and $\bigoplus_{i=e}^{D-e} E_i^\star W$ gives the weight space decomposition of W, where e denotes the endpoint of W. In particular, $H(D, 2)$ is thin. We refer the reader to Terwilliger [618] and Go [260] for more details.

4.4 Equitable partitions and completely regular codes

4.4.1 Interlacing, the quotient matrix, and the quotient graph

Eigenvalue interlacing is a useful tool in studying distance-regular graphs, and more generally, in spectral graph theory; see the survey by Haemers [283]. A sequence of numbers $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$ is said to *interlace* a sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, with $n > m$, if $\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i}$ for all $i = 1, 2, \ldots, m$. The interlacing is called *tight* if for some

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[4]The definition of the i-thin condition here is taken from [197, 199]. This is slightly different from the standard one for the case when the association scheme is P-polynomial, where it is called *i-thin with respect to* x if every irreducible $\mathbb{T}(x)$-module with endpoint at most i is thin. On the other hand, the present definition of course has the advantage that it makes sense for general association schemes. We shall be careful below not to cause any confusion when we discuss results involving this concept.

$k \in \{0, 1, \ldots, m\}$ the equalities $\lambda_i = \mu_i, i = 1, \ldots, k$ and $\lambda_{n-m+i} = \mu_i, i = k+1, \ldots, m$ hold.

An elementary interlacing result states that the eigenvalues of a principal submatrix B of a symmetric matrix A interlace the eigenvalues of A itself. When applied to graphs: the eigenvalues of an induced subgraph of a graph Γ interlace the eigenvalues of Γ.

A somewhat more complicated — but very useful — result concerns the so-called quotient matrix. Let $\Pi = \{P_1, P_2, \ldots, P_t\}$ be a partition of the vertex set of a graph Γ. Let f_{ij} be the average number of neighbors in P_j of a vertex in P_i, for $i, j = 1, 2, \ldots, t$. The matrix $F = (f_{ij})$ is called the *quotient matrix* of Π. The partition Π is called *equitable* if every vertex in P_i has exactly f_{ij} neighbors in P_j. Also the eigenvalues of F interlace the eigenvalues of Γ. Moreover, if the interlacing is tight, then the partition is equitable. In this case, it can easily be seen that an eigenvector $\mathbf{u}$ of F can be 'blown up' to an eigenvector $\mathbf{v}$ of Γ (with the same eigenvalue) by setting $v_x = u_i$ if $x \in P_i$. An example of an equitable partition in a distance-regular graph Γ is the distance partition $\Pi = \{\Gamma_0(z), \Gamma_1(z), \ldots, \Gamma_D(z)\}$ of a vertex z, and its quotient matrix is the intersection matrix L as in (3).

Given a partition $\Pi = \{P_1, P_2, \ldots, P_t\}$ of the vertex set of a graph Γ, we define the *quotient graph* Γ/Π with vertex set Π, where $P_i \sim P_j$ if $i \neq j$ and there exist $x \in P_i$ and $y \in P_j$ such that $x \sim y$ in Γ.

We call an equitable partition Π *uniformly regular* if its quotient matrix F and the adjacency matrix B of Γ/Π are related as $F = fI + \tilde{f}B$, for some numbers f and $\tilde{f} \neq 0$. It is clear that in this case, the eigenvalues of the quotient Γ/Π follow in a straightforward way from the eigenvalues of F, and the latter are eigenvalues of Γ, as we just observed. An example of a uniformly regular partition is given by the partition into fibres of an antipodal distance-regular graph. In this case, the corresponding quotient graph is the folded graph.

4.4.2 Completely regular codes

Let Γ be a connected graph, say with diameter D, and let C be a subset of $V = V_\Gamma$. For $i \geqslant 0$, let $C_i = \{x \in V : d(x, C) = i\}$, where $d(x, C) = \min\{d(x, c) : c \in C\}$. The *covering radius* of C, denoted by $\rho = \rho(C)$, is the maximum i such that $C_i \neq \emptyset$. The subset (or code) C is called *completely regular* if the distance partition $\Pi = \{C_i : i = 0, 1, \ldots, \rho\}$ is equitable. Note that the corresponding quotient matrix is tridiagonal; it is therefore common to denote $f_{i,i-1}, f_{i,i}$ and $f_{i,i+1}$ by γ_i, α_i, and β_i, respectively. These numbers are called the *intersection numbers* of C. This definition of a completely regular subset (or code) was introduced by Neumaier [512] and he showed that for distance-regular graphs it is equivalent to Delsarte's definition [189, p. 67] in terms of the so-called outer distribution. It is clear that if C is completely regular then so is C_ρ. Note that for a distance-regular graph, every singleton $\{z\}$ is a completely regular code with $\gamma_i = c_i, \alpha_i = a_i$, and $\beta_i = b_i$. In general, the behavior of the intersection numbers of a completely regular code can however be quite different from that of the intersection numbers of a distance-regular graph. For example, it is not true in general that the γ_i are non-decreasing; see [409]. For more background information on completely regular codes, we refer to the work of Martin

[462, 466].

A partition $\Pi = \{P_1, P_2, \ldots, P_t\}$ of V is called a *completely regular partition* if it is equitable and all of the P_i are completely regular with the same intersection numbers. It is known that a completely regular partition is uniformly regular. A typical (and motivating) example of a completely regular partition is the partition into cosets of a linear code C of length n over $GF(q)$ that is completely regular in the Hamming graph $H(n, q)$. (More generally, we can consider a completely regular additive code in a translation distance-regular graph.[5]) In this case we call Γ/Π the coset graph of C. This coset graph is distance-regular by the following result.

Theorem 4.1. [78, Thm. 11.1.5, 11.1.8] *Let Γ be a distance-regular graph and Π a uniformly regular partition of Γ with quotient matrix F. Then Γ/Π is distance-regular if and only if Π is completely regular. Moreover, if so, then the intersection numbers of Γ/Π can be explicitly calculated from the intersection numbers of Γ and F.*

Delsarte cliques are examples of completely regular codes. Indeed, the following result characterizes such cliques.

Proposition 4.2. [264, Lemmas 13.7.2, 13.7.4] *Let Γ be a distance-regular graph with valency k, diameter D and smallest eigenvalue $\theta_{\min}$. Let C be a clique in Γ with c vertices. Then C is a Delsarte clique if and only if C is a completely regular code with covering radius $D - 1$. Moreover, if so, then $\phi_i u_i + (c - \phi_i)u_{i+1} = 0$, where $(u_i)_{i=0}^{D}$ is the standard sequence for $\theta_{\min}$ and $\phi_i = |\Gamma_i(x) \cap C|$ for a vertex x at distance i from C.*

Note that the equation $\phi_i u_i + (c - \phi_i)u_{i+1} = 0$ follows from the fact that $E\chi = 0$ (with E and χ as in the proof of Proposition 2.11). Indeed, if $E = UU^\top$, then $U^\top \chi = 0$, and hence for the corresponding representation associated to $\theta_{\min}$ (see Section 2.5) we have that

$$\sum_{z \in C} \hat{z} = 0. \tag{12}$$

Taking the inner product with $\hat{x}$, where x is a vertex at distance i from C gives the required equation. This implies (by using [512, Thm. 4.1]) that the intersection numbers of a Delsarte clique can be explicitly calculated from the intersection numbers of Γ.

For a subset of the vertex set of an association scheme, with characteristic vector χ, the *degree* and *dual degree* are defined by $|\{i \neq 0 : \chi^\top A_i \chi \neq 0\}|$ and $|\{i \neq 0 : \chi^\top E_i \chi \neq 0\}|$, respectively.

4.5 Distance-biregular graphs and weakly geometric graphs

For an arbitrary graph with vertices x and y at distance i, we define $c_i(x, y)$, $a_i(x, y)$, and $b_i(x, y)$ as the numbers of neighbors of y that are at distance $i-1$, i, and $i+1$, respectively. Thus, a connected graph with diameter D is distance-regular if these numbers do not depend on x and y (but only on their distance i). If in an arbitrary graph the numbers

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[5]A *translation* distance-regular graph is a distance-regular Cayley graph on an abelian group. An *additive* code in such a graph is just a subgroup of the abelian group (= vertex set).

$c_i(x, y)$, $a_i(x, y)$, or $b_i(x, y)$ do not depend on x and y, for some i, then we will write c_i, a_i, or b_i, respectively (as in distance-regular graphs). For example, in an arbitrary bipartite graph, one has $a_i = 0$ for all i.

For ease of notation and formulation, we will call the two biparts of a bipartite graph its color classes R and B, and say that a vertex in R is red, and a vertex in B is blue.

Now a connected bipartite graph is called *distance-biregular* if the numbers $c_i(x, y)$ and $b_i(x, y)$ depend only on i and the color of x. We denote these numbers by c_i^R, b_i^R, c_i^B, and b_i^B. Straightforward examples are the complete bipartite graphs.

We say that a graph Γ is distance-regular around a vertex x if the singleton $\{x\}$ is a completely regular code in Γ. A well-known result by Godsil and Shawe-Taylor [277] states that if Γ is a connected graph that is distance-regular around every vertex, then Γ is distance-regular or distance-biregular.

A bipartite graph is called *semiregular* (or biregular) if the valency of a vertex only depends on its color. We denote these valencies by k_R and k_B.

Powers [543] used the term semiregular for a concept that he introduced, and what we now call distance-semiregular (following Suzuki [581, 587]). A connected bipartite graph is called *distance-semiregular* with respect to one of its color classes, R say, if it is distance-regular around all red vertices, with the same parameters (i.e, there are b_i^R and c_i^R such that $b_i(x, y) = b_i^R$ and $c_i(x, y) = c_i^R$ if $x \in R$ and $d(x, y) = i$). Note that every distance-biregular graph is distance-semiregular, and in turn, each distance-semiregular graph is semiregular, with valencies $k_R = b_0^R$ and $k_B = 1 + b_1^R$. The Hoffman graph [328] (the unique graph cospectral but not isomorphic to $H(4, 2)$) is an example of a (regular!) distance-semiregular graph that is not distance-biregular.

Let Γ be distance-semiregular with respect to R, then its halved graph Γ_2^R (i.e., the distance-2 graph of Γ, induced on R) is distance-regular. Let $C = \Gamma(x)$ for some blue vertex x. Then C is a clique in Γ_2^R, that is also a completely regular code in Γ_2^R. This leads to the following definition.

A distance-regular graph Δ is called *weakly geometric* (with respect to $\mathcal{C}$) if it contains a collection $\mathcal{C}$ of cliques such that each edge is contained in a unique $C \in \mathcal{C}$ and all $C \in \mathcal{C}$ are completely regular codes with the same parameters. Thus, a geometric distance-regular graph (see Section 2.8) is weakly geometric. Because of the property that each edge is contained in a unique clique, there is a naturally associated partial linear space, whose points are the vertices of Δ and whose lines are the cliques of $\mathcal{C}$, and incidence is defined by containment. The point (or collinearity) graph of this partial linear space is Δ. The bipartite (point-line) incidence graph of the partial linear space is a distance-semiregular graph with girth at least 6; in fact, this gives a one-to-one correspondence between the latter type of graphs and weakly geometric distance-regular graphs. The partial linear space has also been studied by De Clerck, De Winter, Kuijken, and Tonesi [186, 427] under the name *distance-regular geometry*.

Using the same correspondence, certain distance-semiregular graphs with girth 4 correspond to Delsarte graphs and Delsarte clique graphs as introduced by Bang, Hiraki, and Koolen [28] (see also [29]). Delsarte graphs and Delsarte clique graphs are closely related to the geometric distance-regular graphs of Section 2.8.

We remark that the Johnson graphs $J(n, D)$ and Grassmann graphs $J_q(n, D)$ are not just (weakly) geometric with respect to a set of Delsarte cliques (the $(D-1)$-sets or $(D-1)$-dimensional subspaces; that is, the set of vertices containing a fixed $(D-1)$-set or $(D-1)$-space is a Delsarte clique), but also weakly geometric with respect to another set of cliques, namely the $(D+1)$-sets or $(D+1)$-dimensional subspaces (i.e., the sets of vertices contained in these), respectively. The corresponding incidence graphs are distance-biregular; for $n = 2D+1$, we obtain the distance-regular Doubled Odd graph and Doubled Grassmann graph, respectively. In Section 9 we will discuss geometric distance-regular graphs in more detail.

Following Suzuki [587], we say a distance-regular graph Γ is *of order* (s, t) (for some integers s and t) if it is locally the disjoint union of $t + 1$ cliques of size s. This is equivalent to the property that Γ contains no induced complete tripartite graph $K_{2,1,1}$ (a kite of length 2).

A distance-regular graph Γ of order (s, t) with diameter D is called a *regular near polygon* if $a_i = c_i a_1$ for all $i = 1, 2, \ldots, D-1$. If $a_d = c_D a_1$ we call Γ a regular near $2D$-gon; otherwise it is called a regular near $(2D+1)$-gon. A regular near polygon of diameter D is geometric if and only if it is a regular near $2D$-gon. We say Γ is *thick* if $s \geqslant 2$ (the regular near polygons with $s = 1$ are exactly the bipartite distance-regular graphs and the generalized odd graphs).

Weng [650] defined a distance-regular graph to have *geometric parameters* (D, b, α) if it has classical parameters (D, b, α, β) with $b \neq 1$ and $\beta = \alpha \frac{1+b^D}{1-b}$. He used this concept in the partial classification of distance-regular graphs with classical parameters with $b < -1$. This does not seem to be related to geometric distance-regular graphs.

4.6 Homogeneity

Let Γ be a connected graph. For two distinct vertices x and y, define $\Gamma_{i,j}(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$. If it is clear (or irrelevant) which pair x, y is meant we will write $\Gamma_{i,j}$ instead of $\Gamma_{i,j}(x, y)$. For $u \in \Gamma_{i,j}$, let $p_{i,j;r,s}(u) = |\{z \in \Gamma_{r,s} : z \sim u\}|$. We say the parameter $p_{i,j;r,s}$ exists with respect to the pair x, y if $p_{i,j;r,s}(u) = p_{i,j;r,s}(u')$ for all $u, u' \in \Gamma_{i,j}(x, y)$.

A connected graph Γ with diameter D is called *i-homogeneous* (in the sense of Nomura), $i = 0, 1, \ldots, D$ if for all pairs x, y at distance i and all $r, s, r', s' \in \{0, 1, \ldots, D\}$, the parameter $p_{r,s;r',s'}$ exists and does not depend on the pair x, y, or in other words, the partition $\{\Gamma_{i,j}(x, y) : \Gamma_{i,j}(x, y) \neq \emptyset, i, j = 0, 1, \ldots, D\}$ is equitable for each pair x, y at distance i and the parameters do not depend on the pair x, y.[6]

Note that a 0-homogeneous graph is distance-regular, and a 1-homogeneous graph is distance-regular. Examples of 1-homogeneous distance-regular graphs are the Johnson graphs $J(2D, D)$, the bipartite distance-regular graphs, and the regular near $2D$-gons. To study i-homogeneous graphs, it is sometimes useful to draw intersection diagrams with respect to two vertices x and y. In Figure 3 we have an example of such an intersection diagram for the Johnson graph $J(6, 3)$.

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[6]In most of the literature $\Gamma_{i,j}$ is denoted as D_i^j. We chose different notation because D stands for the diameter, and the superscript-subscript notation seems useful only in intersection diagrams.
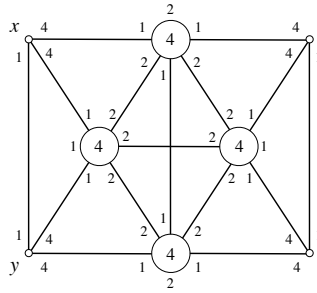
Figure 3: Intersection diagram of $J(6,3)$

4.7 Designs

Consider an association scheme with primitive idempotents E_i $(i = 0, 1, \ldots, D)$. Let T be a subset of $\{1, 2, \ldots, D\}$. A set Y of vertices of the association scheme with characteristic vector χ is called a (*Delsarte*) *T-design* if $E_i\chi = 0$ for all $i \in T$. This definition is due to Delsarte [189].

Suppose that the association scheme is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$. In this case, a $\{1, 2, \ldots, t\}$-design is simply called a t-design. The *strength* of Y is then defined by $\min\{i \neq 0 : E_i\chi \neq 0\} - 1$, i.e., it is the maximum integer t for which Y is a t-design. Delsarte [189] showed that the t-designs in the Johnson graphs and Hamming graphs are precisely the combinatorial block t-designs and the orthogonal arrays of strength t, respectively. A similar interpretation was established for the other classical families of distance-regular graphs by Delsarte [190], Munemasa [502], and Stanton [572].

For more results on T-designs in association schemes, see the recent survey by Martin and Tanaka [470] and the references therein.

5 Q-polynomial distance-regular graphs

In this section, we collect (relatively new) results on Q-polynomial distance-regular graphs. Throughout this section, we shall use the following notation unless otherwise stated. Let Γ denote a distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$. Let θ be an eigenvalue of Γ, E the corresponding primitive idempotent, and $(u_i)_{i=0}^{D}$ the standard sequence with respect to θ.

Suppose for the moment that E is Q-polynomial, and let $E_0, E_1 = E, E_2, \ldots, E_D$ be the corresponding Q-polynomial ordering. Then by Leonard's theorem (cf. [78, §8.1]) there exist $p, r, r^\star \in \mathbb{C}$ such that

$$u_{i-1} + u_{i+1} = pu_i + r, \quad \theta_{i-1} + \theta_{i+1} = p\theta_i + r^\star \quad (i = 1, 2, \ldots, D-1). \tag{13}$$

It should be remarked that the sequence of polynomials $(v_i)_{i=0}^{D}$ (see (2)) belongs to the terminating branch of the Askey scheme [403, 402] of (basic) hypergeometric orthogonal

polynomials. (We also allow the specialization[7] $q \to -1$.) See also [627]. We call E *classical* if $(u_{i-1} - u_i)/(u_i - u_{i+1})$ is independent of $i = 1, 2, \ldots, D - 1$. It follows that Γ has a classical Q-polynomial idempotent if and only if it has classical parameters such that $p = b + b^{-1}$; cf. [78, Thm. 8.4.1], [598, Prop. 6.2].

We begin with discussions on graphs with classical parameters.

5.1 The graphs with classical parameters with $b = 1$

All graphs with classical parameters with $b = 1$ have been determined: the Hamming graphs, Doob graphs, halved cubes, Johnson graphs, and the Gosset graph; cf. [78, Thm. 6.1.1] or [510]. Main contributors to this classification were Egawa [207], who characterized the Hamming and Doob graphs, and Terwilliger [608, 610] and Neumaier [510], who used the classification of root lattices and the representation with respect to the second largest eigenvalue to come to the final classification. We note that $b = 1$ implies $\theta_1 = b_1 - 1$, and the graphs satisfying the latter have been classified; cf. [78, Thm. 4.4.11]. In [424, 561], Koolen and Shpectorov used metric theory to classify the distance-regular graphs whose distance-matrix has exactly one positive eigenvalue. The distance-regular graphs with classical parameters with $b = 1$ have this property. Godsil [267] considered the convex hull of the representation with respect to a fixed eigenvalue. He classified when the 1-skeleton of this polytope with respect to the second largest eigenvalue is isomorphic to the original distance-regular graph. In his classification he again finds all distance-regular graphs with classical parameters with $b = 1$.

5.2 Recent results on graphs with classical parameters

Metsch [481, Cor. 1.3] showed that if Γ has classical parameters and is not a Johnson, Grassmann, Hamming, or bilinear forms graph, then the parameter β is bounded in terms of D, b, and α.

Terwilliger [619] showed that if Γ has classical parameters with $b < -1$ then Γ has no kites of any length $i = 2, 3, \ldots, D$. This result, combined with earlier work of Ivanov and Shpectorov [364], proves that the Hermitian forms graphs $Her(D, q^2)$ with $D \geqslant 3$ are uniquely determined by their intersection arrays. See also [647]. A related result by Weng [649] is as follows (cf. Section 11.1).

Proposition 5.1. *Suppose Γ is Q-polynomial with $D \geqslant 3$, $c_2 > 1$, and $a_1 \neq 0$. Then the following are equivalent:*

(i) *Γ has classical parameters, and either $b < -1$, or Γ is a dual polar graph or a Hamming graph,*

(ii) *Γ has no parallelogram of length 2 or 3,*

(iii) *Γ is D-bounded.*

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[7]The polynomials corresponding to the case $q = -1$ have recently been receiving considerable attention; see, e.g., [249] and the references therein.

Liang and Weng [448] showed that if Γ is Q-polynomial and $D \geqslant 4$ then Γ is parallelogram-free if and only if either (i) Γ is bipartite, (ii) Γ is a generalized odd graph, or (iii) Γ has classical parameters and either $b < -1$ or Γ is a Hamming graph or a dual polar graph. Weng [648] showed that if Γ has classical parameters with $b < -1, a_1 \neq 0, c_2 > 1$, and $D \geqslant 4$, then Γ has geometric parameters (cf. Section 4.5). Building on this, he showed among other results that there are no distance-regular graphs with classical parameters with $D \geqslant 4$, $c_2 = 1$, and $a_2 > a_1 > 1$, and that under the assumption $D \geqslant 4$ and $c_2 > 1$, the dual polar graphs $^2\mathcal{A}_{2D-1}(-b)$ are the only graphs with classical parameters with $b = -a_1 - 1$. The latter characterizes the dual polar graphs $^2\mathcal{A}_{2D-1}(\sqrt{q})$ by their intersection arrays for $D \geqslant 4$. Weng [650] also showed the following result.

Theorem 5.2. *If Γ has classical parameters with $b < -1$, $a_1 \neq 0$, $c_2 > 1$, and $D \geqslant 4$, then either Γ is a dual polar graph $^2\mathcal{A}_{2D-1}(-b)$ or a Hermitian forms graph $Her(D, (-b)^2)$, or $\alpha = (b-1)/2$, $\beta = -(1 + b^D)/2$, and $-b$ is a power of an odd prime.*

Vanhove [641] showed that a $((1-b)/2)$-*ovoid* (i.e., a $(D-1)$-design with index $(1-b)/2$) in the dual polar graph $^2\mathcal{A}_{2D-1}(-b)$ with b odd would induce a distance-regular graph having classical parameters of the latter case. For $D = 2$, such $((1-b)/2)$-ovoids are better known as *hemisystems* and these were constructed by Cossidente and Penttila [144] for every odd prime power $-b$; see also [20]. No construction of a $((1-b)/2)$-ovoid is known for $D \geqslant 3$.

Triangle-free distance regular graphs with classical parameters have been studied by Pan, Lu, and Weng [527, 528, 529] and Hiraki [317]. One of the results is that if Γ has classical parameters and $a_1 = 0, a_2 \neq 0, D \geqslant 3$ then either (i) $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$ $(c_2 = 1)$, or (ii) $(b, \alpha, \beta) = (-2, -3, -1 - (-2)^D)$ $(c_2 = 2)$, or (iii)$(b, \alpha, \beta) = (-3, -2, -(1 + (-3)^D)/2)$ $(c_2 = 2)$; cf. [529, 317]. Case (i) with $D = 3$ is uniquely realized by the Witt graph M_{23} [78, §11.4B], whereas Huang, Pan, and Weng [343] ruled out case (i) with $D \geqslant 4$. Case (ii) is uniquely realized by the Hermitian forms graph $Her(D, 4)$.

5.3 Imprimitive graphs with classical parameters and partition graphs

It is known ([78, Prop. 6.3.1]) when a distance-regular graph with classical parameters (D, b, α, β) with $D \geqslant 3$ is imprimitive: it is bipartite if and only if $\alpha = 0$ and $\beta = 1$, whereas it is antipodal if and only if $b = 1$ and $\beta = 1 + \alpha(D - 1)$, in which case it is an antipodal double cover of its folded graph. This folded graph has diameter D' and intersection numbers $b_i = (D - i)(1 + \alpha(D - 1 - i))$ and $c_i = i(1 + \alpha(i - 1))$ for $i < D'$, $b_{D'} = 0$, and $c_{D'} = \gamma D'(1 + \alpha(D' - 1))$, where $\gamma = 1$ if $D = 2D' + 1$ and $\gamma = 2$ if $D = 2D'$. The distance-regular graphs with such intersection numbers are called *pseudo partition graphs*. Bussemaker and Neumaier [98, Thm. 3.3] showed that pseudo partition graphs with diameter $D' \geqslant 3$ must have the same intersection arrays as in one of the three families of partition graphs: the folded cubes ($\alpha = 0$), the folded Johnson graphs ($\alpha = 1$), and the folded halved cubes ($\alpha = 2$).

The folded cubes are determined by their intersection arrays, except for the folded 6-cube, which has two mates (i.e., non-isomorphic distance-regular graphs with the same

intersection array) in the form of (other) incidence graphs of 2-$(16, 6, 2)$ designs (cf. [78, Thm. 9.2.7]). The characterization of the other two families of partition graphs is now complete, due to work by Metsch, Gavrilyuk and Koolen:

Proposition 5.3. [479, 480, 241] *The folded Johnson graphs with diameter at least three are uniquely determined as distance-regular graphs by their intersection arrays.*

Proposition 5.4. [479, 482, 241] *The folded halved cubes with diameter at least three are uniquely determined as distance-regular graphs by their intersection arrays.*

Thus, all pseudo partition graphs with diameter at least three are known.

5.4   Characterizations of the *Q*-polynomial property

Bannai and Ito [38, p. 312] conjectured that every *primitive* distance-regular graph with sufficiently large diameter is Q-polynomial. We note that the Doubled Odd graphs are not Q-polynomial yet have arbitrarily large diameter, so that the 'primitivity' condition in the conjecture is necessary. Currently we know of no (real) progress towards proving the conjecture; however there are several new characterizations of the Q-polynomial property (since 'BCN' [78]). For completeness and because of its importance, we begin with Terwilliger's *balanced set condition* [611, 620]; cf. [78, §2.11, §8.3]. For distinct $x, y \in V$ and for $i, j = 0, 1, \ldots, D$, we let $\chi_{i,j}(x, y) = \sum_{z \in \Gamma_{i,j}(x,y)} \mathbf{e}_z$ denote the characteristic vector of $\Gamma_{i,j}(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$; cf. Section 4.6.

Theorem 5.5. (Balanced set condition [611, 620]) *The primitive idempotent E is Q-polynomial if and only if $u_i \neq 1$ for all $i = 1, 2, \ldots, D$ and*

$$E\chi_{i,j}(x, y) - E\chi_{j,i}(x, y) = p_{ij}^h \frac{u_i - u_j}{1 - u_h}(E\mathbf{e}_x - E\mathbf{e}_y) \qquad (14)$$

for all $i, j = 0, 1, \ldots, D$, $h = 1, 2, \ldots, D$, and $x, y \in V$ with $d(x, y) = h$.

Terwilliger [620] obtained an inequality for every $\ell = 3, 4, \ldots, D$ involving only the intersection numbers, θ, and $(u_i)_{i=0}^D$, by applying Cauchy-Schwarz to $E\chi_{i,1}(x, y) - E\chi_{1,i}(x, y)$ and $E\mathbf{e}_x - E\mathbf{e}_y$ with $\{i, h\} = \{\ell, \ell - 1\}$, and averaging over $x, y \in V$ with $d(x, y) = h$. Equality is attained for all $\ell = 3, 4, \ldots, D$ (or just for $\ell = 3$) if and only if E is Q-polynomial; cf. [78, §8.3]. Instead of the four vectors in (14), we may also consider the linear dependency of $E\chi_{i,j}(x, y)$, $E\mathbf{e}_x$, and $E\mathbf{e}_y$. This was worked out in detail by Terwilliger [614].[8] In particular, he applied Cauchy-Schwarz to $E\chi_{1,1}(x, y)$ and $E\mathbf{e}_x + E\mathbf{e}_y$, and took the average over each of the sets $\{(x, y) : d(x, y) = h\}$ ($h = 1, 2$) to obtain an inequality involving only a_1, b_1, c_2, u_1, u_2; in this case, equality is attained if and only if E is Q-polynomial with $a_0^\star = a_1^\star = \cdots = a_{D-1}^\star = 0$. The linear dependency among $E\chi_{1,1}(x, y)$, $E\mathbf{e}_x$, and $E\mathbf{e}_y$ for adjacent x and y is also relevant to the property of being tight; cf. Section 6.1.1. There is also a 'symmetric' version of (14) due to Terwilliger

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[8]The term 'balanced set' was introduced in [614] in this context, but many authors now refer to (14) as *the* balanced set condition.

[619, Thm. 2.6]. This lead, in particular, to the characterization of the Hermitian forms graphs by their intersection arrays; cf. Section 5.2. Tonejc [636] recently presented several inequalities by considering the vectors $E\chi_{i,1}(x,y) + E\chi_{1,i}(x,y)$ and $E\mathbf{e}_x + E\mathbf{e}_y$.

The following result is due to Pascasio [536] and may be viewed as an extension of [78, Thm. 8.2.1] for the bipartite case.

Proposition 5.6. *E is Q-polynomial if and only if all following properties hold:*

(i) *there exist $p, r \in \mathbb{C}$ such that $u_{i-1} + u_{i+1} = pu_i + r$ $(i = 1, 2, \ldots, D-1)$,*

(ii) *there exist $\xi, \omega, \eta^\star \in \mathbb{C}$ such that $a_i(u_i - u_{i-1})(u_i - u_{i+1}) = \xi u_i^2 + \omega u_i + \eta^\star$ $(i = 0, 1, \ldots, D)$, where u_{-1} and u_{D+1} are defined by (i) with $i = 0$ and $i = D$, respectively,*

(iii) *$u_i \neq 1$ $(i = 1, 2, \ldots, D)$.*

We call E a *tail* [434] if $E \circ E$ is a linear combination of E_0, E, and at most one other primitive idempotent of $\mathbb{A}$. Jurišić, Terwilliger, and Žitnik [386] established a characterization similar to Proposition 5.6, where property (ii) is replaced by E being a tail. We shall discuss tails in detail in Section 6.3.

The following characterization is due to Kurihara and Nozaki [429]; cf. [525].

Proposition 5.7. *Let F be a primitive idempotent other than E. Then there is a Q-polynomial ordering $(E_i)_{i=0}^D$ such that $E = E_1$ and $F = E_D$ if and only if $u_0, u_1, \ldots, u_D$ are distinct, and for $i = 0, 1, \ldots, D$, the eigenvalue of A_i for F is*

$$\frac{(1 - u_1)(1 - u_2) \cdots (1 - u_D)}{(u_i - u_0) \cdots (u_i - u_{i-1})(u_i - u_{i+1}) \cdots (u_i - u_D)}.$$

This result originated in an investigation of the D distances occurring in the spherical embedding $\{E\mathbf{e}_x : x \in V\} \subset \mathbb{R}^{m(\theta)}$, extending a similar observation by Bannai and Bannai [36] for strongly regular graphs.[9] Nozaki [526] recently showed that E is Q-polynomial provided that $v > \binom{m(\theta)+D-2}{D-1} + \binom{m(\theta)+D-3}{D-2}$ and $u_i \neq 1$ for $i = 1, 2, \ldots, D$.

There are also many results characterizing Q-polynomial graphs within certain subclasses of distance-regular graphs, such as bipartite graphs and tight graphs (cf. Section 6.1.1); see, e.g., [620, 533, 436, 629, 590]. For example, if Γ is a thick regular near polygon with $D \geqslant 3$, then Γ is Q-polynomial if and only if Γ has classical parameters; cf. [78, Thm. 8.5.1]. It should be remarked that De Bruyn and Vanhove [185] recently showed that for $D \geqslant 4$ there are no Q-polynomial thick regular near polygons, apart from the Hamming graphs and dual polar graphs. See also [648, Thm. C] and Theorem 9.11.

5.5 Classification results

In this section, suppose that E is a Q-polynomial idempotent, and let $p, r, r^\star$ be as in (13). Note that these scalars depend on E. We note also that, in the notation of Bannai

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[9]See e.g., [106, 503, 573] for some results about spherical designs (cf. [37]) obtained in this way from Q-polynomial distance-regular graphs.

and Ito [38, §III.5] (cf. [616, §2]), E is classical if and only if the Q-polynomial structure satisfies either type I with $s^* = 0$ or one of types IA, IIA, IIC; see [598, Prop. 6.2] (cf. [78, Thm. 8.4.1]). It turns out that most graphs with $p = \pm 2$ already appeared in Sections 5.1 and 5.3.

5.5.1   Case $p \neq \pm 2$

The Q-polynomial structure is type I or type IA in [38]. The following result is due to Terwilliger [unpublished].

Proposition 5.8. *Type IA does not occur.*

Proof. If the Q-polynomial structure is type IA then E is classical and we have

$$\theta_i = \theta_0 - sb(1 - b^i), \quad b_i = -tb^{i+1}(1 - b^{i-D}), \quad c_i = b(1 - b^i)(s - tb^{i-D-1})$$

for $i = 0, 1, \ldots, D$, where $b, s, t \in \mathbb{C} \backslash \{0\}$ and $p = b + b^{-1}$; cf. [38, §III.5], [616, §2]. The corresponding classical parameters are (D, b, α, β), where $\alpha = tb^{1-D}(1 - b)^2$ and $\beta = tb^{1-D}(1 - b)$. In particular, b is an integer distinct from $0, \pm 1$, and thus $s, t \in \mathbb{R}$. From $\theta_0 > \theta_1, \theta_2$, it follows that $b \geqslant 2$ and $s < 0$. Moreover, because $c_2 > 0$ we have $sb^D \leqslant 2sb^{D-1} < 2t$. But then $\theta_0 + \theta_D = 2b_0 - sb(1 - q^D) = b(1 - b^{-D})(sb^D - 2t) < 0$, so that $\theta_D < -\theta_0$, a contradiction. $\square$

It follows that all graphs having classical parameters with $b \neq 1$ fall into type I with $s^* = 0$ (with respect to the associated Q-polynomial ordering).

5.5.2   Case $p = 2$, $r \neq 0$, $r^\star \neq 0$

The Q-polynomial structure is type II in [38]. Terwilliger [609] showed that if $D \geqslant 14$ then either Γ is the halved $(2D+1)$-cube, or Γ has the same intersection array as a folded Johnson graph or a folded halved cube. By Propositions 5.3 and 5.4, the classification is now complete for $D \geqslant 14$.

5.5.3   Case $p = 2$, $r = 0$, $r^\star \neq 0$

E is classical and the Q-polynomial structure is type IIA in [38]. It follows that Γ is either a Johnson graph, a halved cube, or the Gosset graph; cf. Section 5.1.

5.5.4   Case $p = 2$, $r \neq 0$, $r^\star = 0$

The Q-polynomial structure is type IIB in [38]. Terwilliger [613] showed that Γ is either a folded cube or one of the other two non-isomorphic graphs with the intersection array $\{6, 5, 4; 1, 2, 6\}$ of the folded 6-cube; cf. [78, §9.2D].

5.5.5   Case $p = 2$, $r = r^\star = 0$

E is classical and the Q-polynomial structure is type IIC in [38]. Egawa [207] showed that Γ is either a Hamming graph or a Doob graph; cf. [78, §9.2B].

5.5.6 Case $p = -2$

The Q-polynomial structure is type III in [38]. Terwilliger [612] showed that Γ is either the D-cube (D even), the Odd graph O_{D+1}, or the folded $(2D+1)$-cube.

Next, we move on to (almost) imprimitive graphs.

5.5.7 Bipartite graphs

Suppose Γ is bipartite. Then $r^\star = 0$ by [78, Thm. 8.2.1]. If $p = \pm 2$ then it follows from the above results that Γ is either the D-cube, the folded $2D$-cube, or one of the other two graphs with intersection array $\{6, 5, 4; 1, 2, 6\}$. Caughman [114] showed that if $p \neq \pm 2$ and $D \geqslant 12$ then Γ has classical parameters $(D, b, 0, 1)$ where b is an integer at least 2. These parameters are realized by the dual polar graphs $\mathcal{D}_D(b)$ and the Hemmeter graphs.

5.5.8 Antipodal graphs

Curtin [146] showed that bipartite Q-polynomial antipodal (double) covers are precisely the bipartite 2-homogeneous distance-regular graphs, and the latter graphs were classified by Nomura [517]; cf. Section 6.1.3. These are the D-cube, the regular complete bipartite graphs minus a perfect matching, the Hadamard graphs, and the graphs with intersection arrays satisfying

$$(c_1, c_2, \ldots, c_5) = (1, \mu, k - \mu, k - 1, k), \quad b_i = c_{5-i} \ (i = 0, 1, \ldots, 4),$$

where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $\mu = \gamma(\gamma + 1)$, and $\gamma \geqslant 2$ is an integer. The last case is uniquely realized for $\gamma = 2$ by the double cover of the Higman-Sims graph.

Dickie and Terwilliger [198] gave a classification of non-bipartite Q-polynomial antipodal distance-regular graphs as follows: the Johnson graph $J(2D, D)$, the halved $2D$-cube, the non-bipartite Taylor graphs, and the graphs satisfying

$$(c_1, c_2, c_3, c_4) = (1, p\eta, (p^2 - 1)(2\eta - p + 1), p(2\eta + 2\eta p - p^2)), \tag{15}$$
$$b_i = c_{4-i} \ (i = 0, 1, 2, 3),$$

where $p \geqslant 3$, $\eta \geqslant 3p/4$ are integers and η divides $p^2(p^2 - 1)/2$. An example of the last case is the Meixner double cover ($p = 4, \eta = 6$); cf. Section 3.2.4. The array (15) with p, η odd has been ruled out by Jurišić and Koolen [375, Cor. 3.2].

5.5.9 Almost bipartite graphs

The Q-polynomial generalized odd graphs have been classified by Lang and Terwilliger [438]: the folded $(2D+1)$-cube, the Odd graph O_{D+1}, and the graphs with $D = 3$ satisfying

$$k = 1 + (p^2 - 1)(p(p + 2) - (p + 1)c_2), \quad c_3 = -(p + 1)(p^2 + p - 1 - (p + 1)c_2),$$

where $p < -2$ is an integer. No example is known for the last case. We recall that the distance-2 graph Γ_2 is again distance-regular, as it is the halved graph of the bipartite double of Γ; cf. Section 13.2.

5.5.10 Almost Q-bipartite graphs

Suppose $D \geqslant 4$ and Γ is almost Q-bipartite, i.e., $a_i^\star = 0$ for $i < D$ and $a_D^\star > 0$. Dickie [197] showed that Γ is either the halved $(2D + 1)$-cube, the folded $(2D + 1)$-cube, or a dual polar graph $^2\mathcal{A}_{2D-1}(\sqrt{q})$. We note that the '$Q$-bipartite double' of Γ is a cometric association scheme, and that E_2 is again a Q-polynomial idempotent; cf. Sections 5.7.2 and 16.8.

5.6 The Terwilliger algebras of Q-polynomial distance-regular graphs

Below we collect 'handy' sufficient conditions for Γ being thin when it is Q-polynomial.

Proposition 5.9. [616, §5] *Suppose Γ is Q-polynomial with respect to the ordering $(E_i)_{i=0}^D$. Then the following properties hold.*

(i) Γ *is thin with respect to $x \in V$ if for $i = 1, 2, \ldots, D$ and for every $y, z \in \Gamma_i(x)$, there is an automorphism π of Γ such that $\pi(x) = x$, $\pi(y) = z$, and $\pi(z) = y$,*

(ii) Γ *is thin if $a_2 = a_3 = \cdots = a_{D-1} = 0$,*

(iii) Γ *is thin if $a_2^\star = a_3^\star = \cdots = a_{D-1}^\star = 0$.*

In particular, a Q-polynomial distance-regular graph is thin provided that it is bipartite (=Q-antipodal), almost bipartite, antipodal (=Q-bipartite), or almost Q-bipartite. It also follows that many of the known graphs with classical parameters as well as partition graphs (cf. Section 5.3) are thin; see [616, Ex. 6.1] for details. The following graphs are known to be *non*-thin: Doob graphs, (bilinear, alternating, Hermitian, quadratic) forms graphs, and the twisted Grassmann graphs. The irreducible $\mathbb{T}$-modules of the Doob graphs were determined by Tanabe [593]. Concerning the twisted Grassmann graph (cf. Section 3.2.1), Bang, Fujisaki, and Koolen [25] showed that it is thin with respect to any base vertex x which is an $(e - 1)$-dimensional subspace of the fixed hyperplane H, by verifying a different combinatorial criterion for thinness [616, Thm. 5.1(v)]. However, they also showed that if x is not contained in H then the twisted Grassmann graph is not 1-thin with respect to x.

The irreducible $\mathbb{T}$-modules of bipartite (resp. almost bipartite) Q-polynomial distance-regular graphs were described by Caughman [111] (resp. Caughman, MacLean, and Terwilliger [115]). For these graphs, it turns out that the intersection array completely determines the structure of $\mathbb{T}$. In particular, explicit formulas for the multiplicities of the irreducible $\mathbb{T}$-modules in $\mathbb{C}^v$ with small endpoints were successfully used in the classification of these graphs; cf. Section 5.5. Curtin and Nomura [155] and Curtin [151] studied the Terwilliger algebra of bipartite Q-polynomial antipodal (double) covers which are not the D-cube; in this case, it follows that $\mathbb{T}$ is a homomorphic image of the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$; cf. Section 5.8.

In general, if Γ is Q-polynomial then the structure of irreducible $\mathbb{T}$-modules with endpoint 1 is determined by the intersection array and the spectrum of the local graph $\Upsilon(x)$ with respect to the base vertex x; cf. [618, Lecture 35]. To be more precise, suppose

for the moment that Γ is Q-polynomial, and let W be an irreducible $\mathbb{T}$-module with endpoint 1. Then $\dim E_1^\star W = 1$, so that $E_1^\star W$ is an eigenspace for $E_1^\star A E_1^\star$; let η denote the corresponding eigenvalue. Then the isomorphism class of W is determined by η. Moreover, W is thin if and only if η is a root of a polynomial T of degree 4, which we call the *Terwilliger polynomial* of Γ; if Γ has classical parameters (D, b, α, β) then its four roots are $-1, -b-1, \beta - \alpha - 1$, and $\alpha b \begin{bmatrix} D-1 \\ 1 \end{bmatrix} - 1$. See also [241, Lemma 4.7] and [616, Cor. 4.12(5)]. If W is non-thin then it follows that[10] W has diameter $D-1$ and that $\dim E_1^\star W = \dim E_D^\star W = 1$, $\dim E_2^\star W = \cdots = \dim E_{D-1}^\star W = 2$. Hobart and Ito [327] studied in detail the structure of such a non-thin irreducible $\mathbb{T}$-module with endpoint 1. Miklavič [486, 491] showed that Γ is 1-homogeneous if it is Q-polynomial with $a_1 = 0$, and described the unique irreducible $\mathbb{T}$-module with endpoint 1 (with $\eta = 0$) when $a_2 \neq 0$, which turns out to be non-thin. Miklavič [490] also described the irreducible $\mathbb{T}$-modules with endpoint 1 when Γ has classical parameters with $b < -1$, $a_1 \neq 0$, and is not a near polygon; there are exactly two isomorphism classes, and the first one is thin with $\eta = -1$ and the second one is non-thin with $\eta = a_1$.

Suppose again that Γ is Q-polynomial, and let η be a local eigenvalue of Γ (with respect to the base vertex x), i.e., an eigenvalue of $\Upsilon(x)$. We call η *non-degenerate* if it has an eigenvector orthogonal to the all-ones vector, and *degenerate* otherwise. We note that a_1 is the only possible degenerate local eigenvalue and that it is non-degenerate precisely when $\Upsilon(x)$ is disconnected. The Terwilliger polynomial T mentioned above depends only on the intersection array of Γ and the Q-polynomial ordering, and has the property that $T(\eta) \geqslant 0$ for every non-degenerate local eigenvalue η for every base vertex x. We note that if Γ has two Q-polynomial orderings then T may be different for the different ordering. Using the polynomial T, Gavrilyuk and Koolen [241] recently showed the uniqueness of the folded halved $2m$-cube for $m \geqslant 6$; cf. Section 5.3. With the same approach we can also show the uniqueness of the folded Johnson graphs. For the Grassmann graphs $J_q(2D, D)$ ($D \geqslant 3$), Gavrilyuk and Koolen also obtained partial results. See also [602, §4.3] for more discussions on the Terwilliger polynomial.

See Section 6.2 for more results on the irreducible $\mathbb{T}$-modules with endpoint 1 of general distance-regular graphs.

5.7 Further results on Q-polynomial distance-regular graphs

In this section, we always assume that Γ is Q-polynomial.

5.7.1 Antipodal covers

Van Bon and Brouwer [59] determined the distance-regular antipodal covers of the classical families of distance-regular graphs; cf. [78, §6.12]. Suppose E is Q-polynomial, and recall

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[10]In fact, Terwilliger [618, Lectures 34–37] stated this result as a conjecture, and showed that $\dim E_i^\star W \leqslant 2$ for $i = 2, 3, \ldots, D-1$, $\dim E_D^\star W \leqslant 1$, and that W is thin if $\dim E_2^\star W = 1$. Now, that W has diameter $D-1$ follows from a result of Go and Terwilliger [261, Thm. 9.8], and the values of the $\dim E_i^\star W$ follow from their symmetric and unimodal properties proved by Ito, Tanabe, and Terwilliger [349] in the more general context of tridiagonal systems; cf. Section 5.8.

(cf. (13)) that there exist $p, r \in \mathbb{C}$ such that $u_{i-1} + u_{i+1} = pu_i + r$ for $i = 1, 2, \ldots, D-1$. Terwilliger [617] showed that if Γ has an antipodal cover of diameter $\tilde{D} \geqslant 7$, then this three-term recurrence extends to $i = 1, 2, \ldots, \tilde{D} - 1$, where we formally define $u_i = u_{\tilde{D}-i}$ ($i = D + 1, D + 2, \ldots, \tilde{D}$). This parametric condition provides simple proofs of some of the (non-existence) results in [59], and may be applied to the twisted Grassmann graphs as well; cf. [235]. Caughman [112] used the condition to show that if Γ is bipartite with $D \geqslant 4$ and has an antipodal cover then Γ is the folded $2D$-cube; cf. [436, Cor. 12.3].

5.7.2 Distance-regular graphs with multiple Q-polynomial orderings

An association scheme can have at most two P-polynomial orderings, except for those coming from the polygons; cf. [78, §4.2D]. Bannai and Ito [38, pp. 354–360] showed that if $k \geqslant 3$ and $D \geqslant 34$ then Γ has at most two Q-polynomial idempotents and moreover all eigenvalues are integral. Brouwer, Cohen, and Neumaier [78, p. 247] conjectured that the assumption $D \geqslant 34$ can be replaced by $D \neq 4$. Dickie [197, pp. 69–70] established the result under the assumption $D \geqslant 5$. Indeed, he showed that if $k \geqslant 3$ and $D \geqslant 5$ then Γ has more than one Q-polynomial idempotent if and only if Γ is either the D-cube (D even), the halved $(2D + 1)$-cube, the folded $(2D + 1)$-cube, or a dual polar graph $^2\mathcal{A}_{2D-1}(\sqrt{q})$, and these graphs have precisely two Q-polynomial idempotents but no non-integral eigenvalues. (Note that if Γ has non-integral eigenvalues and E is Q-polynomial then E^σ is again Q-polynomial for any $\mathbb{Q}$-automorphism σ of the splitting field over $\mathbb{Q}$.) Building on work by Dickie [197], Suzuki [586] showed that every association scheme has at most two Q-polynomial idempotents, again except for those coming from the polygons; cf. Section 16.8. For $D \in \{2, 3, 4\}$, the known Q-polynomial distance-regular graphs with $k \geqslant 3$ and with non-integral eigenvalues belong to the following four families: the conference graphs ($D = 2$), the incidence graphs of symmetric designs ($D = 3$), the Taylor graphs ($D = 3$), and the Hadamard graphs ($D = 4$).[11] Note that the graphs in these families always have two Q-polynomial idempotents. The other candidate intersection arrays $\{\mu(2\mu + 1), (\mu - 1)(2\mu + 1), \mu^2, \mu; 1, \mu, \mu(\mu - 1), \mu(2\mu + 1)\}$ ($\mu \geqslant 2$) of primitive Q-polynomial distance-regular graphs with non-integral eigenvalues given by Brouwer et al. [78, pp. 247–248] were ruled out by Godsil and Koolen [270]; cf. Section 17.2.2. Ma and Koolen [452] recently classified the distance-regular graphs with $k \geqslant 3$, $D = 4$, and with two Q-polynomial idempotents; these are the 4-cube, the halved 9-cube, the folded 9-cube, the dual polar graphs $^2\mathcal{A}_7(\sqrt{q})$, and the Hadamard graphs.

5.7.3 Bounds for the girth

Brouwer, Cohen, and Neumaier [78, p. 248] conjectured that Γ has girth at most 6, with equality only for the Odd graph O_{D+1}, and showed that the numerical girth g of Γ is at most 7. Lewis [447] showed $c_3 \geqslant 2$, proving $g \leqslant 6$. We note that if Γ has girth 6, i.e., $a_1 = a_2 = 0$ and $c_2 = 1$, then it follows from Proposition 5.6 (or [486, Thm. 6.3]) that

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[11]It seems that the above conjecture by Brouwer et al. was not properly stated, because we already have counterexamples with $D \in \{2, 3\}$. We note that the graphs in the last three families are imprimitive.

$a_1 = a_2 = \cdots = a_{D-1} = 0$, so that Γ is bipartite or almost bipartite. Miklavič [488] showed that if Γ is bipartite and $D = 4$ then $c_2 \geqslant 2$, i.e., $g = 4$.

5.7.4 The Erdős-Ko-Rado theorem

At the end of each of Sections 9.1-9.4 and 9.5A in [78] there is a remark about the Erdős-Ko-Rado theorem for the graph in question. See [594, 541, 599, 345, 600, 273] for recent results on this topic.

5.7.5 Unimodality of the multiplicities

We recall from Proposition 2.4 (iv) that the k_i are unimodal. Concerning the multiplicities m_i, Pascasio [534] showed that if Γ is Q-polynomial with respect to the ordering $(E_i)_{i=0}^{D}$ then $m_{i-1} \leqslant m_i \leqslant m_{D-i}$ for $i = 1, 2, \ldots, \lfloor D/2 \rfloor$. This result was originally conjectured by Dennis Stanton in 1993, and is a simple application of the theory of tridiagonal systems; cf. Section 5.8. We note that Bannai and Ito [38, p. 205] earlier conjectured that the multiplicities of a cometric association scheme satisfy the unimodal property.

5.7.6 Posets associated with Q-polynomial distance-regular graphs

There are several classes of finite ranked posets that are closely related to Q-polynomial distance-regular graphs: *regular semilattices* [190, 571], *uniform posets* [615], *quantum matroids* [621]. (For definitions, see the references given.) Many of the known families of Q-polynomial distance-regular graphs arise as the top fibers of these posets, where two vertices are adjacent if and only if they cover a common element.

Concerning quantum matroids, Terwilliger [621, Thm. 38.2] showed that if a quantum matroid is 'non-trivial' and 'regular', then the graph on the top fiber with the above adjacency is distance-regular. Moreover, in this case, the graph has classical parameters if its diameter is equal to the rank of the quantum matroid. The culmination of the study of quantum matroids is the classification ([621, Thm. 39.6]) of non-trivial regular quantum matroids with rank at least four: they are precisely those posets naturally associated with Johnson, Hamming, Grassmann, bilinear forms, and dual polar graphs. We may use this classification as follows.

Fix a Q-polynomial ordering $(E_i)_{i=0}^{D}$ of Γ. Let Y be a non-empty subset of V and let χ be its characteristic vector. Brouwer, Godsil, Koolen, and Martin [81] defined the *width* and *dual width* of Y by $w = \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}$ and $w^\star = \max\{i : \chi^{\mathsf{T}} E_i \chi \neq 0\}$, respectively. They showed among other results that $w + w^\star \geqslant D$, and we call Y a *descendent* (cf. [598]) of Γ if equality holds. It follows that every descendent is completely regular, and that the induced subgraph is a Q-polynomial distance-regular graph if it is connected; cf. [81, Thm. 1–3].[12] We say that a set $\mathscr{D}$ of descendents of Γ *satisfies* $(\mathrm{UD})_i$ if each two vertices $x, y \in V$ at distance i are contained in a unique descendent in $\mathscr{D}$ with width i.

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[12]The results in [81] are in contrast with Delsarte theory [189] based on the minimum distance and (maximum) strength of a subset. We may remark that Suda [574] recently developed a theory which unifies and 'interpolates' some of the theorems in [189] and [81] to a certain extent.

Proposition 5.10. [598] *Let $\mathscr{D}$ be a set of descendents of Γ. Suppose that the following properties hold.*

(i) *Γ has classical parameters,*

(ii) *$\mathscr{D}$ satisfies $(\mathrm{UD})_i$ for all i,*

(iii) *$Y_1 \cap Y_2 \in \mathscr{D} \cup \{\emptyset\}$ for all $Y_1, Y_2 \in \mathscr{D}$.*

Then $\mathscr{D}$, together with the partial order defined by reverse inclusion, forms a non-trivial regular quantum matroid. In particular, if $D \geqslant 4$ then Γ is either a Johnson, Hamming, Grassmann, bilinear forms, or dual polar graph.

It was also shown that if $\mathscr{D}$ is the set of *all* descendents of Γ then condition (iii) in the above proposition is implied by the other two. See [81, 594, 337, 598] for more information on descendents.

Unlike regular semilattices and quantum matroids, uniform posets are not assumed to be semilattices, but give rise to at least 13 infinite families of Q-polynomial distance-regular graphs with unbounded diameter, rather than just five as above; cf. [615, §4]. Suppose Γ is (Q-polynomial and) bipartite, and fix $x \in V$. Then we may view Γ as the Hasse diagram of a ranked poset with $D + 1$ fibers $\Gamma_i(x)$ ($i = 0, 1, \ldots, D$). Miklavič and Terwilliger [496] recently showed that this poset is uniform.[13] Caughman [113] showed that the graph on the top fiber $\Gamma_D(x)$ defined in the previous manner (which is in this case the induced subgraph of the distance-2 graph of Γ) is distance-regular and Q-polynomial. See [630] and the references therein for more results on uniform posets.

The poset $\mathscr{S}$ consisting of all strongly closed subgraphs of Γ with partial order defined by reverse inclusion plays an important role in the study of distance-regular graphs having classical parameters with $b < -1$. Suppose Γ has geometric parameters (D, b, α) (cf. Section 4.5) with $D \geqslant 4$ and is D-bounded in the sense of Weng [648, 650], i.e., every $\Delta \in \mathscr{S}$ is assumed to be regular. Then $b < -1$ by [650, Lemma 5.5]. (Conversely, if Γ has classical parameters with $b < -1, D \geqslant 4, a_1 \neq 0, c_2 > 1$ then Γ is D-bounded and has geometric parameters; cf. [650, Thm. 5.7, 5.8].) In this case, Weng [648] showed that $\mathscr{S}$ is a ranked (meet) semilattice and every interval is a modular atomic lattice which is isomorphic to a projective geometry over $GF(b^2)$.

5.8 Tridiagonal systems

Let W be a finite dimensional vector space over $\mathbb{C}$. Let $\mathfrak{a} \in \mathrm{End}_{\mathbb{C}}(W)$ be diagonalizable, and let $(\theta_i)_{i=0}^{\delta}$ be an ordering of the distinct eigenvalues of $\mathfrak{a}$. Then there is a sequence of elements $(\mathfrak{e}_i)_{i=0}^{\delta}$ in $\mathrm{End}_{\mathbb{C}}(W)$ such that (i) $\mathfrak{a}\mathfrak{e}_i = \theta_i \mathfrak{e}_i$; (ii) $\mathfrak{e}_i \mathfrak{e}_j = \delta_{ij} \mathfrak{e}_i$; (iii) $\sum_{i=0}^{\delta} \mathfrak{e}_i = \mathbf{1}$, where $\mathbf{1}$ is the identity element in $\mathrm{End}_{\mathbb{C}}(W)$. (Specifically, $\mathfrak{e}_i = \prod_{j \neq i} \frac{\mathfrak{a} - \theta_j \mathbf{1}}{\theta_i - \theta_j}$ ($i = 0, 1, \ldots, \delta$).) We call $\mathfrak{e}_i$ the *primitive idempotent* of $\mathfrak{a}$ associated with θ_i ($i = 0, 1, \ldots, \delta$). Let $\mathfrak{a}^{\star}$ be another

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[13]See [496] for the precise statement of the result, noting that the hypercube $H(D, 2)$ with D even has two Q-polynomial structures. They also introduced the concept of *strongly uniform* and investigated when the poset Γ has that property.

diagonalizable element in $\text{End}_{\mathbb{C}}(W)$. Let $(\theta_i^{\star})_{i=0}^{\delta^{\star}}$ be an ordering of the distinct eigenvalues of $\mathfrak{a}^{\star}$ and let $(\mathfrak{e}_i^{\star})_{i=0}^{\delta^{\star}}$ be the corresponding sequence of the primitive idempotents. The sequence $\Phi = (\mathfrak{a}; \mathfrak{a}^{\star}; (\mathfrak{e}_i)_{i=0}^{\delta}; (\mathfrak{e}_i^{\star})_{i=0}^{\delta^{\star}})$ is a *tridiagonal system* (or *TD system*) if

$$\mathfrak{e}_i^{\star} \mathfrak{a} \mathfrak{e}_j^{\star} = 0 \quad \text{if } |i - j| > 1 \quad (i, j = 0, 1, \ldots, \delta^{\star}),$$

$$\mathfrak{e}_i \mathfrak{a}^{\star} \mathfrak{e}_j = 0 \quad \text{if } |i - j| > 1 \quad (i, j = 0, 1, \ldots, \delta),$$

and W is irreducible as a $\mathbb{C}[\mathfrak{a}, \mathfrak{a}^{\star}]$-module. This definition is due to Ito, Tanabe, and Terwilliger [349].[14] Note that if Γ is a Q-polynomial distance-regular graph then it follows from (10) that every irreducible $\mathbb{T}$-module naturally has the structure of a TD system.

Suppose that Φ is a TD system. Ito et al. [349] showed $\delta = \delta^{\star}$. Define $U_i = (\sum_{h=0}^{i} \mathfrak{e}_h^{\star} W) \cap (\sum_{\ell=i}^{\delta} \mathfrak{e}_{\ell} W)$ $(i = 0, 1, \ldots, \delta)$. Note that $U_0 = \mathfrak{e}_0^{\star} W$, and that $(\mathfrak{a} - \theta_i \mathbf{1}) U_i \subseteq U_{i+1}$, $(\mathfrak{a}^{\star} - \theta_i^{\star} \mathbf{1}) U_i \subseteq U_{i-1}$ $(i = 0, 1, \ldots, \delta)$, where $U_{-1} = U_{\delta+1} = 0$. They showed $W = \bigoplus_{i=0}^{\delta} U_i$. It also turns out that $\dim \mathfrak{e}_i W = \dim \mathfrak{e}_i^{\star} W = \dim U_i$ $(i = 0, 1, \ldots, \delta)$. The sum $W = \bigoplus_{i=0}^{\delta} U_i$ is called the *split decomposition* and plays a crucial role in the theory of TD systems. Let $\rho_i = \dim \mathfrak{e}_i W$ $(i = 0, 1, \ldots, \delta)$ and call the sequence $(\rho_i)_{i=0}^{\delta}$ the *shape* of Φ. They showed that the shape is symmetric and unimodal: $\rho_i = \rho_{\delta-i}$ $(i = 0, 1, \ldots, \delta)$ and $\rho_{i-1} \leqslant \rho_i$ $(i = 1, 2, \ldots, \lfloor \delta/2 \rfloor)$. A TD system with $\rho_0 = \cdots = \rho_{\delta} = 1$ is called a *Leonard system* [623]. Leonard systems provide a linear algebraic framework for Leonard's theorem and have been extensively studied; see [627] and the references therein. Note that if the TD system Φ is afforded on an irreducible $\mathbb{T}$-module of a Q-polynomial distance-regular graph, then the $\mathbb{T}$-module W is thin if and only if Φ is a Leonard system. See [117] for a detailed description of thin irreducible $\mathbb{T}$-modules motivated by the theory of Leonard systems.

Ito et al. [349] showed that there exist scalars $p, \gamma, \gamma^{\star}, \varrho, \varrho^{\star} \in \mathbb{C}$ such that

$$0 = [\mathfrak{a}, \mathfrak{a}^2 \mathfrak{a}^{\star} - p \mathfrak{a} \mathfrak{a}^{\star} \mathfrak{a} + \mathfrak{a}^{\star} \mathfrak{a}^2 - \gamma(\mathfrak{a} \mathfrak{a}^{\star} + \mathfrak{a}^{\star} \mathfrak{a}) - \varrho \mathfrak{a}^{\star}], \tag{16}$$

$$0 = [\mathfrak{a}^{\star}, \mathfrak{a}^{\star 2} \mathfrak{a} - p \mathfrak{a}^{\star} \mathfrak{a} \mathfrak{a}^{\star} + \mathfrak{a} \mathfrak{a}^{\star 2} - \gamma^{\star}(\mathfrak{a}^{\star} \mathfrak{a} + \mathfrak{a} \mathfrak{a}^{\star}) - \varrho^{\star} \mathfrak{a}], \tag{17}$$

where $[\mathfrak{b}, \mathfrak{c}] := \mathfrak{b}\mathfrak{c} - \mathfrak{c}\mathfrak{b}$, and (cf. (13))

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{i-2}^{\star} - \theta_{i+1}^{\star}}{\theta_{i-1}^{\star} - \theta_i^{\star}} = p + 1 \quad (i = 2, 3, \ldots, \delta - 1). \tag{18}$$

The relations (16) and (17) generalize the q-Serre relations (which are among the defining relations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$) and the Dolan-Grady relations (which are the defining relations of the Onsager algebra); cf. [622]. It is conjectured ([349, Conj. 13.7]) that there exist positive integers $\delta_1, \delta_2, \ldots, \delta_n$ such that $\sum_{i=0}^{\delta} \rho_i t^i = \prod_{j=1}^{n} (1 + t + \cdots + t^{\delta_j})$, where t is an indeterminate. This conjecture in fact suggests that Φ would be regarded as a 'tensor product' of Leonard systems. Let q be a nonzero scalar in $\mathbb{C}$ such that $p = q^2 + q^{-2}$. Using the representation theory of $U_q(\widehat{\mathfrak{sl}}_2)$ (cf. [122]), Ito and Terwilliger

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[14]TD systems can be defined on vector spaces over arbitrary fields, and many of the results are valid over wider classes of fields. However, for simplicity and in view of the connections to the theory of Q-polynomial distance-regular graphs, we shall only discuss TD systems over $\mathbb{C}$.

[351, 358] indeed constructed *all* TD systems (up to isomorphism[15]) explicitly as tensor products of Leonard systems (i.e., *evaluation modules*), under the assumption that q is not a root of unity. We remark that in this case the split decomposition corresponds to the weight space decomposition. See also [350, 352, 356, 357, 236, 295]. Ito [private communication] pointed out that the proofs of most of the results in [358] work under the weaker assumption $q^2 \neq \pm 1$, i.e., $p \neq \pm 2$. It seems that the above conjecture is still open for general TD systems, but Nomura and Terwilliger [521, 523] showed among other results that $\rho_0 = 1$, and more generally, $\rho_i \leqslant \binom{\delta}{i}$ $(i = 0, 1, \ldots, \delta)$, a result which would follow directly from the conjecture. See, e.g., [291, 353, 348] for some results on TD systems with $p = 2$.

Observe now that the 1-dimensional subspace $\mathfrak{e}_0^\star W$ is invariant under

$$(\mathfrak{a}^\star - \theta_1^\star \mathbf{1})(\mathfrak{a}^\star - \theta_2^\star \mathbf{1}) \ldots (\mathfrak{a}^\star - \theta_i^\star \mathbf{1})(\mathfrak{a} - \theta_{i-1}\mathbf{1}) \ldots (\mathfrak{a} - \theta_1\mathbf{1})(\mathfrak{a} - \theta_0\mathbf{1})$$

for $i = 0, 1, \ldots, \delta$, and let ζ_i be the corresponding eigenvalue $(i = 0, 1, \ldots, \delta)$. The sequence $((\theta_i)_{i=0}^\delta; (\theta_i^\star)_{i=0}^\delta; (\zeta_i)_{i=0}^\delta)$ is called the *parameter array* of Φ. Nomura and Terwilliger [521] showed that the parameter array is a complete invariant for a TD system. Ito, Nomura, and Terwilliger [347] established the following theorem:

Theorem 5.11. [347, Thm. 3.1] *Let* $\pi = ((\theta_i)_{i=0}^\delta; (\theta_i^\star)_{i=0}^\delta; (\zeta_i)_{i=0}^\delta)$ *be a sequence of scalars in* $\mathbb{C}$ *such that* $\theta_i \neq \theta_j$, $\theta_i^\star \neq \theta_j^\star$ *if* $i \neq j$ $(i, j = 0, 1, \ldots, \delta)$, *and suppose that* (18) *holds for some* $p \in \mathbb{C}$. *Then there exists a (unique) TD system with parameter array* π *if and only if* $\zeta_0 = 1$, $\zeta_\delta \neq 0$, *and* $\sum_{i=0}^\delta \zeta_i \prod_{\ell=i+1}^\delta (\theta_0 - \theta_\ell)(\theta_0^\star - \theta_\ell^\star) \neq 0$.

We remark that the left-hand side of the last condition on the ζ_i is a certain value of the *Drinfel'd polynomial* of the corresponding TD system; cf. [356, 358]. See, e.g., [522, 524, 57] for more results on TD systems.

Given the above progress in the theory of TD systems, it is important to 'pull back' the results to the study of Q-polynomial distance-regular graphs. For example, Pascasio [534] used the symmetric and unimodal property of the shape of Φ to study the multiplicities m_i of a Q-polynomial distance-regular graph Γ; cf. Section 5.7.5. Terwilliger [626] 'extended', so to speak, the split decompositions of the TD systems on the irreducible $\mathbb{T}$-modules to the entire standard module $\mathbb{C}^v$, and obtained the *split* and *displacement decompositions for* Γ. Ito and Terwilliger [354] used these decompositions to show that for the forms graphs there are four natural algebra homomorphisms from $U_q(\widehat{\mathfrak{sl}}_2)$ to $\mathbb{T}$ via the so-called q-*tetrahedron algebra* $\boxtimes_q$ [352], and that $\mathbb{T}$ is generated by each of their images together with the center $Z(\mathbb{T})$. Corresponding results for the case $p = 2$, i.e., for Hamming and Doob graphs, were recently obtained by Morales and Pascasio [501]. See also [355, 393, 394] for more results on the split and displacement decompositions. Worawannotai [652] applied a similar idea to dual polar graphs to show (among other results) that there are two algebra homomorphisms from the quantum algebra $U_q(\mathfrak{sl}_2)$ to $\mathbb{T}$, and that $\mathbb{T}$ is again generated by

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[15]A TD system $\Phi' = (\mathfrak{a}'; \mathfrak{a}^{\star\prime}; (\mathfrak{e}_i')_{i=0}^\delta; (\mathfrak{e}_i^{\star\prime})_{i=0}^\delta)$ on a vector space W' is *isomorphic* to Φ if there is an isomorphism of vector spaces $\sigma : W \to W'$ such that $\sigma\mathfrak{a} = \mathfrak{a}'\sigma$, $\sigma\mathfrak{a}^\star = \mathfrak{a}^{\star\prime}\sigma$, and $\sigma\mathfrak{e}_i = \mathfrak{e}_i'\sigma$, $\sigma\mathfrak{e}_i^\star = \mathfrak{e}_i^{\star\prime}\sigma$ for $i = 0, 1, \ldots, \delta$.

each of their images together with $Z(\mathbb{T})$. The split and displacement decompositions have also been applied to the Assmus-Mattson theorem for codes in Q-polynomial distance-regular graphs [595]; cf. [78, §2.8].

6 The Terwilliger algebra and combinatorics

In this section, let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant 3$, and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Concerning 1-homogeneity of distance-regular graphs, we shall occasionally consider the following weaker concepts. We say Γ is 1-*homogeneous with respect to an edge xy* if the parameters $p_{i,j;r,s}$ exist with respect to x, y for all $i, j, r, s = 0, 1, \ldots, D$; cf. Section 4.6. We say Γ is 1-*homogeneous with respect to a vertex $x \in V$* if it is 1-homogeneous with respect to the edge xy for every $y \in \Gamma(x)$ and the parameters $p_{i,j;r,s}$ do not depend on the choice of y.

6.1 Homogeneity and tight distance-regular graphs

6.1.1 Tight distance-regular graphs

Jurišić, Koolen, and Terwilliger [382] showed the following so-called 'fundamental bound':

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right)\left(\theta_D + \frac{k}{a_1 + 1}\right) \geqslant -\frac{ka_1 b_1}{(a_1 + 1)^2}. \tag{19}$$

For $a_1 = 0$, equality holds if and only if Γ is bipartite. One way to prove this bound is to use the fact ([78, Thm. 4.4.3]) that $(\eta_i - \tilde{\theta}_1)(\eta_i - \tilde{\theta}_D) \leqslant 0$ for $i = 2, 3, \ldots, k$, where $\tilde{\theta}_1 = -1 - \frac{b_1}{1+\theta_1}$, $\tilde{\theta}_D = -1 - \frac{b_1}{1+\theta_D}$, and $a_1 = \eta_1 \geqslant \eta_2 \geqslant \ldots \geqslant \eta_k$ are the eigenvalues of a local graph; cf. [375]. This immediately shows that if $a_1 \neq 0$ then equality holds if and only if every (or at least one) local graph is connected strongly regular with non-trivial eigenvalues $\tilde{\theta}_1, \tilde{\theta}_D$. We may also prove (19) by considering the determinants of the Gram matrices of the three vectors $E\mathbf{e}_x, E\mathbf{e}_y, E\chi_{1,1}(x, y)$ for adjacent vertices $x, y \in V$ and $E \in \{E_1, E_D\}$, where $\chi_{1,1}(x, y)$ is the characteristic vector of $\Gamma_{1,1}(x, y) = \Gamma(x) \cap \Gamma(y)$; cf. [382]. See also [535] for another proof. We say Γ is *tight* if $a_1 \neq 0$ and equality holds in (19). Jurišić et al. [382] showed that Γ is tight if and only if $a_1 \neq 0$, $a_D = 0$, and Γ is 1-homogeneous. To be more precise, call an edge xy *tight with respect to* a non-trivial eigenvalue θ if $E\mathbf{e}_x, E\mathbf{e}_y, E\chi_{1,1}(x, y)$ are linearly dependent, where E is the primitive idempotent corresponding to θ. Then the following properties are all equivalent: (i) Γ is tight; (ii) $a_1 \neq 0$ and every (or at least one) edge of Γ is tight with respect to both θ_1 and θ_D; (iii) $a_1 \neq 0$, $a_D = 0$, and Γ is 1-homogeneous (or 1-homogeneous with respect to an edge). Pascasio [533] showed that if Γ is Q-polynomial then the following properties are equivalent: (i) Γ is tight; (ii) Γ is non-bipartite and $a_D = 0$; (iii) Γ is non-bipartite and $a_D^\star = 0$. More characterizations of the tightness property will be given in the next sections. The fundamental bound inspired quite a bit of the later research by Terwilliger and his students.

It follows from the above result of Pascasio that the non-bipartite antipodal Q-polynomial distance-regular graphs are tight; examples are the Johnson graph $J(2D, D)$,

the halved $2D$-cube, the non-bipartite Taylor graphs and the Meixner 2-cover; cf. Section 5.5.8. There are several sporadic examples known, all of which have diameter 4, and of which only one is primitive, namely the Patterson graph. Jurišić and Koolen [376, Thm. 3.2] showed that tight distance-regular graphs with $D = 3$ are precisely the non-bipartite Taylor graphs. Suda [575] recently gave a simple proof of this result by looking at the intersection matrix L; cf. (3).

Using the fact that the Patterson graph, the Meixner 4-cover, the $3.O_7(3)$-graph, and the $3.O_6^-(3)$-graph are tight and hence 1-homogeneous, one can easily show that the minimal convex subgraph of two vertices at distance two is a complete multipartite graph $K_{n \times t}$ with $n \geqslant 2$, $t \geqslant 2$. This leads in each of the cases to its uniqueness as a distance-regular graph; cf. [377, 378, 379, 380, 88].

The family of tight antipodal distance-regular graphs with $D = 4$ is called the AT4-family. That they are 1-homogeneous gives rise to several feasibility conditions; cf. [375]. Jurišić and Koolen [379] classified the members of the AT4-family with complete multipartite μ-graphs. Jurišić, Munemasa, and Tagami [384] simplified, generalized, and strengthened some of the results in [379].

Vidali and Jurišić [643] recently showed the non-existence of primitive tight distance-regular graphs with classical parameters $(D, b, b - 1, b^{D-1})$, where $D \geqslant 4$ and $b > 1$.

6.1.2 The CAB condition and 1-homogeneous distance-regular graphs

Jurišić and Koolen [374] introduced the CAB_j condition. For vertices $x, y \in V$ at distance $i = 0, 1, \ldots, D$, define the sets $C_i(x, y) = \Gamma_{i-1}(x) \cap \Gamma(y)$, $A_i(x, y) = \Gamma_i(x) \cap \Gamma(y)$, and $B_i(x, y) = \Gamma_{i+1}(x) \cap \Gamma(y)$ (with $\Gamma_{-1}(x) = \Gamma_{D+1}(x) = \emptyset$). For $j = 0, 1, \ldots, D$, we say Γ *satisfies* CAB_j, if for all $i = 0, 1, \ldots, j$ and $x, y \in V$ at distance i, the partition $\{C_i(x, y), A_i(x, y), B_i(x, y)\}$ of the local graph $\Upsilon(y)$ is equitable (where we assume that empty sets are excluded from the partition). It is clear that Γ satisfies CAB_0, and that Γ satisfies CAB_1 if and only if it is locally strongly regular. Note that if Γ satisfies CAB_2 then the μ-graph $\Upsilon(x, y)$ for vertices $x, y \in V$ at distance 2 is regular. Jurišić and Koolen [374] showed that if Γ satisfies CAB_j then for all $i = 0, 1, \ldots, j$ and $x, y \in V$ at distance i, the quotient matrix of $\{C_i(x, y), A_i(x, y), B_i(x, y)\}$ does not depend on the pair x, y, but only on i. They also showed that if $a_1 \neq 0$ then Γ satisfies CAB_D if and only if it is 1-homogeneous. Note that if $a_1 = 0$ then Γ always satisfies CAB_D. Nomura [516] showed that the 1-homogeneous distance-regular graphs of order (s, t) with $s \geqslant 2$, $t \geqslant 1$ are exactly the regular near $2D$-gons, a result that can be shown easily using the CAB_D condition. Jurišić and Koolen [374] also determined the 1-homogenous Terwilliger graphs, and gave an algorithm to determine all 1-homogeneous distance-regular graphs that are locally a given strongly regular graph. See also [377].

6.1.3 More results on homogeneity

Miklavič [486] showed that the triangle-free Q-polynomial distance-regular graphs are 1-homogeneous. Note that if $a_1 = 0$ then the multiplicity of an eigenvalue distinct from $\pm k$ is at least k by Terwilliger's tree bound; cf. Section 14.1. Coolsaet, Jurišić, and Koolen

[142] showed among other results that Γ is 1-homogeneous if it has an eigenvalue with multiplicity k, $a_1 = 0$, $a_2 \neq 0$, and $a_4 = 0$ (when $D \geqslant 4$), and then ruled out the infinite family of intersection arrays $\{2\mu^2 + \mu, 2\mu^2 + \mu - 1, \mu^2, \mu, 1; 1, \mu, \mu^2, 2\mu^2 + \mu - 1, 2\mu^2 + \mu\}$ ($\mu \geqslant 2$). For $\mu = 1$, this intersection array is uniquely realized by the dodecahedron. Jurišić, Koolen, and Žitnik [383] showed among other results that if Γ is primitive and has an eigenvalue with multiplicity k, $a_1 = 0$, and $D = 3$, then the association scheme underlying Γ is formally self-dual and thus Γ is Q-polynomial and 1-homogeneous.

Nomura [517] classified the 2-homogeneous bipartite distance-regular graphs; cf. Section 5.5.8. Nomura [518] also classified the 2-homogeneous generalized odd graphs. Yamazaki [654] observed that if Γ is bipartite then Γ has an eigenvalue with multiplicity k if and only if it is 2-homogeneous, while Curtin [146] showed that if Γ is bipartite then Γ is 2-homogeneous if and only if it is Q-polynomial and antipodal.

6.2 Thin modules

Thin irreducible $\mathbb{T}$-modules with endpoint 1 have been extensively studied; see e.g., [618, 261, 624, 625] and Section 5.6. For example, let $\mathbf{v}$ be a nonzero vector in $E_1^\star \mathbb{C}^v$ which is orthogonal to $A_1 \mathbf{e}_x$, so that $E_0 \mathbf{v} = 0$. Go and Terwilliger [261] showed that if $E_i \mathbf{v}$ vanishes for some $i = 1, 2, \ldots, D$ then $i \in \{1, D\}$ and $\mathbb{A}\mathbf{v}$ is a thin irreducible $\mathbb{T}$-module with endpoint 1 and diameter $D - 2$. There is also a characterization of thin irreducible $\mathbb{T}$-modules with endpoint 1, involving the pseudo primitive idempotents introduced by Terwilliger and Weng [628]. Let $\theta \in \mathbb{C}$ (not necessarily an eigenvalue of Γ). The *pseudo cosine sequence for θ* is the sequence $(\sigma_i)_{i=0}^D$ defined by $\sigma_0 = 1$ and the recursion $c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i$ for $i = 0, 1, \ldots, D - 1$; cf. (4). A *pseudo primitive idempotent E_θ associated with θ* is then any nonzero scalar multiple of $\sum_{i=0}^D \sigma_i A_i$. We also define E_∞ to be any nonzero scalar multiple of A_D. Let $\mathbf{v}$ be as above, and let $(\mathbb{A}; \mathbf{v}) = \{M \in \mathbb{A} : M\mathbf{v} \in E_D^\star \mathbb{C}^v\}$. Note that $J \in (\mathbb{A}; \mathbf{v})$. Terwilliger and Weng [628] showed that $\mathbb{T}\mathbf{v}$ is a thin irreducible $\mathbb{T}$-module (with endpoint 1) if and only if $\dim(\mathbb{A}; \mathbf{v}) \geqslant 2$. Moreover, if this is the case, then $\dim(\mathbb{A}; \mathbf{v}) = 2$ and we have $(\mathbb{A}; \mathbf{v}) = \operatorname{span}_{\mathbb{C}} \{J, E_{\tilde{\eta}}\}$, where η is the local eigenvalue corresponding to $\mathbb{T}\mathbf{v}$ and

$$
\tilde{\eta} = \begin{cases} \infty & \text{if } \eta = -1, \\ -1 & \text{if } \eta = \infty, \\ -1 - \frac{b_1}{1+\eta} & \text{if } \eta \neq -1, \infty. \end{cases}
$$

Terwilliger [625] obtained an inequality[16] involving the local eigenvalues of Γ, and showed that equality is attained if and only if Γ is 1-thin with respect to the base vertex x. Go

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[16]We may remark that the discussions in [625] and those in the proof of the spectral excess theorem (Theorem 10.2) given by Fiol and Garriga [226] are similar in nature. In [625], Terwilliger is concerned with the thinness of irreducible $\mathbb{T}$-modules with endpoint 1 of a distance-regular graph, whereas Fiol and Garriga [226] take a "local approach", which can be understood as being concerned with the thinness of the primary $\mathbb{T}$-module of a general (finite, simple, and connected) graph. (See [618] for the basic theory about the Terwilliger algebra of a general graph.) In both cases, the characterization of the thinness as equality in a bound is obtained by focusing on two specific vectors in $E_D^\star \mathbb{C}^v$.

and Terwilliger [261, Thm. 13.7] showed that the following properties are all equivalent: (i) Γ is tight; (ii) Γ is non-bipartite, $a_D = 0$, and Γ is 1-thin; (iii) Γ is non-bipartite, $a_D = 0$, and Γ is 1-thin with respect to at least one vertex.

We have some comments. It is well known that $a_1 \neq 0$ implies $a_i \neq 0$ for $i = 1, 2, \ldots, D-1$; cf. [78, Prop. 5.5.1]. Dickie and Terwilliger [199] showed that if Γ is 1-thin with respect to at least one vertex then $a_1 = 0$ implies $a_i = 0$ for $i = 1, 2, \ldots, D-1$. We note that these results have dual versions for Q-polynomial association schemes; cf. [197, 199].

Collins [136] showed that Γ is thin with $c_3 = 1$ if and only if it is a generalized octagon of order $(1, t)$. This shows that if Γ is thin then the numerical girth g is at most 8 (and cannot be 7). (Collins [136] only mentioned the implication for the girth of Γ.) Suzuki [589] strengthened this result as follows. Suppose Γ is of order (s, t), and recall that g coincides with the geometric girth in this case. Suzuki showed among other results that (i) $g \leqslant 11$ if there is a thin irreducible $\mathbb{T}$-module with endpoint 3; (ii) Γ is a regular near polygon[17] if and only if it is 1-thin; (iii) if $g \geqslant 8$ then Γ is a generalized $2D$-gon of order $(1, t)$ if and only if it is 1- and 2-thin; (iv) if $g \geqslant 8$ then Γ is a generalized octagon of order $(1, t)$ if and only if it is 1-, 2-, and 3-thin.

Curtin [147] studied the Terwilliger algebras of bipartite distance-regular graphs. Suppose for the moment that Γ is bipartite. Then he showed among other results that Γ is always 1-thin with a unique irreducible $\mathbb{T}$-module with endpoint 1 up to isomorphism, and that if Γ is 2-thin with respect to the base vertex x then the intersection array is determined by D and the multiplicity in $\mathbb{C}^v$ of each of the irreducible $\mathbb{T}$-modules W with endpoint 2, together with the scalar $\psi(W) = -\frac{b_2 b_3}{c_2(\eta(W)+1)} - 1$, where $\eta(W)$ is the eigenvalue of $E_2^* A_2 E_2^*$ on $E_2^* W$, which is an eigenvalue of the local graph of x in the halved graph of Γ. See also [148]. In particular, if Γ is 2-thin with respect to x with (at most) two irreducible $\mathbb{T}$-modules W_1, W_2 with endpoint 2 up to isomorphism, then it turns out that the intersection array is determined by $D, k, c_2, \psi(W_1)$, and $\psi(W_2)$.

Collins [137] studied in detail the relation between the irreducible $\mathbb{T}$-modules of an almost bipartite distance-regular graph Γ and those of its bipartite double $\tilde{\Gamma}$. In particular, he showed that Γ is thin if and only if $\tilde{\Gamma}$ is thin.

6.3 Vanishing Krein parameters

Vanishing of Krein parameters often leads to strong (combinatorial) consequences. A classical example is a result of Cameron, Goethals, and Seidel [107] which states that if a strongly regular graph satisfies either $q_{11}^1 = 0$ or $q_{22}^2 = 0$ then for every vertex, the induced subgraphs on both of the subconstituents are strongly regular.[18] See [263, 376, 373] for similar results for antipodal distance-regular graphs with diameter 3 or 4. In this section, we discuss more results on this topic.

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[17]The referee kindly pointed out an error in [589, Thm. 1.2(ii)].

[18]In passing, by the results of [107] one can quickly find all the irreducible $\mathbb{T}$-modules of a strongly regular graph. In particular, it is always thin; cf. [635].

6.3.1 Triple intersection numbers

Coolsaet, Jurišić, and others used vanishing Krein parameters to obtain information on triple intersection numbers as follows. Let $x, y, z \in V$. For $r, s, t = 0, 1, \ldots, D$, let $p_{r,s,t}^{x,y,z} = |\{u \in V : d(x, u) = r, d(y, u) = s, d(z, u) = t\}|$. Now, if $q_{ij}^h = 0$ then it follows from (8) that

$$\sum_{r,s,t=0}^{D} Q_{ri} Q_{sj} Q_{th} p_{r,s,t}^{x,y,z} = 0. \tag{20}$$

This equation gives some extra information on the triple intersection numbers. We note that (8) was also used earlier by Terwilliger [607] to study the number of 4-vertex configurations with given mutual distances; he showed that if Γ is Q-polynomial then such numbers can be computed from the intersection array and the numbers of 4-vertex cliques in $\Gamma_1, \Gamma_2, \ldots, \Gamma_{\lfloor D/2 \rfloor}$. Using (20), Coolsaet and Jurišić [141] ruled out the infinite family of intersection arrays $\{4r^3 + 8r^2 + 6r + 1, 2r(r+1)(2r+1), 2r^2 + 2r + 1; 1, 2r(r+1), (2r+1)(2r^2 + 2r + 1)\}$ $(r \geqslant 2)$. The case $r = 1$, i.e., $\{19, 12, 5; 1, 4, 15\}$, was eliminated by Neumaier; cf. [74, §5.5A]. Coolsaet and Jurišić also ruled out the intersection array $\{74, 54, 15; 1, 9, 60\}$. Jurišić and Vidali [388] used the above idea of triple intersection numbers to show that there exists a set of vertices mutually at distance 3 of size $p_{33}^3 + 2$ for distance-regular graphs with intersection arrays $\{(2r^2-1)(2r+1), 4r(r^2-1), 2r^2; 1, 2(r^2-1), r(4r^2-2)\}$ or $\{2r^2(2r+1), (2r-1)(2r^2+r+1), 2r^2; 1, 2r^2, r(4r^2-1)\}$ $(r \geqslant 2)$, and showed that consequently such graphs do not exist. Urlep [637] used (20) to rule out the intersection arrays $\{(r+1)(r^3-1), r(r-1)(r^2+r-1), r^2-1; 1, r(r+1), (r^2-1)(r^2+r-1)\}$ $(r \geqslant 3)$. For $r = 2$, this intersection array is uniquely realized by the halved 7-cube. Vidali [642] recently used (20) again to rule out the intersection array $\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}$.

6.3.2 Hadamard products of two primitive idempotents

Another important use of vanishing Krein parameters is the study of pairs of non-trivial primitive idempotents E, F such that $E \circ F$ is a linear combination of a small number of primitive idempotents; cf. (6). For convenience, we define $e(M) = \{E_i : ME_i \neq 0\}$ for $M \in \mathbb{A}$. Pascasio [532] showed that non-trivial primitive idempotents E, F satisfy $|e(E \circ F)| = 1$ precisely when one of the following holds: (i) Γ is tight, $\{E, F\} = \{E_1, E_D\}$, and $e(E \circ F) = \{E_{D-1}\}$; (ii) Γ is bipartite and $E_D \in \{E, F\}$.

Suppose for the moment that Γ is bipartite with $D \geqslant 4$. Let θ, θ' be eigenvalues of Γ other than $\pm k$, and let E, F be the primitive idempotents associated with θ, θ'. Then $|e(E \circ F)| > 1$ by (ii) above. MacLean [455] called the pair $\{E, F\}$ *taut* if $|e(E \circ F)| = 2$. He showed that $|e(E \circ F)| = 2$ if and only if θ, θ' attain equality in what he called the 'bipartite fundamental bound'. We comment on the proof of this result. Let $E = E_i$ and $F = E_j$, and for $t = 0, 1, 2$, let $\mathbf{f}_t$ be the vector in $\mathbb{R}^{D+1}$ with h-coordinate

$$\theta_h^t \sqrt{\frac{q_{ij}^h m_h}{m_i m_j}} \quad (h = 0, 1, \ldots, D). \tag{21}$$

Then $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2$ are linearly dependent if and only if $|e(E \circ F)| = 2$, and computing the determinant of the (positive semidefinite) Gram matrix of $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2$ yields the bipartite fundamental bound. See [459, 458] for more proofs of this bound. We say Γ is *taut* if it has a taut pair of primitive idempotents and is not 2-homogeneous. MacLean [455, Thm. 1.4] showed that $\{E, F\}$ is taut precisely when one of the following holds: (i) Γ is taut and $\{E, F\} \in \{\{E_h, E_\ell\} : h \in \{1, D-1\}, \ell \in \{\tau, D-\tau\}\}$ where $\tau = \lfloor D/2 \rfloor$; (ii) Γ is 2-homogeneous and $\{E, F\} \cap \{E_1, E_{D-1}\} \neq \emptyset$. For $D = 4, 5$, Γ is taut or 2-homogeneous if and only if Γ is antipodal [455, §§7–8]. MacLean and Terwilliger [459] showed among other results that if D is odd then the following are equivalent: (i) Γ is taut or 2-homogeneous; (ii) Γ is antipodal and 2-thin; (iii) Γ is antipodal and 2-thin with respect to at least one vertex; see also [460]. Examples of taut graphs with odd $D \geqslant 5$ are the Doubled Odd graphs, the Doubled Hoffman-Singleton graph, the Doubled Gewirtz graph, and the Doubled 77-graph; cf. [456, p. 131]. For D even and at least 6, MacLean [455, Thm. 5.8] showed that Γ is taut or 2-homogeneous if and only if its halved graphs are tight. If Γ is taut in this case, then it turns out however that $D \neq 6$ and that no known example of a tight distance-regular graph with diameter at least 4 can be a halved graph of Γ; cf. [457].

Retaining the situation of the last paragraph, let Δ_E be the representation diagram[19] of $E = E_i$, and let $(u_h)_{h=0}^D$ be the standard sequence associated with E. Note that 0 and D are leaves (i.e., terminal vertices) in Δ_E, and that j is a leaf in Δ_E precisely when $|e(E \circ F)| = 2$ and $F \in e(E \circ F)$. Lang [436] showed that

$$(u_1 - u_{h+1})(u_1 - u_{h-1}) \geqslant (u_2 - u_h)(u_0 - u_h) \quad (h = 1, 2, \ldots, D-1), \qquad (22)$$

with equality for every $h = 1, 2, \ldots, D-1$ (or just for $h = 3$) if and only if $u_{h-1} - pu_h + u_{h+1}$ is independent of $h = 1, 2, \ldots, D-1$ for some $p \in \mathbb{R}$. When E attains equality, Lang [434] showed that (i) $u_D \neq 1$ if and only if Δ_E is a path (i.e., E is Q-polynomial); (ii) $u_D = 1$ if and only if Δ_E is the disjoint union of two paths. It follows that if case (ii) occurs then Γ is antipodal and the folded graph is Q-polynomial; cf. [436, Thm. 10.2, 10.4]. Note that in both cases (i) and (ii), E is a *tail*, i.e., $|e(E \circ E)| \leqslant 3$ and $|e(E \circ E)\backslash\{E_0, E\}| \leqslant 1$. Conversely, Lang [434] showed that if E is a tail and $D \neq 6$ then E attains equality in (22). Lang [435] also showed that Δ_E has a leaf other than $0, D$ if and only if E attains equality in (22) and case (ii) occurs above. One of the other results in [436] is that if $D \geqslant 6$ and Γ has more than one primitive idempotent that attains equality in (22), then Γ is the D-cube.

Suppose now that Γ is arbitrary (i.e., $D \geqslant 3$ and not necessarily bipartite). By considering the Gram matrix of $\mathbf{f}_0$ and $\mathbf{f}_1$, Pascasio [535] later extended some of the results in [532], as well as the fundamental bound, to the level of P-polynomial character algebras. Tomiyama [634] considered the situation where one of $1, D$ is a leaf in Δ_E and generalized some of the results in [435, 436, 532, 533].

Assume $E = F$ (so $i = j$) and $\theta(= \theta') \neq \pm k$. Then $|e(E \circ E)| \geqslant 2$. We call E a *light*

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[19]The *representation diagram* of $E = E_i$ is the simple graph with vertex set $\{0, 1, \ldots, D\}$, where two distinct vertices h, ℓ are adjacent whenever $q_{ih}^\ell \neq 0$.

tail[20] [387] if $|e(E \circ E)| = 2$. Let $\mathbf{f}_0', \mathbf{f}_1'$ be the vectors obtained from $\mathbf{f}_0, \mathbf{f}_1$, respectively, by the removal of the 0-coordinate. Note that $\mathbf{f}_0', \mathbf{f}_1'$ are linearly dependent if and only if E is a light tail. Jurišić, Terwilliger, and Žitnik [387] considered the Gram matrix of $\mathbf{f}_0', \mathbf{f}_1'$. In this case, the resulting inequality gives a lower bound on the multiplicity of θ; cf. Proposition 14.5. They showed among other results that distance-regular graphs with a light tail are close to being 1-homogeneous, i.e., the parameters $p_{i,j;r,s}$ exist with respect to, and are independent of, every pair of adjacent vertices $x, y \in V$ for all $i, j, r, s = 0, 1, \ldots, D$ except possibly $i = j = 2, 3, \ldots, D - 1$. In particular, the local graphs are strongly regular. We note that these results generalize those of Cameron, Goethals, and Seidel [107] mentioned at the beginning of Section 6.3. They indeed showed that primitive strongly regular graphs with a light tail (and $k \geqslant 3$) are precisely the Smith graphs.

6.4 Relaxations of homogeneity

In the previous sections, we explored connections among homogeneity, thin modules, tightness, local graphs, Hadamard products of two primitive idempotents, and so on. In fact, many of these results can be generalized in several directions, as we discuss below.

We say Γ is *pseudo 1-homogeneous with respect to an edge* xy [385] if the parameters $p_{i,j;r,s}$ exist with respect to x, y for all $i, j, r, s = 0, 1, \ldots, D$ except possibly $i = j = D$. Let $\theta \in \mathbb{R} \backslash \{k\}$, and let E_θ be a pseudo primitive idempotent associated with θ; cf. Section 6.2. We say the edge xy is *tight with respect to* θ [385] if a non-trivial linear combination of $E_\theta \mathbf{e}_x, E_\theta \mathbf{e}_y, E_\theta \chi_{1,1}(x, y)$ is contained in the subspace $\mathrm{span}_{\mathbb{R}}\{\mathbf{e}_z : z \in \Gamma_{D,D}(x, y)\}$. Jurišić and Terwilliger [385] showed among other results that if $a_1 \neq 0$ then the edge xy is tight with respect to two distinct real numbers if and only if Γ is pseudo 1-homogeneous with respect to xy and the induced subgraph on $\Gamma_{1,1}(x, y)$ is not a clique. Under the condition $a_1 \neq 0$, Curtin and Nomura [157] characterized the situation where Γ is 1-thin with respect to x with precisely two non-isomorphic irreducible $\mathbb{T}(x)$-modules with endpoint one, in terms of the pseudo 1-homogeneous property[21] of the edges xy ($y \in \Gamma(x)$). They studied in detail the case where $a_1 = 0$ as well. Extending the work of Pascasio [532] on the tightness property, Pascasio and Terwilliger [537] described exactly when $E_\theta \circ E_{\theta'}$ with $\theta, \theta' \in \mathbb{R}$ is a scalar multiple of E_τ for some $\tau \in \mathbb{R}$.

Suppose for the moment that Γ is bipartite with $D \geqslant 4$, and let x, y be vertices with $d(x, y) = 2$. Curtin [146, §§4–5] showed that Γ is 2-homogeneous if and only if $|\Gamma_{1,1}(x, y) \cap \Gamma_{i-1}(z)|$ depends only on $i = 1, 2, \ldots, D - 1$ and is independent of $z \in \Gamma_{i,i}(x, y)$. We say Γ is *almost 2-homogeneous* [150] if the same condition holds for $i = 1, 2, \ldots, D - 2$. Recall that Γ is 1-thin with a unique irreducible $\mathbb{T}$-module with endpoint 1 up to isomorphism. Curtin [150] showed among other results that Γ is almost 2-homogeneous if and only if it is 2-thin with a unique irreducible $\mathbb{T}$-module with endpoint 2 up to isomorphism. Curtin [150] and Jurišić, Koolen, and Miklavič [381] classified the almost 2-homogeneous bipartite

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[20]It is easy to see that if $e(E \circ E) = \{E_0, E\}$ then Γ is antipodal with $D = 3$; cf. [387, Thm. 4.1(b)]. We view this case as degenerate, so we propose to assume $E \notin e(E \circ E)$ as well in the definition of a light tail.

[21]It should be remarked that Curtin and Nomura [157] do not require the existence of the parameter $p_{D,D-1;r,s}$ with respect to x, y.

distance-regular graphs: the 2-homogeneous graphs (cf. Section 5.5.8), the generalized $2D$-gons of order $(1, k-1)$, the folded $2D$-cube, and the coset graph of the extended binary Golay code.[22] Lang [437] considered when $E_\theta \circ E_\theta$ with $\theta \in \mathbb{C} \backslash \{k, -k\}$ is a linear combination of J and E_τ for some $\tau \in \mathbb{C}$, and showed that this occurs precisely when Γ is almost 2-homogeneous and $c_2 \geqslant 2$.

In some cases, it is possible to get an equitable partition of V from $\Pi = \{\Gamma_{i,j}(x, y) : \Gamma_{i,j}(x, y) \neq \emptyset, i, j = 0, 1, \ldots, D\}$, where $d(x, y) = h \in \{1, 2\}$, by refining some of the $\Gamma_{i,i}(x, y)$ $(i = 2, 3, \ldots, D)$ into two cells, even when Π itself is not equitable. This was worked out in detail by Miklavič for distance-regular graphs having classical parameters with $b < -1$, $a_1 \neq 0$ [487] $(h = 1)$, for bipartite Q-polynomial distance-regular graphs with $c_2 = 1$ [488] $(h = 2)$, and for the bipartite dual polar graphs $\mathcal{D}_D(q)$ [492] $(h = 2)$. See also [489] for a description of the $\mathbb{A}$-module spanned by $\{\chi_{i,j}(x, y) : i, j = 0, 1, \ldots, D\}$ with $h = 2$ (cf. Section 5.4) for bipartite Q-polynomial distance-regular graphs. The parameters of the new equitable partition give rise to additional integrality conditions, and he used these conditions to show that there is no bipartite Q-polynomial distance-regular graph with $D = 4$ and girth 6; cf. Section 5.7.3.

7   Growth of intersection numbers and bounds on the diameter

In this section we will look at the growth of intersection numbers and its consequences for bounds on the diameter.

7.1   The Ivanov bound

Ivanov [359] obtained the first general diameter bound for distance-regular graphs.

Theorem 7.1. (The Ivanov bound) *Let Γ be a distance-regular graph with diameter $D \geqslant 2$, head h, and valency k. Let $2 \leqslant i \leqslant i + j \leqslant D - 1$. If $(c_{i-1}, a_{i-1}, b_{i-1}) \neq (c_i, a_i, b_i) = (c_{i+j}, a_{i+j}, b_{i+j})$, then $j \leqslant i$. In particular, $D < 2^{k-1}(h + 1)$.*

Suzuki [587, p. 67] gave a proof of this bound using so-called intersection diagrams. Bang, Hiraki, and Koolen [27, 319] improved the Ivanov bound, as we shall discuss below. One of the tools that they used is the following result of Koolen [404], [407, Prop. 2.3].

Proposition 7.2. *Let Γ be a distance-regular graph with diameter D.*

(i) *If $c_i > c_{i-1}$ for some $i = 2, \ldots, D$, then $c_{i-j} + c_j \leqslant c_i$ for all $j = 1, \ldots, i-1$,*

(ii) *If $b_i > b_{i+1}$ for some $i = 0, \ldots, D-2$, then $b_i \geqslant b_{i+j} + c_j$ for all $j = 1, \ldots, D-i$.*

Wajima [644] also obtained Proposition 7.2 (but with a completely different method), and Hiraki [315] obtained slight improvements of this result. Another tool by Bang et al. [27] is the following.

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[22]See also [591] for a generalization of this result (as well as Nomura's classification [517, 518] of bipartite or almost bipartite 2-homogeneous distance-regular graphs) to triangle-free distance-regular graphs. That the coset graph of the extended binary Golay code is almost 2-homogeneous was pointed out by Lang [437, Lemma 3.4].

Proposition 7.3. *Let Γ be a distance-regular graph with valency k and diameter D. For $1 \leqslant c \leqslant k$, define $\xi_c = \min\{i : c_i \geqslant c\}$ and $\eta_c = |\{i : c_i = c\}|$. Then $\eta_c \leqslant \xi_c - 1$.*

Using a combination of Propositions 7.2 and 7.3, Bang et al. [27] improved the Ivanov bound as follows (using the notation as introduced in Proposition 7.3):

Proposition 7.4. *Let Γ be a distance-regular graph with valency k and diameter D. Then*

$$D < \frac{1}{2}k^\alpha \eta_1 + 1,$$

where $\alpha = \inf\{x > 0 : 4^{\frac{1}{x}} - 2^{\frac{1}{x}} \leqslant 1\} \approx 1.441$.

Hiraki [311] showed, using earlier work from [123, 303, 308, 320], that if $h \geqslant 2$, then $c_{2h+3} \geqslant 2$ or, in other words, $\eta_1 \leqslant 2h + 2$. This immediately implies that if $h \geqslant 2$, then

$$D \leqslant k^\alpha(h+1) + 1.$$

For $h = 1$, it is conjectured by Hiraki [304] that there exists a constant C such that $\eta_1 \leqslant C$. Chen, Hiraki, and Koolen [124] showed that if $a_1 \neq 2$ and $a_1 \leqslant 100$, then $c_4 \geqslant 2$.

In the next sections we present better diameter bounds for certain subclasses of distance-regular graphs.

7.2 Distance-regular graphs of order (s, t)

The following result was first shown by Terwilliger [605] for distance-regular graphs with $a_1 = 0$ or $c_2 \geqslant 2$. Later it was generalized by Faradjev, Ivanov, and Ivanov [212] to distance-regular graphs with $a_1 > 0$. We present their bound for the case that Γ is locally a disjoint union of cliques and $c_{h+1} \geqslant 2$ holds. In the next section we will also present the bound of Terwilliger for the case $c_2 \geqslant 2$.

Proposition 7.5. *(cf. [587, Thm. 1.4.3, Cor. 1.4.4]) Let Γ be a distance-regular graph of order (s, t) with head h, valency k, and diameter $D \geqslant 2$. If $c_{h+1} > 1$, then $b_i > b_{i+h}$ and $c_i < c_{i+h}$ for all $i = 0, 1, \ldots, D - h$, and in particular, $D \leqslant th + 1$.*

For the bipartite case, Koolen [404] and Hiraki [315] made some improvements. Hiraki [315] showed that if Γ is a bipartite distance-regular graph with head $h \geqslant 2$ and diameter D, then Γ is a Doubled Odd graph or $D \leqslant \lfloor \frac{k+2}{2} \rfloor h$; see also Section 9.2. For the weakly geometric case, Suzuki obtained the following.

Proposition 7.6. *(cf. [587, Prop. 3.1.6]) Let Γ be a weakly geometric distance-regular graph of order (s, t) with head h and diameter D. Then $b_i > b_{i+h+1}$ and $c_i < c_{i+h+1}$ for all $i = 0, 1, \ldots, D - h - 1$. In particular, $D \leqslant t(h+1) + 1$.*

Corollary 7.7. *Let Γ be a distance-regular graph of order (s, t) with head h and diameter D. If $s > t$, then Γ is geometric and hence $b_i > b_{i+h+1}$ and $c_i < c_{i+h+1}$ for all $i = 0, 1, \ldots, D - h - 1$ and in particular, $D \leqslant t(h+1) + 1$.*

Corollary 7.8. *For all integer $t \geqslant 1$ there exists a constant C_t such that for all distance-regular graphs Γ of order (s, t), the diameter of Γ is bounded by $C_t h(\Gamma)$, where $h(\Gamma)$ is the head of Γ.*

7.3   A bound for distance-regular graphs with $c_2 \geqslant 2$

Terwilliger [605] obtained the following bound for distance-regular graphs with $c_2 \geqslant 2$.

Proposition 7.9. (cf. [78, Thm. 5.2.5, Prop. 1.9.1]) *Let Γ be a distance-regular with diameter $D \geqslant 2$ and $c_2 \geqslant 2$. If $c_2 \geqslant 2(a_1+1)$, then $c_i - b_i \geqslant c_{i-2} - b_{i-2} + 2$ $(i = 2, 3, \ldots, D)$, and in particular $D \leqslant k$. Moreover, if $\max\{a_1, 2\} \leqslant c_2$, then $c_i \geqslant c_{i-1} + 1$ $(i = 2, 3, \ldots, D)$.*

Caughman [109] improved this result for bipartite distance-regular graphs as follows.

Proposition 7.10. *Let Γ be a bipartite distance-regular graph with valency k, diameter $D \geqslant 3$, and $c_2 \geqslant 2$. Let $i = 1, 2, \ldots, D-1$. If $k > c_i((c_2-1)(c_2-2)(c_i - c_{i-1} - 1)/2 + 1)$, then $c_{i+1} \geqslant c_i(c_2 - 1) + 1$.*

Moreover, Terwilliger [606] obtained the following diameter bound.

Proposition 7.11. (cf. [78, Thm. 5.2.1, Cor. 5.2.2]) *Let Γ be a distance-regular graph with diameter D. If Γ contains an induced quadrangle, then $c_i - b_i \geqslant c_{i-1} - b_{i-1} + a_1 + 2$ and, in particular, $D \leqslant \frac{k+c_D}{a_1+2}$.*

The distance-regular graphs with diameter $\frac{k+c_D}{a_1+2}$ and containing a quadrangle have second largest eigenvalue $b_1 - 1$ and have been classified: besides the strongly regular graphs with smallest eigenvalue -2, these are the Hamming graphs, Doob graphs, halved cubes, Johnson graphs, locally Petersen graphs, and the Gosset graph, see [78, Thm. 5.2.3]; also cf. Section 5.1. Note that if a distance-regular graph contains a quadrangle then the second largest eigenvalue is at most $b_1 - 1$.

Neumaier [513] showed among other results that if there are infinitely many distance-regular graphs with fixed a_1, c_2, a_i, c_i containing an induced quadrangle then necessarily $c_{i+1} \geqslant 1 + (c_2 - 1)c_i$. For dual polar graphs, equality holds.

7.4   The Pyber Bound

Using a slightly weaker result than Proposition 7.2, Pyber [544] showed that $D \leqslant 5 \log_2 v$ for a distance-regular graph with v vertices and diameter D. This essentially settles a problem in 'BCN' [78, p. 189]. Pyber's bound was improved by Bang, Hiraki, and Koolen [27] to $D < \frac{8}{3} \log_2 v$.

8   The Bannai-Ito conjecture

In 1984, Bannai and Ito [38, p. 237] made the following conjecture.

Bannai-Ito conjecture. There are finitely many distance-regular graphs with fixed valency at least three.

This Bannai-Ito conjecture has recently been proved by Bang, Dubickas, Koolen, and Moulton [24]. In the next section, we will give an outline of this proof.

8.1 Proof of the Bannai-Ito conjecture

Let Γ be a distance-regular graph with valency $k \geqslant 3$, head h, and diameter D. The Ivanov bound (Theorem 7.1) tells us that $D \leqslant 4^k h$. So in order to prove the conjecture it suffices to bound h as a function of k.

Bannai and Ito [40] obtained the following result, using head and tail. Recall that the latter is defined by $t = \ell(b_1, a_1, c_1)$.

Theorem 8.1. *Let $M \geqslant 1$ and $k \geqslant 3$. Then there are finitely many triangle-free distance-regular graphs with valency k, and diameter $D \leqslant h + t + M$.*

The key idea of the proof of this theorem is as follows. By interlacing, Γ has an eigenvalue θ in the interval $(2\sqrt{k-1}\cos\frac{3\pi}{h+1}, 2\sqrt{k-1}\cos\frac{\pi}{h+1})$. Let Θ be the set of algebraic conjugates of θ. Then

$$\prod_{\theta' \in \Theta} ((\theta')^2 - k + 1)$$

is a nonzero integer. Let $S = \{x \in [-k, k] : |x^2 - k + 1| > 1\}$. If h is large enough, then $\theta \notin S$ and there is an algebraic conjugate θ' of θ which is in S. Now it can be shown, using Biggs' formula (Theorem 2.8), that the multiplicity of θ is of order $\frac{v}{h^3}$. If $\theta' \in (-2\sqrt{k-1}, +2\sqrt{k-1})$, then the multiplicity of θ' is $\Omega(\frac{v}{h})$, and else the multiplicity of θ' is $O(\frac{v}{a^h})$ for some fixed real $a > 1$. This shows that the multiplicities of θ and θ' are not the same if h large, which is a contradiction to the fact that they are algebraic conjugates. Therefore h is bounded.

Suzuki [580] generalized this result by replacing the triangle-free condition by the condition $(a_1 + 1)(a_1 + 2) \leqslant k$. Bang, Koolen, and Moulton [34] extended the result as follows.

Proposition 8.2. *Let $k \geqslant 3$. Then there exists a positive $\epsilon = \epsilon_k$ such that there are finitely many distance-regular graphs Γ with valency k, and diameter $D \leqslant h + t + \epsilon h$.*

The proof of Proposition 8.2 closely follows the proof of Theorem 8.1. Instead of considering an eigenvalue in the above mentioned interval close to $2\sqrt{k-1}$, so-called indicator intervals are used.

Let $G = \{(c_i, a_i, b_i) : i = 1, 2, \ldots, D - 1\}$ and $g = |G|$; note that $g \leqslant 2k - 3$. We will assume that $G = \{(\gamma_i, \alpha_i, \beta_i) : i = 1, 2, \ldots, g\}$ is ordered by $\gamma_{i+1} \geqslant \gamma_i$ and $\beta_{i+1} \leqslant \beta_i$. Let $\ell_i = \ell(\gamma_i, \alpha_i, \beta_i)$ for $i = 1, 2, \ldots, g$, whence $h = \ell_1$. Let $\mathcal{L}_i = \alpha_i - 2\sqrt{\gamma_i \beta_i}$ and $\mathcal{R}_i = \alpha_i + 2\sqrt{\gamma_i \beta_i}$ be the *left* and *right indicator points*, respectively. The *indicator interval* is defined as the open interval $\mathcal{I}_i = (\mathcal{L}_i, \mathcal{R}_i)$, $i = 1, 2, \ldots, g$. Using the fact that the c_is are non-decreasing and the b_is are non-increasing, it is fairly easy to see that $(\mathcal{R}_i)_i$ is a unimodal sequence. The fact that $b_i + a_i \geqslant a_1 + 2$ $(i = 1, 2, \ldots, D - 1)$ implies that $\mathcal{R}_g \geqslant \mathcal{R}_1$.

By removing from the (tridiagonal) intersection matrix L rows and columns $0, \ell_1, \ell_1 + \ell_2, \ldots, \ell_1 + \cdots + \ell_{g-1}, D$ and using interlacing, it follows that at most $2g + 2$ eigenvalues of Γ (in general this is a relatively small number compared to the total) do not lie in any of the indicator intervals. Instead of taking θ close to $\mathcal{R}_1$, one can show that there must

exist an eigenvalue θ close to a right indicator point different from $\mathcal{R}_1$ if $D - h - t$ is large enough. Then it is shown in a similar way as in Theorem 8.1 that there exists an algebraic conjugate of θ whose multiplicity is different from the multiplicity of θ; again a contradiction, so h is bounded.

Until now the approach was to calculate the multiplicity of a specific eigenvalue precisely and then show that this eigenvalue has an algebraic conjugate with a different multiplicity. For a proof of the Bannai-Ito conjecture one needs to use another tactic, especially in the case that $D - h - t$ is large. The idea here is to find an interval $\mathcal{I}$ in which there are at least δh eigenvalues (where δ is a positive real number only depending on k) and in which every two algebraic conjugate eigenvalues θ and θ' satisfy $|\theta - \theta'| \leqslant f(h)$, where $f(h) \to 0 \;\; (h \to \infty)$. The main reason that one can find such an interval $\mathcal{I}$ is that the right indicator points form a unimodal sequence. Although to calculate the multiplicities of the eigenvalues only involves three-term recurrence relations, to show that $\mathcal{I}$ really exists and that we can approximate the multiplicities in $\mathcal{I}$ well enough is extremely technical and subtle. Using some elementary number theory, it then follows that the number of algebraic conjugates of eigenvalues in $\mathcal{I}$ (which must all be eigenvalues of Γ) is at least $z(h)h$, where $z(h) \to \infty \;\; (h \to \infty)$. But as the the number of eigenvalues besides the valency is exactly D, which — by the Ivanov bound — is at most $4^k h$, we see that this is a contradiction if h is large. Again, this means that h is bounded, which proves the Bannai-Ito conjecture.

8.2 Extensions of the Bannai-Ito conjecture

Bannai and Ito [39] showed that the length $\ell(c, k - 2c, c)$ is bounded by $10k2^k$ for every c. This inspired Hiraki, Suzuki, and others to obtain bounds for $\ell(1, k - 2, 1)$. The current best bound is by Hiraki [314], who obtained $\ell(1, k-2, 1) \leqslant 14$ if $k \geqslant 3$ and $\ell(1, k-2, 1) \leqslant 1$ if $k \geqslant 58$. Inspired by this, Bang, Koolen, and Moulton [33] showed that if b and c are positive integers, then there exists a constant $k_{\min} \geqslant \max\{b + c, 3\}$ such that if Γ is a distance-regular graph with valency $k \geqslant k_{\min}$ and $h \geqslant 2$, then $\ell(c, k - b - c, b) \leqslant 1$. This implies, by using the validity of the Bannai-Ito conjecture, that if b and c are positive integers, then there exists a constant $\ell_{\max}$ such that for every distance-regular graph Γ with valency $k \geqslant \max\{b + c, 3\}$ and $h \geqslant 2$, we have that $\ell(c, k - b - c, b) \leqslant \ell_{\max}$. It is still an open problem whether this is true for $h = 1$ and $c_2 = 1$. This has been conjectured by Bang et al. [33]. Park, Koolen, and Markowsky [531] extended the Bannai-Ito conjecture as follows.

Proposition 8.3. *Let M be a positive integer. Then there are finitely many distance-regular graphs with valency $k \geqslant 3$, diameter $D \geqslant 6$, and $\frac{k_2}{k} \leqslant M$.*

For diameter at most four, the analogous result is not true. For diameter two this is clear. For diameter three, the Taylor graphs have $k_2 = k$ and the incidence graphs of the complements of projective planes of order t have $k = t^2$ and $k_2 = t^2 + t$. For diameter four, the Hadamard graphs have $k_2 = 2(k - 1)$.

Koolen and Park [419] showed that the only primitive distance-regular graphs with $\frac{k_2}{k} \leqslant 1.5$, and diameter at least three are the Johnson graph $J(7, 3)$ and the halved 7-cube.

8.3 The distance-regular graphs with small valency

The edge is the only distance-regular graph with valency one, and the polygons are the distance-regular graphs with valency two. The distance-regular graphs with valency three have been classified by Biggs, Boshier, and Shawe-Taylor [52] (see also [78, Thm. 7.5.1]): There are exactly 13 of them and all have diameter at most 8.

The intersection arrays of the distance-regular graphs with valency four have been classified by Brouwer and Koolen [89]. There are exactly 17 such intersection arrays and all have diameter at most 7 (all graphs are known, except perhaps for point-line incidence graphs of a generalized hexagon of order three).

The distance-regular graphs with valency 6 and $a_1 = 1$ (i.e., of order $(2, 2)$) have been classified by Hiraki, Nomura, and Suzuki [325]. There are exactly five of them and they are all geometric. This last result also completes the classification of all distance-regular graphs with valency at most 7 and $a_1 \geqslant 1$ (see [325] for a complete list).

The larger t is, the more difficult it is to classify the distance-regular graphs of order (s, t). For example, it is much harder to classify the distance-regular graphs with valency 5 than the distance-regular with valency 6 and $a_1 = 1$. For $t = 1$ we have the line graphs, and Yamazaki [653] developed some theory for the case $t = 2$. It is not known whether for a distance-regular graph with order (s, t), one can bound the diameter in terms of t only, if $t \geqslant 2$ (see also Corollary 7.8).

9 Geometric distance-regular graphs

9.1 Metsch's characterizations

As mentioned before, Metsch characterized most of the Grassmann graphs and bilinear forms graphs by their intersection arrays. From these intersection arrays, he recovers the geometric properties of these graphs. An important ingredient for this is the following proposition, which is used to construct lines — large cliques — that partition the edge set.

Proposition 9.1. [478, Result 2.2] *Let $\mu \geqslant 1$, λ_1, λ_2, and m be integers. Assume that Γ is a connected graph with the following properties:*

  (i) *Every two adjacent vertices have at least λ_1 and at most λ_2 common neighbors,*

 (ii) *Every two nonadjacent vertices have at most μ common neighbors,*

(iii) $2\lambda_1 - \lambda_2 > (2m - 1)(\mu - 1) - 1$,

 (iv) *Every vertex has fewer than $(m + 1)(\lambda_1 + 1) - \frac{1}{2}m(m + 1)(\mu - 1)$ neighbors.*

Define a line to be a maximal clique C satisfying $|C| \geqslant \lambda_1 + 2 - (m - 1)(\mu - 1)$. Then every vertex is on at most m lines, and every two adjacent vertices lie in a unique line.

In [478], Metsch used Proposition 9.1 and a characterization of projective incidence structures by Ray-Chaudhuri and Sprague [546] (see also [78, Thm. 9.3.9], and [158] for a generalization by Cuypers) to characterize the Grassmann graphs. The interesting thing is that he hardly required any of the regularity conditions that follow from the intersection array. The only conditions that Metsch used were the intersection number c_2 and upper and lower bounds on the number $b_0(x)$ of neighbors of a vertex x, and on the number of common neighbors $a_1(x,y)$ of two adjacent vertices x and y, and a lower bound on the number $b_2(x,y)$ of vertices x and y at distance two. In weaker form, the characterization is as follows.

Proposition 9.2. [478, Thm. 1.1] *Let $q \geqslant 2$ be an integer, and let D and n be integers satisfying $2D \leqslant n$. Let $s + 1 = (q^{n-D+1} - 1)/(q-1)$ and $m = (q^D - 1)/(q-1)$. Let Γ be a connected ms-regular graph with the property that every two adjacent vertices have $a_1 = s - 1 + (m-1)q$ common neighbors and every two vertices at distance two have $c_2 = (q+1)^2$ common neighbors, and such that every two vertices x and y at distance two have $b_2(x,y) > (m-q-1)(s-q^2-q)$. If $D \neq 2, \frac{n}{2}$, or $\frac{n-1}{2}$ (for all q) and $(D,q) \neq (\frac{n-2}{2}, 2), (\frac{n-2}{2}, 3)$, or $(\frac{n-3}{2}, 2)$, then q is a prime power and Γ is the Grassmann graph $J_q(n, D)$.*

Building on work by Huang [341] and a characterization of attenuated spaces by Sprague [569], Metsch [481] also used Proposition 9.1 to characterize the bilinear forms graphs.

Proposition 9.3. *Let Γ be a distance-regular graph with classical parameters (D, q, α, β), where $\alpha = q - 1$ and $D \geqslant 3$. Suppose that either $q = 2$ and $\beta \geqslant q^{D+4} - 1$ or $q \geqslant 3$ and $\beta \geqslant q^{D+3} - 1$. Then q is a prime power, $\beta = q^e - 1$ for some integer e, and Γ is the bilinear forms graph $Bil(D \times e, q)$.*

Proposition 9.1 can be used further in characterizing other geometric distance-regular graphs. We will get back to this in Section 9.4.

9.2 Characterization of Doubled Odd and Doubled Grassmann graphs

In Section 4.5 we mentioned the distance-biregular graphs that arise as incidence graphs between the vertices and the cliques coming from the $(D+1)$-subspaces in the Grassmann graph $J_q(n, D)$, and the similar one (with $(D+1)$-subsets) from the Johnson graph $J(n, D)$. Cuypers [158] classified the distance-biregular graphs with diameter at least 5 and $c_2^R = 1 < c_3^R = c_4^R$ (R being one of the color classes): the only ones are the above mentioned graphs and the Doubled Moore graphs. This implies that the Doubled Grassmann graphs, the Doubled Odd graphs (the case $n = 2D + 1$), and also the Doubled Hoffman-Singleton graph are determined as distance-regular graphs by their intersection arrays.

Hiraki [315] (also) characterized the Odd graphs and the Doubled Odd graphs among the distance-regular graphs by a few of their intersection numbers, as we already mentioned for the Odd graphs in Section 2.3. Moreover, Hiraki [312] characterized the Doubled Grassmann graphs, the Doubled Odd graphs, and the Odd graphs by their strongly closed subgraphs.

9.3 Bounds on claws

An m-claw in a graph Γ is an induced m-star $K_{1,m}$, or in other words, a coclique of size m in one of the local graphs $\Upsilon(x)$, $x \in V$.

If Γ is a geometric distance-regular graph (with respect to a set of Delsarte cliques $\mathcal{C}$) with smallest eigenvalue $-m$, then it follows easily that each vertex is in m cliques of $\mathcal{C}$, and hence Γ has no $(m+1)$-claws. Under some conditions, a reverse statement can be made, as we shall see at the end of this section. Besides this, the existence of claws of certain size gives rise to new parameter conditions. But as we shall see, sometimes the intersection numbers force the existence of claws, thus giving some nonexistence results.

A special case of a result of Metsch's work [477, Lemma 1.1.b] on the existence of large cliques in graphs is the following (see also work by Godsil [265, Lemma 2.3] or Koolen and Park [418, Lemma 2]).

Lemma 9.4. *Let Γ be a distance-regular graph. Let x be a vertex and let μ be the maximum size of $\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$ where $y \sim x \sim z$ and $y \not\sim z$. If the local graph $\Upsilon(x)$ contains a coclique of size m, then $\mu \geqslant \frac{m(a_1+1)-k}{\binom{m}{2}}$.*

The following consequence of this lemma was observed by Koolen and Park [418, Thm. 4], using that a 'greedy' coclique in $\Upsilon(x)$ has at least $k/(a_1 + 1)$ vertices.

Proposition 9.5. *Let Γ be a distance-regular graph with valency k and diameter $D \geqslant 2$, and let $m' = \lceil \frac{k}{a_1+1} \rceil$. Then*

$$c_2 - 1 \geqslant \frac{m'(a_1+1) - k}{\binom{m'}{2}} \tag{23}$$

with equality implying that Γ is a Terwilliger graph.

This result shows that there are no distance-regular graphs with intersection arrays $\{44, 30, 5; 1, 3, 40\}$, $\{65, 44, 11; 1, 4, 55\}$, $\{81, 56, 24, 1; 1, 3, 56, 81\}$, $\{117, 80, 30, 1; 1, 6, 80, 117\}$, $\{117, 80, 32, 1; 1, 4, 80, 117\}$ and $\{189, 128, 45, 1; 1, 9, 128, 189\}$ (the last four were also ruled out by Jurišić and Koolen [375]).

Gavrilyuk [239] showed that the only distance-regular graphs with $c_2 > 1$ for which equality holds in (23) are the Icosahedron, the Conway-Smith graph, and the Doro graph. Gavrilyuk [240] also extended the above by using Brooks' theorem to eliminate the existence of a distance-regular graph with intersection array $\{55, 36, 11; 1, 4, 45\}$. Brooks' theorem (see [60, Thm. 14.4]) states that the chromatic number of a connected graph is at most its maximum valency, except for the odd cycles and complete graphs. For a distance-regular graph Γ, this implies that $\Upsilon(x)$ has a coclique of size at least k/a_1, unless possibly when $\Upsilon(x)$ contains an odd cycle (if $a_1 = 2$) or an $(a_1 + 1)$-clique as one of its components. This means that Proposition 9.5 can be sharpened a bit.

Proposition 9.6. *Let Γ be a distance-regular graph with valency k and diameter $D \geqslant 2$, and let $m'' = \lceil \frac{k}{a_1} \rceil$. If for some vertex x, the local graph $\Upsilon(x)$ does not contain an odd cycle or an $(a_1 + 1)$-clique as one of its components, then*

$$c_2 - 1 \geqslant \frac{m''(a_1+1) - k}{\binom{m''}{2}}.$$

In particular, if Γ has smallest eigenvalue $\theta_{\min}$, then this proposition can be applied when $-1 - \frac{b_1}{\theta_{\min}+1} < a_1$ (in which case the local graph is connected) and $a_1 \neq 2$.

In her work on distance-regular graphs without 4-claws, Bang [21] also ruled out the intersection array $\{55, 36, 11; 1, 4, 45\}$. Moreover, she related such graphs to geometric distance-regular graphs with smallest eigenvalue -3. This led to the following more general result by Bang and Koolen [32].

Proposition 9.7. *Let $m \geqslant 3$ be an integer, and let Γ be a distance-regular graph with diameter $D \geqslant 2$ and valency larger than $\max\{m^2 - m, \frac{m^2-1}{m}(a_1 + 1)\}$. Then Γ has no $(m+1)$-claws if and only if Γ is a geometric distance-regular graph with smallest eigenvalue $-m$.*

For $m = 3$, a slightly stronger result was obtained by Bang [21], in the sense that the corresponding result holds for valency larger than $\max\{3, \frac{8}{3}(a_1 + 1)\}$. We finally note that the distance-regular graphs without 3-claws have been determined by Blokhuis and Brouwer [54], and that some more work on distance-regular graphs without 4-claws has been done by Guo and Makhnev [281] and Bang, Gavrilyuk, and Koolen [26].

9.4 Sufficient conditions

Proposition 9.8. *Let Γ be a distance-regular graph with diameter D with the property that there exists a positive integer m and a set $\mathcal{C}$ of cliques in Γ such that every edge is contained in exactly one clique of $\mathcal{C}$ and every vertex x is contained in exactly m cliques of $\mathcal{C}$. If $|\mathcal{C}| < |V|$, then Γ is geometric with smallest eigenvalue $-m$. In particular, this is the case if $\min\{|C| : C \in \mathcal{C}\} > m$.*

Proof. Consider the $|V| \times |\mathcal{C}|$ incidence matrix N, where $N_{xC} = 1$ if $x \in C$ and 0 otherwise. Then $NN^\top = mI + A$, so the smallest eigenvalue $\theta_{\min}$ of Γ satisfies $\theta_{\min} \geqslant -m$. Suppose now that $|V| > |\mathcal{C}|$. Then $NN^\top$ is singular and hence $\theta_{\min} = -m$. By the Delsarte bound, every clique C has size at most $1 - \frac{k}{\theta_{\min}} = 1 + \frac{k}{m}$. On the other hand, by considering the cliques $C \in \mathcal{C}$ containing a fixed vertex, we see that they have $1 + \frac{k}{m}$ vertices on average. This means that all cliques in $\mathcal{C}$ contain exactly $1 + \frac{k}{m}$ vertices, and hence Γ is a geometric distance-regular graph. In particular, if $\min\{|C| : C \in \mathcal{C}\} > m$, then it follows by counting the number of incident pairs (x, C) in two different ways that $|\mathcal{C}| < |V|$, so Γ is geometric. $\qquad\square$

This proposition implies that distance-regular graphs of order (s, t) are geometric with smallest eigenvalue $-t - 1$ if $s > t$; see also Corollary 7.7. Using Proposition 9.1 we now obtain the following result.

Proposition 9.9. *Let $m \geqslant 2$ be an integer, and let Γ be a distance-regular graph with $(m-1)(a_1 + 1) < k < m(a_1 + m)$ and diameter $D \geqslant 2$. If $a_1 \geqslant \frac{1}{2}m(m+1)(c_2 + 1)$, then Γ is geometric with smallest eigenvalue $-m$.*

Proof. The two given lower bounds on a_1 assure that Proposition 9.1 can be applied, i.e., that $a_1 \geqslant (2m-1)(c_2 - 1)$ and $k < (m+1)(a_1 + 1) - \frac{1}{2}m(m+1)(c_2 - 1)$. Thus the set $\mathcal{C}$

of maximal cliques of size at least $a_1 + 2 - (m-1)(c_2 - 1)$ forms a set of lines such that each vertex is in at most m lines, and each edge is in exactly one line. The given lower bound on k assures that each vertex is in exactly m lines. Because the minimal line size is at least $a_1 + 2 - (m-1)(c_2 - 1)$, which is at least $m + 1$ by one of the given inequalities, it follows from Proposition 9.8 that Γ is geometric with smallest eigenvalue $-m$. $\square$

We note that the assumption $k < m(a_1 + m)$ holds for all distance-regular graphs with smallest eigenvalue $-m$ (see [410]).

9.5 Distance-regular graphs with a fixed smallest eigenvalue

Generalizing results by Neumaier [509] and Godsil [265], Koolen and Bang [410] showed the following.

Theorem 9.10. *For given $m \geqslant 2$, there are only finitely many non-geometric distance-regular graphs with both valency and diameter at least 3 and smallest eigenvalue at least $-m$.*

Note that valency 2 is excluded because of the odd polygons; and diameter 2 because of the complete multipartite graphs. Koolen and Bang [410] did not quite prove this result, as they restricted themselves to graphs with $c_2 \geqslant 2$. The graphs with $c_2 = 1$ are of order (s, t), with $s = a_1 + 1$ and $t = k/s - 1$. If such a graph has smallest eigenvalue at least $-m$, then by interlacing (see Section 4.4.1), the existence of a $(t+1)$-claw implies that $t + 1 \leqslant m^2$. Because a distance-regular graph of order (s, t) that is not geometric has $s \leqslant t$, it follows that $k = s(t+1) \leqslant (m^2 - 1)m^2$. Therefore the result for the case $c_2 = 1$ follows from the Bannai-Ito conjecture. Note that the (general) result was known for $m = 2$, see [78, Thm. 3.12.4, 4.2.16].

Because the smallest eigenvalue of a geometric graph is always integral, this theorem also gives a partial answer to the question from [78, p. 130] whether every distance-regular graph with valency at least three and diameter at least three has an integral eigenvalue besides the valency.

One may also wonder whether it is true that for a given integer $m \geqslant 2$, there are only finitely many geometric distance-regular graphs with $D \geqslant 3$, $c_2 \geqslant 2$, and smallest eigenvalue $-m$, besides the Grassmann graphs, Johnson graphs, bilinear forms graphs, and Hamming graphs. For $D = 2$ this is not true, but Neumaier [509] showed that in essence the geometric strongly regular graphs fall into two infinite classes.

Concerning large smallest eigenvalue, we know that the distance-regular graphs with smallest eigenvalue -1 are exactly the complete graphs. The ones with smallest eigenvalue -2 are either strongly regular (and classified by Seidel [556]) or line graphs (and classified by Mohar and Shawe-Taylor [497]) [78, Thm. 3.12.14]. Among these, the only geometric distance-regular graphs with $D \geqslant 3$ and $k \geqslant 3$ are the generalized $2D$-gons of order $(s, 1)$, $s \geqslant 2$ and $D = 3, 4, 6$, and the line graphs of the Petersen graph, the Hoffman-Singleton graph, and putative Moore graphs on 3250 vertices (see [78, Thm. 4.2.16]). Bang and Koolen [31] finished the classification of the geometric distance-regular graphs with diameter at least 3, $c_2 \geqslant 2$, and smallest eigenvalue -3, by showing that such a

graph is a Hamming graph, Johnson graph, or a generalized quadrangle of order $(s, 3)$ minus a spread (with $s = 3$ or 5). Yamazaki [653] obtained strong restrictions on distance-regular graphs of order $(s, 2)$, $s \geqslant 3$. The (five) distance-regular graphs of order $(2, 2)$ were classified by Hiraki, Nomura, and Suzuki [325]. All of these graphs are geometric with smallest eigenvalue -3.

9.6 Regular near polygons

Recall from Section 4.5 that a distance-regular graph Γ of order (s, t) with diameter D is called a regular near $2D$-gon if $a_i = c_i a_1$ for all $i = 1, 2, \ldots, D$, and that such a graph is geometric. We call Γ thick if $s \geqslant 2$.

Theorem 9.11. (cf. [78, Thm. 6.6.1, 9.4.4]) *Let Γ be a thick regular near $2D$-gon with $D \geqslant 4$. If $c_2 \geqslant 3$ or $c_i = i$ $(i = 2, 3)$, then Γ is either a dual polar graph or a Hamming graph.*

Proof. (sketch) For $c_2 \geqslant 3$, the proof is implicitly given in [78]. Brouwer and Wilbrink [94] showed that a thick regular near $2D$-gon with $D \geqslant 4$ and $c_2 \geqslant 3$ satisfies $c_3 = c_2^2 - c_2 + 1$ (the gap as mentioned in [78, p. 206] is repaired by De Bruyn [181]). From [78, p. 277, Rem. ii] (a remark on a a result by Brouwer and Cohen [77]), it follows that a thick regular near $2D$-gon with $c_3 = c_2^2 - c_2 + 1$ and $c_2 \geqslant 3$ is a dual polar graph.

For the case $c_i = i$ $(i = 2, 3)$, we will give a sketch of the proof, as it is not in the literature. Let Γ be a thick regular near $2D$-gon of order (s, t) with $D \geqslant 4$ and $c_i = i$ $(i = 2, 3)$. First, by a result of Brouwer and Wilbrink [94], one may assume that $c_i = i$ for $i \leqslant D - 1$. Second, it can be shown that if Γ is of order (s, t) with $c_i = i$ $(i = 2, 3)$ and $a_2 = c_2 a_1$, then there exists a map $\phi : H(t + 1, s - 1) \rightarrow \Gamma$, such that the partition $\{\phi^{-1}(x) : x \in V\}$ is completely regular (cf. [515, Thm. 3]). Using Theorem 4.1, one can show that $\phi^{-1}(x)$ is a completely regular code with minimum distance $2D$. Now its truncated code is a perfect $(D-1)$-error-correcting code and by the perfect code theorem (see for example [330]), the only such codes with $D \geqslant 4$ (that are relevant to us; $s \geqslant 2$) are the codes consisting of exactly one code word. This shows that $c_D = D$ and that Γ is the Hamming graph $H(D, s + 1)$. This finishes the proof of the theorem. $\qquad \square$

In some cases, the intersection numbers of a distance-regular graph imply that it must be a regular near $2D$-gon; if $c_2 = 1$, $a_1 \leqslant 1$, or the graph has classical parameters $(D, -a_1 - 1, \alpha, \beta)$, see [619]. This for example implies that there can be no distance-regular graphs with intersection array $\{147, 144, 135; 1, 4, 49\}$ (and classical parameters $(3, -3, -3, 21)$), because it would yield a regular near hexagon with $(s, c_2, c_3) = (3, 4, 49)$ and this was ruled out by Shult according to Brouwer [69].

Let Γ be a thick regular near polygon with diameter D and head h. Hiraki [306] showed that if $D \geqslant 2h + 1$, then $h \in \{1, 2, 3\}$ (he mentions also the possibility $h = 5$, but this would lead to a thick generalized 12-gon, a contradiction). This result also follows from Proposition 11.3 (ii) ($m = h - 1$), as we obtain a thick generalized $2(h + 1)$-gon as strongly closed subgraph, and by the Feit-Higman theorem (cf. [78, Thm. 6.5.1]), it follows that $h + 1 \in \{2, 3, 4\}$. Hiraki [310] conjectured that if $D > 2h + 1$, then $h = 1$.

For a thick regular near $2D$-gon with $D \leqslant 2h$, one can bound the valency in terms of a_1, see [322].

De Bruyn and Vanhove [184] (see also [641]) obtained that for a regular near $2D$-gon with $a_1 > 0$, the intersection numbers satisfy $c_2 \leqslant (a_1 + 1)^2 + 1$ and

$$\frac{((a_1 + 1)^i - 1)(c_{i-1} - (a_1 + 1)^{i-2})}{(a_1 + 1)^{i-2} - 1} \leqslant c_i \leqslant \frac{((a_1 + 1)^i + 1)(c_{i-1} + (a_1 + 1)^{i-2})}{(a_1 + 1)^{i-2} + 1}$$

for $i = 3, 4, \ldots, D$. Neumaier [511] obtained the upper bound for odd i and the lower bound for even i as a specialization of the balanced set condition of Section 5.4 for the smallest eigenvalue of a regular near $2D$-gon. For $D = i = 3$, the upper bound is the Haemers-Mathon bound [282, p. 60][23] for regular near hexagons. The upper bound for even i and the lower bound for odd i can be seen as a specialization of Tonejc's [636] modification of the balanced set condition.

The following is a slight extension of a result due to Brouwer, Godsil, Koolen, and Martin [81, Thm. 10]:

Proposition 9.12. *Let Γ be a thick regular near $2D$-gon with quads (i.e., geodetically closed subgraphs with diameter two). Then the second smallest eigenvalue θ_{D-1} of Γ satisfies*

$$\theta_{D-1} \geqslant a_1 + 1 - \frac{b_1}{(a_1 + 1)(c_2 - 1)},$$

with equality if and only if every quad has width and dual degree summing to D. Equality occurs only for the dual polar graphs and Hamming graphs.

The last sentence of the above proposition follows from the following. Let H be a sub-hexagon of Γ and Q be a subquadrangle in H. Brouwer and Wilbrink [94] showed that $c_3 \geqslant c_2(c_2 - 1) + 1$ with equality if and only if there is no vertex at distance 2 from Q in H; see also [181, p. 26]. Suppose there is a vertex x at distance 2 from Q in H. If $D \geqslant 4$, this means that Q cannot be a completely regular code in Γ, as this vertex has distance at most 3 to all vertices in Q, while there also exists a vertex y at distance 2 from Q with distance 4 to some vertex in Q. If $D = 3$, then the dual degree is at least the covering radius of Q in H, which is at least two, and therefore the sum of the width and dual degree is at least 4. Therefore $c_3 = c_2(c_2 - 1) + 1$, and hence Γ is a dual polar graph or a Hamming graph (for $D \geqslant 4$, this follows from Theorem 9.11, whereas for $D = 3$, it follows from [78, Thm. 9.4.4]).

For more results on regular near polygons, we refer to [321, 323, 324, 629].

10 Spectral characterizations

It is known that distance-regularity of a graph is in general not determined by the spectrum of the graph; see below and the overview by Van Dam, Haemers, Koolen, and Spence

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[23]This bound is also called the Mathon bound. It was obtained jointly by Haemers and Mathon. Yanushka recognized that the bound can be obtained from a Krein condition. Besides the remark in [282], this is all unpublished.

[175]. See also the survey by Fiol [222] on algebraic characterizations of distance-regular graphs, and the surveys by Van Dam and Haemers [172, 173] on spectral characterizations of graphs.

10.1 Distance-regularity from the spectrum

The following proposition surveys the cases for which it is known that distance-regularity follows from the spectrum.

Proposition 10.1. *If Γ is a distance-regular graph with diameter D, valency k, girth g, and distinct eigenvalues $k = \theta_0, \theta_1, \ldots, \theta_D$, satisfying one of the following properties, then every graph cospectral with Γ is also distance-regular, with the same intersection array as Γ:*

(i) *$g \geqslant 2D - 1$ [82],*

(ii) *$g \geqslant 2D - 2$ and Γ is bipartite [171],*

(iii) *$g \geqslant 2D - 2$ and $c_{D-1}c_D < -(c_{D-1} + 1)(\theta_1 + \cdots + \theta_D)$ [171],*

(iv) *Γ is a generalized odd graph, that is, $a_1 = \cdots = a_{D-1} = 0$, $a_D \neq 0$ [174, 342],*

(v) *$c_1 = \cdots = c_{D-1} = 1$ [171],*

(vi) *Γ is the dodecahedron, or the icosahedron [285],*

(vii) *Γ is the coset graph of the extended ternary Golay code [171],*

(viii) *Γ is the Ivanov-Ivanov-Faradjev graph [175],*

(ix) *Γ is the Hamming graph $H(3, e)$, with $e \geqslant 36$ [23].*

In fact, more general results hold, because it is actually not in all cases (explicitly) required that the graph is cospectral to a distance-regular graph. Instead, for the graph to be distance-regular, it suffices that a similar spectral condition holds, where the diameter D is replaced by the number of distinct eigenvalues minus one, and the intersection numbers by the so-called *preintersection numbers*; for details, we refer to Abiad, Van Dam, and Fiol [2].

Note that the polygons, strongly regular graphs, and bipartite distance-regular graphs with diameter three are special cases of (i) and (ii). We also refer to the survey paper by Van Dam and Haemers [172], where a list of distance-regular graphs that are known to be determined by the spectrum is included (except that the antipodal 7-cover of K_9 is not mentioned). Van Dam, Haemers, Koolen, and Spence [175] give a list of graphs cospectral with distance-regular graphs on at most 70 vertices (where Hadamard graphs on 64 vertices are missing). Note that Van Dam and Haemers [172] conjectured that almost all graphs are determined by the spectrum. It follows from the prolific constructions of distance-regular graphs by Fon-Der-Flaass [232] (see also Section 3.2.3) that almost all distance-regular graphs are *not* determined by the spectrum.

For (ix), we refer to Bang, Van Dam, and Koolen [23], who showed that the Hamming graph $H(3, e)$ with diameter three is uniquely determined by its spectrum for $e \geqslant 36$. Moreover, it is shown that for given $D \geqslant 2$, every graph cospectral with the Hamming graph $H(D, e)$ is locally the disjoint union of D copies of the complete graph of size $e - 1$, that is, it is geometric, for e large enough. The latter is obtained by bounding the number of common neighbours of two vertices in terms of the spectrum, and applying Proposition 9.1. The result on the Hamming graphs with diameter three then follows from a result by Bang and Koolen [30] who showed that if a graph cospectral with $H(3, e)$ has the same local structure as $H(3, e)$, i.e., if it is geometric, then it is either the Hamming graph $H(3, e)$ or the dual graph of $H(3, 3)$. Furthermore, it is known that for $D \geqslant e \geqslant 3$, ($D \geqslant 4$ and $e = 2$), or ($D \geqslant 2$ and $e = 4$), the Hamming graph $H(D, e)$ is not uniquely determined by its spectrum, whereas for ($2 \leqslant D \leqslant 3$ and $e = 2$) or ($e \geqslant D = 2$ and $e \neq 4$), the Hamming graph $H(D, e)$ is uniquely determined by its spectrum (cf. [78, 175, 285, 328]).

Van Dam, Haemers, Koolen, and Spence [175] showed that the Ivanov-Ivanov-Faradjev graph is determined by its spectrum, whereas the Johnson graphs, the Doubled Odd graphs, the Grassmann graphs, the Doubled Grassmann graphs, the antipodal covers of complete bipartite graphs, and many of the Taylor graphs are shown to have cospectral mates that are not distance-regular. These mates are usually obtained by Godsil-McKay switching or by constructing partial linear spaces that resemble the structure of the distance-regular graphs in question. Van Dam and Haemers [171] also used switching to construct cospectral mates that are not distance-regular for the Wells graph, the bipartite double of the Hoffman-Singleton graph, the triple cover of $GQ(2, 2)$, and the Foster graph.

10.2 The p-rank

The p-ranks of Γ, that is, the ranks over $GF(p)$ of matrices of the form $A + \alpha I + \beta J$ with α, β integral (and A the adjacency matrix), can sometimes be used to distinguish cospectral graphs. Peeters [539] studied these p-ranks of distance-regular graphs. He showed among other results that for odd e, the Hamming graphs $H(3, e)$ are determined by the spectrum and the 2-rank of $A + I$. On the other hand, he showed that the p-ranks of the Doob graphs and the Hamming graphs (with the same intersection array) are the same.

10.3 Spectral excess theorem

The spectral excess theorem by Fiol and Garriga [226] states that a connected regular graph with $d + 1$ distinct eigenvalues is distance-regular (with diameter d) if and only if for every vertex, the number of vertices at distance d from that vertex (the excess) equals a given expression in terms of the spectrum (the spectral excess). So a simple 'quasi-spectral' property suffices for a graph to be distance-regular. To specify the result, one should know that from the spectrum of a regular graph, a system of orthogonal polynomials $v_i, i = 0, 1, \ldots, d$ — the so-called *predistance polynomials* — can be con-

structed. For distance-regular graphs, this system is well-known, and satisfies $A_i = v_i(A)$, for $i = 0, 1, \ldots, d$, where A_i is the distance-i adjacency matrix; see (2).

Theorem 10.2. (Spectral excess theorem) *Let Γ be a connected k-regular graph on n vertices with $d + 1$ distinct eigenvalues and corresponding orthogonal polynomials $v_i, i = 0, 1, \ldots, d$, and let $k_d(x)$ be the number of vertices at distance d from x. Then Γ is distance-regular if and only if $k_d(x) = v_d(k)$ for all x.*

In fact, the theorem can be stated a bit stronger: instead of requiring that $k_d(x) = v_d(k)$ for all x, it is sufficient to require that the harmonic mean of $n - k_d(x)$ equals $n - v_d(k)$. Another remark is that the spectral excess $v_d(k)$ can be computed from the spectrum $\{k = \theta_0^1, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$ directly as

$$v_d(k) = \frac{n}{\pi_0^2} \left[\sum_{i=0}^{d} \frac{1}{m_i \pi_i^2} \right]^{-1},$$

where $\pi_i = \prod_{j \neq i} |\theta_i - \theta_j|$ for $i = 0, 1, \ldots, d$.

The first result of this kind was obtained by Cvetković [159] and by Laskar [439], who showed that for a Hamming or Doob graph with diameter three, distance-regularity is determined by the spectrum and having the correct number of vertices at distance two from each vertex. This result was generalized to all distance-regular graphs with diameter three by Haemers [284], and subsequently by Van Dam and Haemers [170], who proved the spectral excess theorem for graphs with four distinct eigenvalues (not assuming that the graph has the spectrum of a distance-regular graph).

At the same time, Fiol, Garriga, and Yebra [229] showed that a graph with $d + 1$ distinct eigenvalues is distance-regular if each vertex has at least one vertex at distance d and its distance-d adjacency matrix A_d is a polynomial of degree d in the adjacency matrix A, which is the first important step towards the spectral excess theorem, which was then proved by Fiol and Garriga in [226]. The improvement to considering the above mentioned harmonic mean was later proved in [222] (see also [165]). Fiol also obtained more specific results for antipodal distance-regular graphs [219] and for strongly distance-regular graphs [220] (a distance-regular graph with diameter D is *strongly distance-regular* if its distance-D graph is strongly regular; examples are the connected strongly regular graphs, antipodal distance-regular graphs, and distance-regular graphs with $D = 3$ and $\theta_2 = -1$). Elementary proofs of the spectral excess theorem are given by Van Dam [165] and Fiol, Gago, and Garriga [225]. The original proof by Fiol et al. [226, 229] has a local approach and, because of that, it is quite technical.[24] We remark however that by this local approach, Fiol et al. manage to prove more related results. Van Dam and Fiol [167] generalized the spectral excess theorem by dropping the regularity condition and using the Laplacian eigenvalues. We refer the interested reader also to surveys by Fiol [222, 223].

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[24]In the language of the Terwilliger algebra, (part of) this local approach can be interpreted as finding a condition on the thinness of the primary $\mathbb{T}$-module; see Footnote 16.

A useful application of the spectral excess theorem is, for example, given by the construction by Van Dam and Koolen [176] of a new family of distance-regular graphs with the same intersection array as certain Grassmann graphs, see Section 3.2.1. Distance-regularity of these graphs is proved by showing that they have the same spectrum as the Grassmann graphs, and then checking the number of vertices at extremal distance from each vertex. The spectral excess theorem was also used by Van Dam and Haemers [174] to show that each regular graph with $d+1$ distinct eigenvalues and shortest odd cycle of length $2d+1$ is a distance-regular generalized odd graph. Lee and Weng [441] generalized this by dropping the regularity condition, using a version of the spectral excess theorem for nonregular graphs. Van Dam and Fiol [166] obtained the same result by an alternative method that avoids the spectral excess theorem; these results generalize Proposition 10.1 (iv) above.

Kurihara [428] obtained a dual version of the spectral excess theorem, in the sense that it characterizes when a spherical 2-design generates a cometric association scheme. Kurihara and Nozaki [430] and Nomura and Terwilliger [525] independently derived a spectral characterization of P-polynomial schemes (and hence distance-regular graphs) among symmetric association schemes that is closely related to the spectral excess theorem.

10.4 Almost distance-regular graphs

Motivated by spectral and other algebraic characterizations of distance-regular graphs, Dalfó, Van Dam, Fiol, Garriga, and Gorissen [163] studied *almost* distance-regular graphs. They used the spectrum and the predistance polynomials of a graph to discuss concepts such as m-walk-regularity and partial distance-regularity. It was shown by Rowlinson [552] that a graph is distance-regular if and only if the number of walks of given length between vertices depends only on the distance between these vertices. Godsil and McKay [272] called a graph walk-regular if the number of closed walks of given length is constant. The concept of m-walk-regularity, as introduced by Dalfó, Fiol, and Garriga [164], generalizes both, and requires the invariance of the number of walks of each given length between vertices at each given distance at most m. Algebraically, this is equivalent to $A_i \circ E_j = \frac{1}{v} Q_{ij} A_i$ for all $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, d$ (and some Q_{ij}), where the notation is as usual (cf. Section 2.5). An interesting problem raised in [163] is to determine the smallest $m = m(D)$ such that each m-walk-regular graph with diameter D is distance-regular. Informally, the question is till what distance m one needs to check m-walk-regularity to assure distance-regularity. We expect that $m(D)$ is approximately $D/2$.

Dalfó, Van Dam, and Fiol [161] showed that m-walk-regular graphs can be characterized through the cospectrality of certain perturbations of such graphs. As a consequence, some new characterizations of distance-regularity in terms of certain perturbations are obtained. Cámara, Van Dam, Koolen, and Park [102] observed a structural gap between 1-walk-regularity and 2-walk-regularity. They showed among other results that Godsil's bound on the valency in terms of a multiplicity (in Theorem 14.3), Terwilliger's bounds on the local eigenvalues [78, Thm. 4.4.3], and the fundamental bound (19) generalize to 2-walk-regular graphs. Moreover, they show that there are finitely many non-geometric

2-walk-regular graphs with given smallest eigenvalue and given diameter (in the same spirit as Theorem 9.10).

Another concept is that of m-partial distance-regularity (distance-regularity up to distance m). This means that for $i \leqslant m$, the distance-i matrix can be expressed as a polynomial of degree i in the adjacency matrix, which is equivalent to saying that the intersection numbers c_i, a_i, b_i are well defined up to c_m. We note that there are $(D-1)$-partially distance-regular graphs with diameter D that are not distance-regular; for example the direct product of an edge and the folded cube. Lee and Weng [442] used 2-partial distance-regularity to characterize the distance-regular graphs among the bipartite graphs whose halved graphs are distance-regular (cf. Proposition 2.13).

Related to these concepts are two other generalizations of distance-regular graphs. Weichsel [646] called a graph distance-polynomial if each distance-i matrix can be expressed as a polynomial in the adjacency matrix. A graph is called distance degree regular if each distance-i graph is regular. Such graphs were studied by Bloom, Quintas, and Kennedy [56], Hilano and Nomura [302], and also by Weichsel [646] (as super-regular graphs). A concept that is dual to partial distance-regularity was introduced by Dalfó, Van Dam, Fiol, and Garriga [162].

11   Subgraphs

Let Γ be a distance-regular graph. In this section, a subgraph in Γ will always be an induced subgraph. Recall that a code in Γ is simply a non-empty subset of V_Γ. Therefore, subgraphs, codes, and (vertex) subsets will be virtually the same objects in this section, and we shall adopt one of these names depending on the context. Completely regular codes will be separately discussed in Section 12.

11.1   Strongly closed subgraphs

Suzuki [582, Thm. 1.1] showed that strongly closed subgraphs of distance-regular graphs are usually distance-regular.

Theorem 11.1. *Let Δ be a strongly closed subgraph of a distance-regular graph Γ. Let h be the head of Γ and k be the valency of Γ. Then one of the following holds:*

(i) *Δ is distance-regular,*

(ii) *$2 \leqslant D_\Delta \leqslant h$,*

(iii) *h and D_Δ are even, and Δ is a distance-biregular graph with $c_{2i-1} = c_{2i}$ for all $i = 1, 2, \dots, \frac{1}{2} D_\Delta$,*

(iv) *$h = 3$, $D_\Delta = 5$, and Δ is isomorphic to the graph obtained by replacing each edge in a complete graph $K_{\ell+1}$, $\ell \geqslant 3$, by a path of length 3,*

(v) *$h = 6$, $D_\Delta = 8$, and Δ is isomorphic to the graph obtained by replacing each edge in a Moore graph with valency $\ell \in \{3, 7, 57\}$ by a path of length 3.*

It follows that (i) holds above precisely when $b_{D_\Delta - 1} > b_{D_\Delta}$, and that (iv) or (v) hold above precisely when $a_1 = 0$ and $(c_{D_\Delta - 1}, a_{D_\Delta - 1}) = (c_{D_\Delta}, a_{D_\Delta}) = (1, 1)$. The Biggs-Smith graph is the only known example of a distance-regular graph with $h \geqslant 2$ which satisfies $(c_{h+1}, a_{h+1}) = (c_{h+2}, a_{h+2}) = (1, 1)$. We note that if $c_2 \geqslant 2$ then every strongly closed subgraph of Γ is distance-regular.

Hiraki [308] introduced the *condition* $(\mathrm{SC})_m$ as the condition[25] that for all vertices x and y at distance m there exists a strongly closed subgraph $\Delta(x, y)$ with diameter m containing x and y. Hiraki [311, Thm. 1] showed that $(\mathrm{SC})_m$ is equivalent to m-boundedness for $m = 1, 2, \ldots, D_\Gamma - 1$.

It is clear that if Γ is m-bounded then it is $(m + 1)$-parallelogram-free. The converse is not true in general because every bipartite distance-regular graph is parallelogram-free, but it does not even need to be 2-bounded, as the incidence graph of a 2-$(11, 6, 3)$-design shows. In some cases, however, $(m + 1)$-parallelogram-freeness is known to be equivalent to m-boundedness.

Proposition 11.2. *Let Γ be a distance-regular graph with diameter D and let $m \in \{1, 2, \ldots, D - 1\}$. Suppose one of the following holds:*

(i) $m = 1$,

(ii) $c_2 > 1$ *and* $a_1 > 0$,

(iii) $c_2 = 1$ *and* $a_2 > a_1 > 0$,

(iv) $m = 2$ *and* $a_2 > a_1 = 0$,

(v) $c_{m+1} = 1$ *and* $a_2 > a_1$.

Then Γ is $(m + 1)$-parallelogram-free if and only if Γ is m-bounded.

We remark that (i) is obvious, (ii) was shown by Weng [649, Thm. 6.4], (iii) was shown by Suzuki [583], (iv) was shown by Suzuki [583] for the case $c_2 = 1$ (extending [78, Lemma 4.3.13]) and by Weng [649, Prop. 6.7] for the case $c_2 > 1$, and (v) was shown by Hiraki [305]. Hiraki [316] also obtained other sufficient conditions for a distance-regular graph to be m-bounded.

The following proposition summarizes the known results on Γ being m-bounded for some m that have been obtained by using combinatorial methods. Some more results are known under the assumption that Γ is Q-polynomial; cf. Section 5.2.

Proposition 11.3. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and head $h \geqslant 1$. Let $m \in \{1, 2, \ldots, D - h\}$. Then Γ is m-bounded if one of the following holds:*

(i) $c_{m+h} = 1$ *and* $a_{m-1} < a_m$,

(ii) Γ *is* $K_{2,1,1}$-*free,* $a_1 > 0$, $a_i = c_i a_1$ *for* $i = 1, 2, \ldots, m + h - 1$, *and* $c_{m-1} < c_m$.

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[25]We note that $(\mathrm{SC})_m$ for some $m \in \{1, 2, \ldots, D_\Gamma - 1\}$ implies $K_{2,1,1}$-freeness, which in turn implies h-boundedness, where h is the head of Γ; cf. [311, p. 129, Remarks].

Result (i) was obtained by Ivanov and Brouwer (cf. [78, Prop. 4.3.11]) for $m = 2$, and by Hiraki [308, Thm. 1.3] for the other cases. Result (ii) was obtained by Hiraki [310, Thm. 1.1], generalizing a result of Brouwer and Wilbrink [94] for thick regular near polygons with $h = 1$. We remark that each of the assumptions (i) and (ii) implies $b_{m-1} > b_m$, so that if x and y are at distance m then $\Delta = \Delta(x, y)$ is distance-regular with valency $a_m + c_m$. In particular, if $c_{2h+1} = 1$ and $m = h + 1$, then Δ is a Moore geometry and it is known that such a graph is either an odd polygon or has diameter at most 2; cf. [78, Thm. 6.8.1]. This shows the following proposition in the case $a_1 > 0$. The case $a_1 = 0$ uses results by Chen, Hiraki, and Koolen [123, 303, 320].

Proposition 11.4. [311, Thm. 2] *Let Γ be a distance-regular graph with head $h \geqslant 1$ and diameter $D \geqslant 2h + 3$. Then $h = 1$ or $c_{2h+3} \geqslant 2$.*

We note that Wang [645] did related work. We remark also that if Γ is a distance-regular graph with $h = 1$ and $c_4 = 1$, then by Proposition 11.3(i) and a result from 'BCN' [78, Thm. 5.9.9(i)], Γ has a distance-regular subgraph with diameter 3 and $c_3 = 1$. No such (latter) graph is known, however. Chen, Hiraki, and Koolen [124] in fact showed that no such graph with $a_1 \neq 3$ and $a_1 \leqslant 30$ exists.

Let Γ be a distance-regular graph with diameter D. Suppose Γ is D-bounded and every strongly closed subgraph is regular. In particular, we have $b_i > b_{i+1}$ for $i = 0, 1, \ldots, D-1$. Let $\mathscr{S}$ be the poset consisting of all strongly closed subgraphs of Γ with partial order defined by reverse inclusion. Weng [648] showed that $\mathscr{S}$ is a ranked meet semilattice and every interval in $\mathscr{S}$ is atomic and lower semimodular. He also showed the inequalities

$$\frac{b_{D-i-1} - b_{D-i+1}}{b_{D-i-1} - b_{D-i}} \geqslant \frac{b_{D-i-2} - b_{D-i}}{b_{D-i-2} - b_{D-i-1}} \quad (i = 1, 2, \ldots, D-2),$$

with equality for all $i = 1, 2, \ldots, D-2$ if and only if every interval in $\mathscr{S}$ is a modular atomic lattice. See also Section 5.7.6.

For some more work on strongly closed subgraphs in distance-regular graphs, we refer to [318] and the references therein.

11.2 Bipartite closed subgraphs

Let Γ be a distance-regular graph with diameter D. We say the *condition* $(\mathrm{BGC})_j$ holds if for every pair of vertices at distance j there exists a bipartite closed subgraph with diameter j containing this pair. This condition was introduced by Hiraki [313], and he showed that $(\mathrm{BGC})_j$ with $j \in \{1, 2, \ldots, D-1\}$ implies $(\mathrm{BGC})_i$ for all $i = 1, 2, \ldots, j$. By combining results of Hiraki [313] and Koolen [404, 405], we have the following.

Proposition 11.5. [313, Cor. 4.8] *Let Γ be a distance-regular graph with diameter $D \geqslant 3$. Let $t \in \{2, 3, \ldots, D-1\}$ be such that $c_t = c_{t-1} + 1$ and $a_1 = a_2 = \cdots = a_{t-1} = 0$. Then the condition $(\mathrm{BGC})_t$ holds if and only if one of the following holds:*

(i) $(c_1, c_2, \ldots, c_t) = (1, 1, \ldots, 1, 2)$ *and every bipartite closed subgraph with diameter t is the ordinary $2t$-gon,*

(ii) $(c_1, c_2, \ldots, c_t) = (1, 2, \ldots, t)$ *and every bipartite closed subgraph with diameter t is the t-cube,*

(iii) $t = 2s + 1$ *is odd,* $(c_1, c_2, \ldots, c_t) = (1, 1, 2, 2, \ldots, s, s, s+1)$, *and every bipartite closed subgraph with diameter t is the Doubled Odd graph with valency $s + 1$,*

(iv) $t = 4$, $(c_1, c_2, c_3, c_4) = (1, 1, 2, 3)$, *and every bipartite closed subgraph with diameter 4 is the Pappus graph.*

Some more general results are obtained by Hiraki [313].

11.3   Maximal cliques

In most cases, it is easy to determine the maximal cliques of classical distance-regular graphs; cf. [297]. However, the structure of the maximal cliques of the quadratic forms graphs turns out to be extremely complicated. Hemmeter, Woldar, and Brouwer completed the classification of the maximal cliques in this case in a series of papers [298, 299, 300, 87, 301]. Brouwer and Hemmeter [86] classified the maximal cliques of half dual polar graphs and Ustimenko graphs (which are the distance 1-or-2 graphs of dual polar graphs $\mathcal{B}_m(q)$ and $\mathcal{C}_m(q)$, respectively). The maximal cliques of twisted Grassmann graphs were described by Van Dam and Koolen [176].

Hemmeter [297] observed that if Γ is a bipartite distance-regular graph with diameter $D \geqslant 4$, then $\Gamma_1(x)$ is a maximal clique of the halved graph for every $x \in V_\Gamma$. Using this fact, he was able to determine all bipartite distance-regular graphs whose halved graphs belong to one of the known (at the time) infinite families with unbounded diameter; cf. [296, 298]. Brouwer, Godsil, Koolen, and Martin [81, Cor. 2] showed that if a distance-regular graph Γ has a Delsarte clique then it cannot have an antipodal cover of odd diameter. Van Dam and Koolen [176] looked at the structure of the maximal cliques of the twisted Grassmann graphs to show that these graphs are not vertex-transitive.

11.4   Convex subgraphs

Lambeck [432] studied in detail the noncomplete convex subgraphs of classical distance-regular graphs. He classified such subgraphs in Johnson, Hamming, Grassmann, dual polar, bilinear forms, Hermitian forms, alternating forms graphs, and also quadratic forms graphs $Qua(n, q)$ with q odd. The noncomplete convex subgraphs of $Qua(n, q)$ with q even were classified by Munemasa, Pasechnik, and Shpectorov [505]. It turns out that if Γ is one of these graphs then its noncomplete convex subgraphs are distance-regular and belong to the same family as Γ, with the exception of $Her(D, 4)$, which has $K_{2,2}$ as a convex subgraph.

Tanaka [598] used the above results to describe the descendents (cf. Section 5.7.6) of these graphs.

11.5 Designs

For recent updates on the study of combinatorial block designs and orthogonal arrays (i.e., t-designs in the Johnson and Hamming graphs), we refer the reader to [135]. It should be remarked here that Keevash [390] has recently proved that, given t, k, and λ, the natural divisibility conditions for the existence of a block t-(v, k, λ) design are also sufficient, provided that v is large enough. This generalizes the result of Wilson [651] for the case $t = 2$ and that of Teirlinck [603] which establishes the existence of block t-designs for all t.

A number of simple t-designs over finite fields (i.e., t-designs in the Grassmann graphs) with t at most 3 have been constructed by many researchers; see, e.g., [68] and the references therein. Recently, Fazeli, Lovett, and Vardy [214] showed that non-trivial simple t-designs over finite fields $\mathbb{F}_q$ exist for all t and q.

Delsarte T-designs in a distance-regular graph with $|T| = D - 1$ (where D is the diameter of the graph) have dual degree 1. Such designs are necessarily completely regular with covering radius 1, and will be briefly discussed in Section 12.4.

11.6 The Terwilliger algebra with respect to a code

Let Γ be a distance-regular graph with diameter $D \geqslant 3$, adjacency matrix A, and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$, and let C be a non-empty subset of V_Γ (i.e., a code) with covering radius ρ. Let $\{C_0 = C, C_1, \ldots, C_\rho\}$ be the distance partition with respect to C, and let χ_i be the characteristic vector of C_i ($i = 0, 1, \ldots, \rho$). For each $i = 0, 1, \ldots, \rho$, let $E_i^\star = E_i^\star(C)$ be the diagonal matrix in $M_{v \times v}(\mathbb{C})$ with diagonal entries $(E_i^\star)_{yy} = (\chi_i)_y$. The *Terwilliger algebra* $\mathbb{T} = \mathbb{T}(C)$ *with respect to* C is the subalgebra of $M_{v \times v}(\mathbb{C})$ generated by $A, E_0^\star, E_1^\star, \ldots, E_\rho^\star$. The algebra $\mathbb{T}(C)$ was first introduced and studied by Martin and Taylor [471] for binary Hamming graphs. We shall use the same terminology as in the case of the ordinary Terwilliger algebra (i.e., with respect to a vertex); cf. Section 4.3. However, as observed by Martin and Taylor [471] and Suzuki [588], the primary $\mathbb{T}$-module is thin (and is therefore equal to $\mathrm{span}_{\mathbb{C}}\{\chi_0, \chi_1, \ldots, \chi_\rho\}$) precisely when C is a completely regular code.

Suzuki [588] studied irreducible $\mathbb{T}$-modules in detail. The results in [588] generalize (to some extent) both the theory of tight graphs (cf. Go and Terwilliger [261]) and the theory of the width of a code (cf. Brouwer et al. [81]). Suzuki [588] showed that a $\mathbb{T}(C)$-module with endpoint ν is also a $\mathbb{T}(C_\nu)$-module with endpoint 0, where irreducibility and thinness are also preserved. This allows us to focus on the irreducible modules with endpoint 0.

Recall that the width of C is defined by $w = \max\{i : \chi_0^\mathsf{T} A_i \chi_0 \neq 0\}$, where A_i is the distance-i matrix of Γ ($i = 0, 1, \ldots, D$); cf. Section 5.7.6. For $i = 0, 1, \ldots, D$, let E_i be the primitive idempotent associated with θ_i. Let $\mathbf{v}$ be a nonzero vector in $E_0^\star \mathbb{C}^v$. Then it is easy to see that there is a polynomial f of degree at most w such that $||E_i\mathbf{v}||^2 = f(\theta_i)m(\theta_i)$ ($i = 0, 1, \ldots, D$), where $m(\theta)$ denotes the multiplicity of an eigenvalue θ of Γ. This immediately gives the inequality $w \geqslant D - r(\mathbf{v})$, where $r(\mathbf{v}) = |\{i : E_i\mathbf{v} \neq 0\}| - 1$. (Note that $r(\chi_0)$ is the dual degree of C.) The vector $\mathbf{v}$ is said to be *tight* (with respect to C) if $w = D - r(\mathbf{v})$. Suzuki [588] showed among other results that if $\mathbf{v}$ is tight then $\mathbb{T}\mathbf{v}$

is a thin irreducible $\mathbb{T}$-module with endpoint 0. This result was previously obtained by Brouwer, Godsil, Koolen, and Martin [81] for $\mathbf{v} = \chi_0$, and generalizes a theorem of Go and Terwilliger [261, Thm. 9.8]. An important consequence is that if Γ is Q-polynomial then every irreducible module of the ordinary Terwilliger algebra $\mathbb{T}(x)$ with *displacement* 0 is thin.[26] This fact was used, e.g., to extend the Assmus-Mattson theorem; cf. [595, §5].

Hosoya and Suzuki [337] called Γ *tight with respect to C* if the orthogonal complement of $\mathrm{span}_{\mathbb{C}}\{\chi_0\}$ in $E_0^\star \mathbb{C}^v$ is spanned by tight vectors. By [261, Thm. 13.6], Γ is tight in the sense of Section 6.1.1 if and only if Γ is non-bipartite and tight with respect to $\Gamma_1(x)$ for some (or all) $x \in V_\Gamma$. Hosoya and Suzuki also introduced a homogeneity *with respect to C* in terms of the partition of V_Γ by the distances from both C and a fixed vertex in C, and studied the relation between these two concepts. They moreover showed that if Γ is Q-polynomial then the dual eigenmatrix of the association scheme induced on a descendent (cf. Section 5.7.6) of Γ satisfies a certain system of linear equations, which in particular implies that Γ is tight with respect to every descendent. This system of linear equations turned out to be fundamental to the study of descendents; cf. [597, 598]. Lee [445] studied the above partition for Delsarte cliques (which are descendents of width 1) in Q-polynomial distance-regular graphs that have the most general *q-Racah type*,[27] and showed among other results that there is a natural action of the double affine Hecke algebra of type $(C_1^\vee, C_1)$ on the subspace of $\mathbb{C}^v$ spanned by the characteristic vectors of the cells of the partition. See [602] for detailed information on the Terwilliger algebra with respect to a descendent.

12 Completely regular codes

In this section, we discuss completely regular codes in distance-regular graphs. Many combinatorial configurations can be viewed as completely regular codes with certain additional properties and/or special parameters in their underlying distance-regular graphs; cf. Section 12.4.

As we have seen, a Delsarte clique in a distance-regular graph Γ with diameter D is a completely regular code in Γ with covering radius $D-1$, and all the geometric distance-regular graphs have plenty of Delsarte cliques. Martin [462, Thm. 2.3.3] showed that if two distinct vertices x, y in a distance-regular graph Γ with diameter D form a completely regular code then either $d(x, y) = 1$ and $a_1 = a_2 = \cdots = a_{D-1} = 0$, i.e., Γ is bipartite or almost bipartite, or $d(x, y) = D$ and Γ is antipodal. Cámara, Dalfó, Delorme, Fiol, and Suzuki [101] showed that all the edges of a connected graph are completely regular codes with the same parameters if and only if the graph is a bipartite or almost bipartite distance-regular graph. (The assumption on the parameters of the edges was later dropped by Suzuki [592].) See [227, 228, 103, 104, 105] and also Section 11.6 for some algebraic

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[26]The *displacement* ([626]) of an irreducible $\mathbb{T}(x)$-module W is $\eta = e + e^\star + \delta - D$, where $e, e^\star, \delta$ are the endpoint, dual endpoint, and the diameter of W, respectively. It follows from Caughman's results [111, Lemmas 5.1, 7.1] that $0 \leqslant \eta \leqslant D$.

[27]In the notation of Bannai and Ito [38, §III.5] (cf. [616]), this Q-polynomial structure satisfies type I with $s \neq 0$ and $s^* \neq 0$. The polygons are the only known examples of this type.

characterizations of complete regularity.

For certain distance-regular graphs, we can show that they come from completely regular partitions in Hamming graphs.

Theorem 12.1. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k, intersection numbers $c_i = i$ for $i = 1, 2, 3$, and $a_2 = 2a_1$. Then $a_1 + 1$ divides k and there exists a completely regular partition of the Hamming graph $H(k/(a_1+1), a_1+2)$ or a Doob graph with valency k, with covering radius D and parameters $\gamma_i = c_i$, $\beta_{i-1} = b_{i-1}$ for $i = 1, 2, \ldots, D$. The latter case only occurs when $a_1 = 2$.*

This theorem was shown by Rifà and Huguet [547] when $a_1 = 0$ following ideas of Brouwer [70] (cf. [78, Prop. 4.3.6, Thm. 11.3.2]), by Nomura [515] when $a_1 \neq 2$, and by Koolen [408] when $a_1 = 2$.

12.1 Parameters

It is known that the sequence $(c_i)_i$ in a distance-regular graph Γ is non-decreasing, but this is not true in general for the sequence $(\gamma_i)_i$ of a completely regular code in Γ. Koolen [409] gave an infinite family of completely regular codes in the Doubled Odd graphs with the property that the sequence $(\gamma_i)_i$ is not necessarily increasing, disproving a conjecture of Martin [462]. Koolen also gave a sufficient condition for Γ that the sequence $(\gamma_i)_i$ is increasing for every completely regular code in Γ. Martin [private communication] showed that the sequence $(\gamma_i)_i$ is strictly increasing for any completely regular code in a Hamming graph.

12.2 Leonard completely regular codes

Let Γ be a distance-regular graph with diameter D, valency k, and eigenvalues $k = \theta_0, \theta_1, \ldots, \theta_D$ (not necessarily in decreasing order). Let E_i be the primitive idempotent associated with θ_i for $i = 0, 1, \ldots, D$. Let C be a completely regular code in Γ with covering radius ρ. Let $\{C_0 = C, C_1, \ldots, C_\rho\}$ be the distance partition with respect to C, and let $\mathbf{x}_i$ be the characteristic vector of C_i for $i = 0, 1, \ldots, \rho$. Let $\mathrm{Spec}(C) = \{\theta_{i_0} = k, \theta_{i_1}, \ldots, \theta_{i_\rho}\}$ be (an ordering of) the spectrum of the quotient matrix of the corresponding distance partition. We say C is *Leonard* (with respect the above ordering) if

$$(E_{i_1}\mathbf{x}_0)^{\circ \ell} \in \mathrm{span}_{\mathbb{C}}\{E_{i_0}\mathbf{x}_0, \ldots, E_{i_\ell}\mathbf{x}_0\} \setminus \mathrm{span}_{\mathbb{C}}\{E_{i_0}\mathbf{x}_0, \ldots, E_{i_{\ell-1}}\mathbf{x}_0\}$$

for $\ell = 1, 2, \ldots, \rho$. This definition is due to Koolen, Lee, and Martin [412]. Let A be the adjacency matrix of Γ, and let $A^\star = A^\star(C)$ be the diagonal matrix in $M_{v \times v}(\mathbb{C})$ with diagonal entries $(A^\star)_{yy} = \frac{v}{|C|}(E_{i_1}\mathbf{x}_0)_y$. They showed among other results that C is Leonard if and only if the matrices A and $A^\star$ act on $\mathrm{span}_{\mathbb{C}}\{\mathbf{x}_0, \ldots, \mathbf{x}_\rho\} = \mathrm{span}_{\mathbb{C}}\{E_{i_0}\mathbf{x}_0, \ldots, E_{i_\rho}\mathbf{x}_0\}$ as a Leonard pair [623]. If Γ is a translation distance-regular graph and C is additive, then it follows that C is Leonard if and only if its coset graph is a Q-polynomial distance-regular graph.

Next we consider a weaker condition than being Leonard:

$$(E_{i_1}\mathbf{x}_0)^{\circ \ell} \in \mathrm{span}_{\mathbb{C}}\{E_{i_0}\mathbf{x}_0, \ldots, E_{i_\ell}\mathbf{x}_0\} \quad (\ell = 1, 2, \ldots, \rho).$$

As a class of completely regular codes satisfying this condition, Koolen, Lee, and Martin [412] also introduced *harmonic* completely regular codes as follows. Suppose Γ is Q-polynomial with respect to the ordering $\theta_0, \theta_1, \ldots, \theta_D$. We say C is *harmonic* if there is a positive integer t such that $i_\ell = t\ell$ for $\ell = 0, 1, \ldots, \rho$. Descendents (cf. Section 5.7.6) are examples of harmonic completely regular codes with $t = 1$. Tanaka [598, Prop. 4.6] showed that a descendent in Γ with width w and dual width $w^* = D - w(= \rho) > 1$ is *not* Leonard (with respect to this ordering) *precisely when w is odd* and the Q-polynomial structure satisfies type III in the notation of Bannai and Ito [38, §III.5] (cf. [616]).

12.3 Completely regular codes in the Hamming graphs

Neumaier [512] conjectured that the only completely regular codes (with at least two words) in the Hamming graphs with minimum distance at least 8 are the extended binary Golay code and the (binary) repetition codes of length at least 8. But he forgot to mention the even subcode of the binary Golay code (i.e., the subcode of the Golay code consisting of the codewords with even weight), as remarked by Borges, Rifà, and Zinoviev [63], which was implicitly known to be completely regular. (The bipartite double of the coset graph of the binary Golay code is distance-regular and has intersection array $\{23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23\}$, and it follows from Theorem 12.1 that there is a completely regular partition of the 23-cube corresponding to this graph. It is easy to check that this partition corresponds to the cosets of the even subcode of the Golay code, because all distances are even and it has exactly half the number of codewords of the Golay code; see also [78, p. 362]). So we would like to rephrase Neumaier's conjecture as follows.

Conjecture 12.2. *The only completely regular codes (with at least two words) in the Hamming graphs with minimum distance at least 8 are the extended binary Golay code, the even subcode of the binary Golay code, and the repetition codes.*

Gillespie [253] showed that the only completely regular codes in the binary Hamming graphs $H(D, 2)$ with minimum distance greater than $\max\{2, D/2\}$ are the repetition codes and the dual code of the binary $[7, 4, 3]$-Hamming code. Meyerowitz [485] described all the completely regular codes with strength 0 in the Hamming graphs.

Brouwer [72] showed that any truncation of an even and almost even binary completely regular code is again completely regular. Brouwer [73] also gave a necessary and sufficient condition on when the extension of a binary completely regular code is again completely regular.

We say that a binary code C of length D is *self-complementary* if $\mathbf{1} + c \in C$ for all $c \in C$, and *non-self-complementary* otherwise,[28] where $\mathbf{1} = (1, 1, \ldots, 1)$ denotes the all-

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[28]Self-complementary codes and non-self-complementary codes are sometimes called antipodal codes and non-antipodal codes, respectively, in the literature. However, it seems that these are somewhat confusing names.

ones vector in $GF(2)^D$. Borges, Rifà, and Zinoviev [63] showed among other results that if C is a binary non-self-complementary completely regular code with covering radius ρ, minimum distance at least 3, and distance partition $\{C_0 = C, C_1, \ldots, C_\rho\}$, then $C_\rho = C + \mathbf{1}$, from which it follows that $C \cup C_\rho$ is again a completely regular code. This is a special case of the following construction: if C is a completely regular code with covering radius ρ and if $\gamma_i = \beta_{\rho-i}$ for $i = 1, 2, \ldots, \rho$ with $2i \neq \rho$, then $C \cup C_\rho$ is also completely regular.

12.3.1   Completely transitive codes and generalizations

Giudici and Praeger [259] defined the notion of a completely transitive code in a graph. A code C in a graph Γ is called *completely transitive* if there is a group H of automorphisms of Γ such that every cell of the distance-partition of C is an orbit of H. It is clear that a completely transitive code is completely regular. Next suppose that Γ is a Cayley graph $\mathrm{Cay}(G, S)$. Let C be a subgroup of G with covering radius ρ (as a code in Γ). We say C is *coset-completely transitive* if the subgroup $\mathfrak{T}$ of the automorphism group of G consisting of the elements that stabilize both C and S has exactly $\rho + 1$ orbits on G/C. This is an extension of the notion of coset-completely transitive codes in Hamming graphs $H(D, q)$ defined by Giudici and Praeger [259],[29] which in turn generalizes a concept of Solé [567]. It is easy to see that if C is coset-completely transitive and is in the center of G, then C is completely transitive with $H = C \rtimes \mathfrak{T}$.

There are many examples of coset-completely transitive additive codes in Hamming graphs, such as the codes in the Golay family: the binary Golay code, the extended binary Golay code, the even subcode of the binary Golay code, the punctured code of the binary Golay code, the even subcode of the punctured Golay code, the twice punctured binary Golay code, the ternary Golay code, and the extended ternary Golay code. Rifà and Zinoviev [550] showed that the lifts of the perfect Hamming codes are coset-completely transitive and that the coset graphs of these codes are the bilinear forms graphs. Rifà and Zinoviev [548] also constructed an infinite family of binary linear completely transitive codes whose coset graphs are the halved cubes. Borges, Rifà, and Zinoviev constructed many more linear completely regular and completely transitive codes; see, e.g., [65, 549]. Gillespie and Praeger [256, 257] also considered generalizations of completely transitive codes.

There are only a few completely transitive binary codes known which are not coset-completely transitive, among them are the Hadamard code of length 12 and its punctured code. Gillespie and Praeger [255] showed that these codes are characterized as binary completely regular codes by their lengths and minimum distances. See also [258]. Borges, Rifà, and Zinoviev [61, 62] showed that the only binary coset-completely transitive codes with minimum distance at least 9 are the binary repetition codes. Gillespie, Giudici, and Praeger [254] showed that the only completely transitive codes in the Hamming graphs with minimum distance at least 5 such that the corresponding groups H of automorphisms

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[29]To be more precise, they defined the concept for linear codes and considered the stabilizer of C in the group of weight-preserving sesquilinear automorphisms of $GF(q)^D$.

are faithful on coordinates are the binary repetition codes.

12.3.2 Arithmetic completely regular codes

Harmonic completely regular codes in Hamming graphs were called *arithmetic* completely regular codes and studied in detail by Koolen, Lee, Martin, and Tanaka [413]. Let C be a completely regular code in $H(n, q)$ with covering radius 1. Then the cartesian product $C \times C \times \cdots \times C$ (t times) is an arithmetic completely regular code with covering radius t in $H(nt, q)$. Next, let C be a completely regular code in $H(n, q)$ with covering radius $\rho \geqslant 1$ and parameters β_i, γ_i ($i = 0, 1, \ldots, \rho$). Similarly, let C' be a completely regular code in $H(n', q')$ with covering radius $\rho' \geqslant 1$ and corresponding parameters β_i', γ_i' ($i = 0, 1, \ldots, \rho'$). Koolen, Lee, Martin, and Tanaka [413, Prop. 3.4] showed that $C \times C'$ is completely regular in $H(n, q) \times H(n', q')$ if and only if there are integers β, γ such that $\beta_{\rho-i} = \beta i$, $\gamma_i = \gamma i$ for $i = 0, 1, \ldots, \rho$, and $\beta_{\rho'-i}' = \beta i$, $\gamma_i' = \gamma i$ for $i = 0, 1, \ldots, \rho'$. We note that completely regular codes having parameters of this form are arithmetic. From this result it follows that if we take C' to be a perfect binary 1-error correcting code with length $2^t - 1$ which is not isomorphic to the binary Hamming code C of the same length, then the cartesian product of C' and s copies of C is completely regular with covering radius $s + 1$, but this code is certainly not completely transitive, answering a problem of Gillespie [252, Problem 11.6].

Koolen, Lee, Martin, and Tanaka [413, Thm. 3.16] also classified all arithmetic completely regular linear codes. This is a generalization of a result of Bier [46]. Borges, Rifà, and Zinoviev [64] classified all completely regular linear codes with covering radius 1 (which are clearly arithmetic) using a different approach.

Fon-Der-Flaass [233] showed that, for fixed positive integers β_0 and γ_1, there is a completely regular code in $H(D, 2)$ with covering radius 1 and parameters β_0 and γ_1 for *some* D if and only if $\frac{\beta_0 + \gamma_1}{(\beta_0, \gamma_1)}$ is a power of 2. (He attributed this result to S. Avgustinovich and A. Frid.) Note that if C is such a code in $H(D, 2)$ then so is $C \times GF(2)$ in $H(D+1, 2)$. Fon-Der-Flaass [233] also obtained lower and upper bounds on the smallest diameter $D = D_0(\beta_0, \gamma_1)$ for which such a code exists. A code in $H(D, 2)$ is called *degenerated* if it is isomorphic to $C \times GF(2)$ for some code C in $H(D-1, 2)$, and *non-degenerated* otherwise. Simon [562] showed that for any non-degenerated bipartition of $GF(2)^D (= V_{H(D,2)})$ there is a vertex adjacent to at least $\Omega(\log_2 D)$ vertices in the other cell. This gives an upper bound on the maximum diameter $D = D_1(\beta_0, \gamma_1)(\geqslant D_0(\beta_0, \gamma_1))$ for which there is a non-degenerated binary completely regular code with covering radius 1 and parameters β_0 and γ_1. We remark that the method of Simon works in general for binary completely regular codes, not only for covering radius 1.

12.4 Completely regular codes in other distance-regular graphs

The completely regular codes with strength 0 in the Johnson graphs as well as Hamming graphs were described by Meyerowitz [484, 485]. Note that descendents (cf. Section 5.7.6) in Q-polynomial distance-regular graphs are examples of completely regular codes with strength 0. Brouwer, Godsil, Koolen, and Martin [81] used Meyerowitz's results to determine all the descendents in the Johnson and Hamming graphs. Tanaka [594, 598] extended

the classification of descendents to all of the 15 known infinite families of distance-regular graphs having classical parameters with unbounded diameter.

Martin [463] determined the completely regular codes with strength 1 and minimum distance at least 2 in the Johnson graphs. Martin [464] also studied general completely regular t-designs in the Johnson graphs in detail. Sporadic examples include the 5-$(24, 8, 1)$, 4-$(23, 7, 1)$, and 3-$(12, 6, 2)$ designs. Completely transitive codes (cf. Section 12.3.1) in the Johnson graphs were studied by Godsil and Praeger [274]. Liebler and Praeger [449] also considered generalizations of completely transitive codes in the Johnson graphs. Completely regular codes in the Odd graphs were studied by Martin [468]. Koolen [409] classified the completely regular codes in the Biggs-Smith graph.

The completely regular codes of a distance-regular graph with covering radius 1 are exactly the same as (non-trivial) *intriguing sets* studied by De Bruyn and Suzuki [183]. Note that a code in a distance-regular graph is an intriguing set if and only if it has dual degree 1. Tight sets and m-ovoids in finite polar spaces are examples of intriguing sets. Hemisystems (cf. Section 5.2) are 1-designs in the dual polar graphs $^2\mathcal{A}_3(\sqrt{q})$ with q odd, and are therefore intriguing sets. Gavrilyuk and Mogilnykh [248] showed among other results the non-existence and uniqueness of certain Cameron-Liebler line classes in $PG(3, q)$, which are intriguing sets with dual width 1 in $J_q(4, 2)$. See also [483, 247, 218, 180].

Recall that the incidence graph of a symmetric design is a Q-polynomial bipartite distance-regular graph with diameter 3. Martin [465] observed that several geometric substructures in finite projective spaces are Delsarte T-designs with $T \in \{\{1, 3\}, \{2, 3\}\}$ in the corresponding bipartite distance-regular graphs, so that they provide more examples of intriguing sets.

Vanhove [640] showed among other results that partial spreads with maximum size $\sqrt{q^3} + 1$ in $^2\mathcal{A}_5(\sqrt{q})$ as well as spreads in $\mathcal{B}_D(q)$ and $\mathcal{C}_D(q)$ with $D \in \{3, 5\}$ are completely regular. See also [639] for more results.

Perfect codes in a distance-regular graph Γ are completely regular, but non-trivial ones are very rare.[30] It is well known that the only non-trivial perfect codes in the Hamming graphs $H(D, q)$ with minimum distance $\delta \geqslant 7$ (or $\delta = 5$ and q a prime power) are the binary Golay code and the ternary Golay code; cf. [78, §11.1D]. See, e.g., [210] and the references therein for recent progress towards proving a longstanding conjecture of Delsarte [189, p. 55] that there are no non-trivial perfect codes in the Johnson graphs. Chihara [125] showed that there are no non-trivial perfect codes in the Grassmann graphs, dual polar graphs, and the forms graphs, except possibly $\mathcal{B}_D(q)$ and $\mathcal{C}_D(q)$ with $D = 2^m - 1$ for some positive integer m. Her proof depends only on a detailed analysis of the orthogonal polynomials $(v_i)_{i=0}^D$ associated with these graphs (see (2) and the remark after (13)), so that we also obtain, e.g., the non-existence for the twisted Grassmann graphs. Martin and Zhu [474] gave a simple proof of the non-existence for the Grassmann and bilinear forms graphs using Delsarte's 'Anticode Bound' (cf. [78, Prop. 2.5.3]). We note that the maximum anticodes in this case are precisely the descendents, in view

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[30]Here, 'non-trivial' means that the minimum distance is at least 3, and also at most $D_\Gamma - 2$ if Γ is an antipodal 2-cover with D_Γ odd.

of the Erdős-Ko-Rado theorem for these graphs; cf. [594]. Koolen and Munemasa [417] constructed perfect codes with minimum distance 3 in the two Doob graphs with diameter 5. Krotov [426] recently showed among other results the existence of perfect codes with minimum distance 3 in infinitely many Doob graphs.

13   More combinatorial properties

13.1   Distance-regular graphs with a relatively small number of vertices

The Taylor graphs and Hadamard graphs form infinite families of graphs that have a relative small number of vertices compared to the valency k. Indeed, Taylor graphs have $2k + 2$ vertices and Hadamard graphs have $4k$ vertices. The following result, obtained by Koolen and Park [421], shows that these two families are exceptions.

Theorem 13.1. *Let $\alpha > 2$. Then there are finitely many distance-regular graphs with v vertices, valency k, diameter $D \geqslant 3$ satisfying $v \leqslant \alpha k$, besides imprimitive distance-regular graphs with diameter 3 and antipodal bipartite distance-regular graphs with diameter 4.*

As a consequence, they also obtained the following.

Theorem 13.2. *Let $0 < \epsilon < 1$. Then there are finitely many distance-regular graphs with valency $k \geqslant 3$, diameter $D \geqslant 3$ satisfying $c_2 \geqslant \epsilon k$, besides imprimitive distance-regular graphs with diameter 3 and antipodal bipartite distance-regular graphs with diameter 4.*

For $k \leqslant 1/\epsilon$, this result follows from the Bannai-Ito conjecture (see Section 8.1). If $k > 1/\epsilon$, then $c_2 \geqslant 2$ and one can use the Ivanov bound (Theorem 7.1) to bound the diameter, and hence one can bound the number of vertices by a constant times k.

In the case that Γ contains a quadrangle, Koolen and Park [420] obtained the following bound on c_2 in terms of the valency and the diameter.

Proposition 13.3. *Let Γ be a distance-regular graph with valency $k \geqslant 3$ and diameter $D \geqslant 4$. If Γ contains an induced quadrangle, then $c_2 \leqslant \frac{2}{D}k$ with equality if and only if $D \geqslant 5$ and Γ is a D-cube or $D = 4$ and Γ is a Hadamard graph.*

The assumption of having induced quadrangles is necessary as the Foster graph and the Biggs-Smith graph have $k = 3$, $c_2 = 1$ and $D \geqslant 7$. We wonder whether the assumption can be removed for k large enough.

Note also that diameter three is exceptional, because the complete bipartite graph $K_{k+1,k+1}$ minus a perfect matching has valency k and $c_2 = k - 1$. Koolen and Park [419] showed that if a distance-regular graph has diameter three, then $c_2 \leqslant k/2$ or it is bipartite or a Taylor graph. They also showed that if a distance-regular graph with diameter at least three and valency k has $a_1 \geqslant \frac{k-2}{2}$, then it is a Taylor graph, a line graph, the Johnson graph $J(7,3)$, or the halved 7-cube.

For $4 \leqslant D \leqslant 6$, Koolen and Park [420] strengthened Proposition 13.3 as follows:

Proposition 13.4. *Let Γ be a distance-regular graph with valency $k \geqslant 3$ and diameter D. Then the following hold:*

(i) *If $D \geqslant 6$ and $c_2 \geqslant 2$, then $c_2 \leqslant k/3$,*

(ii) *If $D = 4$ and $c_2 > k/3$ then Γ is a Hadamard graph (and hence $c_2 = k/2$),*

(iii) *If $D = 5$ and $c_2 > k/3$, then Γ is the 5-cube,*

(iv) *If $D = 6$ and $c_2 > 2k/7$, then Γ is the 6-cube or the generalized dodecagon of order $(1, 2)$.*

13.2 Distance-regular graphs with multiple *P*-polynomial orderings

In the following, we assume that both the diameter and valency of Γ are at least three, and we follow 'BCN' [78, §4.2.D]. If a distance-regular graph Γ has a second P-polynomial ordering then the corresponding distance-regular graph Δ with the second ordering is either the distance-2 graph Γ_2, the distance-$(D-1)$ graph Γ_{D-1}, or the distance-D graph Γ_D.

The first case ($\Delta = \Gamma_2$) is only possible if Δ is a Taylor graph (with $c_2 < k - 1$) or a generalized odd graph.

The second case ($\Delta = \Gamma_{D-1}$) occurs if and only if Γ is an antipodal 2-cover with diameter D such that $a_i = 0$ for all $i < (D-1)/2$. In this case, the folded graph is either bipartite or a generalized odd graph. If the folded graph is bipartite, then Γ is bipartite and the diameter D is even. For diameter 4 only the Hadamard graphs occur; for diameter 6 only the 6-cube. For larger diameter only the D-cubes are known. If the folded graph is a generalized odd graph with diameter 3 then Γ is a Taylor graph. For larger diameter and bipartite Γ only the Doubled Odd graphs and the cubes are known. If Γ is not bipartite, then only three graphs are known: the Wells graph ($D = 4$), the dodecahedron ($D = 5$), and the coset graph of the truncated even subcode of the binary Golay code (see [78, p. 365]).

For the last case ($\Delta = \Gamma_D$) either Δ is a generalized odd graph or $a_D = 0$. In the latter case ($a_D = 0$), it holds that $p_{2D}^D \neq 0$ and if $D = 4$ then $p_{34}^4 = 0$; moreover, Suzuki [584] showed that $D \leqslant 4$.

13.3 Characterizing antipodality and the height

Let Γ be a distance-regular graph with valency at least 3. It is well-known that Γ is an antipodal 2-cover if and only if $b_{D-i} = c_i$ for all $i = 1, 2, \ldots, D$ [238]. Araya and Hiraki [7] improved this by showing that Γ is an antipodal 2-cover if and only if $b_{D-i} = c_i$ for $i = 1, 2, \ldots, \lceil D/2 \rceil$. This also improved earlier work of Araya, Hiraki, and Jurišić [8], who showed that a distance-regular graph is an antipodal 2-cover if there is a j with $b_j = 1$ and $D \geqslant 2j$. Araya, Hiraki, and Jurišić [9] also showed that if $b_2 = 1$, then Γ is an antipodal cover, in particular it is either an antipodal cover of a complete graph ($D = 3$), an antipodal 2-cover of a strongly regular graph with $\lambda = 0$ and $\mu = 2$ ($D = 4$), or the dodecahedron ($D = 5$). This solved one of the problems in 'BCN' [78, Prob. (i), p. 182].

Suzuki [579] showed that if $k_i = k_j$ for some $i < j$ with $i + j \leqslant D$, then either $k_D = 1$ or $k_i = k_{i+1} = \cdots = k_j$, thus solving another problem in 'BCN' [78, Prob. (ii), p. 168]. The only known distance-regular graphs with $k_i = k_j$ for some $i < j$ with $i + j \leqslant D$ and $k_D \geqslant 2$ are the odd polygons, but it is unknown whether any others could exist. Hiraki, Suzuki, and Wajima [326] showed that if $k_2 = k_j$ for $2 + j \leqslant D$ and $2 < j$, then Γ is indeed a polygon ($k = 2$) or an antipodal 2-cover ($k_D = 1$). In order to show this result, the height of a distance-regular graph was used; a notion that we will introduce next.

The *height* ht(Γ) of a distance-regular graph Γ with diameter D is defined as the maximal i such that the intersection number p_{Di}^D is nonzero. Note that Γ is an antipodal 2-cover if and only if ht(Γ) = 0. The case ht(Γ) = 1 occurs exactly when the distance-D graph Γ_D is a generalized odd graph; see [78, Prop. 4.2.10]. Nakano [508] strengthened some of the results in [326] by showing that if $k_i = k_j$ for some i and j such that $i < j \leqslant D - i$ and ht(Γ) is even and at most $2(D - 2i)$, then this height must be zero, that is, $k_D = 1$.

Suzuki [580] asked whether Γ can be characterized by its induced subgraphs on $\Gamma_D(x)$, for $x \in V$. Some results in this direction were obtained by Hiraki [309], who showed that if every induced subgraph on $\Gamma_{D,\mathrm{ht}(\Gamma)}(x, y)$ is a clique whenever $d(x, y) = D$, then ht(Γ) = D, $D - 1$, or 1. He also showed that if $p_{D,\mathrm{ht}(\Gamma)}^D = 1$, then ht($\Gamma$) equals D, 1, or 0. Also Tomiyama [632, 633] gave some results in this direction, that is, for the case ht(Γ) = 2.

13.4 Bounds on k_D for primitive distance-regular graphs

Let Γ be distance-regular with diameter D and valency k. Brouwer et al. [78, Prop. 5.6.1] showed that if Γ is not antipodal, then $k \leqslant k_D(k_D - 1)$. Suzuki [578, 580] showed that in this case also the diameter is bounded by a function of k_D. This now also follows from the above statement and the validity of the Bannai-Ito conjecture (see Section 8).

Park [530] showed that if Γ has valency and diameter at least 3 and satisfies $k_{D-1} + k_D \leqslant 2k$, then Γ is an antipodal 2-cover, Γ is bipartite with $D = 3$, Γ is the Johnson graph $J(7, 3)$, or Γ is the halved 7-cube. In the case $D = 3$, there are infinitely many bipartite non-antipodal distance-regular graphs with $k_2 + k_3 \leqslant 2k$, for example the incidence graphs of the complements of projective planes of order at least 3. This result also confirms a conjecture by Bendito, Carmona, Encinas, and Mitjana [44] that states that no primitive distance-regular graph with diameter three has the so-called M-property.

13.5 Terwilliger graphs and existence of quadrangles

Recall that a distance-regular graph without induced quadrangles is called a *Terwilliger graph*. In this section we collect some sufficient conditions for a distance-regular graph with $c_2 \geqslant 2$ to contain induced quadrangles. Note that by Proposition 7.11, the existence of a quadrangle implies that $c_i - b_i \geqslant c_{i-1} - b_{i-1} + a_1 + 2$. On the other hand, it is shown in the proof of [78, Thm. 5.4.1] that a distance-regular graph with $c_3 < 2c_2$ and $c_2 \geqslant 2$ has an induced quadrangle. The following are some more such combinatorial conditions.

Proposition 13.5. *Let Γ be an amply regular Terwilliger graph with diameter $D \geqslant 2$ and with parameters (v, k, λ, μ) such that $\mu \geqslant 2$.*

(i) *If $k \leqslant (6 + \frac{8}{57})(\lambda + 1)$, then Γ is the icosahedron, the Doro graph (see [78, §12.1]), or the Conway-Smith graph (see [78, §13.2]) [419, Prop. 6],*

(ii) *If $k < 50(\mu - 1)$, then Γ is the icosahedron, the Doro graph, or the Conway-Smith graph [78, Cor. 1.16.6(ii)],*

(iii) *If $24\mu > 10(\lambda + 1)$, then Γ is the icosahedron, the Doro graph, or the Conway-Smith graph [395].*

In [78, Thm. 4.4.11], the distance-regular graphs with second largest eigenvalue $b_1 - 1$ are classified. For Terwilliger graphs we can go a little further.

Proposition 13.6. [395] *Let Γ be a distance-regular Terwilliger graph with diameter $D \geqslant 3$ and distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Then $\theta_1 \leqslant b_1/2 - 1$ and $\theta_D \geqslant -b_1/3 - 1$, unless Γ is the icosahedron, the Doro graph, or the Conway-Smith graph.*

Note that if we would know the Terwilliger distance-regular graphs that are locally Hoffman-Singleton, then we could improve the above results. Note that there are two feasible intersection arrays known which could be locally Hoffman-Singleton: $\{50, 42, 9; 1, 2, 42\}$ and $\{50, 42, 1; 1, 2, 50\}$, see [78, p. 36]. Gavrilyuk and Makhnev [244] have worked on the classification of these graphs.

13.6 Connectivity and the second eigenvalue

13.6.1 Connectivity and matchings

Brouwer and Koolen [90] showed that a (non-complete) distance-regular graph Γ with valency $k > 2$ is k-connected and that the only way to disconnect Γ by removing k vertices is to remove the neighborhood of some vertex. This implies that also the edge-connectivity of Γ equals its valency, and consequently, that every distance-regular graph on an even number of vertices has a perfect matching. This had been derived before by Brouwer and Haemers [84], who also showed that the only way to disconnect Γ by removing k edges is to remove the edges through some vertex.

It was noted by Beezer and Farrell [41] that in general, the number of perfect matchings does not follow from the intersection array. They showed that the numbers of matchings consisting of i edges are determined by the intersection array for $i = 1, 2, \ldots, 5$; however the Hamming graph $H(2, 4)$ and the Shrikhande graph (which have the same intersection array) have different numbers of matchings with i edges for every $i > 5$.

13.6.2 The second largest eigenvalue

Koolen, Park, and Yu [422] showed that for given $\alpha > 1$, there are only finitely many distance-regular graphs with $k \geqslant 3$ and $D \geqslant 3$ whose second largest eigenvalue θ_1 satisfies $\alpha \geqslant \theta_1 > 1$. Note that the (infinite family of) regular complete bipartite graphs minus a

perfect matching are the only distance-regular graphs with $D \geqslant 3$ and $\theta_1 = 1$, and there are no distance-regular graphs with $D \geqslant 3$ and $\theta_1 < 1$. The distance-regular graphs with $D \geqslant 3$ and $\theta_1 \leqslant 2$ were also classified.

The distance-regular graphs with $D \geqslant 3$ and $a_1 \geqslant 2$ such that all local graphs have second largest eigenvalue at most one have been classified by Koolen and Yu [425]. One may wonder whether, given $\alpha \geqslant 1$, there are only finitely many distance-regular graphs with $D \geqslant 3$ and $a_1 > \alpha$ such that each local graph has second largest eigenvalue at most α. The condition $a_1 > \alpha$ ensures that the local graphs are connected, and thus excludes infinite families such as the Hamming graphs $H(D, \alpha + 2)$.

13.6.3 The standard sequence

Let Γ be a distance-regular graph with diameter D and let

$$
L(i) = \begin{bmatrix}
0 & b_0 & & & & & \\
c_1 & a_1 & b_1 & & & & \\
 & c_2 & a_2 & b_2 & & & \\
 & & \cdot & \cdot & \cdot & & \\
 & & & \cdot & \cdot & \cdot & \\
 & & & & c_{i-1} & a_{i-1} & b_{i-1} \\
 & & & & & c_i & a_i
\end{bmatrix},
$$

$$
M(i) = \begin{bmatrix}
a_i & b_i & & & & \\
c_{i+1} & a_{i+1} & b_{i+1} & & & \\
 & \cdot & \cdot & \cdot & & \\
 & & \cdot & \cdot & \cdot & \\
 & & & c_{D-1} & a_{D-1} & b_{D-1} \\
 & & & & c_D & a_D
\end{bmatrix}.
$$

Cioabă and Koolen [130] studied the eigenvalues of these matrices in order to answer a question by Brouwer. Note that the eigenvalues of $L = L(D) = M(0)$ are the $D + 1$ distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$ of Γ. Let ρ_i be the largest eigenvalue of $L(i)$, let σ_i be the largest eigenvalue of $M(i)$, and let $u_0 = 1, u_1, \ldots, u_D$ be the standard sequence of the second largest eigenvalue θ_1 of Γ. By the theory of orthogonal polynomials, it follows that this sequence has one sign change. Also, for $i = 2, 3, \ldots, D - 1$ and $\varepsilon \in \{+1, -1\}$, if $\varepsilon u_i > 0$, then $\varepsilon \rho_{i-1} < \varepsilon \theta_1 < \varepsilon \sigma_{i+1}$. If $u_i = 0$, then $\theta_1 = \rho_{i-1} = \sigma_{i+1}$.

For $D = 3$, this means that θ_1 lies between a_3 and $\frac{a_1 + \sqrt{a_1^2 + 4k}}{2}$, and if two of these three numbers are equal, then they are all equal. The latter case defines the class of Shilla graphs introduced by Koolen and Park [418].

Cioabă and Koolen [130] used the above to derive that the induced subgraph $\Xi(j)$ on $\Gamma_j(x) \cup \Gamma_{j+1}(x) \cup \ldots \cup \Gamma_D(x)$ is connected if $j \leqslant D/2$, and that $\Xi(D/2 + 1)$ is not connected if and only if Γ is an antipodal r-cover with $r \geqslant 3$. This answers a question by Brouwer. It is not clear when $\Xi((D + 1)/2)$ is disconnected.

A final remark on the standard sequence of the second eigenvalue is that Park, Koolen, and Markowsky [531] showed that $u_j > 0$ if $j < D/2$ and $u_j \geqslant 0$ if $j = D/2$. Moreover, they showed that $u_{D/2} = 0$ if and only if Γ is an antipodal cover.

We remark that if $\theta_1 < \alpha k$, then $c_t + a_t < \alpha k$ and hence $b_t = k - (a_t + c_t) > (1 - \alpha)k$ for $2t + 2 \leqslant D$. This implies that $k_t > k\frac{(1-\alpha)^{t-1}}{\alpha^{t-1}}$ for $2t + 2 \leqslant D$. So if $\theta_1 < k/2$, then $k_t > k_{t-1}$, which gives a partial answer to a problem in 'BCN' [78, p. 189]. If $\theta_1 < k/3$, then we can improve Pyber's bound of Section 7.4.

13.7   The distance-D graph

The spectral excess theorem (cf. Theorem 10.2) states that a connected regular graph with $d + 1$ distinct eigenvalues is distance-regular precisely when the distance-d graph is regular with the 'right' valency determined by the spectrum of the graph. As mentioned in Section 10.3, Fiol [220] specialized this theorem to strongly distance-regular graphs. Fiol [221, Conj. 3.6] also conjectured that a distance-regular graph with diameter at least 4 is strongly distance-regular if and only if it is antipodal.[31]

Fiol [221] showed that a distance-regular graph with diameter D and distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$ is strongly distance-regular if and only if, for $i = 1, 2, \ldots, D$, the multiplicity m_i of θ_i is expressed as a certain rational function in $\theta_0, \theta_1, \ldots, \theta_D$, and the number of vertices, v. Brouwer and Fiol [80] among other results strengthened this result for the case $D = 4$ as follows:

Proposition 13.7. *Let Γ be a distance-regular graph with diameter* 4. *Then the following are equivalent:*

(i) *Γ is strongly distance-regular, i.e., the distance-4 graph Γ_4 is strongly regular,*

(ii) *$b_3 = a_4 + 1$ and $b_1 = b_3 c_3$,*

(iii) *$(\theta_1 + 1)(\theta_3 + 1) = (\theta_2 + 1)(\theta_4 + 1) = -b_1$.*

See also [224]. In [80], Brouwer and Fiol in fact studied the more general situation where the distance-D graph has at most D distinct eigenvalues; or equivalently, where the distance-D matrix A_D generates a proper subalgebra of the adjacency algebra $\mathbb{A}$. They showed for example that a distance-regular graph with diameter D belongs to this class provided that D is odd and the distance 1-or-2 graph is distance-regular (e.g., the Odd graph O_{D+1}, the folded $(2D+1)$-cube, and the dual polar graphs $\mathcal{B}_D(q)$ and $\mathcal{C}_D(q)$).

14   Multiplicities

14.1   Terwilliger's tree bound

Terwilliger [604] showed that if a distance-regular graph Γ, say with valency k, contains an isometric subgraph that is also a tree, then the multiplicity of each eigenvalue $\theta \neq \pm k$

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[31]We note that a distance-regular graph with diameter 3 is strongly distance-regular precisely when it has -1 as an eigenvalue; cf. [78, Prop. 4.2.17]. Some non-antipodal examples are the Odd graph O_4 and the Sylvester graph.

of Γ is at least the number of leaves (i.e., vertices of valency one) of that tree. This has been generalized by Hiraki and Koolen [320, Prop. 3.1] to the case where the subgraph is a block graph (i.e., a graph whose 2-connected components are complete). Their result is too technical to state here, however, we mention its following consequence.

Proposition 14.1. (cf. [320, Prop. 3.3]) *Let Γ be a distance-regular graph of order (s,t) and with head h. Let $\theta \neq k$ be an eigenvalue with multiplicity m, and let $n = \lfloor \frac{h+1}{2} \rfloor$. Then the following hold:*

(i) *If h is odd and $\theta \neq -t-1$, then $m \geqslant (t+1)t^{n-1}s^n$,*

(ii) *If h is even and $\theta \neq -t-1$, then $m \geqslant (s+1)(st)^n$,*

(iii) *If h is odd and $\theta = -t-1$, then $m \geqslant 1 + (t+1)(s-1)\frac{(st)^n-1}{st-1}$,*

(iv) *If h is even and $\theta = -t-1$, then $m \geqslant \frac{(s-1)(s+1)((st)^{n+1}-1)}{(st-1)s} + \frac{1}{s}$.*

This generalizes a result of Zhu [658, Prop. 3.5] who obtained that $m \geqslant (t+1)(s-1)$ for $n = 1$. It also generalizes a result of Bannai and Ito [39] who showed that if $a_1 \neq 0$, then $m \geqslant (k/2)^n$.

Cámara, Van Dam, Koolen, and Park [102] showed that in a 1-walk-regular graph with valency k and an eigenvalue $\theta \neq k$ with multiplicity m, a clique can have size at most $m + 1$. This result is well-known for distance-regular graphs. Powers [542] already observed earlier that for distance-regular graphs equality in this clique bound cannot occur if θ is the second eigenvalue, except for the complete graph. We can generalize this as follows.

Proposition 14.2. *Let Γ be a distance-regular graph with valency k. If Γ contains a clique with c vertices, and $\theta \neq k$ is an eigenvalue of Γ with multiplicity m, then $c \leqslant m+1$, with equality only if $\theta = \theta_{\min}$ and Γ is complete, complete multipartite, or bipartite.*

Proof. Consider a clique C with c vertices, the idempotent $E = UU^\top$ and standard sequence $(u_i)_{i=0}^D$ corresponding to θ. Recall from the proof of Biggs' formula (Theorem 2.8) that $E = \sum_{i=0}^D \nu_i A_i$, where $\nu_i = \nu_0 u_i$ for $i = 0, 1, \ldots, D$. The submatrix of E indexed by the vertices of C equals $\nu_0(I + u_1(J - I))$, which has rank at least $c - 1$ (recall that $u_1 = \theta/k \neq 1$). Because the rank of E equals m, the bound $c \leqslant m+1$ follows. If equality holds then $1 + u_1(c-1) = 0$, and hence $c = 1 - k/\theta$, which implies that C is a Delsarte clique and $\theta = \theta_{\min}$.

Suppose now that Γ is not a complete graph. We aim to show first that $u_2 = 1$. Consider a representation associated to θ (see Section 2.5); for simplicity we normalize it so that the vectors $\hat{x}$ have length one for all $x \in V$, and the inner products between these vectors are given by the standard sequence. Because the rank of the above submatrix of E is m, it follows that the vectors $\hat{z}$, with $z \in C$ span the row space of U. In particular, if we consider a vertex x at distance one from C, then $\hat{x} = \sum_{z \in C} \alpha_z \hat{z}$ for certain α_z. By taking inner products with $\hat{z}$, it follows that α_z depends only on whether x is adjacent to

z or not. Hence, because $\sum_{z \in C} \hat{z} = 0$, see (12), we may assume that $\alpha_z = 0$ for $z \not\sim x$. Now let y be a vertex in C that is not adjacent to x. We then obtain that

$$1 - u_2 = \langle \hat{x}, \hat{x} \rangle - \langle \hat{x}, \hat{y} \rangle = \sum_{z \in C} \alpha_z (\langle \hat{z}, \hat{x} \rangle - \langle \hat{z}, \hat{y} \rangle) = 0,$$

and hence indeed $u_2 = 1$.

From (4), it now follows that $a_1 = k + \theta$, and hence the polynomial $v_2(z) = \frac{1}{c_2}(z^2 - a_1 z - k)$ from (2) satisfies $v_2(\theta) = v_2(k)$, which implies that Γ_2 is disconnected. If the diameter equals two, then G is a complete multipartite graph, so we may now assume that $D > 2$.

Because Γ_2 is disconnected, it follows that $a_2 = 0$, for otherwise $p_{22}^1 > 0$, which would imply that from every path between two given vertices in Γ one can construct a path in Γ_2 between these two vertices. Suppose now that $a_1 > 0$. Let $x_0 \sim x_1 \sim x_2 \sim x_3$ be a shortest path between two vertices x_0 and x_3 at distance three, and let y be a common neighbor of x_1 and x_2. Because a_2 is zero, it follows that y is also adjacent to x_0 and x_3, and so the latter are not at distance three, which is a contradiction. Thus $a_1 = 0$, and because $a_1 = k + \theta$, it follows that $\theta = -k$, and hence Γ is bipartite. $\qquad\square$

We remark that the above proof, and hence the result, is also valid for 2-walk-regular graphs, just like part of Godsil's bound in the next section (cf. Section 10.4).

14.2   Godsil's bound

Godsil [262] obtained the following lower bound on the multiplicity of an eigenvalue.

Theorem 14.3. (Godsil's bound) *Let Γ be a distance-regular graph with valency k and diameter D, and suppose Γ is not a complete multipartite graph. If Γ has an eigenvalue with multiplicity $m \geqslant 3$, then $D \leqslant 3m - 4$ and $k \leqslant \frac{(m+2)(m-1)}{2}$.*

Godsil's diameter bound was improved by Hiraki and Koolen [320].

Proposition 14.4. [320, Thm. 1.1, 1.2] *Let Γ be a distance-regular graph with diameter D. If Γ has an eigenvalue with multiplicity $m \geqslant 3$, then $D \leqslant m + 6$, unless $h = 1$ and $c_2 = 1$, in which case $D < m + 2 + \log_5 m$.*

Note that the Doubled Odd graph with valency k has diameter $2k - 1$ and an eigenvalue $k - 1$ with multiplicity $2k - 2$, so this result is close to the best possible. Yet another lower bound is obtained by Jurišić, Terwilliger, and Žitnik [387] (cf. Section 6.3.2):

Proposition 14.5. *Let Γ be a distance-regular graph with valency k. If $\theta \neq \pm k$ is an eigenvalue of Γ with multiplicity m, then*

$$m \geqslant k - \frac{a_1 k (\theta + 1)^2}{(k + \theta)^2 + a_1 (\theta^2 - k)}.$$

Koolen, Kim, and Park [411] refined the above valency bound of Godsil. By using the theory developed by Jurišić et al. [387], they were able to show that for $k \geqslant 3$ and $m \geqslant 3$, the only possible distance-regular graphs with diameter at least 3 and $k = \frac{(m+2)(m-1)}{2}$ are Taylor graphs with intersection array $\{(2\alpha + 1)^2(2\alpha^2 + 2\alpha - 1), 2\alpha^3(2\alpha + 3), 1; 1, 2\alpha^3(2\alpha + 3), (2\alpha + 1)^2(2\alpha^2 + 2\alpha - 1)\}$, with $m = 4\alpha^2 + 4\alpha - 1$, where α is an integer not equal to 0 and -1 or $\alpha = \frac{-1 \pm \sqrt{5}}{2}$ ($m = 3$).

Terwilliger (cf. [78, Thm. 4.4.4]) showed that if a distance-regular graph with valency k has an eigenvalue $\theta \neq k$ with multiplicity $m < k$, then θ is either the second largest or the smallest eigenvalue. Moreover, in this case $-1 - \frac{b_1}{\theta+1}$ is an algebraic integer as it is an eigenvalue of a local graph. Also, if $m \leqslant (k-1)/2$, then θ is an integer such that $\theta + 1$ divides b_1. Terwilliger's result was slightly improved by Godsil and Hensel [269] for antipodal distance-regular graphs, and by Godsil and Koolen [270] for the case that $a_D = 0$. As a consequence, Godsil and Koolen showed that a distance-regular graph with intersection array $\{\mu(2\mu + 1), (\mu - 1)(2\mu + 1), \mu^2, \mu; 1, \mu, \mu(\mu - 1), \mu(2\mu + 1)\}$ with $\mu \geqslant 2$ does not exist.

14.3 The distance-regular graphs with a small multiplicity

Let Γ be a distance-regular graph with valency k. The eigenvalues k, and $-k$ in case Γ is bipartite, are the only eigenvalues with multiplicity one. Each eigenvalue of a polygon, besides $\pm k$, has multiplicity two; and the polygons are the only distance-regular graphs with an eigenvalue having multiplicity two. The five Platonic solids, i.e., the icosahedron, dodecahedron, cube, octahedron, and tetrahedron are the only distance-regular graphs with an eigenvalue having multiplicity three. Zhu [658] (see also [657]) determined the distance-regular graphs with an eigenvalue having multiplicity four, whereas Martin and Zhu [473] (see also [407, Ch. 7]) determined those with an eigenvalue having multiplicity five, six, or seven. Koolen and Martin [416] (see also [407, Ch. 7]) determined the distance-regular graphs with an eigenvalue having multiplicity eight.

14.4 Integrality of multiplicities

Biggs' formula (Theorem 2.8) for the multiplicities of the eigenvalues and the requirement that these multiplicities are positive integers is a crucial part of Biggs' definition [48, Def. 21.5] of feasible intersection arrays of distance-regular graphs.

Godsil and McKay [272] generalized Biggs' formula to walk-regular graphs, thus obtaining feasibility conditions for such graphs. Recall that a graph is called walk-regular if the number of closed walks of given length from a vertex to itself is independent of the chosen vertex but depends only on the length, ℓ say; in other words, every power A^ℓ of the adjacency matrix has constant diagonal.

Chvátal [128, Thm. 3] showed that for strongly regular graphs, Biggs' feasibility condition of integer multiplicities implies the condition that the number of closed walks of length p is divisible by p for every prime p. The latter condition is essentially a condition on the spectrum because the number of closed walks of length p equals tr $A^p = \sum_{i=0}^{d} m_i \theta_i^p$.

Here we generalize Chvátal's result as follows.

Proposition 14.6. *Let $\{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$ be the multiset of roots of a monic polynomial with coefficients in $\mathbb{Z}$. If $\sum_{i=0}^{d} m_i \theta_i = 0$, then every prime p divides $\sum_{i=0}^{d} m_i \theta_i^p$.*

Proof. By grouping algebraic conjugates, say Θ_i is the set of algebraic conjugates of θ_i, and observing that $\sum_{\theta \in \Theta_i} \theta^p \equiv (\sum_{\theta \in \Theta_i} \theta)^p \equiv (\sum_{\theta \in \Theta_i} \theta) \mod p$ (the latter equality is by Fermat), it follows that $\sum_{i=0}^{d} m_i \theta_i^p \equiv \sum_{i=0}^{d} m_i \theta_i \equiv 0 \mod p$. $\qquad\square$

For a distance-regular graph, both the eigenvalues with multiplicities and the numbers of walks of length p follow from its intersection array. When one wants to generate putative intersection arrays for distance-regular graphs, testing the integrality of multiplicities is an important but computationally expensive part. The proposition indicates that testing integrality of the multiplicities is stronger than testing that the number of closed walks of length p is divisible by p. Note that given the intersection array it is relatively easy to compute the number of closed walks of given length recursively by using the distance polynomials v_i in (2) and their sum, the Hoffman polynomial (hence it is not necessary to first compute the eigenvalues and multiplicities); cf. [128, Thm. 2]. Brouwer, Cohen, and Neumaier [78, p. 134] give the intersection array $\{26, 25, 5, 1; 1, 5, 25, 26\}$ for a distance-regular graph for which some of the eigenvalues have irrational multiplicities. We found that the number of closed walks of length 7 is not divisible by 7 in this example. An example of an intersection array that survives the tests on the number of walks is $\{18, 15; 1, 2\}$; this corresponds to a strongly regular graph with parameters $(v, k, \lambda, \mu) = (154, 18, 2, 2)$. Indeed, the number of closed walks of length ℓ equals $18(18^{\ell-1} - 4^{\ell-1})$ for odd ℓ, and $18^2 + 153 \cdot 4^2$ for $\ell = 2$. However, the eigenvalues 4 and -4 have multiplicities 74.25 and 78.75, respectively.

15 Applications

15.1 Combinatorial optimization

One of the formulations of Lovász's *ϑ-function bound* [450] on the independence number and the Shannon capacity of a graph is as a semidefinite program (SDP). McEliece, Rodemich, and Rumsey [475] and Schrijver [554] observed that if the adjacency matrix of the graph belongs to the Bose-Mesner algebra $\mathbb{A}$ of an association scheme then we may solve the SDP as an ordinary linear program (LP) and the resulting bound 'essentially' coincides with Delsarte's *linear programming bound* [189] on which his theory on codes and designs is based. In fact, the same idea works for any SDP whenever the matrices defining the problem belong to $\mathbb{A}$; Goemans and Rendl [278] applied this to the *max-cut problem*, and Vallentin [638] to finding the least *distortion embeddings* of distance-regular graphs. Delsarte's theory has been most successful when the association scheme is metric and/or cometric (but see also [470, §8]). We especially recommend the survey by Delsarte and Levenshtein [193] on this topic. Besides codes and designs, Delsarte's LP was also used to prove the *Erdős-Ko-Rado theorem* for several families of *Q*-polynomial distance-regular graphs; cf. Section 5.7.4.

In 2005, Schrijver [555] applied a variant of an extension of the ϑ-function bound based on *matrix cuts* [451] to get a new upper bound on the sizes of binary codes. In this case, the matrices defining the SDP belong to the Terwilliger algebra $\mathbb{T}$ of the hypercube $H(D, 2)$. This method has been applied to codes in Johnson graphs by Schrijver [555], to codes in (nonbinary) Hamming graphs by Gijswijt, Schrijver, and Tanaka [250, 251], and also to the *kissing number* problem in the real sphere $S^{n-1} \subset \mathbb{R}^n$ by Bachoc and Vallentin [12]; see [11] for more results on this topic.

Generalizing De Klerk and Sotirov's idea of exploiting group symmetry [399], De Klerk, De Oliveira Filho, and Pasechnik [396] recently proposed an SDP relaxation[32] of the *quadratic assignment problem* (without linear term) for which one of the two defining matrices belongs to the Bose-Mesner algebra $\mathbb{A}$. For example, the polygons and the complete multipartite graphs correspond to the *traveling salesman problem* (cf. [397]) and the *maximum k-partition problem* (cf. [398]), respectively. De Klerk and Sotirov [400] showed that the relaxation can be strengthened further, provided the association scheme is vertex transitive. Their computational results involve instances related to the hypercube $H(D, 2)$, and we again encounter the Terwilliger algebra $\mathbb{T}$; see also [568]. Van Dam and Sotirov [178, 179] exploited symmetry to obtain bounds for the bandwidth of among others Hamming and Johnson graphs, and for the graph partition problem for several other distance-regular graphs.

We finally mention Lee [443, 444], who used the symmetry of the Johnson graph (and other association schemes) to derive results on the dimension of certain polytopes that are relevant to cutting plane algorithms for combinatorial optimization problems such as the bisection problem, the traveling salesman problem, and the perfect matching problem.

15.2 Random walks, diffusion models, and quantum walks

Given a graph Γ, a *random walk* on Γ is the walk of a particle that travels at random upon the vertices of a graph. At each stage the particle moves to a vertex that is adjacent to its current location, and the probabilities that it moves to each of its neighbors are equal. The particle has no memory, so it is as likely to return to a vertex it has just been to as it is to move to a new vertex.

15.2.1 Diffusion models and stock portfolios

Random walks on certain distance-regular graphs correspond to important models for the diffusion of particles. The Ehrenfests' urn model that was proposed to explain the second law of thermodynamics corresponds to random walks on the hypercube $H(D, 2)$. The classical Bernoulli-Laplace diffusion model corresponds to random walks on the Johnson graph $J(2D, D)$. Diaconis and Shahshahani [196] obtained results on the rate of convergence (i.e., total variation distance) to the stationary distribution in the latter model, using the algebraic properties of the Johnson graph. They find a sharp cut-off point at

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[32]It seems that this possible generalization was indicated implicitly in [399, p. 232]. It should be remarked that the Bose-Mesner algebra may also be replaced by a coherent algebra.

about $\frac{1}{4}D \log D$ steps, in the sense that a few steps earlier the variation distance is essentially maximal, while a few steps later it tends to 0 exponentially fast. Belsley [42] obtained similar results for the (non-bipartite) classical families of examples of Section 3.1, and Hora [333] obtained results for the halved cube and the quadratic forms graph (among others). Diaconis and Saloff-Coste [195] studied separation cut-offs for infinite families of Markov chains and applied their results to families of distance-regular graphs with unbounded diameter.

Distance-regular graphs have also been used as examples for interacting particle systems such as the so-called antivoter model; see [3, Ch. 14].

An application in finance is given by Billio, Calès, and Guégan [53], who used random walks on the Johnson graph to study the momentum strategy for stock portfolios.

15.2.2 Chip-firing and the abelian sandpile model

Chip-firing on a graph is a solitaire game that is related to random walks. It is played with a pile of chips at each vertex of the graph. At each step of the game, a vertex is *fired*, in the sense that a chip moves to each of its adjacent vertices (if the vertex has sufficient chips). Chip-firing is related to the abelian sandpile model for self-organized criticality from statistical physics [446], to avalanche models, and the dollar game. Related to these games and models is the critical group (sandpile group, Picard group) of a graph. Biggs [50] observed that the critical group of a distance-regular graph is in general not determined by the intersection array. He also introduced a subgroup of layered configurations that *is* determined by the intersection array. The Shrikhande graph and Hamming graph $H(2, 4)$ were discussed to illustrate these issues.

15.2.3 Biggs' conjecture on resistance and potential

Let x be a vertex of Γ, and suppose we start a random walk at x. For every other vertex y, we let the *hitting time* H_{xy} be the expected number of steps needed to get to y. The *cover time* $C_x(\Gamma)$ is the expected number of steps that a random walk started at x requires before it has visited every vertex of Γ.

The calculations required to determine these notions for arbitrary graphs are often quite intensive, even for moderately-sized graphs. It is therefore desirable to study graphs possessing symmetry properties that make calculations feasible. Van Slijpe [563], Devroye and Sbihi [194], and Biggs [49] all (independently) derived that in distance-regular graphs, the hitting times for vertices x and y at distance j are given in terms of the intersection numbers and valencies by

$$H_{xy} = k \sum_{i=1}^{j} \frac{1}{k_i c_i} \sum_{h=i}^{D} k_h.$$

Biggs [49] did not have this result explicitly, as he stated it in terms of potentials and electric resistance: let us consider a graph to be an electric circuit with edges corresponding to resistors of unit resistance. The effective resistance between two vertices can — in theory — be calculated using the familiar rules for resistances in series and in parallel.

This resistance measures how easily electricity may flow between the vertices, and can likewise be shown (see [200] or [51]) to measure how easily a random walk will move from one vertex to another. Naturally, the higher the resistance between x and y, the more difficult it is for a random walk to pass from x to y, and conversely. For distance-regular graphs it is possible to give an explicit value for the resistance between two vertices, as we shall now describe.

Let Γ be a distance-regular graph with valency at least 3. Using the intersection numbers, define the *Biggs potentials* ϕ_i recursively by $\phi_0 = v - 1$ and $\phi_i = (c_i \phi_{i-1} - k)/b_i$, for $i = 0, 1, \ldots, D - 1$.

The resistance ρ_j between vertices at distance j is then obtained by $\rho_j = 2 \sum_{i=0}^{j-1} \phi_i / vk$; see [49] (and the hitting time is a factor $vk/2$ larger). This shows that understanding the behavior of the Biggs potentials is crucial for the study of electric resistance in distance-regular graphs. Biggs [49] conjectured that $\phi_1 + \phi_2 + \cdots + \phi_{D-1} \leqslant \frac{94}{101} \phi_0$ and thus $\max_j \rho_j = \rho_D \leqslant (1 + \frac{94}{101}) \rho_1$ with equality only in the case of the Biggs-Smith graph. This conjecture was later proved by Markowsky and Koolen [461]. It implies that the resistance between two vertices is always at most $4/k$, and turns out to be a characteristic feature of the Biggs potentials, namely that the sum of the later ϕ_i is dominated by the earlier ones. In particular, Koolen, Markowsky, and Park [415] showed[33] that $\phi_{j+1} + \phi_{j+2} + \cdots + \phi_{D-1} < (3j+3)\phi_j$ for each $j = 0, 1, \ldots, D-2$, and that $\phi_2 + \phi_3 + \cdots + \phi_{D-1} \leqslant \phi_1$ with equality only in the case of the dodecahedron. The latter result can be used to prove Biggs' conjecture, and is a much stronger statement. It also implies that if $D \geqslant 3$, then $\rho_D/\rho_1 \leqslant 1 + 6/k$, which shows that for large k, all vertices become nearly equidistant when measured with respect to the resistance metric.

By applying techniques from [121], the above implies the following for the hitting times and cover times in distance-regular graphs: For all vertices x, y, we have that $H_{xy} \leqslant (1 + \min(\frac{6}{k}, \frac{94}{101}))(v - 1)$ and $C_x(\Gamma) \leqslant (4 + o(1))(v - 1) \ln v$. In fact, Feige [215] showed that for arbitrary graphs, we have $C_x(\Gamma) \geqslant (1 + o(1)) v \ln v$, so that the upper bound for distance-regular graphs is the best possible for large v, up to the multiplicative constant.

We note that the resistances between vertices at distance at most 3 in distance-regular graphs on at most 70 vertices have been calculated explicitly by Jafarizadeh, Sufiani, and Jafarizadeh [369], whereas those in some other families of distance-regular graphs, such as Hadamard graphs, were calculated in [370].

15.2.4 Quantum walks

The above classical random walks can be considered as Markov chains on the set of vertices of a graph, with the stochastic transition matrix $\frac{1}{k} A$. These have applications as described, but also in classical randomized algorithms. Likewise, in quantum information theory and quantum physics, there are applications of *quantum walks* in quantum computing. In the case of quantum walks, the state space is the set of directed edges (where each edge in

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[33]It was even conjectured that the factor $3j + 3$ can be replaced by a universal constant, but this was disproved in [414].

the graph is replaced by two oppositely directed edges), and the transition matrix U is unitary, that is, $UU^* = I$. For details, we refer to the introductory overview by Kempe [391] and the more graph-theoretical description by Emms, Severini, Wilson, and Hancock [209].

The first results on quantum walks on distance-regular graphs concerned the hypercubes, and were obtained by Moore and Russell [499] and Kempe [392]. Jafarizadeh and Salimi [367] studied quantum walks on, among others, Hamming graphs and Johnson graphs, using the quantum decomposition $A = L + F + R$ (see (11)) of the adjacency matrix. See also Section 16.6. Similarly, Salimi [553] considered the Odd graphs.

An important feature of quantum networks is *perfect state transfer*. This occurs for example between the antipodes in the hypercubes. Godsil [268] obtained that if a distance-regular graph Γ has perfect state transfer, then Γ is an antipodal double cover. He also constructed a family of Taylor graphs with perfect state transfer coming from certain Hadamard matrices. Jafarizadeh and Sufiani [368] discussed perfect state transfer in several other distance-regular antipodal double covers. Coutinho, Godsil, Guo, and Vanhove [145] determined for many more distance-regular antipodal double covers whether they have perfect state transfer; among others for all such graphs in the tables of 'BCN' [78]. Chan [119] showed among other results that for arbitrary $\tau > 0$ there exist graphs having perfect state transfer at time less than τ by taking unions of some of the distance-i graphs of the hypercubes.

See [499, 45, 119] for some work on *instantaneous uniform mixing* of continuous-time quantum walks on the Hamming graphs, folded cubes, halved cubes, and the folded halved cubes.

15.3   Miscellaneous applications

New classes of error-correcting pooling designs were constructed by Bai, Huang, and Wang [13] from Johnson graphs, Grassmann graphs, antipodal distance-regular graphs, and distance-regular graphs of order (s, t). Other classes were constructed by Zhang, Guo, and Gao [656], who used D-bounded distance-regular graphs. Gao, Guo, Zhang, and Fu [237] used subspaces in D-bounded distance-regular graphs to construct authentication codes.

Some applications of distance-transitive graphs referred to by Cohen [131] also apply to distance-regular graphs: Driscoll, Healy Jr., and Rockmore [201] apply the discrete polynomial transform to obtain fast algorithms for data analysis on distance-transitive graphs, mainly just by using its three-term recurrence relation; Jwo and Tuan [389] determined the transmitting delay in networks that can be modeled as a distance-transitive antipodal double cover (such as the hypercube).

Distance-regularity is one of the symmetry properties that are studied in a survey paper by Lakshmivarahan, Jwo, and Dhall [431] on interconnection networks. Among others, shortest routing algorithms in such networks are discussed.

Distance-regular graphs, in particular strongly regular graphs, occur in constructions of energy minimizing spherical codes. As shown by Cohn, Elkies, Kumar, and Schürmann

[133], spherical codes obtained from a spectral embedding of a strongly regular graph are *balanced*, that is, they are in equilibrium under all force laws acting between pairs of points with strength given by a fixed function of distance. As a consequence, these spherical codes appear frequently in the study of universally optimal spherical codes; see [134, 19].

16 Miscellaneous

16.1 Distance-transitive graphs

As already expressed in 'BCN' [78, Ch. 7], it seems to be feasible to classify all distance-transitive graphs, starting with the primitive ones, given the classification of finite simple groups. For more information on this, we refer to the historical essay — and then state of the art survey — by Ivanov [360], the introduction to the field — and survey — by Cohen [131], and the (currently) most recent survey by Van Bon [58]. Concerning the classification of imprimitive distance-transitive graphs, we mention the classification of antipodal distance-transitive covers of complete graphs by Godsil, Liebler, and Praeger [271] and the classification of distance-transitive covers of complete bipartite graphs by Ivanov, Liebler, Penttila, and Praeger [361]. Moreover, Alfuraidan and Hall [5] 'finished' the classification of distance-regular graphs whose so-called primitive core (i.e., the primitive graph obtained after halving and/or quotienting) is a known distance-transitive graph with diameter at least three.

16.2 The metric dimension

Given a graph, a *resolving set W* is a set of vertices such that every vertex in the graph is uniquely determined by the distances to the vertices in W. The *metric dimension* of a graph is the size of a smallest resolving set. Babai [10] studied the metric dimension of graphs motivated by the graph isomorphism problem (he actually studied the problem more generally in coherent configurations). His results imply an upper bound on the metric dimension for primitive distance-regular graphs in terms of the number of vertices, the diameter, and the valencies. Chvátal [127] obtained an asymptotic result on the metric dimension of the Hamming graphs, as a result of his work on strategies for the game *Mastermind*. For details, we refer to the survey paper by Bailey and Cameron [17]. For recent results on the metric dimension of Johnson graphs, Grassmann graphs, bilinear forms graphs, and symplectic dual polar graphs, we refer to [16, 279], [18], [217], and [280], respectively. Guo, Wang, and Li [279] also obtained results on the Doubled Odd graphs, Doubled Grassmann graphs, and twisted Grassmann graphs. The metric dimension of all 'small' distance-regular graphs was determined by Bailey [15]. Bailey [14] also related the metric dimension of several families of imprimitive distance-regular graphs to the metric dimension of corresponding primitive distance-regular graphs. The *fractional metric dimension* of vertex-transitive distance-regular graphs, in particular Hamming and Johnson graphs, was studied by Feng, Lv, and Wang [216].

16.3 The chromatic number

The *chromatic number* of a graph is the smallest number of colors needed to color the vertices such that adjacent vertices have different colors. Bipartite graphs clearly have chromatic number 2. It is not hard to see that the chromatic number of the Hamming graph $H(d, q)$ equals q. Blokhuis, Brouwer, and Haemers [55] studied distance-regular graphs with chromatic number 3. They showed that besides the complete tripartite graphs, the intersection number a_1 is at most 1 in such graphs, and they obtained several results for the case $a_1 = 1$; it seems that the triangle-free case is much more difficult. All graphs with chromatic number 3 among the known distance-regular graphs were classified by Blokhuis et al. [55]: these are the complete tripartite graphs, the odd cycles, the Odd graphs, the Hamming graphs $H(D, 3)$, and nine exceptional graphs. It was also shown that the folded cubes have chromatic number 4. Koolen and Qiao [423] classified the non-bipartite distance-regular graphs with diameter three, valency k, and smallest eigenvalue at most $-k/2$. Using these results, they obtained a complete classification of the distance-regular graphs with diameter three and chromatic number 3. Hahn, Kratochvíl, Širáň, and Sotteau [286] obtained results on the chromatic number of the halved cubes; see also [76]. Etzion and Bitan [211] and Brouwer and Etzion [79] provide a summary of results on the chromatic number of the Johnson graphs.

16.4 Cores

A *core* is a graph having no endomorphisms other than automorphisms. Every graph is homomorphically equivalent (i.e., there are homomorphisms in both directions) to a unique core, called the *core of* the graph. We say that a graph is *core-complete* if it is either a core or has a complete core. Cameron and Kazanidis [108] showed among other results that rank 3 graphs are core-complete. Godsil and Royle [276] showed among other results the core-completeness of many infinite families of geometric strongly regular graphs. In particular, by virtue of a result of Neumaier [509], it follows that, for given $m \geqslant 2$, all but finitely many strongly regular graphs with smallest eigenvalue at least $-m$ are core-complete. Roberson [551] finally showed that all strongly regular graphs are core-complete. Concerning general distance-regular graphs, Godsil and Royle [276] showed that distance-transitive graphs are core-complete, and that triangle-free non-bipartite distance-regular graphs are cores. Huang, Lv, and Wang [340] studied cores and endomorphisms of the Grassmann graphs.

16.5 Modular representations

Some work has been done on the adjacency algebra (denoted $\mathbb{A}_K \subset M_{v \times v}(K)$) of a distance-regular graph Γ over a field K of characteristic $p > 0$. Arad, Fisman, and Muzychuk [6, Thm. 1.1] showed among other results that $\mathbb{A}_K$ is semisimple if and only if the *Frame number* $v^{D+1} \prod_{i=1}^{D} (k_i/m_i)$ (which is an integer) is not divisible by p. See also [287, Thm. 4.2]. Hanaki [288] showed that $\mathbb{A}_K$ is a local algebra if v is a power of p. For strongly regular graphs, Hanaki and Yoshikawa [290] determined the structure of $\mathbb{A}_K$ and

studied the modular standard module K^v. In this case, the p-rank of $M \in \mathbb{A}_K$ (cf. Section 10.2) can be interpreted as the dimension of the submodule MK^v, and their results provide us information as to which elements of $\mathbb{A}_K$ we should look at. Yoshikawa [655] determined the structure of $\mathbb{A}_K$ for Hamming graphs. The structure of $\mathbb{A}_K$ for Johnson graphs was studied by Shimabukuro [557, 559]. Shimabukuro [558] also computed the number of irreducible representations of $\mathbb{A}_K$ for the classical families of distance-regular graphs. Shimabukuro and Yoshikawa [560] recently studied the structure of $\mathbb{A}_K$ for Grassmann graphs. For more information on modular representations of general (non-commutative) association schemes (i.e., homogeneous coherent configurations), we refer to the recent survey by Hanaki [289].

16.6   Asymptotic spectral analysis

Let Γ be a distance-regular graph with adjacency matrix A. Observe that the complex Bose-Mesner algebra $\mathbb{A} = \mathbb{A}_\mathbb{C}$ of Γ is a commutative $*$-algebra, and that $\frac{1}{v}\operatorname{tr}$ is a state on $\mathbb{A}$, i.e., a unital positive linear $*$-functional on $\mathbb{A}$. Thus, $(\mathbb{A}, \frac{1}{v}\operatorname{tr})$ is a classical algebraic probability space, and we may view A as an algebraic random variable. From this point of view, Hora [331] obtained, as variations of the central limit theorem, asymptotic spectral distributions for the families of Hamming graphs, Johnson graphs, halved cubes, and Grassmann graphs. He used the information on the spectra of these graphs directly, but then the method of quantum decomposition, first introduced in this context by Hashimoto [292], was applied to Hamming graphs by Hashimoto, Obata, and Tabei [294] and to Johnson graphs (among others) by Hashimoto, Hora, and Obata [293], which provided a more conceptual and succinct (and 'fully-quantum') approach to the results of Hora [331]. See also [334]. Their theory, in the final form given in [335, 336], turns out to be closely related to the Terwilliger algebra (in the case of distance-regular graphs; though they did not use the language of the Terwilliger algebra). Let $\mathbb{T}$ be the Terwilliger algebra of Γ with respect to $x \in V$. Let L, F, and R be the lowering, flat, and raising matrices, respectively; cf. (11). The *quantum decomposition*[34] of A is the expression $A = L + F + R$. The primary $\mathbb{T}$-module, together with L and R, naturally has the structure of a one-mode interacting Fock space,[35] and they took the limit of the coefficients of the three-term recurrence relation of the associated orthogonal polynomials (which are certain normalizations of the v_i from (2)) to get the *quantum central limit theorem*. See [335, 336] for more details. For the above four families of distance-regular graphs, these orthogonal polynomials belong to the Askey scheme by virtue of Leonard's theorem (see also the comments after (13)), and their results agree with the limit relations of the polynomials in the Askey scheme as described in [403, 402]. We note that they also considered some other states as well; see [332, 335]. The case of the Odd graphs was discussed in detail by Igarashi and Obata [344]. Associated to the Odd graphs are the Bannai/Ito polynomials (which are a $q \to -1$ limit of the most general q-Racah polynomials), and the generalized Hermite polynomials

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[34]The quantum decomposition for the complete graph K_2 is sometimes referred to as a *quantum coin-tossing*.

[35]In this context, L and R are called the *annihilation* and the *creation operators*, respectively.

arise as the orthogonal polynomials corresponding to the limit distribution. See also [249].

16.7   Spin models

A (symmetric) *spin model* is a nowhere-zero symmetric matrix $W \in M_{v \times v}(\mathbb{C})$ which satisfies certain 'invariance equations', and was introduced by Jones [371] as a tool for creating invariants of knots and links. Nomura [519] showed that every spin model W belongs to its *Nomura algebra* $\mathcal{N}_W \subset M_{v \times v}(\mathbb{C})$, which is the Bose-Mesner algebra (over $\mathbb{C}$) of a self-dual association scheme; see also [365, 366, 120]. We say that a distance-regular graph Γ with diameter D *supports* a spin model W if its adjacency algebra $\mathbb{A}$ (over $\mathbb{C}$) satisfies $W \in \mathbb{A} \subset \mathcal{N}_W$. In this case, $\mathbb{A}$ inherits the duality of $\mathcal{N}_W$. In particular, Γ is Q-polynomial. Many examples of spin models have been constructed in this situation; cf. [154, §9]. Write $W = \sum_{i=0}^D t_i A_i$, where A_i is the distance-i matrix of Γ for $i = 0, 1, \ldots, D$. Curtin and Nomura [154] showed among other results that if $t_1 \neq \pm t_0$ then the intersection array of Γ is described by t_1/t_0, $t_0 t_2/t_1^2$, and D. Curtin [149] showed that Γ is thin if $t_i \neq \pm t_0$ for $i = 1, 2, \ldots, D$, and Caughman and Wolff [116] determined the structure of the Terwilliger algebra $\mathbb{T}$. Moreover, Curtin [152, Thm. 1.6] showed that, in view of [116, Thm. 5.3], every irreducible $\mathbb{T}$-module affords not just a Leonard system (cf. Section 5.8) but a *Leonard triple system* [153]. See [339, 95, 631] and the references therein for related results. Nomura [517, 518] and Curtin and Nomura [156] studied the homogeneity of Γ. It is known (cf. [365, §4.4]) that the diagonal matrix T of size $D+1$ defined by $T_{ii} = t_i$ ($i = 0, 1, \ldots, D$) satisfies the *modular invariance property*, i.e., $(PT)^3$ is a scalar matrix, where $P = Q$ is the eigenmatrix of Γ. (Recall $P^2 = vI$.) Chihara and Stanton [126] showed that Γ has at most 12 (diagonal) solutions T to $(PT)^3 = I$ in general, and classified the solutions when Γ is a forms graph or a Hamming graph. See also [520].

16.8   Cometric association schemes

Distance-regular graphs form the class of metric (or P-polynomial) association schemes. Cometric (or Q-polynomial) association schemes are the 'dual version' of distance-regular graphs, but the systematic study of cometric (but not necessarily metric) association schemes has begun rather recently. One of the pioneers in this area is Suzuki [585, 586], who studied imprimitive cometric association schemes and association schemes with multiple Q-polynomial orderings, using a method of Dickie [197] based on matrix identities; cf. (8). In particular, he showed that an imprimitive cometric association scheme with $D \geqslant 7$ and with first multiplicity $m_1 > 2$ is Q-bipartite and/or Q-antipodal; cf. Theorem 2.12. In his classification, there remained two cases of open parameter sets with $D \in \{4, 6\}$. These were recently ruled out by Cerzo and Suzuki [118] for $D = 4$, and by Tanaka and Tanaka [601] for $D = 6$. Similarly, there was an open case in the classification of association schemes with multiple Q-polynomial orderings, which was recently ruled out by Ma and Wang [453]. Thus, the situation is dual to that of Section 13.2. Van Dam, Martin, and Muzychuk [177] studied cometric Q-antipodal association schemes and showed that these are uniform and related to linked systems. Three-class cometric Q-

antipodal association schemes for example are equivalent to linked systems of symmetric designs. LeCompte, Martin, and Owens [440] showed that four-class cometric Q-antipodal and Q-bipartite association schemes are equivalent to real mutually unbiased bases. See also [576]. See [469, 177] for a comprehensive study on imprimitive cometric association schemes. Martin and Williford [472] proved the dual of the Bannai-Ito conjecture discussed in Section 8: There are finitely many cometric association schemes with fixed first multiplicity at least three. Kurihara [428] obtained a dual version of the spectral excess theorem discussed in Section 10.3. See also [429, 526].

Concerning constructions of cometric (but not metric) association schemes, the main sources are block designs, spherical designs, real mutually unbiased bases, and hemisystems and other strongly regular decompositions of a strongly regular graph; see a survey by Bannai and Bannai [37], and also the online table by Martin [467]. The 'bipartite doubles' of the association schemes of the Hermitian dual polar graphs ${}^2\mathcal{A}_{2D-1}(\sqrt{q})$ provide an infinite family of cometric but not metric association schemes with unbounded diameter; cf. [38, pp. 313–315]. The 'extended Q-bipartite double' construction was introduced and worked out in detail by Martin, Muzychuk, and Williford [469]. Penttila and Williford [540] constructed the first known infinite family of primitive cometric association schemes that are not metric. Hollmann and Xiang [329] earlier constructed a family of 3-class association schemes that have the same parameters as those found by Penttila and Williford, without realizing it was cometric. See also [143]. Recently, Moorhouse and Williford [500] constructed an infinite family of cometric Q-bipartite association schemes as certain 'double covers' of the association schemes of the symplectic dual polar graphs $\mathcal{C}_D(q)$ with $q \equiv 1 \pmod 4$. These association schemes have two Q-polynomial orderings, and are not metric; cf. Section 5.7.2. Such a 'double cover' has a quadratic splitting field when q is a non-square, and Moorhouse and Williford asked whether or not it is in general the 'extended Q-bipartite double' of a primitive cometric but not metric association scheme when q is a square. If this is indeed the case, then this family would provide counterexamples to the conjecture of Bannai and Ito [38, p. 312] mentioned at the end of Section 2.7.

We refer the reader to [37, 470, 177] for more information and recent updates on cometric association schemes.

17 Tables

In this section, we report progress on the 'feasibility' and 'uniqueness' of the intersection arrays that were listed in the tables of 'BCN' [78, 74], and some additional (larger) ones not in the tables. We note that these tables are also available online in machine readable form (although they are not exactly the same) [75].

17.1 Diameter 3 and primitive

17.1.1 Uniqueness

For the following intersection array, there is a unique distance-regular graph with that array:

$\{6, 5, 2; 1, 1, 3\}$ ($v = 57$): Perkel graph; Coolsaet and Degraer [140].

17.1.2 Existence

For the following intersection arrays, there is a distance-regular graph with that array:

$\{20, 18, 6; 1, 1, 15\}$ ($v = 525$): unitary non-isotropics graph ($q = 5$); [78, Thm. 12.4.1].
$\{26, 24, 19; 1, 3, 8\}$ ($v = 729$): Brouwer graph ($q = 3$); Brouwer and Pasechnik [93], see Section 3.2.2.
$\{31, 30, 17; 1, 2, 15\}$ ($v = 1024$): Kasami graph ($q = 2, j = 2$); [78, Thm. 11.2.1 (13)].
$\{110, 81, 12; 1, 18, 90\}$ ($v = 672$): Moscow-Soicher graph (see Section 3.2.5).

17.1.3 Nonexistence

The following intersection arrays are not feasible:

$\{5, 4, 3; 1, 1, 2\}$ ($v = 56$): Fon-Der-Flaass [230].
$\{13, 10, 7; 1, 2, 7\}$ ($v = 144$): Coolsaet [138].
$\{19, 12, 5; 1, 4, 15\}$ ($v = 96$): Coolsaet and Jurišić [141], Neumaier [74, p. 15].
$\{21, 16, 8; 1, 4, 14\}$ ($v = 154$): Coolsaet [139].
$\{22, 16, 5; 1, 2, 20\}$ ($v = 243$): Sumalroj and Worawannotai [577].
$\{35, 24, 8; 1, 6, 28\}$ ($v = 216$): Jurišić and Vidali [388].
$\{36, 25, 8; 1, 4, 20\}$ ($v = 352$): $\theta_1 = 14$ with multiplicity 32 [78, Thm. 4.4.4].
$\{40, 33, 8; 1, 8, 30\}$ ($v = 250$): Jurišić and Vidali [388].
$\{44, 30, 5; 1, 3, 40\}$ ($v = 540$): Koolen and Park [418], see Section 9.3.
$\{45, 30, 7; 1, 2, 27\}$ ($v = 896$): Gavrilyuk and Makhnev [246].
$\{52, 35, 16; 1, 4, 28\}$ ($v = 768$): Gavrilyuk and Makhnev [245].
$\{55, 36, 11; 1, 4, 45\}$ ($v = 672$): Bang [21] and Gavrilyuk [240].
$\{56, 36, 9; 1, 3, 48\}$ ($v = 855$): Bang [21] and Gavrilyuk [240].
$\{65, 44, 11; 1, 4, 55\}$ ($v = 924$): Koolen and Park [418], see Section 9.3.
$\{69, 48, 24; 1, 4, 46\}$ ($v = 1330$): Gavrilyuk and Makhnev [245].
$\{72, 45, 16; 1, 8, 54\}$ ($v = 598$): $\theta_1 = 26$ with multiplicity 45 [78, Thm. 4.4.4].
$\{74, 54, 15; 1, 9, 60\}$ ($v = 630$): Coolsaet and Jurišić [141].
$\{77, 60, 13; 1, 12, 65\}$ ($v = 540$): Coolsaet and Jurišić [141].
$\{85, 54, 25; 1, 10, 45\}$ ($v = 800$): $\theta_1 = 35$ with multiplicity 34 [78, Thm. 4.4.4].
$\{90, 60, 12; 1, 12, 72\}$ ($v = 616$): $\theta_1 = 27$ with multiplicity 48 [78, Thm. 4.4.4].
$\{104, 66, 8; 1, 12, 88\}$ ($v = 729$): Urlep [637].
$\{105, 102, 99; 1, 2, 35\}$ ($v = 20608$): De Bruyn and Vanhove [185].
$\{112, 77, 16; 1, 16, 88\}$ ($v = 750$): $\theta_1 = 32$ with multiplicity 49 [78, Thm. 4.4.4].

$\{119, 96, 18; 1, 16, 102\}$ ($v = 960$): Jurišić and Vidali [388].
$\{145, 84, 25; 1, 20, 105\}$ ($v = 900$): $\theta_1 = 55$ with multiplicity 29 [78, Thm. 4.4.4].

17.2   Diameter 4 and primitive

17.2.1   Uniqueness

For the following intersection array, there is a unique distance-regular graph with that array:

$\{280, 243, 144, 10; 1, 8, 90, 280\}$ ($v = 22880$): Patterson graph (see [78, §13.7]); Brouwer, Jurišić, and Koolen [88].

17.2.2   Nonexistence

The following intersection arrays are not feasible:

$\{5, 4, 3, 3; 1, 1, 1, 2\}$ ($v = 176$): Fon-Der-Flaass [231].
$\{39, 32, 20, 2; 1, 4, 16, 30\}$ ($v = 768$): Lambeck [433].
$\{\mu(2\mu + 1), (\mu - 1)(2\mu + 1), \mu^2, \mu; 1, \mu, \mu(\mu - 1), \mu(\mu + 1)\}$ ($v = 8\mu^2(\mu + 1)$), $\mu \geqslant 2$: Godsil and Koolen [270] (besides this family, in the tables also those with $\mu = 4, 5, 6, 7$ are explicitly mentioned: $\{36, 27, 16, 4; 1, 4, 12, 36\}$, $\{55, 44, 25, 5; 1, 5, 20, 55\}$, $\{78, 65, 36, 6; 1, 6, 30, 78\}$, $\{105, 90, 49, 7; 1, 7, 42, 105\}$).
$\{50, 48, 48, 32; 1, 1, 9, 25\}$ ($v = 31635$): De Bruyn [182].

17.3   Diameter 4 and bipartite

17.3.1   Existence

For the following intersection array, there is a distance-regular graph with that array:

$\{45, 44, 36, 5; 1, 9, 40, 45\}$ ($v = 486$): Koolen-Riebeek graph (see Section 3.2.5).

17.3.2   Nonexistence

The following intersection arrays are not feasible:

$\{36, 35, 27, 6; 1, 9, 30, 36\}$ ($v = 324$): Galazidis (see [75])[36].
$\{36, 35, 33, 3; 1, 3, 33, 36\}$ ($v = 912$): Huang (see [75])[37].
$\{88, 87, 77, 4; 1, 11, 84, 88\}$ ($v = 1452$): Huang (see [75])[38].

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[36]$\Gamma_1 \cup \Gamma_4$ would be a strongly regular graph with parameters $(324, 57, 0, 12)$, which does not exist [243].
[37]The halved graphs would be the complement of a strongly regular graph with parameters $(456, 35, 10, 2)$, which does not exist [92] by Proposition 9.5.
[38]The halved graphs would be the complement of a strongly regular graph with parameters $(726, 29, 4, 1)$, which does not exist by Proposition 9.5.

17.4 Diameter 4 and antipodal

17.4.1 Uniqueness

For the following intersection arrays, there is a unique distance-regular graph with that array:

$\{32, 27, 8, 1; 1, 4, 27, 32\}$ ($v = 315$): Soicher graph (see Section 3.2.4); Soicher [566].
$\{45, 32, 12, 1, ; 1, 6, 32, 45\}$ ($v = 378$): $3.O_6^-(3)$-graph (see [78, §13.2C]); Jurišić and Koolen [379].
$\{56, 45, 16, 1; 1, 8, 45, 56\}$ ($v = 486$): Soicher graph (see Section 3.2.4); Brouwer [71] (see also [74, Thm. 11.4.6]).
$\{117, 80, 24, 1; 1, 12, 80, 117\}$ ($v = 1134$): $3.O_7(3)$-graph (see [78, §13.2D]); Jurišić and Koolen [380].
$\{176, 135, 36, 1; 1, 12, 135, 176\}$ ($v = 2688$): Meixner 4-cover (see Section 3.2.4 and [74, §12.4A]); Jurišić and Koolen [380].

17.4.2 Existence

For the following intersection arrays, there is a distance-regular graph with that array:

$\{176, 135, 24, 1; 1, 24, 135, 176\}$ ($v = 1344$): Meixner 2-cover (see Section 3.2.4 and [74, §12.4A]).
$\{416, 315, 64, 1; 1, 32, 315, 416\}$ ($v = 5346$): Soicher graph (see Section 3.2.4).

17.4.3 Nonexistence

The following intersection arrays are not feasible:

$\{32, 27, 6, 1; 1, 6, 27, 32\}$ ($v = 210$): Soicher [566].
$\{32, 27, 9, 1; 1, 3, 27, 32\}$ ($v = 420$): Soicher [566].
$\{45, 32, 9, 1; 1, 9, 32, 45\}$ ($v = 252$): Jurišić and Koolen [375].
$\{45, 32, 15, 1; 1, 3, 32, 45\}$ ($v = 756$): Jurišić and Koolen [375].
$\{45, 40, 11, 1; 1, 1, 40, 45\}$ ($v = 2352$): $\overline{\Gamma}$ is of order $(5, 8)$ with $c_2 = 12$, [78, Thm. 4.2.7].
$\{56, 45, 12, 1; 1, 12, 45, 56\}$ ($v = 324$): Brouwer [74, Thm. 11.4.6].
$\{56, 45, 18, 1; 1, 6, 45, 56\}$ ($v = 648$): Brouwer [74, Thm. 11.4.6].
$\{56, 45, 20, 1; 1, 4, 45, 56\}$ ($v = 972$): Brouwer [74, Thm. 11.4.6].
$\{56, 45, 21, 1; 1, 3, 45, 56\}$ ($v = 1296$): Brouwer [74, Thm. 11.4.6].
$\{81, 56, 18, 1; 1, 9, 56, 81\}$ ($v = 750$): Jurišić and Koolen [379].
$\{81, 56, 24, 1; 1, 3, 56, 81\}$ ($v = 2250$): Jurišić and Koolen [375].
$\{96, 75, 24, 1; 1, 8, 75, 96\}$ ($v = 1288$): Jurišić and Koolen [379].
$\{96, 75, 28, 1; 1, 4, 75, 96\}$ ($v = 2576$): Jurišić and Koolen [375].
$\{115, 96, 32, 1; 1, 8, 96, 115\}$ ($v = 1960$): Jurišić and Koolen [375].
$\{115, 96, 35, 1; 1, 5, 96, 115\}$ ($v = 3136$): Jurišić and Koolen [375].
$\{115, 96, 36, 1; 1, 4, 96, 115\}$ ($v = 3920$): Jurišić and Koolen [375].
$\{117, 80, 27, 1; 1, 9, 80, 117\}$ ($v = 1512$): Jurišić and Koolen [375].

$\{117, 80, 30, 1; 1, 6, 80, 117\}$ ($v = 2268$): Jurišić and Koolen [375].
$\{117, 80, 32, 1; 1, 4, 80, 117\}$ ($v = 3402$): Jurišić and Koolen [375].
$\{175, 144, 25, 1; 1, 25, 144, 175\}$ ($v = 1360$): Jurišić and Koolen [375].
$\{175, 144, 40, 1; 1, 10, 144, 175\}$ ($v = 3400$): Jurišić and Koolen [379].
$\{176, 135, 40, 1; 1, 8, 135, 176\}$ ($v = 4032$): Jurišić and Koolen [375].
$\{189, 128, 27, 1; 1, 27, 128, 189\}$ ($v = 1276$): Jurišić and Koolen [375].
$\{189, 128, 36, 1; 1, 18, 128, 189\}$ ($v = 1914$): Jurišić and Koolen [379].
$\{189, 128, 45, 1; 1, 9, 128, 189\}$ ($v = 3828$): Jurišić and Koolen [375].
$\{204, 175, 40, 1; 1, 20, 175, 204\}$ ($v = 2400$): Jurišić and Koolen [375].
$\{204, 175, 45, 1; 1, 15, 175, 204\}$ ($v = 3200$): Jurišić and Koolen [375].
$\{261, 176, 54, 1; 1, 18, 176, 261\}$ ($v = 3600$): Jurišić and Koolen [375].
$\{414, 350, 45, 1; 1, 45, 350, 414\}$ ($v = 4050$): Jurišić and Koolen [375].

17.5   Diameter 5 and antipodal

17.5.1   Uniqueness

For the following intersection array, there is a unique distance-regular graph with that array:

$\{22, 20, 18, 2, 1; 1, 2, 9, 20, 22\}$ ($v = 729$): coset graph of the dual of the ternary Golay code; Blokhuis, Brouwer, and Haemers [55].

17.5.2   Nonexistence

The following intersection arrays are not feasible:

$\{2\mu^2+\mu, 2\mu^2+\mu-1, \mu^2, \mu, 1; 1, \mu, \mu^2, 2\mu^2+\mu-1, 2\mu^2+\mu\}$ ($v = 4\mu^2(2\mu+3)$), $\mu \geqslant 2$: Coolsaet, Jurišić, and Koolen [142] (besides this family, in the tables also those with $\mu = 3, 4, 5, 6, 7$ are explicitly mentioned: $\{21, 20, 9, 3, 1; 1, 3, 9, 20, 21\}$, $\{36, 35, 16, 4, 1; 1, 4, 16, 35, 36\}$, $\{55, 54, 25, 5, 1; 1, 5, 25, 54, 55\}$, $\{78, 77, 36, 6, 1; 1, 6, 36, 77, 78\}$, $\{105, 104, 49, 7, 1; 1, 7, 49, 104, 105\}$).
$\{105, 90, 49, 7, 1; 1, 7, 49, 90, 105\}$ ($v = 2912$): $\theta_1 = 35$ with multiplicity 78 [78, Thm. 4.4.4].

17.6   Diameter 5 and bipartite

17.6.1   Uniqueness

For the following intersection arrays, there is a unique distance-regular graph with that array:

$\{7, 6, 6, 4, 4; 1, 1, 3, 3, 7\}$ ($v = 310$): Doubled Grassmann ($q = 2$); Cuypers [158], see Section 9.2.
$\{13, 12, 12, 9, 9; 1, 1, 4, 4, 13\}$ ($v = 2420$): Doubled Grassmann ($q = 3$); Cuypers [158], see Section 9.2.

17.6.2   Nonexistence

The following intersection array is not feasible:

$\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}$ ($v = 3500$): Vidali [642].

17.7   Diameter 6 and imprimitive

17.7.1   Nonexistence

The following intersection arrays are not feasible:

$\{15, 14, 12, 6, 1, 1; 1, 1, 3, 12, 14, 15\}$ ($v = 1518$, antipodal): Ivanov and Shpectorov [363]
$\{7, 6, 6, 5, 4, 3; 1, 1, 2, 3, 4, 7\}$ ($v = 686$, bipartite): Koolen [405].

18   Open problems and research directions

In this section we mention some important open problems. The most important one is to classify all distance-regular graphs of large enough diameter.

18.1   The classification of distance-regular graphs of large diameter

We first restrict the classification of distance-regular graphs to the following three problems.

Problem 1. Classify the Q-polynomial distance-regular graphs with large enough diameter.

Problem 2. Classify the geometric distance-regular graphs with large enough diameter.

Problem 3. Prove or disprove the following conjecture of Bannai and Ito [38, p. 312]: A primitive distance-regular graph with large enough diameter is Q-polynomial.[39]

18.2   General problems

Problem 4. Generalize results on distance-regular graphs to larger classes of graphs; for example, the Delsarte clique bound.

Problem 5. Which results on distance-regular graphs can be dualized to cometric association schemes? See Section 16.8.

Below, we will give a list of more specific (and typically smaller) problems related to the classification of distance-regular graphs.

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[39]This conjecture is true in the case of distance-transitive graphs and also for thick regular near polygons with $c_2 \geqslant 3$ and diameter at least 4. On the other hand, the recent construction of an infinite family of imprimitive cometric but not metric association schemes by Moorhouse and Williford [500] may be relevant to disproving the dual version of this conjecture on cometric association schemes (which was also raised by Bannai and Ito, see Section 2.7); cf. Section 16.8.

18.3   *Q*-polynomial distance-regular graphs

The *Q*-polynomial distance-regular graphs fall into types I, IA, II, IIA, IIB, IIC, and III from [38]. In Section 5.5, we showed that type IA cannot occur and that the distance-regular graphs of types IIA, IIB, IIC, and III are completely determined. For type II, the classification is known for $D \geqslant 14$.

Problem 6.   (i) Determine the graphs of type II with $D \leqslant 13$.

  (ii) Determine the *Q*-polynomial distance-regular graphs of type I. A subproblem is to determine the distance-regular graphs with classical parameters with $b \neq 1$.

The classification of imprimitive *Q*-polynomial distance-regular graphs is complete for $D \geqslant 12$, except for the classification of the distance-regular graphs with the same intersection array as the bipartite dual polar graphs. See Sections 5.5.7 and 5.5.8.

Problem 7. Classify the graphs that have the same intersection array as the bipartite dual polar graphs and the Hemmeter graphs for $D \geqslant 12$. Also, improve the condition $D \geqslant 12$ for the bipartite case.

Problem 8. Show that a *Q*-polynomial distance-regular graph with diameter D is imprimitive if and only if $a_D = 0$.

Lang and Terwilliger [438] almost classified the *Q*-polynomial generalized odd graphs with diameter at least three, leaving open one set of intersection arrays for $D = 3$. See Section 5.5.9.

Problem 9. Classify the *Q*-polynomial generalized odd graphs with diameter three.

Problem 10. Classify the primitive *Q*-polynomial distance-regular graphs with two *P*-polynomial orderings and diameter three or four. See Section 13.2.

Problem 11. Classify the primitive distance-regular graphs with two *Q*-polynomial orderings and diameter three. See Section 5.7.2.

Problem 12. Let $\eta_0, \eta_1, \ldots, \eta_D$ be a *Q*-polynomial ordering of the eigenvalues (that is, of the corresponding idempotents), and let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the natural ordering of the eigenvalues. What can be said of the relation between the η_i and the θ_j? For example, determine whether $\eta_1 \in \{\theta_1, \theta_D, \theta_{D-1}\}$, or whether $\{\theta_1, \theta_D\} \cap \{\eta_1, \eta_D\} \neq \emptyset$. (For bipartite graphs and antipodal graphs, see [110, 532] and also Section 5.5.8.)

Problem 13. Classify the tight *Q*-polynomial distance-regular graphs with $D = 4$.

18.4   Vanishing Krein parameters

Bannai and Ito [38, p. 312] conjectured that primitive distance-regular graphs with large enough diameter are *Q*-polynomial (see Problem 3), and so for such graphs most Krein parameters vanish.

Problem 14. Show that there exists a constant C such that for every primitive distance-regular graph there exists a primitive idempotent, say, E_1, such that $|\{j : q_{1j}^i \neq 0\}| \leqslant C$ for all i.

Note that this problem is dual to Problem 73.

Problem 15. Let θ_i be a tail (see Section 5.4).

  (i) Determine whether $\theta_i \in \{\theta_1, \theta_D, \theta_{D-1}\}$. This last case ($\theta_i = \theta_{D-1}$) should only occur for bipartite distance-regular graphs.

  (ii) Determine $j \neq 0, i$ such that $q_{ii}^j \neq 0$.

  (iii) Is it possible to classify the distance-regular graphs with a light tail? Besides the antipodal Q-polynomial distance-regular graphs, there seem to be only the halved cubes and the Hermitian dual polar graphs ${}^2\mathcal{A}_{2D-1}(\sqrt{q})$.

Problem 16. Sometimes, one can use the absolute bound to show that some Krein parameters vanish, if one of the non-trivial eigenvalues has a small multiplicity.

  (i) Find more conditions that imply that some Krein parameters vanish.

  (ii) Study distance-regular graphs with no vanishing (non-trivial) Krein parameters. Among the primitive distance-regular graphs with diameter three that are not Q-polynomial, there are few that have a vanishing (non-trivial) Krein parameter (we checked that there is only one on at most 100 vertices: the Sylvester graph).

Problem 17. Jurišić, Coolsaet, and others have used vanishing Krein parameters to show that certain families of intersection arrays are not feasible, and also to show the uniqueness of some distance-regular graphs by their intersection arrays. See Section 6.3.1. But it is not known when the method of vanishing Krein parameters gives enough extra information in order to decide the non-existence of certain intersection arrays. Explore this.

18.5   Classical parameters

Problem 18. Characterize the classical distance-regular graphs by their intersection arrays.

Problem 19. Show that Q-polynomial geometric distance-regular graphs which are not polygons have classical parameters.

Problem 20. Decide whether the Grassmann graphs $J_q(2D, D)$ are determined by their intersection arrays. See Section 3.1. Decide whether there are other distance-regular graphs than the twisted Grassmann graphs with the same intersection arrays as the Grassmann graphs $J_q(2D + 1, D)$.

The following problem was posed by Vanhove in his thesis [639, Pr. 8].

Problem 21. Determine whether all distance-regular graphs with classical parameters $(D, b, \alpha, \beta) = (D, -q, -(q + 1)/2, -((-q)^D + 1)/2)$, q odd, are subgraphs of the Hermitian dual polar graph ${}^2\mathcal{A}_{2D-1}(q)$ (for sufficiently large D). See Theorem 5.2 and the paragraph that follows it.

18.6 Geometric distance-regular graphs

Problem 22. Determine whether for a given integer $m \geqslant 2$, there are only finitely many geometric distance-regular graphs with $D \geqslant 3$, $c_2 \geqslant 2$, and smallest eigenvalue $-m$, besides the Grassmann graphs, Johnson graphs, bilinear forms graphs, and Hamming graphs. See Section 9.5. Note that the generalized $2D$-gons of order $(q, 1)$ for $D = 3, 4, 6$ (which exist for all prime powers q) are geometric with smallest eigenvalue -2, but they have $c_2 = 1$; see also [78, Thm. 4.2.16].

Problem 23. Classify the geometric distance-regular graphs with $a_1 \geqslant 1$ and $c_2 \geqslant 2$.

Problem 24. Classify the (non-bipartite) geometric distance-regular graphs that are also Q-polynomial.

Let Γ be a geometric distance-regular graph with respect to a set of cliques $\mathcal{C}$. We call an induced subgraph Δ of Γ *a subspace* if Δ is closed and for each edge xy contained in Δ, all the vertices of the clique $C \in \mathcal{C}$ containing x and y are in Δ.

Problem 25. Find sufficient and necessary conditions for the existence of subspaces, in a similar fashion as the m-boundedness condition. See Section 11.

Problem 26. Classify the geometric distance-regular graphs having the property that for each pair of distinct vertices x and y there exists a (unique) subspace $\Phi(x, y)$ of diameter $d(x, y)$. This would be an extension of the classification of thick regular near polygons with $c_2 \geqslant 2$.

Problem 27. Complete the classification of thick regular near polygons with diameter at least 4 and $c_2 \geqslant 2$. Only the case $c_2 = 2$ and $c_3 > 3$ needs to be considered. The case $c_2 = 1$ seems to be too difficult at the moment. See also Theorem 9.11.

Problem 28. Let Γ be a geometric distance-regular graph with respect to $\mathcal{C}$. Define the dual graph on vertex set $\mathcal{C}$, where two cliques are adjacent if they intersect. Determine when this dual graph is distance-regular (this happens for the Johnson graphs and the Grassmann graphs).

18.7 The Bannai-Ito conjecture

The Bannai-Ito conjecture can be interpreted as a diameter bound in terms of the valency, but the current proof (cf. Section 8.1) gives a very bad bound. On the other hand, all the known distance-regular graphs with valency k at least three have $D \leqslant 2k + 2$, with equality only for the Foster graph.

Problem 29. Find a good diameter bound in terms of the valency.

Problem 30. Let Γ be a distance-regular graph with diameter D, head h, and valency k at least three.

(i) Prove Ivanov's conjecture that $\ell(c_i, a_i, b_i) \leqslant h + 1$ [78, p. 191],

(ii) Show that $D \leqslant (2k - 3)h + 1$ except if Γ is the dodecahedron (in which case $D = 5$, $h = 1$, and $k = 3$),

(iii) Show that if $c_i \geqslant 2$ for some i then $|\{i : c_i = c\}| \leqslant \min\{i : c_i \geqslant 2\} - 1$ for $c = 2, 3, \ldots, k - 1$. See also Proposition 7.3.

All the known distance-regular graphs except for the polygons have $h \leqslant 5$, with equality for the generalized dodecagons.

Problem 31. Prove the conjecture of Suzuki [587, Conj. 1.5.2] that claims that there exists a constant H such that all distance-regular graphs with valency at least three have head $h \leqslant H$.

Problem 32. Suzuki's conjecture in the above problem would imply that the girth of a distance-regular graph is bounded. Prove the more specific (unpublished) conjecture by Koolen and Suzuki that the girth of a distance-regular graph with valency at least three is at most 12.

Problem 33. Show that every distance-regular graph with valency and diameter at least three has an integral eigenvalue besides the valency. This was posed as a question by 'BCN' [78, p. 130]. Clearly this is the case for bipartite distance-regular graphs and more generally for geometric distance-regular graphs.

Problem 34. Define the degree of an algebraic integer as the degree of its minimal polynomial. All the eigenvalues of the known distance-regular graphs have degree at most three; with the Biggs-Smith graph as the only example having an eigenvalue with degree equal to three. In this light we propose the following conjecture: *Every eigenvalue of a distance-regular graph with valency at least three has degree at most three.* This conjecture could be a first step to show the above conjecture of Suzuki.

Problem 35. Develop theory for distance-regular graphs with only integral eigenvalues. It is easy to show that for such graphs the diameter D is bounded by $2k$, where k is the valency. If possible, improve this bound. Also obtain a good bound for the head.

18.8   Combinatorics

Problem 36. Classify the 1-homogeneous distance-regular graphs that are not bipartite nor a generalized odd graph. See Section 6.1.3.

Problem 37. Study distance-regular graphs that are locally strongly regular.

Problem 38. Determine whether $k_i = k_j$ for some distinct i and j with $i + j \leqslant D$ and $k_D \geqslant 2$ implies that $k = 2$. See Section 13.3.

Problem 39. Given an integer $\alpha \geqslant 1$, determine whether there are only finitely many distance-regular graphs with diameter at least three and $a_1 > \alpha$ such that each local graph has second largest eigenvalue at most α. See Section 13.6.2.

It is known that if $c_2 \geqslant 2$ then $c_3 > c_2$ [78, Thm. 5.4.1].

Problem 40. Show that if $c_2 \geqslant 2$, then the c_i are strictly increasing.

Problem 41. Determine whether one needs to remove at least $2k - 2 - a_1$ vertices in order to disconnect a distance-regular graph with diameter at least three such that each

resulting component has at least two vertices. Note that if this is the case, then this is best possible because $2k-2-a_1$ is the size of the neighborhood of an edge. Cioabă, Kim, and Koolen [129] showed that it is not true for strongly regular graphs, but it is believed it may be true for strongly regular graphs with $k \geqslant 2a_1 + 3$.

In Section 13.2 we discussed distance-regular graphs Γ with multiple P-polynomial orderings.

Problem 42.   (i) Classify the generalized odd graphs.

 (ii) Determine new putative intersection arrays for generalized odd graphs.

(iii) Show that if Γ is a bipartite antipodal 2-cover with diameter $2e$ and $e \geqslant 3$, then Γ is a $2e$-cube.

(iv) Classify the non-bipartite antipodal 2-covers with diameter $D \geqslant 4$ that have a generalized odd graph as folded graph.

 (v) Show that if $\Delta = \Gamma_D$ is also distance-regular with diameter D, then Δ is a generalized odd graph or a Taylor graph.
See also Problem 10.

The following problem is due to Fiol [221, Conj. 3.6].

Problem 43. Show that a distance-regular graph with diameter at least 4 is strongly distance-regular (cf. Section 10.3) if and only if it is antipodal. See Section 13.7.

The following problem is due to Pyber [545, Conj. 1, 3.1] who showed that all but finitely many strongly regular graphs are Hamiltonian. Recall also the well-known Lovász conjecture that all but finitely many connected vertex-transitive graphs are Hamiltonian.

Problem 44.   (i) Show that all but finitely many distance-regular graphs are Hamiltonian.

 (ii) In particular, show that all but finitely many distance-regular graphs with a fixed diameter D are Hamiltonian.
The case $D = 3$ seems to be the way to attack this problem.

Problem 45. Determine which distance-regular graphs are core-complete. Currently no distance-regular graphs are known that are not core-complete. See Section 16.4.

Problem 46.   (i) Show or disprove the conjecture of Neumaier, i.e., that all completely regular codes in the Hamming graphs with minimum distance at least 8 are known.

 (ii) Neumaier [512] challenged his readers to classify the completely regular codes in the Hamming graphs with $nq \leqslant 48$. But there are many feasible intersection arrays with small covering radius, say 2 and 3. We therefore would like to modify the challenge to classify the completely regular codes in the Hamming graphs with $nq \leqslant 48$ with covering radius at least 4.

(iii) Give more results on the intersection array of a completely regular code in a distance-regular graph.

18.9 Uniqueness and non-existence

Problem 47. Decide whether the Livingstone graph is determined by its intersection array $\{11, 10, 6, 1; 1, 1, 5, 11\}$. See [78, §13.5].

Problem 48. Construct a distance-regular graph with intersection array $\{7, 6, 6; 1, 1, 2\}$, or show that none exists. See [78, p. 148].

Problem 49. Classify the non-bipartite distance-regular graphs with diameter at least four with the same intersection array as a regular near polygon. Currently, the only known ones that are not regular near polygons are the Doob graphs. See Section 9.6.

Problem 50. Classify the distance-regular graphs that are locally Hoffman-Singleton, i.e., those with intersection arrays $\{50, 42, 9; 1, 2, 42\}$ and $\{50, 42, 1; 1, 2, 50\}$. See Section 13.5.

The following problem was raised by Bannai [private communication].

Problem 51. Determine whether the following is true: if a distance-regular graph Γ with diameter $D \geqslant 4$ has intersection numbers $c_i = i^2$ and $b_i = (n - d - i)(d - i)$ for $i \leqslant D - 1$ for some positive integers n and d, then Γ is the folded Johnson graph with diameter D. Similar problems can be formulated for other classical families of distance-regular graphs. See, e.g., Theorem 12.1 for the case of Hamming and Doob graphs.

18.10 The Terwilliger algebra

Problem 52. Determine the structure of the Terwilliger algebra for the four families of forms graphs and also for the twisted Grassmann graphs.

Problem 53. Develop theory for 1-thin distance-regular graphs with exactly three irreducible $\mathbb{T}$-modules with endpoint 1 up to isomorphism.

Problem 54. The vectors $\mathbf{f}_t$ (cf. (21)) can be defined for any (i.e., not necessarily bipartite) distance-regular graph. Give more results using the positive semidefiniteness of the Gram matrix of some of the $\mathbf{f}_t$. For example, is it possible to prove the Terwilliger tree bound (cf. Section 14.1) in this way?

Problem 55. An irreducible $\mathbb{T}$-module W is called *sharp* if $\dim E_t^\star W = 1$, where t is its endpoint. Give sufficient and necessary conditions such that all the irreducible $\mathbb{T}$-modules of a distance-regular graph are sharp.

The following problem was raised by Terwilliger [private communication].

Problem 56. Find all the 2-thin bipartite distance-regular graphs with diameter $D \geqslant 4$ with at most two irreducible $\mathbb{T}$-modules with endpoint 2 up to isomorphism. The Q-polynomial bipartite distance-regular graphs are included in this class, and so are the taut graphs; cf. Section 6.3.2. By a recursion obtained by Curtin [147], for these distance-regular graphs the intersection array is determined by at most four parameters (besides D); cf. Section 6.2. A related problem is to find a closed form for the intersection numbers.

The following three problems were also posed by Terwilliger [616, 618].

Problem 57. Suppose Γ is a thin distance-regular graph with diameter D and is not Q-polynomial. Show that if D is sufficiently large then one of the following holds.

(i) Γ is bipartite, and the halved graph is thin and Q-polynomial.

(ii) Γ is antipodal, and the folded graph is thin and Q-polynomial.

Problem 58. Let Γ be a thin non-bipartite Q-polynomial distance-regular graph. Take any two distinct vertices x, y. Show that the minimal convex subgraph containing x and y is a thin Q-polynomial distance-regular graph with diameter $d(x, y)$. If this claim turns out to be false, then find a simple additional assumption on Γ under which it is true. [40]

Problem 59. Classify the thin Q-polynomial distance-regular graphs.

The following problem was raised by Ito [private communication].

Problem 60. Study the structures of the irreducible $\mathbb{T}$-modules of Q-polynomial distance-regular graphs from the point of view of the theory of tridiagonal systems, in particular as 'tensor products' of Leonard systems. See Section 5.8. Is it true that each of the corresponding tridiagonal systems is a 'tensor product' of at most two Leonard systems? We note that this is indeed true for the irreducible $\mathbb{T}$-modules with endpoint 1; see Section 5.6.

Problem 61. Is the isomorphism class of an irreducible $\mathbb{T}$-module for a Q-polynomial distance-regular graph with $c_2 \geqslant 2$ and $a_1 \neq 0$ determined by its local eigenvalue and endpoint?

18.11 Other classification problems

Problem 62. Classify the distance-regular Terwilliger graphs. See Section 13.5.

There are infinitely many putative parameter sets (v, k, λ, μ) for strongly regular graphs with $\mu = 1$ and $\lambda = 2$.

Problem 63. Show that there are only finitely many strongly regular graphs with $\mu = 1$. By Proposition 11.3 ($m = 2, h = 1$), a consequence of this would be that there are finitely many distance-regular graphs with $c_3 = 1$ and $a_1 \neq a_2$. It would also contribute to the classification of Terwilliger graphs with $\mu \geqslant 2$ by considering its local graphs.

Problem 64. Generalize results for strongly regular graphs that do not yet have analogues for distance-regular graphs.

Problem 65. Fuglister [234] uses mod p calculations for the multiplicities to show that if a distance-regular graph has $D = h + 1$, where h is the head, then $h \leqslant 12$. Suzuki [587, p. 87] claims this can be generalized to the case $D \leqslant h + 3$. Brouwer and Koolen [89] use a similar argument for the generation of feasible arrays for the distance-regular graphs with valency 4. Find more instances where this method works.

Problem 66. For a coconnected distance-regular graph Γ, show that the intersection number c_2 is bounded above by a function of $\frac{b_1}{\theta_1 + 1}$. For strongly regular graphs, this is true as

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[40]That Γ is non-bipartite was not assumed in [618]. Terwilliger [private communication] pointed out that the Hemmeter graphs (and all distance-regular graphs with the same intersection array as (but not isomorphic to) the bipartite dual polar graphs) provide counterexamples.

in that case $\frac{b_1}{\theta_1+1}$ is equal to $-\theta_2 - 1$ and hence it follows by Neumaier's [509] μ-bound. It is also true for distance-regular graphs with $\theta_1 = b_1 - 1$ by the classification of such graphs (see [78, Thm. 4.4.11]), as they all have $c_2 \leqslant 10$. But even for Q-polynomial distance-regular graphs, a bound for c_2 in terms of $\frac{b_1}{\theta_1+1}$ is not known to exist.

Problem 67. Classify the distance-regular graphs of order $(s, 2)$. For $s = 1$ these are precisely the distance-regular graphs with valency three. For $s = 2$, they were classified by Hiraki, Nomura, and Suzuki [325]. Yamazaki [653] obtained some results for $s \geqslant 3$.

Problem 68. The fact that the multiplicities of the eigenvalues of a distance-regular graph are positive integers seems to be one of the strongest known conditions for its intersection array. They are however expensive to compute. Find necessary but easy to compute properties of the intersection numbers of distance-regular graphs that follow from the integrality of the multiplicities, such as the fact that for all prime numbers p the number of closed walks of length p is divisible by p. See Section 14.4.

Problem 69. Bang [22] showed that for $g \equiv 3 \pmod 4$ and $g = 5$, there exists a positive ϵ_g such that if Γ is a triangle-free distance-regular graph with girth g and large enough valency k, then the smallest eigenvalue of Γ is at least $(\epsilon_g - 1)k$. Show the same result for distance-regular graphs with odd girth. For non-bipartite distance-regular graphs with even girth g we can only expect that the smallest eigenvalue is at least $-k + C_g$ for a positive constant C_g, as the folded $(2m+1)$-cube has smallest eigenvalue $-2m+1 = -k+2$ and the Odd graph with valency k has smallest eigenvalue $-k + 1$.

Problem 70. Show that for large enough k, the second largest eigenvalue of a distance-regular graph with valency k is at most $k - 1$, as conjectured by Koolen (unpublished). This would be best possible as the Doubled Odd graphs have second largest eigenvalue $k - 1$. For distance-regular graphs with girth 6, it was shown by Bang, Koolen, and Park [35].

Problem 71.   (i) Determine the vertex-transitive distance-regular graphs.

 (ii) Determine the distance-regular Cayley graphs.

 (iii) Determine the arc-transitive distance-regular graphs.
See, e.g., [1, 454, 493, 494, 495] for some results on distance-regular Cayley graphs.

Problem 72. Classify the distance-regular graphs with chromatic number 3 and $a_1 = 1$. See Section 16.3.

We finish with a problem that is dual to Problem 14 and that is relevant for the dual of Problem 3.

Problem 73. Show that there exists a constant C such that for every primitive cometric association scheme there exists an adjacency matrix, say, A_1, such that $|\{j : p_{1j}^i \neq 0\}| \leqslant C$ for all i.

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