A Dynamic Survey of Graph Labeling

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Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the mid-1960s. In the intervening years over 350 graph labelings techniques have been studied in over 3600 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey, I have collected everything I could find on graph labeling. For the convenience of the reader, the survey includes a detailed table of contents and index. This edition has 55 additional pages and 308 new references that are identified with the reference number and the word "new" in the right margin.

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1 Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [2648] in 1967, or one given by Graham and Sloane [1147] in 1980. Rosa [2648] called a function f a *β-valuation* of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [1115] subsequently called such labelings *graceful* and this is now the popular term. Alternatively, Buratti, Rinaldi, and Traetta [631] define a graph G with q edges to be graceful if there is an injection f from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that every possible difference of the vertex labels of all the edges is the set $\{1, 2, \ldots, q\}$. Rosa introduced β-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [2625] that K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges. Independently, Keevash and Staden [1688] in April 2020 and Montgomery, Pokrovskiy, and Sudakov [2188] in January in 2021 proved Ringel's 1963 conjecture that any tree with n edges packs $2n + 1$ times into the complete graph K_{2n+1} for large n. Keevash and Staden used an embedding algorithm in which the various subroutines are analyzed by a wide range of methods, some of which are adaptations of existing methods whereas, Montgomery et al. used probabilistic methods. Although an unpublished result of Erdős says that most graphs are not graceful (see [1147]), most graphs that have some sort of regularity of structure are graceful. Sheppard [2882] has shown that there are exactly $q!$ gracefully labeled graphs with q edges. Rosa [2648] has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges," (2) G "has too many edges," and (3) G "has the wrong parity." The disjoint union of trees is a case where there are too many vertices for the number of edges. An infinite class of graphs that are not graceful for the second reason is given in [562]. As an example of the third condition Rosa [2648] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful. Knuth [1707] has observed the more general condition that in any graceful labeling of a graph with the number of edges congruent to 1 or 2 (mod 4), the number of vertices with an odd degree and an odd label is always odd. Knuth [1707] proved by way a computer search that all cubic graphs on $4, 6, 8, 10, 12$, or 14 vertices, except $2K_4$ and $3K_4$, which was proved by Kotzig, are graceful. He conjectures that that every connected cubic graph is graceful.

It has been known since 1975 [548] that rooted symmetric trees are graceful. However, the proofs that have been presented for this fact are either indirect inductive proofs or algorithmic descriptive proofs showing the many separate steps involved in labelling the vertices. In 2021 Rofa [2639] provided a graceful labeling for any given rooted symmetric tree in the form of a direct algebraic function that algebraically maps each vertex to a unique label. The function is a generalization of the way a path is canonically gracefully labeled as algorithmically described by Rosa in his 1967 classic paper [2648]. Rofa uses his function to show that a class of rooted symmetric trees that contains the class of binomial trees has weakly α-labeling (see Section 3.1) and it can provide a concise practical, computational way of producing graceful labelings of large rooted symmetric trees in relatively minimal time.

Acharya [27] proved that every graph can be embedded as an induced subgraph of a graceful graph and a connected graph can be embedded as an induced subgraph of a graceful connected graph. Acharya, Rao, and Arumugam [47] proved: every triangle-free graph can be embedded as an induced subgraph of a triangle-free graceful graph; every planar graph can be embedded as an induced subgraph of a planar graceful graph; and every tree can be embedded as an induced subgraph of a graceful tree. Sethuraman, Ragukumar, and Slater [2829] show that every tree can be embedded in a graceful tree (see also [2828]) and pose a related open problem toward settling the Graceful Tree Conjecture. Rao and Sahoo [2593] proved that every connected graph can be embedded as an induced subgraph of an Eulerian graceful graph thereby answering a question originally posed by Rao and mentioned by Acharya and Arumugum in [33]. As a consequence they deduce that the problems on deciding whether the chromatic of a graph number is less than or equal to k, for $k \geqslant 3$, and deciding whether the clique number of a graph is greater than or equal to k, for $k \geqslant 3$ are NP-complete even for Eulerian graceful graphs.

Sethuraman and Ragukumar [2827] provided an algorithm that generates a graceful tree from a given arbitrary tree by adding a sequence of new pendent edges to the given arbitrary tree thereby proving that every tree is a subtree of a graceful tree. They ask the question: If G is a graceful tree and v is any vertex of G of degree 1, is it true that $G - v$ is graceful? If the answer is affirmative, then those additional edges of the input arbitrary tree T introduced for constructing the graceful tree T by their algorithm could be deleted in some order so that the given arbitrary tree T becomes graceful. This would imply that the Graceful Tree Conjecture is true. These results demonstrate that there is no forbidden subgraph characterization of these particular kinds of graceful graphs.

Harmonious graphs naturally arose in the study by Graham and Sloane [1147] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be *harmonious* if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod{q}$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. They proved that almost all graphs are not harmonious. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number of edges q and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [1964] have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, \ldots, d_p$ then $\gcd(d_1, d_2, \ldots d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph. More generally, Sethuraman and Elumalai [2812] have shown that any given set of graphs $G_1, G_2, \ldots, G_t$ can be embedded in a graceful or harmonious graph. Determining whether a graph has a harmonious labeling was shown to be NP-complete by Auparajita, Dulawat, and Rathore in 2001 (see [1758]).

In the early 1980s Bloom and Hsu [577], [578],[551], [579], [649] extended graceful labelings to directed graphs by defining a graceful labeling on a directed graph $D(V, E)$ as a one-to-one map θ from V to $\{0, 1, 2, \ldots, |E|\}$ such that $\theta(y) - \theta(x) \bmod (|E| + 1)$ is distinct for every edge xy in E. Graceful labelings of directed graphs also arose in the characterization of finite neofields by Hsu and Keedwell [1264], [1265]. Graceful labelings of directed graphs was the subject of Marr's 2007 Ph.D. dissertation [2093]. In [2093] and [2094] Marr presents results of graceful labelings of directed paths, stars, wheels, and umbrellas. Hegde and Kumudakshi [1221] we use complete mappings to construct graceful [1221] new
labelings of two directed cycles. Siqinbate and Feng [2988] proved that the disjoint union of three copies of a directed cycle of fixed even length is graceful.

Over the past five decades in excess of 3,000 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes of graphs have been done by several authors and in some cases the same terminology is used for different concepts. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two labeling methods, include [544], [573], [1717], [1002], and [1004]. In [2938] Shivarajkumar, Sriraj, and Hegde provided a 2021 survey article graceful labeling of digraphs. The book edited by Acharya, Arumugam, and Rosa [32] includes a variety of labeling methods that we do not discuss in this survey. In 2002 Eshghi [902] wrote a 65 page paper providing an introduction to graceful graphs. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings, and neofields is discussed in [577] and [578]. The connection between graceful labelings and perfect systems of difference sets is given in [547]. The computational complexity of the gracefulness of a graph is not known, but the complexity of finding a harmonious labeling of a graph is in the NP-class [176]. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, secret sharing schemes, cryptology, models for constraint programming over finite domains, [574], [575], [3165], [2529], [3007], [3008], [222], [221], [288], [2995], [2131], and network passwords–see [3424], [3132], [3423], [3425], [2723], [3207], [3583], [802] and [1101] for details. Applications of graph labelings to encryption and decryption schemes are given in [196], [197], [121], and [196] new
[196]. According to Wang, B. Yao, and M. Yao [3427], graph labelings are used for in- [197] new
corporating redundancy in disks, designing drilling machines, creating layouts for circuit [121] new
boards, and configuring resistor networks. In [1262] Hsieh, Chen, Jiang, Liaw, and Shin use graph labelings in algorithms for image processing schemes for monitoring air quality. [196] new
Zhang, Ye, Zhang, and Yao [3582] investigated the use of graph colorings and graph labelings for designing topological passwords that resist attacks. Sivakumar, Vidyanandini, Sreedevi, Nayak, and Bhoi [2991] demonstrated how the notion of total edge irregularity

strength of complete tripartite graphs can be used in anti-theft networks. Nithya and Anitha [2281] investigated how graph labelings can be applied to the study of computer networks.

Terms and notation not defined below follow that used in [684] and [1002].

2 Graceful and Harmonious Labelings

2.1 Trees

The Ringel-Kotzig conjecture (GTC) that all trees are graceful has been the focus of many papers. Kotzig [1268] has called the effort to prove it a "disease." Among the trees known to be graceful are: caterpillars [2648] (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [1268], [3590] and [1546]; trees with diameter at most 5 [3590] and [1259]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [548], [2401], [2705]; rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity [648]; rooted trees with diameter D where every vertex has even degree except for one root and the leaves in level $\lfloor D/2 \rfloor$ [411]; rooted trees with diameter D where every vertex has even degree except for one root and the leaves, which are in level $\lfloor D/2 \rfloor$ [411]; rooted trees with diameter D where every vertex has even degree except for one root, the vertices in level $\lfloor D/2 \rfloor - 1$, and the leaves which are in level $\lfloor D/2 \rfloor$ [411]; the graph obtained by identifying the endpoints any number of paths of a fixed length except for the case that the length has the form $4r + 1$, $r > 1$ and the number of paths is of the form $4m$ with $m > r$ [2736]; regular bamboo trees [2736] (a rooted tree consisting of branches of equal length the endpoints of which are identified with end points of stars of equal size); and olive trees [2355], [13] (a rooted tree consisting of k branches, where the ith branch is a path of length i); Bahls, Lake, and Wertheim [393] proved that spiders for which the lengths of every path from the center to a leaf differ by at most one are graceful. (A *spider* is a tree that has at most one vertex (called the *center*) of degree greater than 2.) Jampachon, Nakprasit, and Poomsa-ard [1353] provide graceful labelings for some classes of spiders. Panpa and Poomsa-ard [2335] showed that all spider graphs with at most four legs of lengths greater than one admit graceful labeling. In [2163], [2164], [2328], [2165], and [2160] Panda and Mishra and Panda, Mishra, and Dash give graceful labelings for some new classes of trees with diameter six. Pradhan and Kumar [2482] proved that all combs $P_n \odot K_1$ with perfect matching are graceful. In [3339] Varadhan and Guruswamy give a method for combining caterpillars in a specific way such that the resulting tree is graceful. Venkatesh1 and Balasubramanian [3374] also create graceful trees by recursively merging caterpillars. In 2006 Wilf and Yoshimura [3476] defined an ordering on the set of all rooted trees of a fixed number of vertices that leads to fast ranking and unranking algorithms. As an application to the graceful tree conjecture, they showed how their method can eliminate repeated isomorphism testing. They investigated graphs with at most 10 vertices. In 2022 Brankovic and Reynolds [609] published a survey of various computer search algorithms for finding graceful labeling of trees.

In 2018 Montgomery, Pokrovskiy, and Sudakov [2187] proved that every tree is almost-harmonious. That is, every n-vertex tree has an injective Γ-harmonious labeling for any Abelian group Γ of order $n + o(n)$. In 2022 Gnang [1105] posted a paper with a proof on arXiv of the Graceful Tree Conjecture. See [1106] for a newer proof by Gnang. In 2022 Gnang and Williams [1107] posted a proof on arXiv of the long standing Graham-Sloane [1106] new conjecture that every tree admits a harmonious labeling.

Motivated by Horton's work [1257], in 2010 Fang [914] used a deterministic backtracking algorithm to prove that all trees with at most 35 vertices are graceful. In 2011 Fang [915] used a hybrid algorithm that involved probabilistic backtracking, tabu searching, and constraint programming satisfaction to verify that every tree with at most 31 vertices is harmonious. In [2067] Mahmoudzadeh and Eshghi treat graceful labelings of graphs as an optimization problem and apply an algorithm based on ant colony optimization metaheuristic to different classes of graphs and compare the results with those produced by other methods. In [3144] Suparta and Agus Ariawan provide two methods for expanding graceful trees from certain graceful trees.

Aldred, Širáň and Širáň [148] have proved that the number of graceful labelings of P_n grows at least as fast as $(5/3)^n$. They mention that this fact has an application to topological graph theory. One such application was provided by Goddyn, Richter, and and Širáň [1109] who used graceful labelings of paths on $2s+1$ vertices ($s \geqslant 2$) to obtain 2^{2s} cyclic oriented triangular embeddings of the complete graph on $12s+7$ vertices. The Aldred, Širáň and Širáň bound was improved by Adamaszek [55] to $(2.37)^n$ with the aid of a computer. Cattell [662] has shown that when finding a graceful labeling of a path one has almost complete freedom to choose a particular label i for any given vertex v. In particular, he shows that the only cases of P_n when this cannot be done are when $n \equiv 3 \pmod 4$ or $n \equiv 1 \pmod{12}$, v is in the smaller of the two partite sets of vertices, and $i = (n-1)/2$. In [3413] Wang enumerated the nonequivalent graceful trees and obtained a closed formula for the number.

Using an algorithm to run through all $n!$ graceful graphs on $n+1$ vertices Anick [213] proves that the average number of graceful labelings grows superexponentially. He provides a simple criterion to predict which trees have an exceptionally large number of graceful labelings and gives evidence that trees with an exceptionally small number of graceful labelings fall into two already known families of caterpillar graphs. Over the full set of graceful labelings for a given n, Anick shows that the distribution of vertex degrees associated with each label is very close to Poisson, with the exception of labels 0 and n. A graph is said to be *k-ubiquitously graceful* (also called "k-rotatable") if for every vertex there is a graceful labeling which assigns that vertex the label k. He also gives two new families of trees that are not k-ubiquitously graceful and includes questions suggested by his results. Pegg [2378] proved that a graceful graph with edges 0 to m can always be constructed with the nearest integer to $\sqrt{3m+9/4}+E$ vertices, where the excess E is a 0 or 1 value. For $m < 51$, $E = 0$. https://oeis.org/A326499

In [903] and [904] Eshghi and Azimi discuss a programming model for finding graceful labelings of large graphs. The computational results show that the models can easily solve the graceful labeling problems for large graphs. They used this method to verify that all trees with 30, 35, or 40 vertices are graceful. Stanton and Zarnke [3051] and Koh, Rogers, and Tan [1718], [1719], [1722] gave methods for combining graceful trees to yield larger graceful trees. In [3447] Wang, Yang, Hsu, and Cheng generalized the constructions of Stanton and Zarnke and Koh, Rogers, and Tan for building graceful trees from two smaller given graceful trees. Rogers in [2642] and Koh, Tan, and Rogers in [1721] provide recursive constructions to create graceful trees. Burzio and Ferrarese [633] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. and trees obtained from a graceful tree by replacing each edge with a path of fixed length is graceful.

The binomial tree B_0 consists of a single vertex. The binomial tree B_k consists of two binomial trees B_{k-1} that are linked together: the root of one is the leftmost child of the root of the other. Ragukumara and Sethuraman [2539] proved that all binomial trees are graceful. Sethuraman and Murugan [2824] introduced a new method of combining graceful trees called the recursive attachment method and showed that the recursively attached tree $T_i = T_{i-1} \oplus T^{A_{i-1}}$ is graceful for $i \geqslant 1$, where the base tree T_0 is a caterpillar and the attachment tree $T^{A_{i-1}}$ is any caterpillar. Here $T_{i-1} \oplus T^{A_{i-1}}$ represents a tree obtained by attaching a copy of $T^{A_{i-1}}$ at each vertex of degree at least two in T_{i-1}, for $i \geqslant 1$. Sethuraman and Murugan [2826] proved that any acyclic graph can be embedded in an unicyclic graceful graph.

It 1999 Broersma and Hoede [615] proved that an equivalent conjecture for the graceful tree conjecture is that all trees containing a perfect matching are strongly graceful (graceful with an extra condition also called an α-labeling–see Section 3.1). Wang, Yang, Hsu, and Cheng [3447] showed that there exist infinitely many equivalent versions of the graceful tree conjecture (GTC). They verify these equivalent conjectures of the graceful tree conjecture are true for trees of diameter at most 7.

In 1979 Bermond [544] conjectured that lobsters are graceful (a *lobster* is a tree with the property that the removal of the endpoints leaves a caterpillar). Morgan [2189] has shown that all lobsters with perfect matchings are graceful. Krop [1761] proved that a lobster that has a perfect matching that covers all but one vertex (i.e., that has an almost perfect matching) is graceful. Ghosh [1096] used adjacency matrices to prove that three classes of lobsters are graceful. Broersma and Hoede [615] proved that if T is a tree with a perfect matching M of T such that the tree obtained from T by contracting the edges in M is caterpillar, then T is graceful. Superdock [3147] used this result to prove that all lobsters with a perfect matching are graceful. Mishra, Panda, and Dash [2160] gave a [2160] new
class of graceful lobsters with an even number of branches incident on the central path. They also provided graceful labelings for a family of lobsters in which one end vertex of the central path is attached to an even number of branches and the remaining vertices are attached to the combinations of branches. Mishra and Panda [2158] and [2162], [2158] new
Mishra and Bhattacharjee [2157], and Mishra, Rout, and Nayak [2161] gave graceful [2157] new
labeling for a general classes of lobsters by applying component moving transformations on graceful caterpillars. More result results on graceful labeling of lobsters are in [2159]. [2161] new
Sathiamoorthy, Natarajan, Ayyaswamy, and Janakiraman [2722] proved that the splitting [2159] new
graph of a caterpillar is graceful.

A *Skolem sequence* of order n is a sequence $s_1, s_2, \ldots, s_{2n}$ of $2n$ terms such that, for each $k \in \{1, 2, \ldots, n\}$, there exist exactly two subscripts $i(k)$ and $j(k)$ with $s_{i(k)} = s_{j(k)} = k$ and $|i(k) - j(k)| = k$. A Skolem sequence of order n exists if and only if $n \equiv 0$ or 1 (mod 4). Morgan [2190] has used Skolem sequences to construct classes of graceful trees. Morgan and Rees [2191] used Skolem and Hooked-Skolem sequences to generate classes of graceful lobsters.

Mishra and Panigrahi [2166] and [2333] found classes of graceful lobsters of diameter at least five. They show other classes of lobsters are graceful in [2167] and [2168]. In [2815] Sethuraman and Jesintha [2815] explores how one can generate graceful lobsters from a graceful caterpillar while in [2819] and [2820] (see also [1379]) they show how to generate graceful trees from a graceful star. More special cases of Bermond's conjecture have been done by Ng [2283], by Wang, Jin, Lu, and Zhang [3414], Abhyanker [12], and by Mishra and Panigrahi [2167]. Renuka, Balaganesan, Selvaraju [2614] proved spider trees with n legs of even length t and odd $n \geqslant 3$ and lobsters for which each vertex of the spine is adjacent to a path of length two are harmonious.

A tree in which all internal vertices have degrees $r+1$ except one, is called an *full r-ary* tree. A uniform full r-ary tree is a full r-ary tree in which all of its leaves are at the same level. A tree that is obtained from copies of a full r-ary tree by identifying each vertex of a fixed path with each vertex of the tree of degree r is called a *uniform-distant* tree. Suparta and Ariawan [3145] gave methods for constructing graceful classes of caterpillars, lobsters, and uniform trees that generalize results in [2205] and [3236].

Barrientos [441] defines a *y-tree* as a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point. He proves that graphs obtained from a y-tree T by replacing every edge e_i of T by a copy of K_{2,n_i} in such a way that the ends of e_i are merged with the two independent vertices of K_{2,n_i} after removing the edge e_i from T are graceful.

Sethuraman and Jesintha [2816], [2817], and [2818] (see also [1379]) proved that rooted trees obtained by identifying one of the end vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. They also proved that rooted trees obtained by identifying either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. In [2816], [2817], and [2818] (see also [1379] and [1410]) Sethuraman and Jesintha prove that all rooted trees in which every level contains pendent vertices and the degrees of the internal vertices in the same level are equal are graceful. Kanetkar and Sane [1628] show that trees formed by identifying one end vertex of each of six or fewer paths whose lengths determine an arithmetic progression are graceful.

Chen, Lü, and Yeh [692] define a *firecracker* as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a *banana tree* as a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Before Sethuraman and Jesintha [2822] and [2821] (see also [1379]) proved that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful, various kinds of bananas trees had been shown to be graceful by Bhat-Nayak and Deshmukh [557], by Murugan and Arumugam [2223], [2221] and by Vilfred [3386].

Consider a set of caterpillars, having equal diameter, in which one of the penultimate vertices has arbitrary degree and all the other internal vertices including the other penultimate vertex is of fixed even degree. Jesintha and Sethuraman [1412] call the rooted tree obtained by merging an end-vertex adjacent to the penultimate vertex of fixed even degree of each caterpillar a *arbitrarily fixed generalized banana tree*. They prove that such trees are graceful. From this it follows that all banana trees are graceful and all generalized banana trees are graceful.

Jeba Jesintha and Subashini proved the following graphs are graceful: the cycle of vertex switching of even cycles [1394]; the path union of vertex switching of odd cycles [1397]; the path union of vertex switching of even cycles in increasing order [1398]; the path union of vertex switching of odd cycles [1396] the path union of vertex switching of even cycles [1401]; the path union of $P_m\theta S_n$ [1403]; and the cycle of $P_m\theta S_n$ [1403]. In [1400] they prove the cycle of caterpillar trees is graceful and as a corollary the cycle of comb graphs, cycle of paths, and the cycle of coconut trees are graceful. In [1402] Jeba Jasintha and Subashini proved the two quadrilateral snake graphs connected by a path, two alternate quadrilateral snake graphs connected by a path, two double quadrilateral snake graphs connected by a path, and two alternate double quadrilateral snake graphs connected by a path, In [1409] Jeba Jesintha, Subashini, and Rashmi Beula proved that the series of isomorphic copies of a star graph connected between two ladder graphs is graceful. In [1406] Jeba Jesintha, Subashini, and Sabu proved that two complete bipartite graphs connected by an arbitrary path of length n is graceful. In [1407] Jeba Jesintha and Subashini, and Sabu proved that the twig diamond graph with pendant edges is graceful. In [1408] Jeba Jesintha, Subashini, and Siddiqa proved that the path union of vertex switching of odd and even cycle graphs alternately is graceful.

Zhenbin [3593] has shown that graphs obtained by starting with any number of identical stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful. He also shows that graphs obtained by starting with any two stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful. In [1411] Jesintha and Sethuraman use a method of Hrnciar and Havier [1259] to generate graceful trees from a graceful star with n edges.

Aldred and McKay [147] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious was by Graham and Sloane [1147]. Ramya and Meenakshi [2590] gave graceful labelings, harmonious labelings, and Zumkeller labelings for ladders, banana trees, and firecrackers. [2590] new

Vietri [3379] utilized a counting technique that generalizes Rosa's graceful parity condition and provides constraints on possible graceful labelings of certain classes of trees. He expresses doubts about the validity of the graceful tree conjecture. In [3362] Vietri introduced a family of homogeneous polynomials (mod 2), one for every degree, having as many variables as the number of vertices, for any fixed graph; a so-called "graceful polynomial" that vanishes (mod 2) that may be useful for proving that the related graph is non-graceful (the degree 1 case dates back to Rosa's work). He also classified graphs whose graceful polynomials vanish for degrees 2 to 4, thereby obtaining some new non-graceful graphs.

Using a variant of the Matrix Tree Theorem, Whitty [3468] specifies an $n \times n$ matrix of indeterminates whose determinant is a multivariate polynomial that enumerates the gracefully labeled $(n+1)$-vertex trees. Whitty also gives a bijection between gracefully labelled graphs and rook placements on a chessboard on the Möbius strip. In [631] Buratti, Rinaldi, and Traetta use graceful labelings of paths to obtain a result on Hamiltonian cycle systems.

In [611] Brankovic and Wanless describe applications of graceful and graceful-like labelings of trees to several well known combinatorial problems including complete graph decompositions, the Oberwolfach problem, which asks for a decomposition of K_v into copies of a given 2-regular graph F, and coloring. They also discuss the connection between α-labeling of paths and near transversals in Latin squares and show how spectral graph theory might be used to further the progress on the graceful tree conjecture.

In [632] Burgess, Danziger, and Traetta show that Oberwolfach problem has a solution whenever F has a sufficiently large cycle which meets a given lower bound and, in addition, has a single-flip automorphism, which is an involutory automorphism acting as a reflection on exactly one of the cycles of F. Furthermore, they prove analogous results for the minimum covering version and the maximum packing version of the problem. They also show a similar result when the edges of K_v have multiplicity 2, but in this case they do not require that F be single-flip. Their approach allows them to explicitly construct solutions to the Oberwolfach Problem with well-behaved automorphisms. Their constructions use graceful labelings of 2-regular graphs with a vertex removed. They show that this class of graphs is graceful as long as the length of the path-component is sufficiently large. A much better lower bound on the length of the path is given for an α-labeling of such graphs to exist.

Arkut, Arkut, and Basak [221] and Basak [288] proposed an efficient method for managing Internet Protocol (IP) networks by using graceful labelings of the nodes of the spanning caterpillars of the autonomous sub-networks to assign labels to the links in the sub-networks. Graceful labelings of trees also have been used in multi protocol label switching (MPLS) routing platforms in IP networks [222], [2992], and [3207].

Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. More specialized results about trees are contained in [544], [573], [1717], [2040], [642], [1545], and [2649]. In [865] Edwards and Howard provide a lengthy survey paper on graceful trees. Robeva [2637] provides an extensive survey of graceful labelings of trees in her 2011 undergraduate honors thesis at Stanford University. Alfalayleh, Brankovic, Giggins, and Islam [149] survey results related to the graceful tree conjecture as of 2004 and conclude with five open problems. Alfalayleh et al.: say "The faith in the [graceful tree] conjecture is so strong that if a tree without a graceful labeling were indeed found, then it probably would not be considered a tree." In his Princeton University senior thesis Superdock [3147] provided an extensive survey of results and techniques about graceful trees. He also obtained some specialized results about the gracefulness of spiders and trees with diameter 6. Arumugam and Bagga [240] discuss computational efforts aimed at verifying the graceful tree conjecture and we survey recent results on generating all graceful labelings of certain families of unicyclic graphs. Sethuraman and

Murugan [2825] construct a graceful unicyclic graph G from every graceful tree T with $V(G) = V(T)$ such that the graceful labeling of G is derived from the graceful labeling of T.

2.2 Cycle-Related Graphs

Cycle-related graphs have been a major focus of attention. Rosa [2648] showed that the n-cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4) and Graham and Sloane [1147] proved that C_n is harmonious if and only if n is odd. Wheels $W_n = C_n + K_1$ are both graceful and harmonious – [984], [1255], and [1147]. As a consequence we have that a subgraph of a graceful (harmonious) graph need not be graceful (harmonious). The *n-cone* (also called the *n-point suspension*; the 1-cone is the wheel; the 2-cone is also called a *double cone* of C_m) $C_m + \overline{K_n}$ has been shown to be graceful when $m \equiv 0$ or 3 (mod 12) by Bhat-Nayak and Selvam [563]. When n is even and m is 2, 6 or 10 (mod 12) $C_m + \overline{K_n}$ violates the parity condition for a graceful graph. Bhat-Nayak and Selvam [563] also prove that the following cones are graceful: $C_4 + \overline{K_n}$, $C_5 + \overline{K_2}$, $C_7 + \overline{K_n}$, $C_9 + \overline{K_2}$, $C_{11} + \overline{K_n}$ and $C_{19} + \overline{K_n}$. The *helm* H_n is the graph obtained from a wheel by attaching a pendent edge at each vertex of the n-cycle. Helms have been shown to be graceful [266] and harmonious [1104], [1976], [1977] (see also [1964], [2802], [1962], [793], and [2555]). Koh, Rogers, Teo, and Yap, [1720] define a *web* graph as one obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. They asked whether such graphs are graceful. This was proved by Kang, Liang, Gao, and Yang [1633]. Yang has extended the notion of a web by iterating the process of adding pendent points and joining them to form a cycle and then adding pendent points to the new cycle. In his notation, $W(2, n)$ is the web graph whereas $W(t, n)$ is the *generalized web* with t n-cycles. Yang has shown that $W(3, n)$ and $W(4, n)$ are graceful (see [1633]), Abhyanker and Bhat-Nayak [14] have done $W(5, n)$ and Abhyanker [12] has done $W(t, 5)$ for $5 \leqslant t \leqslant 13$. Gnanajothi [1104] has shown that webs with odd cycles are harmonious. Seoud and Youssef [2802] define a *closed helm* as the graph obtained from a helm by joining each pendent vertex to form a cycle and a *flower* as the graph obtained from a helm by joining each pendent vertex to the central vertex of the helm. They prove that closed helms and flowers are harmonious when the cycles are odd. A *gear graph* is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the n-cycle. In 1984 Ma and Feng [2043] proved all gears are graceful while in a Master's thesis in 2006 Chen [693] proved all gears are harmonious. Liu [1976] has shown that if two or more vertices are inserted between every pair of vertices of the n-cycle of the wheel W_n, the resulting graph is graceful. Sethuraman and Sankar [2832] showed that the subdivisions of wheels are graceful for even values of $n \geqslant 4$. Liu [1974] has also proved that the graph obtained from a gear graph by attaching one or more pendent edges to each vertex between the vertices of the n-cycle is graceful. Pradhan and Kumar [2482] proved that graphs obtained by adding a pendent edge to each pendent vertex of hairy cycle $C_n \odot K_1$ are graceful if $n \equiv 0 \pmod{4m}$. They further provide a rule for determining the missing numbers in the graceful labeling of $C_n \odot K_1$ and of the graph obtained by adding pendent edges to each pendent vertex of $C_n \odot K_1$.

Kumar, Mishra, Kumar, and Kumar [1774] proved the following: $C_n \odot K_1$, $n \equiv 0 \pmod 4$ possesses an alpha labeling with the missing number $3n/2$; the one-point union of C_{4n} and a path possesses an alpha labeling with an identifiable missing number; and the graphs obtained by joining two isomorphic copies of the one-ponit union of C_{4n} and a path posses an alpha labeling with identifiable missing numbers.

Abhyanker [12] has investigated various unicyclic (that is, graphs with exactly one cycle) graphs. He proved that the unicyclic graphs obtained by identifying one vertex of C_4 with the root of the olive tree with $2n$ branches and identifying an adjacent vertex on C_4 with the end point of the path P_{2n-2} are graceful. He showed that if one attaches any number of pendent edges to these unicyclic graphs at the vertex of C_4 that is adjacent to the root of the olive tree but not adjacent to the end vertex of the attached path, the resulting graphs are graceful. Likewise, Abhyanker proved that the graph obtained by deleting the branch of length 1 from an olive tree with $2n$ branches and identifying the root of the edge deleted tree with a vertex of a cycle of the form C_{2n+3} is graceful. He also has a number of results similar to these. In [391] Bagga, Fotso, Max, and Arumugam investigate the gracefulness of unicyclic graphs with pendent caterpillars at two adjacent vertices of the cycle, and pendent edges at some other vertices of the cycle. In [392] Bagga and Heinz give some properties of graceful graphs obtained by adding pendent edges at each vertex of a cycle.

Delorme, Maheo, Thuillier, Koh, and Teo [798] and Ma and Feng [2042] showed that any cycle with a chord is graceful. This was first conjectured by Bodendiek, Schumacher, and Wegner [583], who proved various special cases. In 1985 Koh and Yap [1723] generalized this by defining a *cycle with a P_k-chord* to be a cycle with the path P_k joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when $k = 3$ and conjectured that all cycles with a P_k-chord are graceful. This was proved for $k \geqslant 4$ by Punnim and Pabhapote in 1987 [2531]. Chen [698] obtained the same result except for three cases which were then handled by Gao [1165]. In 2005, Sethuraman and Elumalai [2811] defined a *cycle with parallel P_k-chords* as a graph obtained from a cycle C_n ($n \geqslant 6$) with consecutive vertices $v_0, v_1, \ldots, v_{n-1}$ by adding disjoint paths P_k, ($k \geqslant 3$), between each pair of nonadjacent vertices $v_1, v_{n-1}, v_2, v_{n-2}, \ldots, v_i, v_{n-i}, \ldots, v_\alpha, v_\beta$ where $\alpha = \lfloor n/2 \rfloor - 1$ and $\beta = \lfloor n/2 \rfloor + 2$ if n is odd or $\beta = \lfloor n/2 \rfloor + 1$ if n is even. They proved that every cycle C_n ($n \geqslant 6$) with parallel P_k-chords is graceful for $k = 3, 4, 6, 8,$ and 10 and they conjecture that the cycle C_n with parallel P_k-chords is graceful for all even k. Xu [3500] proved that all cycles with a chord are harmonious except for C_6 in the case where the distance in C_6 between the endpoints of the chord is 2. The gracefulness of cycles with consecutive chords has also been investigated. For $3 \leqslant p \leqslant n - r$, let $C_n(p, r)$ denote the n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the r chords $v_1v_p, v_1v_{p+1}, \ldots, v_1v_{p+r-1}$ have been added. Koh and Punnin [1712] and Koh, Rogers, Teo, and Yap [1720] have handled the cases $r = 2, 3$ and $n - 3$ where n is the length of the cycle. Goh and Lim [1114] then proved that all remaining cases are graceful. Moreover, Ma [2045] has shown that $C_n(p, n-p)$ is graceful when $p \equiv 0, 3 \pmod 4$ and Ma, Liu, and Liu [2046] have proved other special cases of these graphs are graceful. Ma also proved that if one adds to the graph $C_n(3, n - 3)$ any number k_i of paths of length 2 from the vertex v_1 to the vertex v_i for $i = 2, \ldots, n$, the resulting graph is graceful. Chen [698] has shown that apart from four exceptional cases, a graph consisting of three independent paths joining two vertices of a cycle is graceful. This generalizes the result that a cycle plus a chord is graceful. Liu [1973] has shown that the n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the chords v_1v_k and v_1v_{k+2} $(2 \leqslant k \leqslant n-3)$ are adjoined is graceful.

For the cycle $C_n : v_1v_2v_3 \cdots v_nv_1$ and a cycle with a C_k- chord Venkatesh and Sivagurunathan [3376] let $C_{n,k}$ denote the graph obtained from C_n by adding a cycle C_k of length k between the non-adjacent vertices v_2 and v_n. They define a cycle with a parallel C_k chord as the graph obtained from a cycle C_n by adding a cycle C_k of length k between every pair of non-adjacent vertices $(v_2, v_n), (v_3, v_{n-1}), \ldots, (v_a, v_b)$ where $a = \lfloor \frac{n}{2} \rfloor$, $b = \lfloor \frac{n}{2} \rfloor + 2$, if n is even and $a = \lfloor \frac{n}{2} \rfloor$, $b = \lfloor \frac{n}{2} \rfloor + 3$, if n is odd. They proved that $C_{n,4}$ and $C^+_{n,4}$ are graceful for $n \equiv 0 \pmod 4$ and that $C^+_{n,6}$ is graceful for all odd values of $n \geqslant 5$.

In [794] Deb and Limaye use the notation $C(n, k)$ to denote the cycle C_n with k cords sharing a common endpoint called the *apex*. For certain choices of n and k there is a unique $C(n, k)$ graph and for other choices there is more than one graph possible. They call these *shell-type* graphs and they call the unique graph $C(n, n-3)$ a *shell*. Notice that the shell $C(n, n-3)$ is the same as the fan $F_{n-1} = P_{n-1} + K_1$. Kuppusamy and Guruswamy [1781] show that the subdivision graph of $K_{2,n}$ is graceful for $n \geqslant 1$ and the subdivision graph of the shell graph $C(n, n-3)$ is graceful for $n \geqslant 4$. Deb and Limaye define a *multiple shell* to be a collection of edge disjoint shells that have their apex in common. A multiple shell is said to be *balanced* with width w if every shell has order w or every shell has order w or $w+1$. Deb and Limaye [794] have conjectured that all multiple shells are harmonious, and have shown that the conjecture is true for the balanced double shells and balanced triple shells. Yang, Xu, Xi, and Qiao [3527] proved the conjecture is true for balanced quadruple shells. Liang [1938] proved the conjecture is true when each shell has the same order and the number of copies is odd.

Jeba Jesintha and Hilda [1383] define a *shell-butterfly* graph as a one-point union of two shells of any order with two pendent edges at the apex. They prove that certain shell-butterfly graphs are harmonious. Jeba Jesintha and Ezhilarasi Hilda [1381] proved butterfly graphs with one shell of order m and the other shell of order $2m+1$ are graceful and double shells in which each shell has the same order are graceful. Jeba Jesintha and Hilda [1387] define a *bow graph* as a double shell in which each shell has arbitrary order. A bow graph in which each shell has the same order is called a *uniform bow* graph. They prove that all uniform bow graphs are graceful. Jeba Jesintha and Ezhilarasi Hilda [1389] proved that shell-butterfly graphs are graceful. In [1382] Jeba Jesintha and Hilda prove [1382] new
k copies of $C(4, 1) \cup K_2$, and shellflowers (a double shell with shells of order m and $2m$) are graceful.

In [1196] Haviar and Kurtulík defined a *k-enriched fan graph* kF_n, for integers $k, n \geqslant 2$, as the graph of size $(k+1)n-1$ obtained by connecting n copies of the star S_k of order k to the fan F_n such that one vertex of each copy of the star S_k is identified with one vertex of the main path P_n of F_n. They proved that k-enriched fan graphs are graceful and and provided characterizations of the k-enriched fan graphs among all simple graphs via Sheppard's labeling sequences [2882] introduced in the 1970s, as well as via labeling relations and graph chessboards.

Sethuraman and Dhavamani [2808] use $H(n,t)$ to denote the graph obtained from the cycle C_n by adding t consecutive chords incident with a common vertex. If the common vertex is u and v is adjacent to u, then for $k \geqslant 1,\ n \geqslant 4$, and $1 \leqslant t \leqslant n-3$, Sethuraman and Dhavamani denote by $G(n,t,k)$ the graph obtained by taking the union of k copies of $H(n,k)$ with the edge uv identified. They conjecture that every graph $G(n,t,k)$ is graceful. They prove the conjecture for the case that $t = n-3$.

For $i = 1, 2, \ldots, n$ let $v_{i,1}, v_{i,2}, \ldots, v_{i,2m}$ be the successive vertices of n copies of C_{2m}. Sekar [2736] defines a *chain of cycles* $C_{2m,n}$ as the graph obtained by identifying $v_{i,m}$ and $v_{i+1,m}$ for $i = 1, 2, \ldots, n-1$. He proves that $C_{6,2k}$ and $C_{8,n}$ are graceful for all k and all n. Barrientos [444] proved that all $C_{8,n}$, $C_{12,n}$, and $C_{6,2k}$ are graceful.

Truszczyński [3229] studied unicyclic graphs and proved several classes of such graphs are graceful. Among these are what he calls dragons. A *dragon* is formed by joining the end point of a path to a cycle (Koh, et al. [1720] call these *tadpoles*; Kim and Park [1699] call them *kites*). This work led Truszczyński to conjecture that all unicyclic graphs except C_n, where $n \equiv 1$ or 2 (mod 4), are graceful. Guo [1164] has shown that dragons are graceful when the length of the cycle is congruent to 1 or 2 (mod 4). Lu [2039] uses $C_n^{+(m,t)}$ to denote the graph obtained by identifying one vertex of C_n with one endpoint of m paths each of length t. He proves that $C_n^{+(1,t)}$ (a tadpole) is not harmonious when $a+t$ is odd and $C_n^{+(2m,t)}$ is harmonious when $n = 3$ and when $n = 2k+1$ and $t = k-1, k+1$ or $2k-1$. In his Master's thesis, Doma [843] investigates the gracefulness of various unicyclic graphs where the cycle has up to 9 vertices. Guruswamy and Varadhan [1166] proved that any acyclic graph can be embedded in a unicyclic graceful graph. Because of the immense diversity of unicyclic graphs, a proof of Truszczyński's conjecture seems out of reach in the near future. In [569] Biatch, Baggab, and Arumugam gave a survey of results related to Truszczynski's conjecture on the gracefulness ofunicyclic graphs and provided a new class of graceful unicyclic graphs.

Cycles that share a common edge or a vertex have received some attention. Murugan and Arumugan [2222] have shown that books with n pentagonal pages (i.e., n copies of C_5 with an edge in common) are graceful when n is even and not graceful when n is odd. Lu [2039] uses $\Theta(C_m)^n$ to denote the graph made from n copies of C_m that share an edge (an n page book with m-polygonal pages). He proves $\Theta(C_{2m+1})^{2n+1}$ is harmonious for all m and n; $\Theta(C_{4m+2})^{4n+1}$ and $\Theta(C_{4m})^{4n+3}$ are not harmonious for all m and n. Xu [3500] proved that $\Theta(C_m)^2$ is harmonious except when $m = 3$. ($\Theta(C_m)^2$ is isomorphic to $C_{2(m-1)}$ with a chord "in the middle.") Nurvazly and Sugeng [2315] proved that $\Theta(C_3)^n$ graphs (n copies of C_3 that share an edge) have graceful labelings.

A *kayak paddle* $KP(k,m,l)$ is the graph obtained by joining C_k and C_m by a path of length l. Litersky [1960] proves that kayak paddles have graceful labelings in the following cases: $k \equiv 0$ mod 4, $m \equiv 0$ or 3 (mod 4); $k \equiv m \equiv 2$ (mod 4) for $k \geqslant 3$; and $k \equiv 1$ (mod 4), $m \equiv 3$ (mod 4). She conjectures that $KP(4k+4, 4m+2, l)$ with $2k < m$ is graceful when $l \leqslant 2m$ if l is even and when $l \leqslant 2m+1$ if l is odd; and $KP(10,10,l)$ is graceful when $l \geqslant 12$. The cases are open: $KP(4k, 4m+1, l)$; $KP(4k, 4m+2, l)$; $KP(4k+1, 4m+1, l)$; $KP(4k+1, 4m+2, l)$; $KP(4k+2, 4m+3, l)$; $KP(4k+3, 4m+3, l)$.

Let $C_n^{(t)}$ denote the one-point union of t cycles of length n. Bermond, Brouwer, and Germa [545] and Bermond, Kotzig, and Turgeon [547]) proved that $C_3^{(t)}$ (that is, the *friendship graph* or *Dutch t-windmill*) is graceful if and only if $t \equiv 0$ or 1 (mod 4) while Graham and Sloane [1147] proved $C_3^{(t)}$ is harmonious if and only if $t \not\equiv 2$ (mod 4). Koh, Rogers, Lee, and Toh [1713] conjecture that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0$ or 3 (mod 4). Yang and Lin [3519] have proved the conjecture for the case $n = 5$ and Yang, Xu, Xi, Li, and Haque [3525] did the case $n = 7$. Xu, Yang, Li and Xi [3504] did the case $n = 11$. Xu, Yang, Han and Li [3505] did the case $n = 13$. Qian [2537] verifies this conjecture for the case that $t = 2$ and n is even and Yang, Xu, Xi, and Li [3526] did the case $n = 9$. Figueroa-Centeno, Ichishima, and Muntaner-Batle [935] have shown that if $m \equiv 0$ (mod 4) then the one-point union of 2, 3, or 4 copies of C_m admits a special kind of graceful labeling called an α-labeling (see Section 3.1) and if $m \equiv 2$ (mod 4), then the one-point union of 2 or 4 copies of C_m admits an α-labeling. Bodendiek, Schumacher, and Wegner [589] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. (The other cases violate the necessary parity condition.) Shee [2874] has proved that $C_4^{(t)}$ is graceful for all t. Seoud and Youssef [2800] have shown that the one-point union of a triangle and C_n is harmonious if and only if $n \equiv 1$ (mod 4) and that if the one-point union of two cycles is harmonious then the number of edges is divisible by 4. The question of whether this latter condition is sufficient is open. Figueroa-Centeno, Ichishima, and Muntaner-Batle [935] have shown that if G is harmonious then the one-point union of an odd number of copies of G using the vertex labeled 0 as the shared point is harmonious. Sethuraman and Selvaraju [2838] have shown that for a variety of choices of points, the one-point union of any number of non-isomorphic complete bipartite graphs is graceful. They raise the question of whether this is true for all choices of the common point.

Another class of cycle-related graphs is that of triangular cacti. The *block-cutpoint graph* of a graph G is a bipartite graph in which one partite set consists of the cut vertices of G, and the other has a vertex b_i for each block B_i of G. A *block* of a graph is a maximal connected subgraph that has no cut-vertex. A *triangular cactus* is a connected graph all of whose blocks are triangles. A *triangular snake* is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$). Rosa [2650] conjectured that all triangular cacti with $t \equiv 0$ or 1 (mod 4) blocks are graceful. (The cases where $t \equiv 2$ or 3 (mod 4) fail to be graceful because of the parity condition.) Moulton [2200] proved the conjecture for all triangular snakes. A proof of the general case (i.e., all triangular cacti) seems hopelessly difficult. Liu and Zhang [1964] gave an incorrect proof that triangular snakes with an odd number of triangles are harmonious whereas triangular snakes with $n \equiv 2$ (mod 4) triangles are not harmonious. Xu [3501] subsequently proved that triangular snakes are harmonious if and only if the number of triangles is not congruent to 2 (mod 4).

A *double triangular snake* consists of two triangular snakes that have a common path. That is, a double triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$ and to a new vertex u_i for $i = 1, 2, \ldots, n-1$.

Xi, Yang, and Wang [3496] proved that all double triangular snakes are harmonious.

A *hexagonal snake* is obtained from a path $p_1, p_2, p_3, \ldots, p_n$ by joining p_i, p_{i+1} to new vertices x_i and y_i respectively and adding edges $x_i y_i$ for $i = 1, 2, \ldots, n-1$ and replacing every edge with a 6-cycle; an *alternate hexagonal snake* is obtained from a path $p_1, p_2, p_3, \ldots, p_n$ by joining p_i, p_{i+1} to new vertices x_i and y_i (alternatively) and adding edges $x_i y_i$, where $1 \leqslant i \leqslant n-1$ for even n and $1 \leqslant i \leqslant n-2$ for odd n and replacing each alternate edge with a 6-cycle; a *double hexagonal snake* is obtained from two hexagonal snakes that share the n-path; a *double alternate hexagonal snake* is obtained from two alternative hexagonal snakes that share the n-path. Pattabiraman, Loganathan, and Rao [2373] provided graceful labelings for double hexagonal snakes, alternate hexagonal snakes, odd alternate hexagonal snakes, and double alternate hexagonal snakes.

For any graph G defining G-snake analogous to triangular snakes, Sekar [2736] has shown that C_n-snakes are graceful when $n \equiv 0 \pmod 4$ $(n \geqslant 8)$ and when $n \equiv 2 \pmod 4$ and the number of C_n is even. Gnanajothi [1104, pp. 31-34] had earlier shown that quadrilateral snakes are graceful. Grace [1145] has proved that K_4-snakes are harmonious. Rosa [2650] has also considered analogously defined quadrilateral and pentagonal cacti and examined small cases. Yu, Lee, and Chin [3562] showed that Q_2-snakes and Q_3-snakes are graceful and, when the number of blocks is greater than 1, Q_2-snakes, Q_3-snakes and Q_4-snakes are harmonious.

Barrientos [435] calls a graph a kC_n*-snake* if it is a connected graph with k blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to C_n. (When $n > 3$ and $k > 3$ there is more than one kC_n-snake.) If a kC_n-snake where the path of minimum length that contains all the cut-vertices of the graph has the property that the distance between any two consecutive cut-vertices is $\lfloor n/2 \rfloor$ it is called *linear*. Barrientos proves that kC_4-snakes are graceful and that the linear kC_6-snakes are graceful when k is even. He further proves that kC_8-snakes and kC_{12}-snakes are graceful in the cases where the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph are all even and that certain cases of kC_{4n}-snakes and kC_{5n}-snakes are graceful (depending on the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph).

Badr [270] defines a *linear cyclic snake* $(m,k)C_n$ as the graph consisting of k copies of C_n with two non-adjecent vertices in common where every copy has m copies of C_n and the block-cutpoint graph is not a path. He proves that the linear cyclic snakes $(m,k)C_4$-snake and $(m,k)C_8$-snake are graceful and conjectures that all the linear cyclic snakes $(m,k)C_n$-snakes are graceful for $n \equiv 0 \pmod{4}$ or $n \equiv 3 \pmod 4$.

Several people have studied cycles with pendent edges attached. Frucht [984] proved that any cycle with a pendent edge attached at each vertex (i.e., a *crown*) is graceful (see also [1266]). If G has order n, the *corona of* G *with* H, $G \odot H$ is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H. Barrientos [440] also proved: if G is a graceful graph of order m and size $m-1$, then $G \odot nK_1$ and $G + nK_1$ are graceful; if G is a graceful graph of order p and size q with $q > p$, then $(G \cup (q+1-p)K_1) \odot nK_1$ is graceful; and all unicyclic graphs, other than a cycle, for which the deletion of any edge from the cycle results in a caterpillar are graceful.

For a given cycle C_n with $n \equiv 0$ or 3 (mod 4) and a family of trees $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, let u_i and $v_i, 1 \leqslant i \leqslant n$, be fixed vertices of C_n and T_i, respectively. Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [940] provide two construction methods that generate a graceful labeling of the unicyclic graphs obtained from C_n and $\mathcal{T}$ by amalgamating them at each u_i and v_i. Their results encompass all previously known results for unicyclic graphs whose cycle length is 0 or 3 (mod 4) and considerably extend the known classes of graceful unicyclic graphs. Khairunnisa and Sugeng [1683] let $A_{(m,n)}$ denote the graph obtained from C_m by connecting each two adjacent vertices with P_{n+1}. They prove that the graphs $A_{(3,1)} \odot \overline{K_r}$ are graceful.

In [437] Barrientos proved that helms (graphs obtained from a wheel by attaching one pendent edge to each vertex) are graceful. Grace [1144] showed that an odd cycle with one or more pendent edges at each vertex is harmonious and conjectured that $C_{2n} \odot K_1$, an even cycle with one pendent edge attached at each vertex, is harmonious. This conjecture has been proved by Liu and Zhang [1963], Liu [1976] and [1977], Hegde [1212], Huang [1269], and Bu [618]. Sekar [2736] has shown that the graph $C_m \odot P_n$ obtained by attaching the path P_n to each vertex of C_m is graceful. For any $n \geqslant 3$ and any t with $1 \leqslant t \leqslant n$, let C_n^{+t} denote the class of graphs formed by adding a single pendent edge to t vertices of a cycle of length n. Ropp [2647] proved that for every n and t the class C_n^{+t} contains a graceful graph. Gallian and Ropp [1002] conjectured that for all n and t, all members of C_n^{+t} are graceful. This was proved by Qian [2537] and by Kang, Liang, Gao, and Yang [1633]. Of course, such graphs are just a special case of the aforementioned conjecture of Truszczyński that all unicyclic graphs except C_n for $n \equiv 1$ or 2 (mod 4) are graceful. Sekar [2736] proved that the graph obtained by identifying an endpoint of a star with a vertex of a cycle is graceful. Lu [2039] shows that the graph obtained by identifying each vertex of an odd cycle with a vertex disjoint copy of C_{2m+1} is harmonious if and only if m is odd. Sudha [3069] proved that the graphs obtained by starting with two or more copies of C_4 and identifying a vertex of the i^{th} copy with a vertex of the $i+1^{th}$ copy and the graphs obtained by starting with two or more cycles (not necessarily of the same size) and identifying an edge from the i^{th} copy with an edge of the $i+1^{th}$ copy are graceful. Sudha and Kanniga [3076] proved that the graphs obtained by identifying any vertex of C_m with any vertex of degree 1 of S_n where $n = \lceil (m-1)/2 \rceil$ are graceful.

For a given cycle C_n with $n \equiv 0$ or 3 (mod 4) and a family of trees $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, let u_i and $v_i, 1 \leqslant i \leqslant n$, be fixed vertices of C_n and T_i, respectively. Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [940] provide two construction methods that generate a graceful labeling of the unicyclic graphs obtained from C_n and $\mathcal{T}$ by amalgamating them at each u_i and v_i. Their results encompass all previously known results for unicyclic graphs whose cycle length is 0 or 3 (mod 4) and considerably extend the known classes of graceful unicyclic graphs.

Solairaju and Chithra [3017] defined three classes of graphs obtained by connecting copies of C_4 in various ways. Denote the four consecutive vertices of ith copy of C_4 by $v_{i,1}, v_{i,2}, v_{i,3}, v_{i_4}$. They show that the graphs obtained by identifying $v_{i,4}$ with $v_{i+1,2}$ for $i = 1, 2, \ldots, n-1$ is graceful; the graphs obtained by joining $v_{i,4}$ with $v_{i+1,2}$ for

$i = 1, 2, \ldots, n-1$ by an edge is graceful; and the graphs obtained by joining $v_{i,4}$ with $v_{i+1,2}$ for $i = 1, 2, \ldots, n-1$ with a path of length 2 is graceful.

Venkatesh [3370] showed that for positive integers m and n divisible by 4 the graphs obtained by appending a copy of C_n to each vertex of C_m by identifying one vertex of C_n with each vertex of C_m is graceful.

2.3 Product Related Graphs

Graphs that are cartesian products and related graphs have been the subject of many papers. That planar grids, $P_m \times P_n$ $(m, n \geqslant 2)$, (many authors use $G \square H$ to denote the Cartesian product of G and H) are graceful was proved by Acharya and Gill [41] in 1978. In 1980, Maheo [2053] clarified the complicated-appearing construction of Acharya and Gill for $P_m \times P_2$ that readily extends to all grids. Liu, T. Zou, Y. Lu [1971] proved $P_m \times P_n \times P_2$ is graceful. In 1980 Graham and Sloane [1147] proved ladders, $P_m \times P_2$, are harmonious when $m > 2$ and in 1992 Jungreis and Reid [1561] showed that the grids $P_m \times P_n$ are harmonious when $(m, n) \neq (2, 2)$. A few people have looked at graphs obtained from planar grids in various ways. Kathiresan [1662] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful and that graphs obtained by appending an edge to each vertex of a ladder are graceful [1664]. Barrientos and Minion [469] showed that a graceful graph is obtained when every step of a ladder is subdivided an even number of times. In addition, they proved that when each edge of a ladder is subdivided exactly once, the resulting graph is graceful.

Acharya [30] has shown that certain subgraphs of grid graphs are graceful. Lee [1836] defines a *Mongolian tent* as a graph obtained from $P_m \times P_n$, n odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex. A *Mongolian village* is a graph formed by successively amalgamating copies of Mongolian tents with the same number of rows so that adjacent tents share a column. Lee proves that Mongolian tents and villages are graceful. A *Young tableau* is a subgraph of $P_m \times P_n$ obtained by retaining the first two rows of $P_m \times P_n$ and deleting vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a nonincreasing sequence. Lee and Ng [1860] have proved that all Young tableaus are graceful. Lee [1836] has also defined a variation of Mongolian tents by adding an extra vertex above the top row of a Young tableau and joining every other vertex of that row to the extra vertex. He proves these graphs are graceful. In [3016] and [3015] Solairaju and Arockiasamy prove that various families of subgraphs of grids $P_m \times P_n$ are graceful. Sudha [3069] proved that certain subgraphs of the grid $P_n \times P_2$ are graceful. Knuth [1707] proved that $K_n \times P_3$ is graceful if and only if $n \leqslant 6$.

Prisms are graphs of the form $C_m \times P_n$. These can be viewed as grids on cylinders. In 1977 Bodendiek, Schumacher, and Wegner [583] proved that $C_m \times P_2$ is graceful when $m \equiv 0 \pmod 4$. According to the survey by Bermond [544], Gangopadhyay and Rao Hebbare did the case that m is even about the same time. In a 1979 paper, Frucht [984] stated without proof that he had done all $C_m \times P_2$. A complete proof of all cases and some related results were given by Frucht and Gallian [987] in 1988.

In 1992 Jungreis and Reid [1561] proved that all $C_m \times P_n$ are graceful when m and n are even or when $m \equiv 0 \pmod 4$. They also investigated the existence of a stronger form of graceful labeling called an α-labeling (see Section 3.1) for graphs of the form $P_m \times P_n$, $C_m \times P_n$, and $C_m \times C_n$ (see also [1004]).

Yang and Wang have shown that the prisms $C_{4n+2} \times P_{4m+3}$ [3524], $C_n \times P_2$ [3522], and $C_6 \times P_m$ ($m \geqslant 2$) (see [3524]) are graceful. Singh [2970] proved that $C_3 \times P_n$ is graceful for all n. In their 1980 paper Graham and Sloane [1147] proved that $C_m \times P_n$ is harmonious when n is odd and they used a computer to show $C_4 \times P_2$, the cube, is not harmonious. In 1992 Gallian, Prout, and Winters [1007] proved that $C_m \times P_2$ is harmonious when $m \neq 4$. In 1992, Jungreis and Reid [1561] showed that $C_4 \times P_n$ is harmonious when $n \geqslant 3$. Huang and Skiena [1271] have shown that $C_m \times P_n$ is graceful for all n when m is even and for all n with $3 \leqslant n \leqslant 12$ when m is odd. Abhyanker [12] proved that the graphs obtained from $C_{2m+1} \times P_5$ by adding a pendent edge to each vertex of an outer cycle is graceful.

Torus grids are graphs of the form $C_m \times C_n$ ($m > 2, n > 2$). Very little success has been achieved with these graphs. The graceful parity condition is violated for $C_m \times C_n$ when m and n are odd and the harmonious parity condition [1147, Theorem 11] is violated for $C_m \times C_n$ when $m \equiv 1, 2, 3 \pmod 4$ and n is odd. In 1992 Jungreis and Reid [1561] showed that $C_m \times C_n$ is graceful when $m \equiv 0 \pmod 4$ and n is even. A complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

There has been some work done on prism-related graphs. Gallian, Prout, and Winters [1007] proved that all prisms $C_m \times P_2$ with a single vertex deleted or single edge deleted are graceful and harmonious. The *Möbius ladder* M_n is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of P_n. In 1989 Gallian [1001] showed that all Möbius ladders are graceful and all but M_3 are harmonious. Ropp [2647] has examined two classes of prisms with pendent edges attached. He proved that all $C_m \times P_2$ with a single pendent edge at each vertex are graceful and all $C_m \times P_2$ with a single pendent edge at each vertex of one of the cycles are graceful. Ramachandran and Sekar [2572] proved that the graph obtained from the ladder L_n ($P_n \times P_2$) by identifying one vertex of L_n with any vertex of the star S_m other than the center of S_m is graceful.

Another class of cartesian products that has been studied is that of books and "stacked" books. The *book* B_m is the graph $S_m \times P_2$ where S_m is the star with m edges. In 1980 Maheo [2053] proved that the books of the form B_{2m} are graceful and conjectured that the books B_{4m+1} were also graceful. (The books B_{4m+3} do not satisfy the graceful parity condition.) This conjecture was verified by Delorme [797] in 1980. Maheo [2053] also proved that $L_n \times P_2$ and $B_{2m} \times P_2$ are graceful. Both Grace [1143] and Reid (see [1006]) have given harmonious labelings for B_{2m}. The books B_{4m+3} do not satisfy the harmonious parity condition [1147, Theorem 11]. Gallian and Jungreis [1006] conjectured that the books B_{4m+1} are harmonious. Gnanajothi [1104] has verified this conjecture by showing B_{4m+1} has an even stronger form of labeling – see Section 4.1. Liang [1934] also proved the conjecture. In 1988 Gallian and Jungreis [1006] defined a *stacked book* as a graph of the form $S_m \times P_n$. They proved that the stacked books of the form $S_{2m} \times P_n$ are graceful and posed the case $S_{2m+1} \times P_n$ as an open question. The *n-cube* $K_2 \times K_2 \times \cdots \times K_2$

(n copies) was shown to be graceful by Kotzig [1741]—see also [2053]. Although Graham and Sloane [1147] used a computer in 1980 to show that the 3-cube is not harmonious (see also [2334]), Ichishima and Oshima [1310] proved that the n-cube Q_n has a stronger form of harmonious labeling called an α-labeling (see Section 3.1) for $n \geqslant 4$.

In 1986 Reid [2611] found a harmonious labeling for $K_4 \times P_n$. In 2003 Petrie and Smith [2387] investigated graceful labelings of graphs as an exercise in constraint programming satisfaction. They determined that $K_n \times P_2$ is graceful for $n = 3$, 4 and 5; $K_4 \times P_3$ is graceful; $K_4 \times C_3$ is graceful; $(C_n \cup C_n) + K_1$ (double wheel) is graceful for $n = 4$ and 5; and $(C_3 \cup C_3) + K_1$ is not graceful. That $K_3 \times K_3$ is not graceful follows from the parity condition given in the introduction. Using significantly better methods in 2010, Smith and Puget obtained the results about graceful labelings for $K_m \times K_1$, $K_m \times P_n$, and $K_m \times C_n$ given in Table 1. Their labeling for $K_5 \times P_2$ and $K_6 \times P_3$ are the unique graceful labelings for those graphs. Redl [2610] proved that $K_4 \times P_n$ is graceful for $n = 1, 2, 3, 4,$ and 5 using a constraint programming approach and asked if all graphs of the form $K_4 \times P_n$ are graceful

Vaidya, Kaneria, Srivastav, and Dani [3281] proved that $P_n \cup P_t \cup (P_r \times P_s)$ where $t < \min\{r, s\}$ and $P_n \cup P_t \cup K_{r,s}$ where $t \leqslant \min\{r, s\}$ and $r, s \geqslant 3$ are graceful. Kaneria, Vaidya, Ghodasara, and Srivastav [1624] proved $K_{mn} \cup (P_r \times P_s)$ where $m, n, r, s > 1$; $(P_r \times P_s) \cup P_t$ where $r, s > 1$ and $t \neq 2$; and $K_{mn} \cup (P_r \times P_s) \cup P_t$ where $m, n, r, s > 1$ and $t \neq 2$ are graceful. Xie, Zhao, and Yao [3498] proved that graphs of the form $C_n \odot T$ [3498] new
where T is a graceful tree are graceful.

The *composition* $G_1[G_2]$ is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1, y_1), (x_2, y_2)|\ x_1x_2 \in E(G_1) \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(G_2)\}$. The *symmetric product* $G_1 \oplus G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1, y_1), (x_2, y_2)|\ x_1x_2 \in E(G_1) \text{ or } y_1y_2 \in E(G_2) \text{ but not both}\}$. Seoud and Youssef [2801] have proved that $P_n \oplus \overline{K_2}$ is graceful when $n > 1$ and $P_n[P_2]$ is harmonious for all n. They also observe that the graphs $C_m \oplus C_n$ and $C_m[C_n]$ violate the parity conditions for graceful and harmonious graphs when m and n are odd.

2.4 Complete Graphs

The questions of the gracefulness and harmoniousness of the complete graphs K_n have been answered. In each case the answer is positive if and only if $n \leqslant 4$ ([1115], [2964], [1147], [551]). Both Rosa [2648] and Golomb [1115] proved that the complete bipartite graphs $K_{m,n}$ are graceful while Graham and Sloane [1147] showed they are harmonious if and only if m or $n = 1$. Aravamudhan and Murugan [220] have shown that the complete tripartite graph $K_{1,m,n}$ is both graceful and harmonious while Gnanajothi [1104, pp. 25–31] has shown that $K_{1,1,m,n}$ is both graceful and harmonious and $K_{2,m,n}$ is graceful. Some of the same results have been obtained by Seoud and Youssef [2796] who also observed that when $m, n,$ and p are congruent to 2 (mod 4), $K_{m,n,p}$ violates the parity conditions for harmonious graphs. Beutner and Harborth [551] give graceful labelings for $K_{1,m,n}, K_{2,m,n}, K_{1,1,m,n}$ and conjecture that these and $K_{m,n}$ are the only complete multipartite graphs that are graceful. They have verified this conjecture for graphs with up to 23 vertices via computer.

Beutner and Harborth [551] also show that $K_n - e$ (K_n with an edge deleted) is graceful only if $n \leqslant 5$; any $K_n - 2e$ (K_n with two edges deleted) is graceful only if $n \leqslant 6$; and any $K_n - 3e$ is graceful only if $n \leqslant 6$. They also determine all graceful graphs of the form $K_n - G$ where G is $K_{1,a}$ with $a \leqslant n-2$ and where G is a matching M_a with $2a \leqslant n$.

The *windmill* graph $K_n^{(m)}$ ($n > 3$) consists of m copies of K_n with a vertex in common. A necessary condition for $K_n^{(m)}$ to be graceful is that $n \leqslant 5$ – see [1720]. Bermond [544] has conjectured that $K_4^{(m)}$ is graceful for all $m \geqslant 4$. The gracefulness of $K_4^{(m)}$ is equivalent to the existence of a $(12m+1, 4, 1)$-perfect difference family, which are known to exist for $m \leqslant 1000$ (see [1271], [7], [3454], and [1065]). Bermond, Kotzig, and Turgeon [547] proved that $K_n^{(m)}$ is not graceful when $n = 4$ and $m = 2$ or 3, and when $m = 2$ and $n = 5$. Stones [3055] proved that $K_5^{(3)}$ and $K_5^{(4)}$ are graceful. In 1982 Hsu [1263] proved that $K_4^{(m)}$ is harmonious for all m. Graham and Sloane [1147] conjectured that $K_n^{(2)}$ is harmonious if and only if $n = 4$. They verified this conjecture for the cases that n is odd or $n = 6$. Liu [1962] has shown that $K_n^{(2)}$ is not harmonious if $n = 2^a p_1^{a_1} \cdots p_s^{a_s}$ where $a, a_1, \ldots, a_s$ are positive integers and $p_1, \ldots, p_s$ are distinct odd primes and there is a j for which $p_j \equiv 3 \pmod 4$ and a_j is odd. He also shows that $K_n^{(3)}$ is not harmonious when $n \equiv 0 \pmod 4$ and $3n = 4^e(8k+7)$ or $n \equiv 5 \pmod 8$. Koh, Rogers, Lee, and Toh [1713] and Rajasingh and Pushpam [2556] have shown that ${K_{m,n}}^{(t)}$, the one-point union of t copies of $K_{m,n}$, is graceful. Sethuraman and Selvaraju [2834] have proved that the one-point union of graphs of the form K_{2,m_i} for $i = 1, 2, \ldots, n$, where the union is taken at a vertex from the partite set with exactly 2 vertices is graceful if at most two of the m_i are equal. They conjecture that the restriction that at most two of the m_i are equal is not necessary. Sudha [3070] proved that two or more complete bipartite graphs having one bipartite vertex set in common are graceful. Mitra and Bhoumik [2170] proved that $K_{2n,2n} \odot K_2$ is graceful.

Koh, Rogers, Lee, and Toh [1720] introduced the notation $B(n, r, m)$ for the graph consisting of m copies of K_n with a K_r in common ($n \geqslant r$). (We note that Guo [1165] has used the notation $B(n, r, m)$ to denote the graph obtained by joining opposite endpoints of three disjoint paths of lengths n, r and m.) Bermond [544] raised the question: "For which m, n, and r is $B(n, r, m)$ graceful?" Of course, the case $r = 1$ is the same as $K_n^{(m)}$. For $r > 1$, $B(n, r, m)$ is graceful in the following cases: $n = 3,\ r = 2,\ m \geqslant 1$ [1714]; $n = 4,\ r = 2,\ m \geqslant 1$ [797]; $n = 4,\ r = 3,\ m \geqslant 1$ (see [544]), [1714]. Seoud and Youssef [2796] have proved $B(3, 2, m)$ and $B(4, 3, m)$ are harmonious. Liu [1961] has shown that if there is a prime p such that $p \equiv 3 \pmod 4$ and p divides both n and $n-2$ and the highest power of p that divides n and $n-2$ is odd, then $B(n, 2, 2)$ is not graceful. Smith and Puget [3008] has shown that up to symmetry, $B(5, 2, 2)$ has a unique graceful labeling; $B(n, 3, 2)$ is not graceful for $n = 6, 7, 8, 9$, and 10; $B(6, 3, 3)$ and $B(7, 3, 3)$ are not graceful; and $B(5, 3, 3)$ is graceful. Combining results of Bermond and Farhi [546] and Smith and Puget [3008] show that $B(n, 2, 2)$ is not graceful for $n > 5$. Lu [2039] obtained the following results: $B(m, 2, 3)$ and $B(m, 3, 3)$ are not harmonious when $m \equiv 1 \pmod 8$; $B(m, 4, 2)$ and $B(m, 5, 2)$ are not harmonious when m satisfies certain special conditions; $B(m, 1, n)$ is not harmonious when $m \equiv 5 \pmod 8$ and $n \equiv 1, 2, 3 \pmod 4$;

$B(2m+1, 2m, 2n+1) \cong K_{2m} + \overline{K_{2n+1}}$ is not harmonious when $m \equiv 2 \pmod 4$.

More generally, Bermond and Farhi [546] have investigated the class of graphs consisting of m copies of K_n having exactly k copies of K_r in common. They proved such graphs are not graceful for n sufficiently large compared to r. Barrientos [441] proved that the graph obtained by performing the one-point union of any collection of the complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_t,n_t}$, where each K_{m_i,n_i} appears at most twice and $\gcd(n_1, n_2, \ldots, n_t) = 1$, is graceful.

Sethuraman and Elumalai [2810] have shown that $K_{1,m,n}$ with a pendent edge attached to each vertex is graceful and Jirimutu [1551] has shown that the graph obtained by attaching a pendent edge to every vertex of $K_{m,n}$ is graceful (see also [179]). In [2823] Sethuraman and Kishore determine the graceful graphs that are the union of n copies of K_4 with i edges deleted for $1 \leqslant i \leqslant 5$ and with one edge in common. The only cases that are not graceful are those graphs where the members of the union are C_4 for $n \equiv 3$ (mod 4) and where the members of the union are P_2. They conjecture that these two cases are the only instances of edge induced subgraphs of the union of n copies of K_4 with one edge in common that are not graceful.

Renuka, Balaganesan, Selvaraju [2614] proved the graphs obtained by joining a vertex of $K_{1,m}$ to a vertex of $K_{1,n}$ by a path are harmonious. Sethuraman and Selvaraju [2840] have shown that union of any number of copies of K_4 with an edge deleted and one edge in common is harmonious.

Clemens, Coulibaly, Garvens, Gonnering, Lucas, and Winters [749] investigated the gracefulness of the one-point and two-point unions of graphs. They show the following graphs are graceful: the one-point union of an end vertex of P_n and K_4; the graph obtained by taking the one-point union of K_4 with one end vertex of P_n and the one-point union of the other end vertex of P_n with the central vertex of $K_{1,r}$; the graph obtained by taking the one-point union of K_4 with one end vertex of P_n and the one-point union of the other end of P_n with a vertex from the partite set of order 2 of $K_{2,r}$; the graph obtained from the graph just described by appending any number of edges to the other vertex of the partite set of order 2; the two-point union of the two vertices of the partite set of order 2 in $K_{2,r}$ and two vertices from K_4; and the graph obtained from the graph just described by appending any number of edges to one of the vertices from the partite set of order 2.

A *Golomb ruler* is a marked straightedge such that the distances between different pairs of marks on the straightedge are distinct. If the set of distances between marks is every positive integer up to and including the length of the ruler, then ruler is a called a *perfect Golomb ruler*. Golomb [1115] proved that perfect Golomb rulers exist only for rulers with at most 4 marks. Beavers [522] examines the relationship between Golomb rulers and graceful graphs through a correspondence between rulers and complete graphs. He proves that K_n is graceful if and only if there is a perfect Golomb ruler with n marks and Golomb rulers are equivalent to complete subgraphs of graceful graphs.

2.5 Disconnected Graphs

There have been many papers dealing with graphs that are not connected. For any graph G the graph mG denotes the disjoint union of m copies of G. In 1975 Kotzig [1740] investigated the gracefulness of the graphs rC_s. When $rs \equiv 1$ or 2 (mod 4), these graphs violate the gracefulness parity condition. Kotzig proved that when $r = 3$ and $4k > 4$, then rC_{4k} has a stronger form of graceful labeling called α-labeling (see §3.1) whereas when $r \geqslant 2$ and $s = 3$ or 5, rC_s is not graceful. In 1984 Kotzig [1742] once again investigated the gracefulness of rC_s as well as graphs that are the disjoint union of odd cycles. For graphs of the latter kind he gives several necessary conditions. His paper concludes with an elaborate table that summarizes what was then known about the gracefulness of rC_s. M. He [1198] has shown that graphs of the form $2C_{2m}$ and graphs obtained by connecting two copies of C_{2m} with an edge are graceful. Cahit [645] has shown that rC_s is harmonious when r and s are odd and Seoud, Abdel Maqsoud, and Sheehan [2762] noted that when r or s is even, rC_s is not harmonious. Seoud, Abdel Maqsoud, and Sheehan [2762] proved that $C_n \cup C_{n+1}$ is harmonious if and only if $n \geqslant 4$. They conjecture that $C_3 \cup C_{2n}$ is harmonious when $n \geqslant 3$. This conjecture was proved when Yang, Lu, and Zeng [3520] showed that all graphs of the form $C_{2j+1} \cup C_{2n}$ are harmonious except for $(n, j) = (2, 1)$. As a consequence of their results about super edge-magic labelings (see §5.2) Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [939] have that $C_n \cup C_3$ is harmonious if and only if $n \geqslant 6$ and n is even. Renuka, Balaganesan, Selvaraju [2614] proved that for odd n $C_n \cup P_3$ (see also [2284]) and $C_n \odot \overline{K_m} \cup P_3$ are harmonious. Ng, Alwie, Marjadi, and Sugeng [2284] proved: $C_m \cup P_1$ is harmonious if and only if $m \neq 2$ mod 4, $C_m \cup P_2$ $(m \geqslant 3)$, and they conjectured that $C_m \cup P_3$ is harmonious for all $m \geqslant 3$. Youssef [3543] has shown that if G is harmonious then mG is harmonious for all odd m.

In 1978 Kotzig and Turgeon [1745] proved that mK_n is graceful if and only if $m = 1$ and $n \leqslant 4$. Liu and Zhang [1964] have shown that mK_n is not harmonious for n odd and $m \equiv 2$ (mod 4) and is harmonious for $n = 3$ and m odd. They conjecture that mK_3 is not harmonious when $m \equiv 0$ (mod 4). Bu and Cao [619] give some sufficient conditions for the gracefulness of graphs of the form $K_{m,n} \cup G$ and they prove that $K_{m,n} \cup P_t$ and the disjoint union of complete bipartite graphs are graceful under some conditions.

Recall a *Skolem sequence* of order n is a sequence $s_1, s_2, \ldots, s_{2n}$ of $2n$ terms such that, for each $k \in \{1, 2, \ldots, n\}$, there exist exactly two subscripts $i(k)$ and $j(k)$ with $s_{i(k)} = s_{j(k)} = k$ and $|i(k) - j(k)| = k$. (A Skolem sequence of order n exists if and only if $n \equiv 0$ or 1 (mod 4)). Abrham [19] has proved that any graceful 2-regular graph of order $n \equiv 0$ (mod 4) in which all the component cycles are even or of order $n \equiv 3$ (mod 4), with exactly one component an odd cycle, can be used to construct a Skolem sequence of order $n + 1$. Also, he showed that certain special Skolem sequences of order n can be used to generate graceful labelings on certain 2-regular graphs.

The graph H_n obtained from the cycle with consecutive vertices $u_1, u_2, \ldots, u_n$ $(n \geqslant 6)$ by adding the chords $u_2u_n, u_3u_{n-1}, \ldots, u_\alpha u_\beta$, where $\alpha = (n-1)/2$ for all n and $\beta = (n-1)/2 + 3$ if n is odd or $\beta = n/2 + 2$ if n is even is called the *cycle with parallel chords*. In Elumalai and Sethuraman [880] prove the following: for odd $n \geqslant 5$, $H_n \cup K_{p,q}$ is graceful; for even $n \geqslant 6$ and $m = (n-2)/2$ or $m = n/2$ $H_n \cup K_{1,m}$ is graceful; for $n \geqslant 6$, $H_n \cup P_m$ is graceful, where $m = n$ or $n - 2$ depending on $n \equiv 1$ or 3 (mod 4) or $m \equiv n-1$ or $n-3$ depending on $n \equiv 0$ or 2 (mod 4). Elumali and Sethuraman [882] proved that every n-cycle $(n \geqslant 6)$ with parallel chords is graceful and every n-cycle with parallel

P_k-chords of increasing lengths is graceful for $n = 2 \pmod 4$ with $1 \leqslant k \leqslant (\lfloor n/2 \rfloor - 1)$.

In 1985 Frucht and Salinas [988] conjectured that $C_s \cup P_n$ is graceful if and only if $s + n \geqslant 6$ and proved the conjecture for the case that $s = 4$. The conjecture was proved by Traetta [3222] in 2012 who used his result to get a complete solution to the well known two-table Oberwolfach problem; that is, given odd number of people and two round tables when is it possible to arrange series of seatings so that each person sits next to each other person exactly once during the series. The t-table Oberwolfach problem $\mathrm{OP}(n_1, n_2, \ldots, n_t)$ asks to arrange a series of meals for an odd number $n = \sum n_i$ of people around t tables of sizes $n_1, n_2, \ldots, n_t$ so that each person sits next to each other exactly once. A solution to $\mathrm{OP}(n_1, n_2, \ldots, n_t)$ is a 2–factorization of K_n whose factors consists of t cycles of lengths $n_1, n_2, \ldots, n_t$. The λ–fold Oberwolfach problem $\mathrm{OP}_\lambda(n_1, n_2, \ldots, n_t)$ refers to the case where K_n is replaced by λK_n. Traetta used his proof of the Frucht and Salinas conjecture to provide a complete solutions to both $OP(2r + 1, 2s)$ and $OP(2r + 1, s, s)$, except possibly for $OP(3, s, s)$. He also gave a complete solution of the general λ–fold Oberwolfach problem $OP_\lambda(r, s)$.

Seoud and Youssef [2803] have shown that $K_5 \cup K_{m,n}, K_{m,n} \cup K_{p,q}$ $(m, n, p, q \geqslant 2), K_{m,n} \cup K_{p,q} \cup K_{r,s}$ $(m, n, p, q, r, s \geqslant 2,\ (p, q) \neq (2, 2))$, and $pK_{m,n}$ $(m, n \geqslant 2, (m, n) \neq (2, 2))$ are graceful. They also prove that $C_4 \cup K_{1,n}$ $(n \neq 2)$ is not graceful whereas Choudum and Kishore [721], [1704] have proved that $C_s \cup K_{1,n}$ is graceful for $s \geqslant 7$ and $n \geqslant 1$. Lee, Quach, and Wang [1876] established the gracefulness of $P_s \cup K_{1,n}$. Seoud and Wilson [2795] have shown that $C_3 \cup K_4, C_3 \cup C_3 \cup K_4$, and certain graphs of the form $C_3 \cup P_n$ and $C_3 \cup C_3 \cup P_n$ are not graceful. Abrham and Kotzig [24] proved that $C_p \cup C_q$ is graceful if and only if $p + q \equiv 0$ or $3 \pmod 4$. Zhou [3597] proved that $K_m \cup K_n$ $(n > 1, m > 1)$ is graceful if and only if $\{m, n\} = \{4, 2\}$ or $\{5, 2\}$. Knuth [1707] used a computer to show that $K_5 \cup K_2$ has a unique graceful labeling up to a complement. (C. Barrientos has called to my attention that $K_1 \cup K_n$ is graceful if and only if $n = 3$ or 4.) Shee [2873] has shown that graphs of the form $P_2 \cup C_{2k+1}$ $(k > 1)$, $P_3 \cup C_{2k+1}$, $P_n \cup C_3$, and $S_n \cup C_{2k+1}$ all satisfy a condition that is a bit weaker than harmonious. Bhat-Nayak and Deshmukh [558] have shown that $C_{4t} \cup K_{1,4t-1}$ and $C_{4t+3} \cup K_{1,4t+2}$ are graceful. Section 3.1 includes numerous families of disconnected graphs that have a stronger form of graceful labelings.

For $m = 2p + 3$ or $2p + 4$, Wang, Liu, and Li [3440] proved the following graphs are graceful: $W_m \cup K_{n,p}$ and $W_{m,2m+1} \cup K_{n,p}$; for $n \geqslant m$, $W_{m,2m+1} \cup K_{1,n}$; for $m = 2n + 5$, $W_{m,2m+1} \cup (C_3 + \overline{K_n})$. If G_p is a graceful graph with p edges, they proved $W_{2p+3} \cup G_p$ is graceful.

In considering graceful labelings of the disjoint unions of two or three stars S_e with e edges Yang and Wang [3523] permitted the vertex labels to range from 0 to $e + 1$ and 0 to $e + 2$, respectively. With these definitions of graceful, they proved that $S_m \cup S_n$ is graceful if and only if m or n is even and that $S_m \cup S_n \cup S_k$ is graceful if and only if at least one of m, n, or k is even $(m > 1, n > 1, k > 1)$.

Seoud and Youssef [2799] investigated the gracefulness of specific families of the form $G \cup K_{m,n}$. They obtained the following results: $C_3 \cup K_{m,n}$ is graceful if and only if $m \geqslant 2$ and $n \geqslant 2$; $C_4 \cup K_{m,n}$ is graceful if and only if $(m, n) \neq (1, 1)$; $C_7 \cup K_{m,n}$ and $C_8 \cup K_{m,n}$ are graceful for all m and n; $mK_3 \cup nK_{1,r}$ is not graceful for all m, n and r; $K_i \cup K_{m,n}$

is graceful for $i \leqslant 4$ and $m \geqslant 2, n \geqslant 2$ except for $i = 2$ and $(m, n) = (2, 2)$; $K_5 \cup K_{1,n}$ is graceful for all n; $K_6 \cup K_{1,n}$ is graceful if and only if n is not 1 or 3. Youssef [3545] completed the characterization of the graceful graphs of the form $C_n \cup K_{p,q}$ where $n \equiv 0$ or 3 (mod 4) by showing that for $n > 8$ and $n \equiv 0$ or 3 (mod 4), $C_n \cup K_{p,q}$ is graceful for all p and q (see also [439]). Note that when $n \equiv 1$ or 2 (mod 4) certain cases of $C_n \cup K_{p,q}$ violate the parity condition for gracefulness.

For $i = 1, 2, \ldots, m$ let $v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ be a 4-cycle. Yang and Pan [3518] define $F_{k,4}$ to be the graph obtained by identifying $v_{i,3}$ and $v_{i+1,1}$ for $i = 1, 2, \ldots, k-1$. They prove that $F_{m_1,4} \cup F_{m_2,4} \cup \cdots \cup F_{m_n,4}$ is graceful for all n. Pan and Lu [2327] have shown that $(P_2 + \overline{K_n}) \cup K_{1,m}$ and $(P_2 + \overline{K_n}) \cup T_n$ are graceful.

Barrientos [439] has shown the following graphs are graceful: $C_6 \cup K_{1,2n+1}$; $\bigcup_{i=1}^{t} K_{m_i,n_i}$ for $2 \leqslant m_i < n_i$; and $C_m \cup \bigcup_{i=1}^{t} K_{m_i,n_i}$ for $2 \leqslant m_i < n_i, m \equiv 0$ or 3 (mod 4), $m \geqslant 11$. In [1608] Kaneria, Makadia, and Viradia proved that the union of three grid graphs, $\bigcup_{l=1}^{3}$ $(P_{m_l} \times P_{n_l})$, is graceful, the union of finitely many copies of $P_m \times P_n$ is graceful, and provided two new graceful labeling for $P_m \times P_n$.

Wang and Li [3438] use $St(n)$ to denote the star $K_{n,1}$, F_n to denote the fan $P_n \odot K_1$, and $F_{m,n}$ to denote the graph obtained by identifying the vertex of F_m with degree m and the vertex of F_n with degree n. They showed: for all positive integers n and p and $m \geqslant 2p + 2$, $F_m \cup K_{n,p}$ and $F_{m,2m} \cup K_{n,p}$ are graceful; $F_m \cup St(n)$ is graceful; and $F_{m,2m} \cup St(n)$ and $F_{m,2m} \cup G_r$ are graceful. In [3444] Wang, Wang, and Li gave a sufficient condition for the gracefulness of graphs of the form $(P_3 + \overline{K_m}) \cup G$ and $(C_3 + \overline{K_m}) \cup G$. Wei, Wang, and Sun [3461] provided graceful labelings for the unions of some families of wheels related graphs and complete bipartite graphs. They also gave graceful labelings for some graphs of the form $G \cup (C_3 + \overline{K_m}) \cup S_n$ where G is wheel related. In [3563] Yu, Wang, and Song proved the following graphs are graceful: $K_{n,m} \cup (\overline{K_2} + \overline{K_n})$, $K_{n,m} \cup (P_3 + \overline{K_n})$, $K_{n,m} \cup (P_1 + P_{2n+2})$, and $K_{n,m} \cup K_{1,2n}$. They proved the gracefulness of such graphs for a variety of cases when G involves stars and paths. More technical results like these are given in [3446], [3445], and [651].

2.6 Joins of Graphs

A number of classes of graphs that are the join of graphs have been shown to be graceful or harmonious. Koh, Rogers, and Lim [1714] proved $G + H$ is graceful if G is a graceful tree and H is one of $\overline{K_n}$, $P_n \cup K_1$, or a star. Koh, Phoon, and Soh [1710] point out that some versions of this survey prior to 2017 incorrectly stated that Acharya [27] proved that if G is a connected graceful graph, then $G + \overline{K_n}$ is graceful. Redl [2610] showed that the double cone $C_n + \overline{K_2}$ is graceful for $n = 3, 4, 5, 7, 8, 9, 11$. That $C_n + \overline{K_2}$ is not graceful for $n \equiv 2$ (mod 4) follows that Rosa's parity condition. Redl asks what other double cones are graceful. Bras, Gomes, and Selman [612] showed that double wheels $(C_n \cup C_n) + K_1$ are graceful. Koh, Phoon, and Soh [1710] prove that $K_3 + \overline{K_n}$ is graceful. Reid [2611] proved that $P_n + \overline{K_t}$ is harmonious. Sethuraman and Selvaraju [2839] and [2744] have shown that $P_n + K_2$ is harmonious. They ask whether $S_n + P_n$ or $P_m + P_n$ is harmonious. As stated in an earlier section, wheels are of the form $C_n + K_1$ and are graceful and harmonious. In 2006 Chen [693] proved that multiple wheels $nC_m + K_1$ are harmonious for all $n \not\equiv 0$ mod 4. She believes that the $n \not\equiv 0 \pmod 4$ case is also harmonious. Chen also proved that if H has at least one edge, $H + K_1$ is harmonious, and if n is odd, then $nH + K$ is harmonious.

For $n \geqslant t+2$ and $t \geqslant 1$, Koh, Phoon, and Soh [1711] use $P(n,t)$ to denote the graph of order n consisting of a path of length t and $n-(t+1)$ isolated vertices. For $n \geqslant 2t+1$ and $t \geqslant 1$, they use $I(n,t)$ to denote the disjoint union of tK_2 and $\overline{K_{n-2t}}$. They proved: $\overline{K_p} + P(n,t)$ is graceful for all $p \geqslant 1, n \geqslant t+2$ and $t \geqslant 1$; $\overline{K_p} + I(n,t)$ is graceful for all $p \geqslant 1,\ n \geqslant 2t+1$ and $t \geqslant 1$; and for $s,t \in \{1,2\}$, $P(m,s) + P(n,t)$ is graceful for all $m \geqslant s+2$ and $n \geqslant t+2$. In [1711] Koh, Phoon, and Soh ask "What can be said about the gracefulness of $C_m + P(n,t)$ where $n \geqslant t+2$" and is "Is $P(m,s) + P(n,t)$ always graceful for all $m \geqslant s+2,\ n \geqslant t+2$, where $s \geqslant 3$ or $t \geqslant 3$?" In [1710] they state as problems about graceful graphs: $C_m + P_n$ $(m \geqslant 3,\ n \geqslant 3)$; $C_m + C_n$ $(m \geqslant 3,\ n \geqslant 3)$ and $K_{1,p} + P(n,t)$ and prove that $C_3 + P(n,t)$ is graceful for all $n \geqslant t+2$, where $1 \leqslant t \leqslant 3$ and $C_5 + P(n,1)$ is graceful for all $n \geqslant 3$.

Shee [2873] has proved $K_{m,n} + K_1$ is harmonious and observed that various cases of $K_{m,n} + K_t$ violate the harmonious parity condition in [1147]. Liu and Zhang [1964] have proved that $K_2 + K_2 + \cdots + K_2$ is harmonious. Youssef [3543] has shown that if G is harmonious then G^m is harmonious for all odd m. He asks the question of whether G is harmonious implies G^m is harmonious when $m \equiv 0 \pmod 4$. Yuan and Zhu [3565] proved that $K_{m,n} + K_2$ is graceful and harmonious. Gnanajothi [1104, pp. 80–127] obtained the following: $C_n + \overline{K_2}$ is harmonious when n is odd and not harmonious when $n \equiv 2,4,6$ (mod 8); $S_n + \overline{K_t}$ is harmonious; and $P_n + \overline{K_t}$ is harmonious. Balakrishnan and Kumar [413] have proved that the join of $\overline{K_n}$ and two disjoint copies of K_2 is harmonious if and only if n is even. Ramírez-Alfonsín [2577] has proved that if G is graceful and $|V(G)| = |E(G)| = e$ and either 1 or e is not a vertex label then $G + \overline{K_t}$ is graceful for all t. Sudha and Kanniga [3073] proved that the graph $P_m + \overline{K_n}$ is graceful.

Seoud and Youssef [2801] have proved: the join of any two stars is graceful and harmonious; the join of any path and any star is graceful; and $C_n + \overline{K_t}$ is harmonious for every t when n is odd. They also prove that if any edge is added to $K_{m,n}$ the resulting graph is harmonious if m or n is at least 2. Deng [800] has shown certain cases of $C_n + \overline{K_t}$ are harmonious. Seoud and Youssef [2798] proved: the graph obtained by appending any number of edges from the two vertices of degree $n \geqslant 2$ in $K_{2,n}$ is not harmonious; dragons $D_{m,n}$ (i.e., an endpoint of P_m is appended to C_n) are not harmonious when $m+n$ is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has odd order. Youssef [3542] has shown that if G is a graceful graph with p vertices and q edges with $p = q+1$, then $G + S_n$ is graceful.

Sethuraman and Elumalai [2814] have proved that for every graph G with p vertices and q edges the graph $G + K_1 + \overline{K_m}$ is graceful when $m \geqslant 2^p - p - 1 - q$. As a corollary they deduce that every graph is a vertex induced subgraph of a graceful graph. Balakrishnan and Sampathkumar [414] ask for which $m \geqslant 3$ is the graph $mK_2 + \overline{K_n}$ graceful for all n. Bhat-Nayak and Gokhale [562] have proved that $2K_2 + \overline{K_n}$ is not graceful. Youssef [3542] has shown that $mK_2 + \overline{K_n}$ is graceful if $m \equiv 0$ or 1 (mod 4) and that $mK_2 + \overline{K_n}$ is not graceful if n is odd and $m \equiv 2$ or 3 (mod 4). Ma [2044] proved that if G is a graceful tree then, $G + K_{1,n}$ is graceful. Amutha and Kathiresan [179] proved that the graph obtained by attaching a pendent edge to each vertex of $2K_2 + \overline{K_n}$ is graceful.

Wu [3487] proves that if G is a graceful graph with n edges and $n + 1$ vertices then the join of G and $\overline{K_m}$ and the join of G and any star are graceful. Wei and Zhang [3460] proved that for $n \geqslant 3$ the disjoint union of $P_1 + P_n$ and a star, the disjoint union of $P_1 + P_n$ and $P_1 + P_{2n}$, and the disjoint union of $P_2 + \overline{K_n}$ and a graceful graph with n edges are graceful. More technical results on disjoint unions and joins are given in [3459], [3460], [3462], [3458], and [651].

2.7 Miscellaneous Results

It is easy to see that P_n^2 is harmonious [1144] while a proof that P_n^2 is graceful has been given by Kang, Liang, Gao, and Yang [1633]. (P_n^k, the kth *power of* P_n, is the graph obtained from P_n by adding edges that join all vertices u and v with $d(u, v) = k$.) This latter result proved a conjecture of Grace [1144]. Seoud, Abdel Maqsoud, and Sheehan [2762] proved that P_n^3 is harmonious and conjecture that P_n^k is not harmonious when $k > 3$. The same conjecture was made by Fu and Wu [991]. However, Youssef [3552] has proved that P_8^4 is harmonious and P_n^k is harmonious when k is odd. Yuan and Zhu [3565] proved that P_n^{2k} is harmonious when $1 \leqslant k \leqslant (n-1)/2$. Selvaraju [2739] has shown that P_n^3 and the graphs obtained by joining the centers of any two stars with the end vertices of the path of length n in P_n^3 are harmonious.

Cahit [645] proves that the graphs obtained by joining p disjoint paths of a fixed length k to single vertex are harmonious when p is odd and when $k = 2$ and p is even. Gnanajothi [1104, p. 50] has shown that the graph that consists of n copies of C_6 that have exactly P_4 in common is graceful if and only if n is even. For a fixed n, let v_{i1}, v_{i2}, v_{i3} and v_{i4} $(1 \leqslant i \leqslant n)$ be consecutive vertices of n 4-cycles. Gnanajothi [1104, p. 35] also proves that the graph obtained by joining each v_{i1} to $v_{i+1,3}$ is graceful for all n and the generalized Petersen graph $P(n, k)$ is harmonious in all cases (see also [1881]). Recall $P(n, k)$, where $n \geqslant 5$ and $1 \leqslant k \leqslant n$, has vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_i a_{i+1} \mid i = 0, 1, \ldots, n-1\} \cup \{a_i b_i \mid i = 0, 1, \ldots, n-1\} \cup \{b_i b_{i+k} \mid i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo n [3457]. The standard Petersen graph is $P(5, 2)$.) Redl [2610] has used a constraint programming approach to show that $P(n, k)$ is graceful for $n = 5, 6, 7, 8, 9$, and 10. In [3361] and [3377] Vietri proved that $P(8t, 3)$ and $P(8t + 4, 3)$ are graceful for all t. He conjectures that the graphs $P(8t, 3)$ have a stronger form a graceful labeling called an α-labeling (see §3.1). The gracefulness of the generalized Petersen graphs is an open problem. Shao, Deng, Li, and Vese [2866] provide an backtracking algorithm that finds graceful labelings for all generalized Petersen graphs $P(n, k)$ with $n \leqslant 75$ within several seconds. The algorithm strongly outperforms the standard backtracking algorithm.

Rao and Sahoo [2593] prove that every connected graph can be embedded as an induced subgraph in an Eulerian graceful graph. They also show that for an integer $k \geqslant 3$, the problems of deciding whether the chromatic number is less than or equal to k and whether the clique number is greater than or equal to k are NP-complete even for Eulerian graceful graphs. Sethuraman, Ragukumar, and Slater [2830] proved that any tree with m edges can be embedded in a graceful tree with less than $4m$ edges and in a graceful planar graph. A conjecture in the graph theory book by Chartrand and Lesniak [684, p. 266] that graceful graphs with arbitrarily large chromatic numbers do not exist was shown to be false by Acharya, Rao, and Arumugam [47] (see also Mahmoody [2065]).

In [447] Barrientos calculates the number of non-isomorphic harmoniously labeled graphs with n edges and at most n vertices. He provides harmonious labelings for certain unicyclic graphs obtained via the corona product and triangular grids obtained via edge amalgamation of copies of C_3 in such a way that each copy of a cycle shares at most two edges with other copies. Moreover, he uses the edge-switching technique on C_{4t} to generate unicyclic graphs with strongly felicitous labelings (see §4.4).

Bača and Youssef [390] investigated the existence of harmonious labelings for the corona graphs of a cycle and a graph G. They proved that if $G+K_1$ is strongly harmonious (that is, a harmonious labeling f for which the edge labels induced by $f(x)+f(y)$ for each edge xy are $1,\dots,q$. with the 0 label on the vertex of K_1, then $C_n \odot G$ is harmonious for all odd $n \geqslant 3$. By combining this with existing results they have as corollaries that the following graphs are harmonious: $C_n \odot C_m$ for odd $n \geqslant 3$ and $m \not\equiv 2 \pmod 3$; $C_n \odot K_{s,t}$ for odd $n \geqslant 3$; and $C_n \odot K_{1,s,t}$ for odd $n \geqslant 3$.

Sethuraman and Selvaraju [2833] define a graph H to be a *supersubdivision* of a graph G, if every edge uv of G is replaced by $K_{2,m}$ (m may vary for each edge) by identifying u and v with the two vertices in $K_{2,m}$ that form the partite set with exactly two members. Sethuraman and Selvaraju prove that every supersubdivision of a path is graceful and every cycle has some supersubdivision that is graceful. They conjecture that every supersubdivision of a star is graceful and that paths and stars are the only graphs for which every supersubdivision is graceful. Barrientos [441] disproved this latter conjecture by proving that every supersubdivision of a y-trees is graceful (recall a y-tree is obtained from a path by appending an edge to a vertex of a path adjacent to an end point). Barrientos asks if paths and y-trees are the only graphs for which every supersubdivision is graceful. This seems unlikely to be the case. The conjecture that every supersubdivision of a star is graceful was proved by Kathiresan and Amutha [1666]. In [2837] Sethuraman and Selvaraju prove that every connected graph has some supersubdivision that is graceful. They pose the question as to whether this result is valid for disconnected graphs. Barrientos and Barrientos [451] answered this question by proving that any disconnected graph has a supersubdivision that admits an α-labeling (see §3.1). They also proved that every supersubdivision of a connected graph admits an α-labeling. Sekar and Ramachandren proved that an arbitrary supersubdivision of disconnected graph is graceful [2737] and supersubdivisions of ladders are graceful [2574]. Sethuraman and Selvaraju also asked if there is any graph other than $K_{2,m}$ that can be used to replace an edge of a connected graph to obtain a supersubdivision that is graceful.

Sethuraman and Selvaraju [2833] call superdivision graphs of G where every edge uv of G is replaced by $K_{2,m}$ and m is fixed an *arbitrary supersubdivision* of G. Barrientos and Barrientos [451] answered the question of Sethuraman and Selvaraju by proving that any graph obtained from $K_{2,m}$ by attaching k pendent edges and n pendent edges to the vertices of its 2-element stable set can be used instead of $K_{2,m}$ to produce an arbitrary supersubdivision that admits an α-labeling (a *stable* set S consists of a set of vertices such that there is not an edge v_iv_j for all pairs v_i, v_j in S).

Kathiresan and Sumathi [1676] affirmatively answer the question posed by Sethuraman and Selvaraju in [2833] of whether there are graphs different from paths whose arbitrary supersubdivisions are graceful.

For a graph G Ambili and Singh [175] call the graph G^* a *strong supersubdivision* of G if G^* is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{r_i,s_i}. A strong supersubdivision G^* of G is said to be an *arbitrary strong supersubdivision* if G^* is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{r,s_i} (r is fixed and s_i may vary). They proved that arbitrary strong supersubdivisions of paths, cycles, and stars are graceful. They conjecture that every arbitrary strong supersubdivision of a tree is graceful and ask if it is true that for any non-trivial connected graph G, an arbitrary strong supersubdivision of G is graceful?

In [2836] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a strong form of graceful labeling called an α-labeling (see §3.1 for the definition).

Kathiresan [1663] uses the notation $P_{a,b}$ to denote the graph obtained by identifying the end points of b internally disjoint paths each of length a. He conjectures that $P_{a,b}$ is graceful except when a is odd and $b \equiv 2 \pmod 4$ and proves the conjecture for the case that a is even and b is odd. Liang and Zuo [1944] proved that the graph $P_{a,b}$ is graceful when both a and b are even. Daili, Wang and Xie [789] provided an algorithm for finding a graceful labeling of $P_{2r,2}$ and showed that a $P_{2r,2(2k+1)}$ is graceful for all positives r and k. Sekar [2736] has shown that $P_{a,b}$ is graceful when $a \neq 4r+1,\ r > 1, b = 4m$, and $m > r$. Yang (see [3521]) proved that $P_{a,b}$ is graceful when $a = 3, 5, 7$, and 9 and b is odd and when $a = 2, 4, 6$, and 8 and b is even (see [3521]). Yang, Rong, and Xu [3521] proved that $P_{a,b}$ is graceful when $a = 10, 12$, and 14 and b is even. Yan [3510] proved $P_{2r,2m}$ is graceful when r is odd. Yang showed that $P_{2r+1,2m+1}$ and $P_{2r,2m}$ ($r \leqslant 7$, and $r = 9$) are graceful (see [2646]). Rong and Xiong [2646] showed that $P_{2r,b}$ is graceful for all positive integers r and b. Kathiresan also shows that the graph obtained by identifying a vertex of K_n with any noncenter vertex of the star with $2^{n-1} - n(n-1)/2$ edges is graceful.

For a family of graphs $G_1(u_1, u_2), G_2(u_2, u_3), \ldots, G_m(u_m, u_{m+1})$ where u_i and u_{i+1} are vertices in G_i Cheng, Yao, Chen, and Zhang [702] define a *graph-block chain* H_m as the graph obtained by identifying u_{i+1} of G_i with u_{i+1} of G_{i+1} for $i = 1, 2, \ldots, m$. They denote this graph by $H_m = G_1(u_1, u_2) \oplus G_2(u_2, u_3) \oplus \cdots \oplus G_m(u_m, u_{m+1})$. The case where each G_i has the form P_{a_i,b_i} they call a *path-block chain.* The vertex u_1 is called the *initial vertex* of H_m. They define a *generalized spider* S_m^* as a graph obtained by starting with an initial vertex u_0 and m path-block graphs and join u_0 with each initial vertex of each of the path-block graphs. Similarly, they define a *generalized caterpillar* T_m^* as a graph obtained by starting with m path-block chains $H_1, H_2, \ldots, H_m$ and a caterpillar T with m isolated vertices $v_1, v_2, \ldots, v_m$ and join each v_i with the initial vertex of each H_i. They prove several classes of path-block chains, generalized spiders, and generalized caterpillars are graceful.

The graph T_n with $3n$ vertices and $6n - 3$ edges is defined as follows. Start with a triangle T_1 with vertices $v_{1,1}, v_{1,2}$ and $v_{1,3}$. Then T_{i+1} consists of T_i together with three new vertices $v_{i+1,1}, v_{i+1,2}, v_{i+1,3}$ and edges $v_{i+1,1}v_{i,2}$, $v_{i+1,1}v_{i,3}$, $v_{i+1,2}v_{i,1}$, $v_{i+1,2}v_{i,3}$, $v_{i+1,3}v_{i,1}$, $v_{i+1,3}v_{i,2}$. Gnanajothi [1104] proved that T_n is graceful if and only if n is odd. Sekar [2736] proved T_n is graceful when n is odd and T_n with a pendent edge attached to the starting triangle is graceful when n is even.

In [526] and [2851] Begam, Palanivelrajan, Gunasekaran, and Hameed give graceful labelings for graphs constructed by combining theta graphs (that is, a collection of edge disjoint paths that have common endpoints) with paths and stars. Khatun and Abu Nayeem [1685] prove that the zero divisor graph of the commutative ring of integers modulo n is graceful if $n = pq$, $4p$ or $9p$, where p and q are prime numbers.

The torch graph O_n is defined by $V(O_n) = \{v_i \mid 1 \leqslant i \leqslant n+4\}, E(O_n) = \{v_iv_{n+1} \mid 2 \leqslant i \leqslant n-2\} \cup \{v_iv_{n+3} \mid 2 \leqslant i \leqslant n-2\} \cup \{v_1v_i \mid 2 \leqslant i \leqslant n+4\} \cup \{v_{n-1}v_n, v_nv_{n+2}, v_nv_{n+4}, v_{n+1}v_{n+3}\}$. Manulang and Sugeng [2077] showed that the torch graph is graceful.

For a graph G, the *splitting graph* of G, $S'(G)$, is obtained from G by adding for each vertex v of G a new vertex $v^{'}$ so that $v^{'}$ is adjacent to every vertex that is adjacent to v. Sekar [2736] has shown that $S'(P_n)$ is graceful for all n and $S'(C_n)$ is graceful for $n \equiv 0, 1 \pmod 4$. Vaidya and Shah [3304] proved that the square graph of a bistar, the splitting graph of a bistar, and the splitting graph of a star are graceful graphs.

In [3074] Sudha and Kanniga proved that fans and the splitting graph of a star are graceful. Sudha and Kanniga [3075] proved that the following graphs are graceful: arbitrary supersubdivisions of wheels; combs ($P_n \odot K_1$); double fans ($P_n \odot \overline{K_2}$); $(P_m \cup P_n) \odot K_1$; and graphs obtained by starting with two star graphs S_m and S_n and identifying some of the pendent vertices of each. Sudha and Kanniga [3076] proved that the graphs obtained from $P_n \odot K_1$ by identifying the center of a S_n with the endpoint of a pendent edge attached to the endpoint of P_n are graceful; and the graphs obtained from a fan $P_n \odot K_1$ by deleting a pendent edge attached to an endpoint of P_n are graceful. Sunda [3069] provided some results on graphs obtained by connecting copies of $K_{m,n}$ in certain ways. Sudha and Kanniga [3072] proved that the graphs obtained by joining the vertices of a path to any number isolated points are graceful. They also proved that the arbitrary supersubdivision of all the edges of helms, combs ($P_n \odot K_1$) and ladders ($P_n \times P_2$) with pendent edges at the vertices of degree 2 by a complete bipartite graphs $K_{2,m}$ are graceful.

The *duplication of an edge* $e = uv$ of a graph G is the graph G' obtained from G by adding an edge $e' = u'v'$ such that $N(u) = N(u')$ and $N(v) = N(v')$. The *duplication of a vertex* of a graph G is the graph G' obtained from G by adding a new vertex v' to G such that $N(v') = N(v)$. Kaneria, Vaidya, Ghodasara, and Srivastav [1624] proved the duplication of a vertex of a cycle, the duplication of an edge of an even cycle, and the graph obtained by joining two copies of a fixed cycle by an edge are graceful.

For a graph G and a vertex v of G, a *vertex switching* G_v is the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. Boxwala and Vashishta [606] show that the graph obtained by switching an arbitrary vertex of C_n ($n > 3$), the duplication of an arbitrary vertex on the rim of a wheel with an even number of vertices, and the mirror graph of a path are graceful. Jeba

Jesintha and Subashini [1413] proved that the path union of vertex switching of even cycles in increasing order is graceful.

The *join sum* of complete bipartite graphs $< K_{m_1,n_1}, \dots, K_{m_t,n_t} >$ is the graph obtained by starting with $K_{m_1,n_1}, \dots, K_{m_t,n_t}$ and joining a vertex of each pair K_{m_i,n_i} and $K_{m_{i+1},n_{i+1}}$ to a new vertex v_i where $1 \leqslant i \leqslant k-1$. The *path union* of a graph G is the graph obtained by adding an edge from n copies $G_1, G_2, \dots, G_n$ of G from G_i to G_{i+1} for $i = 1, \dots, n-1$. We denote this graph by $P(n \cdot G)$. Kaneria, Makadia, and Meghpara [1604] proved the following graphs are graceful: the graph obtained by joining C_{4m} and C_{4n} by a path of arbitrary length; the path union of finite many copies of C_{4n}; and C_{4n} with twin chords. Kaneria, Makadia, Jariya, and Meghpara [1603] proved that the join sum of complete bipartite graphs, the star of complete bipartite graphs, and the path union of a complete bipartite graphs are graceful.

Given connected graphs $G_1, G_2, \dots, G_n$, Kaneria, Makadia, and Jariya [1602] define a *cycle of graphs* $C(G_1, G_2, \dots, G_n)$ as the graph obtained by adding an edge joining G_i to G_{i+1} for $i = 1, \dots, n-1$ and an edge joining G_n to G_1. (The resulting graph can vary depending on which vertices of the G_is are chosen.) When the n graphs are isomorphic to G the notation $C(n \cdot G)$ is used. Kaneria et al. proved that $C(2t \cdot C_{4n})$ and $C(2t \cdot K_{n,n})$ are graceful. In [1605] and [1607] Kaneria, Makadia, and Meghpara prove that the following graphs are graceful: $C(2t \cdot K_{m,n})$; $C(C_{4n_1}, C_{4n_2}, \dots, C_{4n_t})$ when t is even and $\sum_{i=1}^{\frac{t}{2}} n_i = \sum_{i=\frac{t}{2}}^{t} n_i$; $C(2t \cdot P_m \times P_n)$; the star of $P_m \times P_n$; and the path union of t copies of $P_m \times P_n$. Kaneria, Viradia, Jariya, and Makadia [1625] proved the cycle graph $C(t \cdot P_n)$ is graceful.

The *star of graphs* $G_1, G_2, \dots, G_n$, denoted by $S(G_1, G_2, \dots, G_n)$, is the graph obtained by identifying each vertex of $K_{1,n}$, except the center, with one vertex from each of $G_1, G_2, \dots, G_n$. The case that $G_1 = G_2 = \cdots = G_n = G$ is denoted by $S(n \cdot G)$. In [1616] and [1617] Kaneria, Meghpara, and Makadia proved the following graphs are graceful: $S(t \cdot K_{m,n})$; $S(t \cdot P_m \times P_n)$; the barycentric subdivision of $P_m \times P_n$ (that is, the graph obtained from $P_m \times P_n$ by inserting a new vertex in each edge); the graph obtained by replacing each edge of $K_{1,t}$ by P_n; the graph obtained by identifying each end point of $K_{1,n}$ with a vertex of $K_{m,n}$; and the graph obtained by identifying each end point of $K_{1,n}$ with a vertex of $P_m \times P_n$. Kanani and Kaneria [1573] proved that the following graphs are graceful: the barycentric subdivision of C_n-snakes (that is, the graph obtained from the subdivision of C_n by inserting a new vertex in each edge); the barycentric subdivision of alternate C_n-snakes; and quadrilateral snakes.

Kaneria and Makadia [1593] and [1594] proved the following graphs are graceful: $(P_m \times P_n) \cup (P_r \times P_s)$; $C_{2f+3} \cup (P_m \times P_n) \cup (P_r \times P_s)$, where $f = 2(mn + rs) - (m + n + r + s)$; the tensor product of P_n and P_3; the tensor product of P_m and P_n for odd m and n; the star of C_{4n}; the $t-$supersubdivision of $P_m \times P_n$; and the graph obtained by joining C_{4n} and a grid graph with a path. In [1615] Kaneria, Meghpara, and Makadia proved that the star of $K_{1,n}$ is a graceful tree.

The graph P_n^t is obtained by identifying one end point from each of t copies of P_n. The graph $P_n^t(G_1, G_2, \dots, G_{tn})$ obtained by replacing each edge of P_n^t, except those adjacent to the vertex of degree t, by the graphs $G_1, G_2, \dots, G_{tn}$ is called the *one point path union of* $G_1, G_2, \dots, G_{tn}$. The case where $G_1 = G_2 = \cdots = G_{tn} = H$ is denoted by $P_n^t(tn \cdot H)$

. In [1616] and [1617] Kaneria, Meghpara, and Makadia proved P_n^t and $P_n^t(tn \cdot K_{m,r})$ are graceful. In [1614] Kaneria and Meghpara proved P_n^t $(tn \cdot P_r \times P_s)$, $P_n^t(tn \cdot K_{1,m})$, $S(t \cdot C_{4n})$, and $P_n^t(tn \cdot C_{4m})$ are graceful.

A graph H is said to be a m-super subdivision of a simple graph G, if every edge of G is replaced by the complete bipartite graph $K_{m,m}$ with $m > 2$ in such a way that the end vertices of the edge are merged with any two vertices of the same partite set A or B of $K_{m,m}$ after removal of the edge of G. Srinivasan, Chidambaram, Devadoss, Pakkirisamy, and Krishnamoorthi [3046] proved that m-super subdivision of path and cycle are graceful.

Kanneria and Makadia [1595] define a *step grid graph* as the graph obtained by starting with paths $P_n, P_n, P_{n-1}, \ldots, P_2$ $(n \geqslant 3)$ arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. In [1595] and [1596] they prove the following graphs are graceful: step grid graphs; one point union for a path of step grid graphs; cycles of step grid graphs; stars of step grid graphs; $m-$super subdivisions of the step grid graphs; open stars of step grid graphs; one point unions of paths of step grid graphs; and graphs obtained by joining C_{4m} and step grid graphs with a path of arbitrary length.

For n even [1597] Kaneria and Makadia [1597] define a *double step grid graph* of size n (denoted by DSt_n) as the graph obtained by starting with paths $P_n, P_n, P_{n-2}, P_{n-4}, \ldots, P_4, P_2$ arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. They prove the following graphs are graceful: double step grid graphs; path unions of copies of DSt_n; cycles of $r \equiv 0,\ 3 \pmod 4$ copies of double step grid graphs; and stars of double step grid graphs.

In [1609] Kaneria, Makadia and Viradia prove the following graphs are graceful: open stars of double step grid graphs; one point union of paths of double step grid graphs ${P_n}^t(tn \cdot DSt_m)$; graphs obtained by joining C_{4m} and a double step grid graph with a path of arbitrary length; and graphs obtained by starting with a cycle ${C_m}^+$ $(m \equiv 2 \bmod 4)$ with chords that form a triangle with an edge of the cycle and joining ${C_m}^+$ and a double step grid graph with a path of arbitrary length.

For even $n > 2$ Kaneria and Makadia [1598] define a *plus graph* of size n (denoted by Pl_n) as the graph obtained by starting with paths $P_2, P_4, \ldots, P_{n-2}, P_n, P_n, P_{n-2}, \ldots, P_4, P_2$ arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. They prove plus graphs, path unions of copies of Pl_n, cycles of $r \equiv 0,\ 3 \pmod 4$ copies of Pl_n, and stars of plus graphs are graceful. In [1599] Kaneria and Makadia prove the following graphs are graceful: open stars of plus graphs; graphs obtained by joining C_{4m} and a plus graph with a path of arbitrary length; graphs obtained from cycles ${C_m}^+$ $(m \equiv 2 \pmod 4))$ with twin chords that form a triangle with an edge of the cycle by joining ${C_m}^+$ and a plus graph with a path of arbitrary length.

Kaneria and Makadia [1600] define a *swastik graph* as the graph obtained from four copies of C_{4n} $(n > 1)$ with vertices $V_{i,j}$ ($i = 1, 2, 3, 4,\ j = 1, 2, \ldots, 4n$) and identifying $V_{1,4t}$ and $V_{2,1}$, $V_{2,4t}$ and $V_{3,1}$, $V_{3,4t}$ and $V_{4,1}$, and $V_{4,4t}$ and $V_{1,1}$. They proved that path unions of swastik graphs of the same size, cycles of $r \equiv 0,\ 3 \pmod 4$ copies of swastik graphs of the same size, and the star of swastik graphs are graceful. In [1601] Kaneria and

Makadia prove the following graphs are graceful: open stars of swastik graphs; one point unions for paths of swastik graphs; graphs obtain by joining C_{4m} and a swastik graph with a path of arbitrary length; graphs obtained from cycles C_m ($m \equiv 2 \pmod 4$) with twin chords that form a triangle with an edge by joining $C_m \odot K_1$ and a swastik graph with a path of arbitrary length.

In [1588] and [1587] Kaneria and Jariya define a *smooth graceful graph* as a bipartite graph G with q edges with the property that for all positive integers l there exists a map $g : V \longrightarrow \{0, 1, \ldots, \lfloor \frac{q-1}{2} \rfloor, \lfloor \frac{q+1}{2} \rfloor + l, \lfloor \frac{q+3}{2} \rfloor + l, \ldots, q+l\}$ such that the induced edge labeling map $g^\star : E \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined by $g^\star(e) = |g(u) - g(v)|$ is a bijection. Note that by taking $l = 0$ a smooth graceful labeling is a graceful labeling. Kaneria and Jariya proved the following graphs are smooth graceful: P_n; C_{4n}; $K_{2,n}$; $P_m \times P_n$; and the graph obtained by joining a cycle C_{4m+2} with twin chords to C_{4n}. They also proved that the graph obtained by joining C_{4m} to W_n with a path is graceful. They proved that $K_{1,n}$ is semi smooth graceful, the star of $K_{1,n}$ is graceful, the path union of a smooth graceful tree is graceful, and the star of a smooth graceful tree is a graceful tree.

Kaneria, Makadia and Viradia [1610] proved the following: the star of a semi smooth graceful graph is graceful; $K_{m,n}$, $P(t \cdot H)$ are semi smooth graceful where H is a semi smooth graceful graph; step grid graphs; and the cycle graphs $C(t \cdot H)$ are smooth graceful, when $t \equiv \pmod 4$, H is a semi smooth; $C^t(m \cdot C_n)$, $P^t(k \cdot T)$, $< C_{n_1}, P_{n_2}, C_{n_3}, \ldots, P_{n_{2t}}, C_{n_{2t+1}} >$, $< K_{m_1,n_1}, P_{r_1}, K_{m_2,n_2}, P_{r_2}, \ldots, P_{r_{t-1}}, K_{m_t,n_t} >$, $< P_{n_1} \times P_{m_1}, P_{r_1}, P_{n_2} \times P_{m_2}, \ldots, P_{r_{t-1}}, P_{n_t} \times P_{m_t} >$ are graceful when T is semi smooth graceful tree.

Kaneria and Meghpara [1613] prove that $B_{m,n}$, the splitting graphs $S'(B_{m,n})$ and $S'(P_n)$ are semi smooth graceful and if graphs obtained by joining semi smooth graceful graph and $B^2_{m,n}$ by an arbitrary path is graceful.

A *komodo dragon* is formed by attaching a path to a vertex of degree 3 in a cycle with a chord and attaching star graphs to the end points of the path. A *komodo dragon with many tails* is formed by attaching many paths of length two to an endpoint of the path in a komodo dragon. In [2852] and [2854] Shahul Hameed, Palanivelrajan, Gunasekaran and Raziya Begam provide graceful labelings of various komodo dragon graphs and their extensions. In [2853] and [2855] Shahul Hameed et al. investigated the gracefulness of classes of graphs constructed by combining some subdivisions of certain theta graphs with stars.

For a bipartite graph G with partite sets X and Y let G' be a copy of G and X' and Y' be copies of X and Y. Lee and Liu [1853] define the *mirror graph*, $M(G)$, of G as the disjoint union of G and G' with additional edges joining each vertex of Y to its corresponding vertex in Y'. The case that $G = K_{m,n}$ is more simply denoted by $M(m, n)$. They proved that for many cases $M(m, n)$ has a stronger form of graceful labeling (see §3.1 for details).

The *total graph* $T(P_n)$ has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in P_n. Balakrishnan, Selvam, and Yegnanarayanan [415] have proved that $T(P_n)$ is harmonious.

For any graph G with vertices $v_1, \ldots, v_n$ and a vector $\mathbf{m} = (m_1, \ldots, m_n)$ of positive integers the corresponding *replicated graph*, $R_{\mathbf{m}}(G)$, of G is defined as follows. For each v_i form a stable set S_i consisting of m_i new vertices $i = 1, 2, \ldots, n$ (a *stable* set S consists of a set of vertices such that there is not an edge v_iv_j for all pairs v_i, v_j in S); two stable sets $S_i, S_j, i \neq j$, form a complete bipartite graph if each v_iv_j is an edge in G and otherwise there are no edges between S_i and S_j. Ramírez-Alfonsín [2577] has proved that $R_{\mathbf{m}}(P_n)$ is graceful for all $\mathbf{m}$ and all $n > 1$ (see §3.4 for a stronger result) and that $R_{(m,1,\ldots,1)}(C_{4n}), R_{(2,1,\ldots,1)}(C_n)$ $(n \geqslant 8)$ and,$R_{(2,2,1,\ldots,1)}(C_{4n})$ $(n \geqslant 12)$ are graceful.

For any permutation f on $1, \ldots, n$, the *f-permutation* graph on a graph G, $P(G, f)$, consists of two disjoint copies of G, G_1 and G_2, each of which has vertices labeled $v_1, v_2, \ldots, v_n$ with n edges obtained by joining each v_i in G_1 to $v_{f(i)}$ in G_2. In 1983 Lee (see [1919]) conjectured that for all $n > 1$ and all permutations on $1, 2, \ldots, n$, the permutation graph $P(P_n, f)$ is graceful. Lee, Wang, and Kiang [1919] proved that $P(P_{2k}, f)$ is graceful when $f = (12)(34)\cdots(k, k+1)\cdots(2k-1, 2k)$. They conjectured that if G is a graceful nonbipartite graph with n vertices, then for any permutation f on $1, 2, \ldots, n$, the permutation graph $P(G, f)$ is graceful. Fan and Liang [913] have shown that if f is a permutation in S_n where $n \geqslant 2(m-1) + 2l$ then the permutation graph $P(P_n, f)$ is graceful if the disjoint cycle form of f is $\prod_{k=0}^{l-1}(m + 2k, m + 2k + 1)$, and if $n \geqslant 2(m-1) + 4l$ the permutation graph $P(P_n, f)$ is graceful the disjoint cycle form of f is $\prod_{k=0}^{l-1}(m + 4k, m + 4k + 2)(m + 4k + 1, m + 4k + 3)$. For any integer $n \geqslant 5$ and some permutations f in S(n), Liang and Y. Miao, [1941] discuss gracefulness of the permutation graphs $P(P_n, f)$ if $f = (m, m+1, m+2, m+3, m+4), (m, m+2)(m+1, m+3), (m, m+1, m+2, m+4, m+3), (m, m+1, m+4, m+3, m+2), (m, m+2, m+3, m+4, m+1), (m, m+3, m+4, m+2, m+1)$ and $(m, m+4, m+3, m+2, m+1)$. In [1943] Liang, Zhang, Xu, Ye, Fan, and Ge prove the permutation graphs $P(P_n, f)$ where f is one of the permutations (12345), (2345), (234), (123456) and (23)(45) are graceful. Some families of graceful permutation graphs are given in [1846], [1936], and [1174].

In [2640] Rofa defined generalized strongly graceful permutations and discovered two new permutations in addition to the known permutation that is obtained by replacing each vertex label $f(v)$ by $q - f(v)$. He use these permutations to prove, by induction, that a lobster with a perfect matching that consists of the set of end edges of the lobster, is strongly graceful. He further showed that there exist strongly graceful labelings that assign the label 0 to four specific vertices of any tree belonging to this family of lobsters. He provided a tractable way for proving an equivalent form of Bermond conjecture which states that all lobsters are graceful. Two out of a total of three cases of the proposed equivalent form of Bermond's conjecture are completed leaving the third case open for refutation or completion. As an applications of his results, he provided in [2641] number systems that are representable by all vertices of a rooted symmetric tree in such a way that the number representation of each vertex depends on its distance from the root vertex.

In [529] Bell provided methods to combine graceful bipartite graphs to create new graceful graphs. These methods unify and generalize some well-known results in the graceful labeling literature. She also found a new class of graceful trees.

The *power graph* of a finite group G has the elements of G as its vertex set and two distinct vertices of G are adjacent if one is a power of other. Sehgal, Takshak, Maan, and

Malik [2735] proved that the power graph of $Z_2^{k-1} \times Z_4$ has a graceful labeling.

A graph (p,q)-graph $G(V,E)$ is said to be (k,d)-*hooked Skolem graceful* if there exists a bijection f from $V(G)$ to $\{1,2,\ldots,p-1,p+1\}$ such that the induced edge labeling g_f from E to $\{k,k+d,\ldots,k+(n-1)d\}$ defined by $g_f(uv) = |f(u)-f(v)|$ for all uv in E is also bijective. Such a labeling f is called a (k,d)-*hooked Skolem graceful labeling* of G. Note that when $k = d = 1$, this notion coincides with that of hooked Skolem graceful labeling of the graph G. In [2383] Pereira, Singh, and Arumugam present some preliminary results on (k,d)-hooked Skolem graceful graphs and prove that nK_2 is $(2,1)$-hooked Skolem graceful if and only if $n \equiv 1$ or $2 \pmod 4$.

Gnanajothi [1104, p. 51] calls a graph G *bigraceful* if both G and its line graph are graceful. She shows the following are bigraceful: P_m; $P_m \times P_n$; C_n if and only if $n \equiv 0,3 \pmod 4$; S_n; K_n if and only if $n \leqslant 3$; and B_n if and only if $n \equiv 3 \pmod 4$. She also shows that $K_{m,n}$ is not bigraceful when $n \equiv 3 \pmod 4$. (Gangopadhyay and Hebbare [1014] used the term "bigraceful" to mean a bipartite graceful graph.) Murugan and Arumugan [2220] have shown that graphs obtained from C_4 by attaching two disjoint paths of equal length to two adjacent vertices are bigraceful.

Several well-known isolated graphs have been examined. Graceful labelings have been found for the Petersen graph [984], the cube [1032], the icosahedron and the dodecahedron. In [1033] [pp. 163-164] Gardner credits Ashenfelter and Chandra for showing the Platonic solids have graceful labelings.[1] Gardner stated that the icosahedron has only five fundamentally different graceful labelings, whereas in 2021 Knuth [1706] determined the correct number to be 24. Graham and Sloane [1147] showed that all of these solids except the cube are harmonious. Winters [3481] verified that the Grőtzsch graph (see [596, p. 118]), the Heawood graph (see [596, p. 236]), and the Herschel graph (see [596, p. 53]) are graceful. Graham and Sloane [1147] determined all harmonious graphs with at most five vertices. Seoud and Youssef [2800] did the same for graphs with six vertices.

In 2009 Zak [3569] defined the following generalization of harmonious labelings. For a graph $G(V,E)$ and a positive integer $t \geqslant |E|$ a function h from $V(G)$ to Z_t (the additive group of integers modulo t) is called a t-*harmonious labeling* of G if h is injective for $t \geqslant |V|$ or surjective for $t < |V|$, and $h(u) + h(v) \neq h(x) + h(y)$ for all distinct edges uv and xy. The smallest such t for which G has a t-harmonious labeling is called the *harmonious order of* G. Obviously, a graph $G(V,E)$with $|E| \geqslant |V|$ is harmonious if and only if the harmonious order of G is $|E|$. Zak determines the harmonious order of complete graphs, complete bipartite graphs, even cycles, some cases of P_n^k, and $2nK_3$. He presents some results about the harmonious order of the Cartesian products of graphs, the disjoint union of copies of a given graph, and gives an upper bound for the harmonious order of trees. He conjectures that the harmonious order of a tree of order n is $n + o(n)$. Hegde and Murthy [1224] proved Zak's conjecture [3569] using the value sets of polynomials, which partially proves the cordial tree conjecture by Hovey [1258] that all trees of order less than a prime p are p-cordial. (See Section 3.7.)

A graceful labeling of P_n is said to be an $(a,b;n)$-*graceful* labeling if one endpoint is labeled a and the other labeled b. A conjecture made in Gvozdjak's PhD Thesis [1168]

[1] D. Knuth called this to my attention.

on the Oberwolfach Problem in 2004 is: "An $(a, b; n)$-graceful labeling of P_n exists if and only if the integers a, b, n satisfy (1) $b - a$ has the same parity as $n(n+1)/2$; (2) $0 < |b - a| \leqslant (n+1)/2$ and (3) $n/2 \leqslant a + b \leqslant 3n/2$." In [3586] Zhang, Zhang, and Wang showed that the conjecture is true for every n whenever it is true for $n \leqslant 4a + 1$ and a is a fixed value. Moreover, they proved that the conjecture is true for $a = 0, 1, 2, 3, 4, 5, 6$.

For a graph with e edges Vietri [3378] generalizes the notion of a graceful labeling by allowing the vertex labels to be real numbers in the interval $[0, e]$. For a simple graph $G(V, E)$ he defines an injective map γ from V to $[0, e]$ to be a *real-graceful* labeling of G provided that $\sum 2^{\gamma(u)-\gamma(v)} + 2^{\gamma(v)-\gamma(u)} = 2^{e+1} - 2^{-e} - 1$, where the sum is taken over all edges uv. In the case that the labels are integers, he shows that a real-graceful labeling is equivalent to a graceful labeling. In contrast to the case for graceful labelings, he shows that the cycles C_{4t+1} and C_{4t+2} have real-graceful labelings. He also shows that the non-graceful graphs K_5, K_6, and K_7 have real-graceful labelings. With one exception, his real-graceful labels are integers.

The *gamma-number* (or gracefulness) of a graph G, denoted by $\gamma(G)$, is the smallest positive integer n for which there exists an injective function $f : V(G) \to \{0, 1, \ldots, n\}$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. The *strong gamma-number* of a graph G, denoted by $\gamma_s(G)$, is defined to be the smallest positive integer n such that $\gamma(G) = n$ with the additional property that there exists an integer λ so that $\min\{f(u), f(v)\} \leqslant \lambda \max\{f(u), f(v)\}$ for each $uv \in E(G)$. The strong gamma-number is defined to be $+\infty$, otherwise. Ichishima and Oshima [1314] proved that if G is a bipartite graph, then $\gamma(mG) \leqslant m\gamma(G) + m - 1$ for any positive integer m. They also show that $\gamma_s(G) < +\infty$ and $\gamma_s(G) \leqslant 2\gamma(G) + 1$ for any bipartite graph G. Moreover, they provide a sharp upper bound for $\gamma(G \cup H)$ in terms of $\gamma(G)$ and $\gamma_s(H)$ when G and H are graphs such that H is bipartite, and give formulas for the gamma-number of certain forests. In addition to these, they present strong gamma-number analogues to the gamma-number results and determine the exact values of the gamma-number and strong gamma-number for all cycles.

A graph G with m vertices and n edges, is said to be *prime graceful* if there is an injection ϕ from the vertices of G to $\{1, 2, \ldots, k\}$ where $k = \min\{2m, 2n\}$ such that $\gcd(\phi(v_i), \phi(v_j)) = 1$ and the induced injective function ϕ^* from the edges of G to $\{1, 2, \ldots, k-1\}$ defined by $\phi^*(v_i v_j) = |\phi(v_i) - \phi(v_j)|$, the resulting edge labels are distinct. In [2745] Selvarajan and Subramoniam proved paths, cycles, stars, friendship graphs, bistars, $C_4 \cup P_n$, $K_{m,2}$, and $K_{m,2} \cup P_n$ have prime graceful labelings.

In 2020 [3572] Zeen El Deen introduced a new type of labeling of a graph as follows. For any postive integer δ an edge δ-graceful labeling of a graph $G(V, E)$ with p vertices and q edges is a bijective f from E to $\{\delta, 2\delta, \ldots, q\delta\}$ such $f[V] = \{f(u) = \sum f(uv) \bmod (k\delta)$ over all edges v incident to u and $k = \max(p, q)$ are pairwise distinct. He proved the existence of an edge $\delta-$graceful labeling, for any positive integer δ, for wheels, alternate triangular cycles, double wheels, $C_n \times P_2$, $W_n \times P_2$, gears, helms, butterflies, and friendship graphs. In [3573] Zeen El Deen and Elmahdy showed that for any positive integer δ there is an edge δ-graceful labeling for the following graphs: the splitting graphs of cycles, fans, and crowns; the shadow graphs of the paths, cycles, and fans; the middle graphs and the total graphs of paths, cycles, and crowns; twigs; and snails.

In [635] Byers and O'Mellan introduceed a new concept that combines graceful and harmonious labelings as follows. A connected graph G with size m is said to be a *graceful-harmonious* labeling if there exits injection $f : V(G) \to \{0, 1, 2, \dots, m\}$ such that when each edge uv is assigned either $|f(u) - f(v)|$ or $(f(u) + f(v)) \pmod{q}$. each edge receives a distinct label from the set $\{0, 1, 2, \dots, m\}$. They prove that cycles, friendship graphs, and double cones $C_{4k} + \overline{K}_2$ admit graceful-harmonious labelings. [635] new

For a graph $G(V, E)$ without isolated vertices, Pereira, Singh, and Arumugam [2382] defined the gracefulness, $grac(G)$, of G as the smallest positive integer k for which there exists an injective function $f : V \to \{0, 1, 2, \dots, k\}$ such that the edge induced function $g_f : E \to \{1, 2, \dots, k\}$ defined by $g_f(uv) = |f(u) - f(v)|$ is also injective. Let $c(f) = \{1, 2, \dots, i\}$ denote the edge labels and let $m(G) = \max\{c(f)\}$, where the maximum is taken over all injective functions $f : V \to N \cup \{0\}$ such that g_f is also injective. This measure $m(G)$ determines how close G is to being graceful. They determine $m(G)$ for certain cycles and friendship graphs. [2382] new

A number of authors have investigated the gracefulness of the directed graphs obtained from copies of directed cycles $\vec{C}_m$ that have a vertex in common or have an edge in common. A digraph $D(V, E)$ is said to be *graceful* if there exists an injection $f\colon V(G) \to \{0, 1, \dots, |E|\}$ such that the induced function $f'\colon E(G) \to \{1, 2, \dots, |E|\}$ that is defined by $f'(u, v) = (f(v) - f(u)) \pmod{|E| + 1}$ for every directed edge uv is a bijection. The notations $n \cdot \vec{C}_m$ and $n - \vec{C}_m$ are used to denote the digraphs obtained from n copies of $\vec{C}_m$ with exactly one point in common and the digraphs obtained from n copies of $\vec{C}_m$ with exactly one edge in common. Du and Sun [855] proved that a necessary condition for $n - \vec{C}_m$ to be graceful is that mn is even and that $n \cdot \vec{C}_m$ is graceful when m is even. They conjectured that $n \cdot \vec{C}_m$ is graceful for any odd m and even n. This conjecture was proved by Jirimutu, Xu, Feng, and Bao in [1558]. Xu, Jirimutu, Wang, and Min [3502] proved that $n - \vec{C}_m$ is graceful for $m = 4, 6, 8, 10$ and even n. Feng and Jirimutu (see [3589]) conjectured that $n - \vec{C}_m$ is graceful for even n and asked about the situation for odd n. The cases where $m = 5, 7, 9, 11$, and 13 and even n were proved Zhao and Jirimutu [3588]. The cases for $m = 15, 17$, and 19 and even n were proved by Zhao et al. in [3587], and [2989]. Zhao, Siqintuya, and Jirimutu [3589] proved that a necessary condition for $n - \vec{C}_m$ to be graceful is that nm is even.

In a 1985 paper Bloom and Hsu [579] say a directed graph D with e edges has a graceful labeling θ if for each vertex v there is a vertex labeling θ that assigns each vertex a distinct integer from 0 to e such that for each directed edge (u, v) the integers $\theta(v) - \theta(u) \bmod (e + 1)$ are distinct and nonzero. They conjectured that digraphs whose underlying graphs are wheels and that have all directed edges joining the hub and the rim in the same direction and all directed edges in the same direction are graceful. This conjecture was proved in 2009 by Hegde and Shivarajkumar [1236]. Yao, Yao, and Cheng [3532] investigated the gracefulness for many orientations of undirected trees with short diameters and proved some directed trees do not have graceful labelings. Hegde and Kumudkshi [1222] established the gracefulness of the directed graph that is an orientation of the planar grid graph $P_m \times P_n$ in which each cell is a unicycle of length four. A *graceful difference labeling* of a directed graph G with vertex set V is a bijection $f : V \to \{1, \ldots, |V|\}$ such that, when each arc uv is assigned the difference label $f(v) - f(u)$, the resulting arc labels are distinct. Hertz and Picouleau [1246] conjectured that all disjoint unions of circuits have a graceful difference labeling, except in two particular cases. They provided partial results that support this conjecture. A survey of results on graceful digraphs by Feng, Xu, and Jirimutu is given in [921]. Marr [2094] and [2093] summarizes previously known results on graceful directed graphs and presents some new results on directed paths, stars, wheels, and umbrellas.

In [2938] Shivarajkumar, Sriraj, and Hegde provided a 2021 survey results on graceful labeling of digraphs.

2.8 Summary

The results and conjectures discussed above are summarized in the tables following. The letter G after a class of graphs indicates that the graphs in that class are known to be graceful; a question mark indicates that the gracefulness of the graphs in the class is an open problem; we put a question mark after a "G" if the graphs have been conjectured to be graceful. The analogous notation with the letter H is used to indicate the status of the graphs with regard to being harmonious. The tables impart at a glimpse what has been done and what needs to be done to close out a particular class of graphs. Of course, there is an unlimited number of graphs one could consider. One wishes for some general results that would handle several broad classes at once but the experience of many people suggests that this is unlikely to occur soon. The Graceful Tree Conjecture alone has withstood the efforts of scores of people over the past four decades. Analogous sweeping conjectures are probably true but appear hopelessly difficult to prove. I thank Don Knuth for his correspondence about the results of Smith and Puget [3008] in Table 1 regarding the gracefulness $K_m \times K_,$ $K_m \times P_n$, and $K_m \times C_n$.

Table 1: **Summary of Graceful Results**

Graph	*Graceful*
trees	G if $\leqslant 35$ vertices [914]
	G if symmetrical [548]
	G if at most 4 end-vertices [1268]
	G with diameter at most 5 [1259]
	G? Ringel-Kotzig
	G caterpillars [2648]
	G firecrackers [692]
	G bananas [2822], [2821]
	G? lobsters [544]
cycles C_n	G iff $n \equiv 0, 3 \pmod 4$ [2648]

Continued on next page

Table 1 – *Continued from previous page*

Graph	*Graceful*
wheels W_n	G [984], [1255]
helms (see §2.2)	G [266]
webs (see §2.2)	G [1633]
gears (see §2.2)	G [2043]
cycles with P_k-chord (see §2.2)	G [798], [2042], [1723], [2531]
C_n with k consec. chords (see §2.2)	G if $k = 2, 3, n-3$ [1712], [1720]
unicyclic graphs	G? iff G $\neq C_n, n \equiv 1, 2 \pmod 4$ [3229]
P_n^k	G if $k = 2$ [1633]
$C_n^{(t)}$ (see §2.2)	$n = 3$ G iff $t \equiv 0, 1 \pmod 4$ [545], [547] G? if $nt \equiv 0, 3 \pmod 4$ [1713] G if $n = 6, t$ even [1713] G if $n = 4, t > 1$ [2874] G if $n = 5, t > 1$ [3519] G if $n = 7$ and $t \equiv 0, 3 \pmod 4$ [3525] G if $n = 9$ and $t \equiv 0, 3 \pmod 4$ [3526] G if $t = 2$ $n \not\equiv 1 \pmod 4$ [2537], [589] G if $n = 11$ [3504]
triangular snakes (see §2.2)	G iff no. blocks $\equiv 0, 1 \pmod 4$ [2200]
K_4-snakes (see §2.2)	?
quadrilateral snakes (see §2.2)	G [1104], [2537]
crowns $C_n \odot K_1$	G [984]
$C_n \odot P_k$	G [2736]
grids $P_m \times P_n$	G [41]
prisms $C_m \times P_n$	G if $n = 2$ [987], [3522]

Continued on next page

Table 1 – *Continued from previous page*

Graph	*Graceful*
	G if m even [1271] G if m odd and $3 \leqslant n \leqslant 12$ [1271] G if $m = 3$ [2970] G if $m = 6$ see [3524] G if $m \equiv 2 \pmod 4$, $n \equiv 3 \pmod 4$ [3524]
$K_m \times P_n$	G if $(m,n) = (4,2),(4,3),(4,4),(4,5),$ $(5,2),(5,3),(6,3),(4,6),(4,7),(4,8)$ not G if $(3,3),(m,2)$ $m = 6,7,8,9,10,$ 11,12 not G? for $(m,2)$ with $m > 12$ [3008]
$K_m \times C_n$	G if $(m,n) = (4,3),(3,4),(4,4),(4,5),$ $(3,6),(4,6)$ not G for $(m,n) = (6,3)$ [3008]
$K_m \odot K_1$	G if $m = 3,4,5,6,7,8,9$ not G if $m = 10,11,12,13,14,15$ not G? if $m > 15$ [3008]
$K_{m,n} \odot K_1$	G [1551]
$K_m \cup K_n$ $(m,n > 1)$	G iff $\{m,n\} = \{4,2\}$ or $\{5,2\}$
$\bigcup_{i=1}^{t} K_{m_i,n_i}$	G $2 \leqslant m_i < n_i$ [439]
torus grids $C_m \times C_n$	G if $m \equiv 0 \pmod 4$, n even [1561] not G if m,n odd (parity condition)
vertex-deleted $C_m \times P_n$	G if $n = 2$ [1007]
edge-deleted $C_m \times P_n$	G if $n = 2$ [1007]
Möbius ladders M_n (see §2.3)	G [1001]
stacked books $S_m \times P_n$ (see §2.3)	$n = 2$, G iff $m \not\equiv 3 \pmod 4$ [2053], [797], [1006] G if m even [1006]
n-cube $K_2 \times K_2 \times \cdots \times K_2$	G [1741]

Continued on next page

Table 1 – *Continued from previous page*

Graph	*Graceful*
$K_n \times P_3$	G iff $n \leqslant 6$ [1707]
K_n	G iff $n \leqslant 4$ [1115], [2964]
$K_{m,n}$	G [2648], [1115]
$K_{1,m,n}$	G [220]
$K_{1,1,m,n}$	G [1104]
windmills $K_n^{(m)} (n > 3)$ (see §2.4)	G if $n = 4, m \leqslant 1000$ [1271],[7],[3454],[1065] G? if $n = 4, m \geqslant 4$ [544] G if $n = 5, m = 4,\ 5$ [3055] not G if $n = 4, m = 2, 3$ [544] not G if $(m, n) = (2, 5)$ [547] not G if $n > 5$ [1720]
$B(n, r, m)$ $r > 1$ (see §2.4)	G if $(n, r) = (3, 2), (4, 3)$ [1714], (4,2) [797] G $(n, r, m) = (5, 2, 2)$ [3008] not G for $(n, 2, 2)$ for $n > 5$ [546], [3008]
mK_n (see §2.5)	G iff $m = 1, n \leqslant 4$ [1745]
$C_m \cup P_n$	G iff $m + n \geqslant 6$ [3222]
$C_m \cup C_n$	G iff $m + n \equiv 0, 3 \pmod 4$ [24]
$C_n \cup K_{p,q}$	for $n > 8$ G iff $n \equiv 0, 3 \pmod 4$ [3545] G $C_6 \times K_{1,2n+1}$ [439] G $C_3 \times K_{m,n}$ iff $m, n \geqslant 2$ [2799] G $C_4 \times K_{m,n}$ iff $(m, n) \neq (1, 1)$[2799] G $C_7 \times K_{m,n}$ [2799] G $C_8 \times K_{m,n}$ [2799]
$K_i \cup K_{m,n}$	G [439]
$\bigcup_{i=1}^{t} K_{m_i,n_i}$	G $2 \leqslant m_i < n_i$ [439]
$C_m \cup \bigcup_{i=1}^{t} K_{m_i,n_i}$	G $2 \leqslant m_i < n_i$, $m \equiv 0$ or 3 (mod 4), $m \geqslant 11$ [439]

Continued on next page

Table 1 – *Continued from previous page*

Graph	*Graceful*
$G + \overline{K_t}$	G for connected graceful G [1714]
double cones $C_n + \overline{K_2}$	G for $n = 3, 4, 5, 7, 8, 9, 11, 12$ not G for $n \equiv 2 \pmod 4$ [2610]
t-point suspension $C_n + \overline{K_t}$	G if $n \equiv 0$ or 3 (mod 12) [563] not G if t is even and $n \equiv 2, 6, 10 \pmod{12}$ G if $n = 4, 7, 11$ or 19 [563] G if $n = 5$ or 9 and $t = 2$ [563]
P_n^2 (see §2.7)	G [1845]
Petersen $P(n, k)$ (see §2.7)	G for $n = 5, 6, 7, 8, 9, 10$ [2610], $(n, k) = (8t, 3)$ [3361]

Table 2: **Summary of Harmonious Results**

Graph	*Harmonious*
trees	H if $\leqslant 31$ vertices [915] H? [1147] H caterpillars [1147] ? lobsters
cycles C_n	H iff n is odd [1147]
wheels W_n	H [1147]
helms (see §2.2)	H [1104], [1977]
webs (see §2.2)	H if cycle is odd
gears (see §2.2)	H [693]
cycles with P_k-chord (see §2.2)	?
C_n with k consec. chords	?

Continued on next page

Table 2 – *Continued from previous page*

Graph	*Harmonious*
(see §2.2)	
unicyclic graphs	?
P_n^k	H if $k = 2$ [1144], k odd [2762], [3552] H if k is even and $k/2 \leqslant (n-1)/2$ [3565]
$C_n^{(t)}$ (see §2.2)	$n = 3$ H iff $t \not\equiv 2 \pmod 4$ [1147] H if $n = 4,\ t > 1$ [2874]
triangular snakes (see §2.2)	H if number of blocks is odd [3501] not H if number of blocks $\equiv 2$ (mod 4) [3501]
K_4-snakes (see §2.2)	H [1145]
quadrilateral snakes (see §2.2)	?
crowns $C_n \odot K_1$	H [1144], [1963]
grids $P_m \times P_n$ prisms $C_m \times P_n$	H iff $(m, n) \neq (2, 2)$ [1561] H if $n = 2, m \neq 4$ [1007] H if n odd [1147] H if $m = 4$ and $n \geqslant 3$ [1561]
torus grids $C_m \times C_n$,	H if $m = 4$, $n \geqslant 3$ [1561] not H if $m \not\equiv 0 \pmod 4$, n odd [1561]
vertex-deleted $C_m \times P_n$	H if $n = 2$ [1007]
edge-deleted $C_m \times P_n$	H if $n = 2$ [1007]
Möbius ladders M_n (see §2.3)	H iff $n \neq 3$ [1001]
stacked books $S_m \times P_n$ (see §2.3)	$n = 2$, H if m even [1143], [2611] not H $m \equiv 3 \pmod 4$, $n = 2$, (parity condition)

Continued on next page

Table 2 – *Continued from previous page*

Graph	*Harmonious*
	H if $m \equiv 1 \pmod 4$, $n = 2$ [1104]
n-cube $K_2 \times K_2 \times \cdots \times K_2$	H if and only if $n \geqslant 4$ [1310]
$K_4 \times P_n$	H [2611]
K_n	H iff $n \leqslant 4$ [1147]
$K_{m,n}$	H iff m or $n = 1$ [1147]
$K_{1,m,n}$	H [220]
$K_{1,1,m,n}$	H [1104]
windmills $K_n^{(m)}$ $(n > 3)$ (see §2.4)	H if $n = 4$ [1263] $m = 2$, H? iff $n = 4$ [1147] not H if $m = 2, n$ odd or 6 [1147] not H for some cases $m = 3$ [1962]
$B(n, r, m)$ $r > 1$ (see §2.4)	$(n, r) = (3, 2), (4, 3)$ [2796]
mK_n (see §2.5)	H $n = 3$, m odd [1964] not H for n odd and $m \equiv 2 \pmod 4$ [1964]
nG	H when G is harmonious and n odd [3543]
G^n	H when G is harmonious and n odd [3543]
$C_m \cup P_n$	H $n = 1$ iff $m \neq 2$ mod 4 [2284] H $n = 2$ [2284] H $(m, 3)$ odd $m \geqslant 3$ [2614], [2284] H? $(m, 3)$ $m \geqslant 3$
fans $F_n = P_n + K_1$	H [1147]
$nC_m + K_1$ $n \not\equiv 0$ mod 4	H [693]

Continued on next page

Table 2 – *Continued from previous page*

Graph	*Harmonious*
double fans $P_n + \overline{K_2}$	H [1147]
t-point suspension $P_n + \overline{K_t}$ of P_n	H [2611]
$S_m + K_1$	H [1104], [674]
t-point suspension $C_n + \overline{K_t}$ of C_n	H if n odd and $t = 2$ [2611], [1104] not H if $n \equiv 2, 4, 6 \pmod 8$ and $t = 2$ [1104]
Petersen $P(n, k)$ (see §2.7)	H [1104], [1881]

3 Variations of Graceful Labelings

3.1 α-labelings

In 1966 Rosa [2648] defined an *α-labeling* (or *α-valuation*) as a graceful labeling with the additional property that there exists an integer k so that for each edge xy either $f(x) \leqslant k < f(y)$ or $f(y) \leqslant k < f(x)$. (Other names for such labelings are *balanced*, *interlaced*, and *strongly graceful*.) It follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an α-labeling is necessarily bipartite and therefore can not contain a cycle of odd length. Wu [3490] has shown that a necessary condition for a bipartite graph with n edges and degree sequence $d_1, d_2, \ldots, d_p$ to have an α-labeling is that the $\gcd(d_1, d_2, \ldots, d_p, n)$ divides $n(n-1)/2$. Barrientos and Minion [468] proved that any tree of size n and excess ϵ is a spanning tree of a graph of size $n + \epsilon$ that admits an α-labeling.

For a path with consecutive vertices $v_1, v_2, \ldots, v_n$ a *triangular tree* is the tree obtained identifying each v_i to an end vertex of the path P_i. Barrientos [446] proved that all triangular trees admit an α-labeling. He also presented several ways to combine this type of trees to construct new trees and unicyclic graphs that can α-labeled.

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an α-labeling is a typical approach. (See [674], [1144], [692], and [1561].) Moreover, Jungreis and Reid [1561] showed how sequential labelings of graphs (see Section 4.1) can often be obtained by modifying α-labelings of the graphs.

Graphs with α-labelings have proved to be useful in the development of the theory of graph decompositions. Rosa [2648], for instance, has shown that if G is a graph with q edges and has an α-labeling, then for every natural number p, the complete graph K_{2qp+1} can be decomposed into copies of G in such a way that the automorphism group of the decomposition itself contains the cyclic group of order p. In the same vein El-Zanati and Vanden Eynden [886] proved that if G has q edges and admits an α-labeling then $K_{qm,qn}$ can be partitioned into subgraphs isomorphic to G for all positive integers m and n. Although a proof of Ringel's conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have α-labelings are easy to construct (one example is the subdivision graph of $K_{1,3}$ — see [2648]). Kotzig [1739] has shown however that almost all trees have α-labelings. Sethuraman and Ragukumar [2827] have proved that every tree is a subtree of a graph with an α-labeling.

As to which graphs have α-labelings, Rosa [2648] observed that the n-cycle has an α-labeling if and only if $n \equiv 0 \pmod 4$ whereas P_n always has an α-labeling. Other familiar graphs that have α-labelings include caterpillars [2648], the n-cube [1738], Möbius ladders M_n when n is odd (see §2.3) for the definition) [2352], B_{4n+1} (i.e., books with $4n+1$ pages) [1006], $C_{2m} \cup C_{2m}$ and $C_{4m} \cup C_{4m} \cup C_{4m}$ for all $m > 1$ [1740], $C_{4m} \cup C_{4m} \cup C_{4n}$ for all $(m, n) \neq 1, 1)$ [905], $P_n \times Q_n$ [2053], $K_{1,2k} \times Q_n$ [2053], $C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$ [1798], $C_{4m} \cup C_{4n+2} \cup C_{4r+2}, C_{4m} \cup C_{4n} \cup C_{4r}$ when $m + n \leqslant r$ [24], $C_{4m} \cup C_{4n} \cup C_{4r} \cup C_{4s}$ when $m \geqslant n + r + s$ [20], $C_{4m} \cup C_{4n} \cup C_{4r+2} \cup C_{4s+2}$ when $m \geqslant n + r + s + 1$ [20], $((m+1)^2 + 1)C_4$ for all m [3596], $k^2 C_4$ for all k [3596], and $(k^2 + k)C_4$ for all k [3596]. Abrham and Kotzig

[22] have shown kC_4 has an α-labeling for $4 \leqslant k \leqslant 10$ and that if kC_4 has an α-labeling then so does $(4k+1)C_4, (5k+1)C_4$, and $(9k+1)C_4$. Eshghi [898] proved that $3C_{4k}$ and $5C_{4k}$ have an α-labeling for all k. In [905] Eshghi and Carter show several families of graphs of the form $C_{4n_1} \cup C_{4n_2} \cup \cdots \cup C_{4n_k}$ have α-labelings.

In [901] Eshghi provides an integer programming model and a Tabu search algorithm to generate α-labelings of the quadratic graphs $mC_{4k})$ where $6 \geqslant m \geqslant 10$ and $2 \geqslant k \geqslant 10$. (See also [907].) The computational complexity of the gracefulness of a graph is not known, but the complexity of finding a harmonious labeling of a graph is in the NP-class [176]. Research on programming models for finding graceful labelings of graphs can be found in [897], [907], [906], [1795], [2674], [903], [2610], [3008], [2066], and [2844].

In [176] Amini and Eshghi gave a new mathematical integer programming model for the graph labeling graphs of the form mC_n (some authors use the notation $Q(m,n)$). The advantages of this model are linearity and the existence of an objective function. They also gave two constraint programming models and a meta-heuristics algorithm that generate feasible graceful labeling and α-labeling for special classes of quadratic graphs. Their results include: mC_{4k} with $1 \leqslant 11$ and less than 1000 vertices has an α-labeling with the exception of $3C_4$; $12C_{4k}$ has α-labeling for $1 \leqslant k \leqslant 19$; and $13C_{4k}$ has α-labeling for $1 \leqslant k \leqslant 13$. In [906] and [2674] Eshghi and Salarrezaei proved that $7C_{4k}$ has an α-labeling for all k. Lakshmi and Vangipuram [1795] proved that $4C_{4k}$ is graceful.

In [470], Barrientos and Minion investigated series-parallel operations with graphs that admit α-labelings. They provided necessary conditions on the graphs G_1 and G_2 to obtain a new α-labeled graph G through each of these operations. As consequence of the series operation, they proved that the one-point union of three or four copies of $K_{n,n}$ has an α-labeling, and that any tree with maximum degree four that can be decomposed into copies of the path of length eleven has an α-labeling when the distance between any pair of vertices of degree four is even. They also showed that any graph of order $n+1$ and size n with an α-labeling is an induced subgraph of a graph of order $n+3$ and size $2n+1$. Additionally, they presented an α-labeling for any graph of the form $K_{2,n} \times P_m$. In [448] Barrientos used vertex and edge duplications, replications of the entire graph, and k-vertex amalgamations to generate α-labeled graphs. He proved that for some families of graphs, it is possible to duplicate several vertices or edges. Using k-vertex amalgamations he obtained an α-labeling of a graph that can be decomposed into multiple copies of a given α-labeled graph as well as a robust family of irregular grids that can α-labeled.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [935] have shown that if $m \equiv 0 \pmod 4$ then the one-point union of 2, 3, or 4 copies of C_m admits an α-labeling, and if $m \equiv 2 \pmod 4$ then the one-point union of 2 or 4 copies of C_m admits an α-labeling. They conjecture that the one-point union of n copies of C_m admits an α-labeling if and only if $mn \equiv 0 \pmod 4$.

Pei-Shan Lee [1835] proved that $C_6 \times P_{2t+1}$ and gear graphs have α-labelings. He raises the question of whether $C_{4m+2} \times P_{2t+1}$ has an α-labeling for all m. Brankovic, Murch, Pond, and Rosa [608] conjectured that all trees with maximum degree three and a perfect matching have an α-labeling.

In his 2001 Ph. D. thesis Selvaraju [2739] investigated the one-point union of complete bipartite graphs. He proves that the one-point unions of the following forms have an α-labeling: K_{m,n_1} and K_{m,n_2}; K_{m_1,n_1}, K_{m_2,n_2}, and K_{m_3,n_3} where $m_1 \leqslant m_2 \leqslant m_3$ and $n_1 < n_2 < n_3$; $K_{m_1,n}$, $K_{m_2,n}$, and $K_{m_3,n}$ where $m_1 < m_2 < m_3 \leqslant 2n$.

Zhile [3596] uses $C_m(n)$ to denote the connected graph all of whose blocks are C_m and whose block-cutpoint-graph is a path. He proves that for all positive integers m and n, $C_{4m}(n)$ has an α-labeling but $C_m(n)$ does not have an α-labeling when m is odd.

Abrham and Kotzig [24] have proved that $C_m \cup C_n$ has an α-labeling if and only if both m and n are even and $m + n \equiv 0 \pmod 4$. Kotzig [1740] has also shown that $C_4 \cup C_4 \cup C_4$ does not have an α-labeling. He asked if $n = 3$ is the only integer such that the disjoint union of n copies of C_4 does not have an α-labeling. This was confirmed by Abrham and Kotzig in [23]. Eshghi [897] proved that every 2-regular bipartite graph with 3 components has an α-labeling if and only if the number of edges is a multiple of four except for $C_4 \cup C_4 \cup C_4$. In [900] Eshghi gives more results on the existence of α-labelings for various families of disjoint union of cycles.

Jungreis and Reid [1561] investigated the existence of α-labelings for graphs of the form $P_m \times P_n, C_m \times P_n$, and $C_m \times C_n$ (see also [1004]). Of course, the cases involving C_m with m odd are not bipartite, so there is no α-labeling. The only unresolved cases among these three families are $C_{4m+2} \times P_{2n+1}$ and $C_{4m+2} \times C_{4n+2}$. All other cases result in α-labelings.

Let $v_{1,j}, v_{2,j}, \ldots, v_{m,j}$ be the consecutive vertices of the jth copy of P_m in $P_m \times P_n$. An *elementary transformation* of $P_m \times P_n$ is the graph obtained by replacing the edge $v_{i,j}v_{i+1,j}$ by the new edge $v_{i-x,j}v_{i+1+x,j}$. A graph is said to be a *grid-like graph* if it is obtained through a sequence of elementary transformations. In [471] Barrientos and Minion proved the exitence of an α-labeling for any grid-like graph. As consequence of this result, they showed that the graphs $C_{4t} \times P_n \cup P_n$ and $C_{4t} \times P_n \cup P_{t-1} \times P_n$ admit α-labelings.

Balakrishman [408] uses the notation $Q_n(G)$ to denote the graph $P_2 \times P_2 \times \cdots \times P_2 \times G$ where P_2 occurs $n-1$ times. Snevily [3011] has shown that the graphs $Q_n(C_{4m})$ and the cycles C_{4m} with the path P_n adjoined at each vertex have α-labelings. He [3012] also has shown that compositions of the form $G[\overline{K_n}]$ (see §2.3 for the definition) have an α-labeling whenever G does (see §2.3 for the definition of composition). Balakrishman and Kumar [412] have shown that all graphs of the form $Q_n(G)$ where G is $K_{3,3}, K_{4,4}$, or P_m have an α-labeling. Balakrishman [408] poses the following two problems. For which graphs G does $Q_n(G)$ have an α-labeling? For which graphs G does $Q_n(G)$ have a graceful labeling?

Rosa [2648] has shown that $K_{m,n}$ has an α-labeling (see also [436]). In [1313] Ichishima and Oshima proved that if m, s and t are integers with $m \geqslant 1, s \geqslant 2$, and $t \geqslant 2$, then the graph $mK_{s,t}$ has an α-labeling if and only if $(m, s, t) \neq (3, 2, 2)$. Barrientos [436] has shown that for n even the graph obtained from the wheel W_n by attaching a pendent edge at each vertex has an α-labeling. In [443] Barrientos shows how to construct graceful graphs that are formed from the one-point union of a tree that has an α-labeling, P_2, and the cycle C_n. In some cases, P_2 is not needed. Qian [2537] has proved that quadrilateral snakes have α-labelings. Yu, Lee, and Chin [3562] showed that Q_3-and Q_3-snakes have α-labelings. Fu and Wu [991] showed that if T is a tree that has an α-labeling with partite sets V_1 and V_2 then the graph obtained from T by joining new vertices $w_1, w_2, \ldots, w_k$ to every vertex of V_1 has an α-labeling. Similarly, they prove that the graph obtained from T by joining new vertices $w_1, w_2, \ldots, w_k$ to the vertices of V_1 and new vertices $u_1, u_2, \ldots, u_t$ to every vertex of V_2 has an α-labeling. They also prove that if one of the new vertices of either of these two graphs is replaced by a star and every vertex of the star is joined to the vertices of V_1 or the vertices of both V_1 and V_2, the resulting graphs have α-labelings. Fu and Wu [991] further show that if T is a tree with an α-labeling and the sizes of the two partite sets of T differ by at most 1, then $T \times P_m$ has an α-labeling. Zhao, Ma, and Yao [3591] proved that a class of super lobster trees have α-labelings. Ghosh [1096] uses various methods of joining graceful graphs and graphs with α-labelings to obtain some classes of graceful lobsters. Lalitha and Tamilselvi [1796] proved that the hexagonal snake graph has an α-labeling.

Selvaraju and G. Sethurman [2744] prove that the graphs obtained from a path P_n by joining all the pairs of vertices u, v of P_n with $d(u, v) = 3$ and the graphs obtained by identifying one of vertices of degree 2 of such graphs with the center of a star and the other vertex the graph of degree 2 with the center of another star (the two stars needs need not have the same size) have α-labelings. They conjecture that the analogous graphs where 3 is replaced with any t with $2 \leqslant t \leqslant n-2$ have α-labelings.

Makadia, Karavadiya, and Kanerian [2070] proved that the graph obtained by merging t consecutive vertices of two cycle C_{4r} and C_{4s} has an α-labeling when $t \leqslant 2\min\{r, s\}$. They also proved that if G_1 has an α-labeling and G_2 is graceful then there exists a graceful labeling of the graph obtained by joining G_1 and G_2 by any path. Moreover, if both G_1 and G_2 have α-labelings then there exists an α-labeling of the graph obtained by joining G_1 and G_2 by any path. Let $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ be a collection of cycles. In [469], Barrientos and Minion say that a graph G is the *coalescence* of these cycles if for every $2 \leqslant i \leqslant k$, the first t_i vertices of C_{n_i} are identified with the last t_i vertices of $C_{n_{i-1}}$, where $t_i \leqslant n_i/2$. They proved that the coalescence of these cycles admits an α-labeling when each $n_i \equiv 0 (\bmod 4)$.

Lee and Liu [1853] investigated the mirror graph $M(m, n)$ of $K_{m,n}$ (see §2.3 for the definition) for α-labelings. They proved: $M(m, n)$ has an α-labeling when n is odd or m is even; $M(1, n)$ has an α-labeling when $n \equiv 0 \pmod 4$; $M(m, n)$ does not have an α-labeling when m is odd and $n \equiv 2 \pmod 4$, or when $m \equiv 3 \pmod 4$ and $n \equiv 4 \pmod 8$.

Kumar, Mishra, Kumar, and Kumar [1774] proved that the following graphs have alpha labelings: $C_{4n} \odot K_1$, the graph obtained by joining any path to a vertex of C_{4n}, and graphs obtained by joining two isomorphic copies of $C_{4n} \odot K_1$.

Barrientos and Minion [460] proved that the Cartesian product of two α-trees is an α-tree when both trees admit α-labelings and their stable sets are balanced. (A *stable* set S consists of a set of vertices such that there is not an edge $v_i v_j$ for all pairs v_i, v_j in S). In addition, they present a tree that has the property that when any number of pendent vertices are attached to the vertices of any subset of its smaller stable set the resulting graph is an α-tree. They also prove of an α-labeling of three types of graphs obtained by connecting, sequentially, any number of paths of equal size.

Barrientos [437] defines a *chain graph* as one with blocks $B_1, B_2, \ldots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cutpoint graph is a path. He shows that if $B_1, B_2, \ldots, B_m$ are blocks that have α-labelings then there exists a chain graph G with blocks $B_1, B_2, \ldots, B_m$ that has an α-labeling. He also shows that if $B_1, B_2, \ldots, B_m$ are complete bipartite graphs, then any chain graph G obtained by concatenation of these blocks has an α-labeling.

The *symmetric product* $G_1 \oplus G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2)\}$ where u_1u_2 is an edge in G_1 or v_1v_2 is an edge in G_2 but not both u_1u_2 is an edge in G_1 and v_1v_2 is an edge in G_2. A *snake* of length $n > 1$ is a packing of n congruent geometrical objects, called *cells*, such that the first and the last cell each has only one neighbor and all $n-2$ cells in between have exactly two neighbors. In [456] Barrientos and Minion define a *snake polyomino* as a snake with square cells. They prove that given two graphs of sizes m and n with α-labelings, the graph that results from the edge amalgamation (identification of two edges) of the edges of weight 1 and n, also has an α-labeling. They use that result to prove the existence of α-labelings of snake polyominoes and hexagonal chains. The result about snake polyominoes partially answers the question of Acharya. In [457], they prove that the third power of a caterpillar admits an α-labeling and that the symmetric product $G \oplus 2K_1$ has an α-labeling when G does. In addition they prove that $G \cup P_m$ is graceful provided that G admits an α-labeling that does not assign the integer $\lambda + 2$ as a label, where λ is its boundary value. They ask if all triangular chains are graceful.

In [462] Barrientos and Minion proved that under certain conditions, the union $C_r \cup G$ of the cycle C_r and a caterpillar G admits a graceful labeling when r is odd, and an α-labeling when r is even. They also proved the existence of an α-labeling for any tree obtained by connecting with a path of length two the central vertices of G_i and G_{i+1}, where G_i is a caterpillar of diameter $2d$ with bipartite sets A_i and B_i such that $|A_i| = |B_i| + 1$ and A_i contains the vertices of maximum eccentricity in G_i.

Let $T_1, T_2, \ldots, T_s$ be trees. A *chain tree* obtained by identifying, for every $1 \leqslant i \leqslant s-1$, a vertex of T_i with a vertex of T_{i+1}. In [463], Barrientos and Minion prove that if every T_i admits an α-labeling, then there exists a chain tree that also admits an α-labeling. Let T be a tree of size n and v be a fixed vertex of T. The tree T_v^{+r} is obtained by connecting, with a path of length r, two copies of T, by identifying the end-point of this path with the vertices v of each copy of T. They give necessary conditions for the existence of an α-labeling for a tree T_v^{+2}, where v is any of the vertices labeled $\lambda, \lambda-1, \ldots, \lambda-\deg(v)-1$ by an α-labeling with boundary value λ that assigns the labels $\lambda+1, \lambda+2, \ldots, \lambda+\deg(v)$ to leaves of T. In addition they proved that T_v^{+4} has an α-labeling if there exists an α-labeling f of T, with boundary value λ, such that $f(v) = \lambda - 1$. In [463], Barrientos and Minion prove the following.The tree $\oplus(T_1, T_2, T_3, T_4)$ obtained by connecting to a new vertex w, the vertices labeled n in T_1 and T_3 and the vertices labeled $n/2$ in T_2 and T_4, where T_i is an α-labeled tree of even size n that has partite sets of cardinality $n/2$ and $n/2+1$. If G is a graph of order m and size n, with $m < n$, that admits an α-labeling, and H is any graceful graph of size $t-1$, then $tG \cup H$ is a graceful graph. For every $m \geqslant n$, $m \geqslant 3$, $n \geqslant 2$, and $t \geqslant 2$, $tK_{m,n} \cup L_{t-1}$ admits an α-labeling where L_{t-1} is any linear forest of size $t-1$. If G is a graph of order m and size n, with $m < n$, that admits an α-labeling, then $tG \cup L_{t-1}$ also admits an α-labeling when L_{t-1} is a linear forest of size $t-1$. As a consequence of this result they prove that $tG \cup P_t$ admits an α-labeling provided that G does.

Barrientos [467] showed that all lobsters constructed with k copies of any caterpillar of diameter four by connecting the central vertices of all pairs of consecutive copies with an edge have an α-labeling. Additionally, he proved that any chain-tree formed by caterpillars and this type of lobsters admits an α-labeling. Barrientos and Minion [471] say that a tree is *regular* when the cardinalities of its stable sets are equal or differ by one. They prove if S and T are regular trees that admit α-labelings then $S \times T$ also admits an α-labeling. They use this result to prove that $S \times T$ admits a sequential labeling (see Section 4.1) as well as a harmonious labeling. They define a *fence* as the tree obtained by connecting an internal vertex of P_{n_i} with an internal vertex of $P_{n_{i+1}}$ by a path of length l_i for every $1 \leqslant i \leqslant t$. They prove the existence of an α-labeling for any fence constructed with t copies of P_n, where $l_i = 2$. They define a 2-*link fence* as the graph obtained by connecting with an edge, two vertices of the ith copy of P_n, with the corresponding two vertices of the $(i+1)$th copy of P_n. They prove that all such graphs admit α-labelings. In [467] Barrientos says that a fence is *irregular* if two consecutive copies of P_n are connected by one or two pairs of corresponding vertices. He proved that all irregular fences have an α-labeling provided that all their Eulerian subgraphs have size divisible by four. In [472] Barrientos and Minion study subfamilies of 2-link fences, a subfamily of column-convex polyominoes, and a subfamily of irregular cyclic-snakes. They prove that under certain conditions, an α-labelings of these graphs can be transformed into harmonious labelings.

Barrientos and Minion [474] provided new families of harmoniously labeled graphs built on α-labeled tress. Among them are ${P_n}^k$, the join of G and tK_1 where G has a restrictive type of harmonious labeling and its order is different of its size by at most one, $K_{m,n} \cup K_{1,m-1}$, and $G \cup T$ where G is a unicyclic graph and T is a tree built with α-trees. They also showed that almost all trees admit harmonious labelings.

In [466] Barrientos and Minion extend the concept of vertex amalgamation as follows. The *k-vertex amalgamation* of G_1 and G_2 is the graph obtained by identifying k independent vertices of G_1 with k independent vertices of G_2. A *t-fold* of a graph G is obtained using t-copies of G, where the ith copy of G is k-vertex amalgamated with the $(i+1)$th copy of G. They prove that if G admits an α-labeling, then any t-fold of G admits an α-labeling. They consider a more general version of this construction for the case where G is a tree. They also introduce a new family of trees that admit α-labelings; in particular, they prove that any tree of diameter $2n$ formed by identifying the end-vertices of four caterpillars admits an α-labeling.

Fronček, Kingston, and Vezina [973] generalized snake polyomino graphs by introducing *straight simple polyominal caterpillars* and proving that they also admit an alpha labeling. This implies that every straight simple polyominal caterpillar with n edges decomposes the complete graph K_{2kn+1} for any positive integer k. In [967] Fronček introduced a similar family of graphs called *full hexagonal caterpillars* and prove that they admit an alpha labeling. This implies that every full hexagonal caterpillar with n edges decomposes the complete graph K_{2kn+1} for any positive integer k.

Golomb [1116] introduced polyominoes in 1953 in a talk to the Harvard Mathematics Club. *Polyominoes* are planar shapes made by connecting a certain number of equal-sized squares, each joined together with at least one other square along an edge.

A graph $G = (V(G), E(G))$ is *even graceful* if there exists an injection f from the set of vertices $V(G)$ to $\{0, 1, 2, 3, 4, \ldots, 2|E(G)|\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $2, 4, 6, \ldots, 2|E(G)|$. Elsonbaty and Mohamed [876] use even graceful labelings to give a new proof for necessary and sufficient conditions for the gracefulness of cycles. They extend this technique to odd graceful and super Fibonacci graceful labelings of cycle graphs (see §3.3). The *polar grid* graph $P_{m,n}$ consists of n copies of C_m numbered from the inner most cycle to the outer cycle as $C^(1)_m, \ldots, C^(n)_m$ and m copies of paths P_{n+1} intersected at the center vertex v_0 numbered as $P^(1)_{n+1}, \ldots, P^(m)_{n+1}$ In [877] Elsonbaty and Daoud provided edge even graceful labelings for various classes of $P_m \times C_n$. El Dean [3571] obtained an edge even graceful labeling for Y-trees, double stars $B_{n,m}$, $\langle K_{1,2n} : K_{1,2m}\rangle$, $P_{2n-1} \odot \overline{K_{2m}}$, $\overline{K_2} + P_n$, the cycle $v_1, v_2, \ldots, v_{2n}$ with a chord from v_1 to v_n, $P_2 \odot C_n$, flags, and flowers. Zeen El Deen and Omar [3575] gave sufficient conditions for $K_{m,n}$ to have an edge even graceful labeling. They also provided edge even graceful labelings of the join of K_1 with stars, wheels, and sunflowers, and the join of $\overline{K_2}$ with stars and wheels. For results on Fibonacci trees see [1716].

Wu ([3489] and [3491]) has given a number of methods for constructing larger graceful graphs from graceful graphs. Let $G_1, G_2, \ldots, G_p$ be disjoint connected graphs. Let w_i be in G_i for $1 \leqslant i \leqslant p$. Let w be a new vertex not in any G_i. Form a new graph $\oplus_w(G_1, G_2, \ldots, G_p)$ by adjoining to the graph $G_1 \cup G_2 \cup \cdots \cup G_p$ the edges $ww_1, ww_2, \ldots, ww_p$. In the case where each of $G_1, G_2, \ldots, G_p$ is isomorphic to a graph G that has an α-labeling and each w_i is the isomorphic image of the same vertex in G_i, Wu shows that the resulting graph is graceful. If f is an α-labeling of a graph, the integer k with the property that for any edge uv either $f(u) \leqslant k < f(v)$ or $f(v) \leqslant k < f(u)$ is called the *boundary value* or *critical number* of f. Wu [3489] has also shown that if $G_1, G_2, \ldots, G_p$ are graphs of the same order and have α-labelings where the labelings for each pair of graphs G_i and G_{p-i+1} have the same boundary value for $1 \leqslant i \leqslant n/2$, then $\oplus_w(G_1, G_2, \ldots, G_p)$ is graceful. In [3487] Wu proves that if G has n edges and $n + 1$ vertices and G has an α-labeling with boundary value λ, where $|n - 2\lambda - 1| \leqslant 1$, then $G \times P_m$ is graceful for all m.

Given graceful graphs H and G with at least one having an α-labeling Wu and Lu [3492] define four graph operations on H and G that when used repeatedly or in turns provide a large number of graceful graphs. In particular, if both H and G have α-labelings, then each of the graphs obtained by the four operations on H and G has an α-labeling.

Ajitha, Arumugan, and Germina [161] use a construction of Koh, Tan, and Rogers [1722] to create trees with α-labelings from smaller trees with graceful labelings. These in turn allows them to generate large classes of trees that have a type of called edge-antimagic labelings (see §6.1). Shiue and Lu [2937] prove that the graph obtained from $K_{1,k}$ by replacing each edge with a path of length 3 has an α-labeling if and only if $k \leqslant 4$. In [3371] Venkatesh and Bharathi recursively construct new trees starting with caterpillars that admit α-lableings.

Seoud and Helmi [2777] have shown that all gear graphs have an α-labeling, all dragons with a cycle of order $n \equiv 0 \pmod 4$ have an α-labeling, and the graphs obtained by identifying an endpoint of a star S_m $(m \geqslant 3)$ with a vertex of C_{4n} has an α-labeling.

Mavonicolas and Michael [2110] say that trees $\langle T_1, \theta_1, w_1\rangle$ and $\langle T_2, \theta_2, w_2\rangle$ with roots w_1 and w_2 and $|V(T_1)| = |V(T_2)|$ are *gracefully consistent* if either they are identical or they have α-labelings with the same boundary value and $\theta_1(w_1) = \theta_2(w_2)$. They use this concept to show that a number of known constructions of new graceful trees using several identical copies of a given graceful rooted tree can be extended to the case where the copies are replaced by a set of pairwise gracefully consistent trees. In particular, let $\langle T, \theta, w\rangle$ and $\langle T_0, \theta_0, w_0\rangle$ be gracefully labeled trees rooted at w and w_0 respectively. They show that the following four constructions are adaptable to the case when a set of copies of $\langle T, \theta, w\rangle$ is replaced by a set of pairwise gracefully consistent trees. When $\theta(w) = |E(T)|$ the garland construction due to Koh, Rogers, and Tan [1715] gracefully labels the tree consisting of h copies of $\langle T, w\rangle$ with their roots connected to a new vertex r. In the case when $\theta(w) = |E(T)|$ and whenever $uw \in E(T)$ and $\theta(u) \neq 0$, then $vw \in E(T)$ where $\theta(u) + \theta(v) = |E(T)|$, the attachment construction of Koh, Tan and Rogers [1722] gracefully labels the tree formed by identifying the roots of h copies of $\langle T, w\rangle$. A construction given by Koh, Tan and Rogers [1722] gracefully labels the tree formed by merging each vertex of $\langle T_0, w_0\rangle$ with the root of a distinct copy of $\langle T, w\rangle$. When $\theta_0(w_0) = |E(T_0)|$, let N be the set of neighbors of w_0 and let x be the vertex of T at even distance from w with $\theta(x) = 0$ or $\theta(x) = |E(T)|$. Then a construction of Burzio and Ferrarese [633] gracefully labels the tree formed by merging each non-root vertex of T_0 with the root of a distinct copy of $\langle T, w\rangle$ so that for each $v \in N$ the edge vw_0 is replaced with a new edge xw_0 (where x is in the corresponding copy of T).

Snevily [3012] says that a graph G *eventually has an α-labeling* provided that there is a graph H, called a *host* of G, which has an α-labeling and that the edge set of H can be partitioned into subgraphs isomorphic to G. He defines the *α-labeling number of* G to be $G_\alpha = \min\{t :$ there is a host H of G with $|E(H)| = t|G|\}$. Snevily proved that even cycles have α-labeling number at most 2 and he conjectured that every bipartite graph has an α-labeling number. This conjecture was proved by El-Zanati, Fu, and Shiue [883]. There are no known examples of a graph G with $G_\alpha > 2$. In [3012] Snevily conjectured that the α-labeling number for a tree with n edges is at most n. Shiue and Fu [2935] proved that the α-labeling number for a tree with n edges and radius r is at most $\lceil r/2\rceil n$. They also prove that a tree with n edges and radius r decomposes K_t for some $t \leqslant (r+1)n^2+1$.

Ahmed and Snevily [124] investigated the claim that for every tree T there exists an α-labeling of T, or else there exists a graph H_T with an α-labeling such that H_T can be decomposed into two edge-disjoint copies of T. They proved this claim is true for the graphs $C_{m,k}$ obtained from $K_{1,m}$ by replacing each edge in $K_{1,m}$ with a path of length k.

A graph G with vertex set V and edge set E is called *super edge-graceful* if there is a bijection f from E to $\{0, \pm1, \pm2, \ldots, \pm(|E| - 1)/2\}$ when $|E|$ is odd and from E to $\{\pm1, \pm2, \ldots, \pm|E|/2\}$ when $|E|$ is even such that the induced vertex labeling f^* defined by $f^*(u) = \sum f(uv)$ over all edges uv is a bijection from V to $\{0, \pm1, \pm2, \ldots, \pm(|V| - 1)/2\}$

when $|V|$ is odd and from V to $\{\pm 1, \pm 2, \ldots, \pm |V|/2\}$ when $|V|$ is even. Clifton and Khodkar [750] proved that graphs formed by identifying the endpoint of a path P_n and a vertex of a cycle (kites) with $n \geqslant 5$ vertices, $n \neq 6$ are super edge-graceful. Khodkar, Nolen, and Perconti [1691] proved that all complete bipartite graphs except for $K_{2,2}, K_{2,3}$, and $K_{1,n}$ (n odd) are super edge-graceful. Khodkar [1693] and [1692] proved that all complete tripartite graphs except $K_{1,1,2}$ are super edge-graceful and that the union of vertex disjoint 3-cycles is super edge-graceful. Lee, Su, and Wei [1899] provide a family of trees of odd orders which are super edge-graceful.

For a tree T with m edges, the *α-deficit* $\alpha_{def}(T)$ equals $m - \alpha(T)$ where $\alpha(T)$ is defined as the maximum number of distinct edge labels over all bipartite labelings of T. Rosa and Siran [2651] showed that for every $m \geqslant 1$, $\alpha_{def}(C_{m,2}) = \lfloor m/3 \rfloor$, which implies that $(C_{m,2})_\alpha \geqslant 2$ for $m \geqslant 3$. Ahmed and Snevily [124] define the graph $C'_{m,j}$ as a comet-like tree with a central vertex of degree m where each neighbor of the central vertex is attached to j pendent vertices for $1 \leqslant j \leqslant (m-1)$. For $m \geqslant 3$ and $1 \leqslant j \leqslant (m-1)$ they prove: $(C'_{m,j})_\alpha \leqslant 2$; $(C'_{2k+1,j})_\alpha = 2$ for $1 \leqslant j \leqslant 2k$ and conjecture if $\Delta_T = (2k+1)$, then $\alpha_{def}(T) \leqslant k$. Ahmed and Snevily [124] prove that for every comet T (that is, graphs obtained from stars by replacing each edge by a path of some fixed length) there exists an α-labeling of T, or else there exists a graph H_T with an α-labeling such that H_T can be decomposed into two edge-disjoint copies of T. This is particularly noteworthy since comets are known to have arbitrarily large α-deficits.

Given two bipartite graphs G_1 and G_2 with partite sets H_1 and L_1 and H_2 and L_2, respectively, Snevily [3011] defines their *weak tensor product* $G_1 \overline{\bigotimes} G_2$ as the bipartite graph with vertex set $(H_1 \times H_2, L_1 \times L_2)$ and with edge $(h_1, h_2)(l_1, l_2)$ if $h_1 l_1 \in E(G_1)$ and $h_2 l_2 \in E(G_2)$. He proves that if G_1 and G_2 have α-labelings then so does $G_1 \overline{\bigotimes} G_2$. This result considerably enlarges the class of graphs known to have α-labelings. In [1986] López and Muntaner-Batle gave a generalization of Snevily's weak tensor product that allows them to significantly enlarges the classes of graphs admitting α-labelings, near α-labelings (defined later in this section), and bigraceful graphs.

The sequential join of graphs $G_1, G_2, \ldots, G_n$ is formed from $G_1 \cup G_2 \cup \cdots \cup G_n$ by adding edges joining each vertex of G_i with each vertex of G_{i+1} for $1 \leqslant i \leqslant n-1$. Lee and Wang [1908] have shown that for all $n \geqslant 2$ and any positive integers $a_1, a_2, \ldots, a_n$ the sequential join of the graphs $\overline{K}_{a_1}, \overline{K}_{a_2}, \ldots, \overline{K}_{a_n}$ has an α-labeling.

In [1002] Gallian and Ropp conjectured that every graph obtained by adding a single pendent edge to one or more vertices of a cycle is graceful. Qian [2537] proved this conjecture and in the case that the cycle is even he shows the graphs have an α-labeling. He further proves that for n even any graph obtained from an n-cycle by adding one or more pendent edges at some vertices has an α-labeling as long as at least one vertex has degree 3 and one vertex has degree 2.

In [2354] Pasotti introduced the following generalization of a graceful labeling. Given a graph G with $e = dm$ edges, an injective function from $V(\Gamma)$ to the set $\{0, 1, 2, \ldots, d(m+1)-1\}$ such that $\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, 3, \ldots, d(m+1)-1\} - \{m+1, 2(m+1), \ldots, (d-1)(m+1)\}$ is called a *d-divisible graceful labeling* of G. Note that for $d = 1$ and of $d = e$ one obtains the classical notion of a graceful labeling and of an odd-graceful labeling (see §3.6 for the definition), respectively. A d-divisible graceful labeling of a bipartite graph G with the property that the maximum value on one of the two bipartite sets is less than the minimum value on the other one is called a *d-divisible α-labeling* of G. Pasotti proved that these new concepts allow to obtain certain cyclic graph decompositions. In particular, if there exists a d-divisible graceful labeling of a graph G of size $e = dm$ then there exists a cyclic G-decomposition of $K_{\left(\frac{e}{d}+1\right)\times 2d}$ and that if there exists a d-divisible α-labeling of a graph Γ of size e then there exists a cyclic G-decomposition of $K_{\left(\frac{e}{d}+1\right)\times 2dn}$ for any integer $n \geqslant 1$. She also it is proved the following: paths and stars admit a d-divisible α-labeling for any admissible d; C_{4k} admits a 2-divisible α-labeling and a 4-divisible α-labeling for any $k \geqslant 1$; C_{2k} admits a 2-divisible labeling for any odd integer $k > 1$; and the ladder graph L_{2k} has a 2-divisible α-labeling if and only if k is even.

Pasotti [2354] generalized the notion of graceful labelings for graphs G with $e = d \cdot m$ edges by defining a *d-graceful labeling* as an injective function f from $V(G)$ to $\{0, 1, 2, \ldots, d(m+1) - 1\}$ such that $\{|f(x) - f(y)| \mid xy \in E(G)\} = \{1, 2, \ldots, d(m+1) - 1\} - \{m+1, 2(m+1), \ldots, (d-1)(m+1)\}$. The case $d = 1$ is a graceful labeling and the case that $d = e$ is an odd-graceful labeling. A *d-graceful* α-labeling of a bipartite graph is a d-graceful labeling with the property that the maximum value in one of the two bipartite sets is less than the minimum value on the other bipartite set. Pasotti [2354] proved that paths and stars have d-graceful α-labelings for all admissible d, ladders $P_n \times P_2$ have a 2-graceful labeling if and only if n is even, and provided partial results about cycles of even length. He showed that the existence of d-graceful labelings can be used to prove that certain complete graphs have cyclic decompositions. Benini and Pasotti [531] used d-divisible α-labelings to construct an infinite class of cyclic Γ-decompositions of the complete multipartite graphs, where Γ is a caterpillar, a hairy cycle or a cycle. Such labelings imply the existence of cyclic Γ-decompositions of certain complete multipartite graphs. Riasat, Kanwal, and Javed [2622] give odd-graceful labelings for disjoint unions of graphs consisting of generalized combs, ladders, stars, bistars, caterpillars and paths.

In [2353], Pasotti proved the existence of d-divisible α-labelings for $C_{4k} \times P_m$ for any integers $k \geqslant 1$, $m \geqslant 2$ for $d = 2m-1,\ 2(2m-1)$ and $4(2m-1)$. Benini and Pasotti [532] proved that the generalized Petersen graph $P_{8n,3}$ admits an α-labeling for any integer $n \geqslant 1$ confirming that the conjecture posed by A. Vietri in [3361] is true.

For any tree $T(V, E)$ whose vertices are properly 2-colored Rosa and Širáň [2651] define a *bipartite labeling* of T as a bijection $f : V \to \{0, 1, 2, \ldots, |E|\}$ for which there is a k such that whenever $f(u) \leqslant k \leqslant f(v)$, then u and v have different colors. They define the *α-size* of a tree T as the maximum number of distinct values of the induced edge labels $|f(u) - f(v)|, uv \in E$, taken over all bipartite labelings f of T. They prove that the α-size of any tree with n edges is at least $5(n+1)/7$ and that there exist trees whose α-size is at most $(5n+9)/6$. They conjectured that minimum of the α-sizes over all trees with n edges is asymptotically $5n/6$. This conjecture has been proved for trees of maximum degree 3 by Bonnington and Širáň [597]. For trees with n vertices and maximum degree 3 Brankovic, Rosa, and Širáň [610] have shown that the α-size is at least $\lfloor \frac{6n}{7} \rfloor - 1$. In [608] Brankovic, Murch, Pond, and Rose provide a lower bound for the α-size trees with maximum degree three and a perfect matching as a function of a lower bound for minimum order of such a tree that does not have an α-labeling. Using a computer search they showed that all such trees on less than 30 vertices have an α-labeling. This brought the lower bound for the α-size to $14n/15$, for such trees of order n. They conjecture that all trees with maximum degree three and a perfect matching have an α-labeling. Heinrich and Hell [1240] defined the *gracesize* of a graph G with n vertices as the maximum, over all bijections $f\colon V(G) \to \{1, 2, \ldots, n\}$, of the number of distinct values $|f(u) - f(v)|$ over all edges uv of G. So, from Rosa and Širáň's result, the gracesize of any tree with n edges is at least $5(n + 1)/7$.

In [614] Brinkmann, Crevals, Mélot, Rylands, and Steffan define the parameter α_{def} which measures how far a tree is from having an α-labeling as it counts the minimum number of errors, that is, the minimum number of edge labels that are missing from the set of all possible labels. Trees with an α-labeling have deficit 0. For a tree $T = (V, E)$ with bipartition classes V_1 and V_2 and a bipartite labeling $f : V \to \{0, \ldots, |V| - 1\}$ the *edge parity* of T is $(\sum_{i=1}^{|E|} i) \bmod 2 = \frac{1}{2}(|V| - 1)|V| \bmod 2$. So if f is an α-labeling this is the sum of all edge labels modulo 2; it is 0 if $|V| \equiv 0, 1 \bmod 4$ and 1 if $|V| \equiv 2, 3 \bmod 4$. The *vertex parity* is the parity of the number of vertices of odd degree with odd label.

Brinkmann et al. [614] proved: in a tree T with α-deficit 0 the edge parity and the vertex parities are equal; and for all non-negative integers k and d and $n \geqslant k^2 + k$, the number of trees T with n vertices, $\alpha_{\mathrm{def}}(T) = d$ and maximum degree $n - k$ is the same. Furthermore, they provide computer results on the α-deficit of all trees with up to 26 vertices; with maximum degree 3 and up to 36 vertices, with maximum degree 4 and up to 32 vertices, and with maximum degree 5 and up to 31 vertices.

In [1007] Gallian weakened the condition for an α-labeling somewhat by defining a *weakly α-labeling* as a graceful labeling for which there is an integer k so that for each edge xy either $f(x) \leqslant k \leqslant f(y)$ or $f(y) \leqslant k \leqslant f(x)$. Unlike α-labelings, this condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph; namely, that the vertex with the label k must be on every odd cycle. Gallian, Prout, and Winters [1007] showed that the prisms $C_n \times P_2$ with a vertex deleted have α-labelings. The same paper reveals that $C_n \times P_2$ with an edge deleted from a cycle has an α-labeling when n is even and a weakly α-labeling when $n > 3$.

In [458] and [461] Barrientos and Minion focused on the enumeration of graphs with graceful and α-labelings, respectively. They used an extended version of the adjacency matrix of a graph to count the number of labeled graphs. In [458] they count the number of gracefully-labeled graphs of size n and order m, for all possible values of m. In [461] they count the number of α-labeled graphs of size n and order m, for all possible values of m, as well as those α-labeled graphs of size n with boundary value λ. They also count the number of α-labeled graphs of size n, order m, and boundary value λ for all possible values of m and λ.

A special case of α-labeling called strongly graceful was introduced by Maheo [2053] in 1980. A graceful labeling f of a graph G is called *strongly graceful* if G is bipartite with two partite sets A and B of the same order s, the number of edges is $2t + s$, there is an integer k with $t - s \leqslant k \leqslant t + s - 1$ such that if $a \in A, f(a) \leqslant k$, and if $b \in B, f(b) > k$, and there is an involution π that is an automorphism of G such that: π exchanges A and B and the s edges $a\pi(a)$ where $a \in A$ have as labels the integers between $t+1$ and $t+s$. Maheo's main result is that if G is strongly graceful then so is $G \times Q_n$. In particular, she proved that $(P_n \times Q_n) \times K_2$, B_{2n}, and $B_{2n} \times Q_n$ have strongly graceful labelings.

In 1999 Broersma and Hoede [615] conjectured that every tree containing a perfect matching is strongly graceful. Yao, Cheng, Yao, and Zhao [3529] proved that this conjecture is true for every tree with diameter at most 5 and provided a method for constructing strongly graceful trees.

El-Zanati and Vanden Eynden [887] call a strongly graceful labeling a *strong α-labeling.* They show that if G has a strong α-labeling, then $G \times P_n$ has an α-labeling. They show that $K_{m,2} \times K_2$ has a strong α-labeling and that $K_{m,2} \times P_n$ has an α-labeling. They also show that if G is a bipartite graph with one more vertex than the number of edges, and if G has an α-labeling such that the cardinalities of the sets of the corresponding bipartition of the vertices differ by at most 1, then $G \times K_2$ has a strong α-labeling and $G \times P_n$ has an α-labeling. El-Zanati and Vanden Eynden [887] also note that $K_{3,3} \times K_2$, $K_{3,4} \times K_2$, $K_{4,4} \times K_2$, and $C_{4k} \times K_2$ all have strong α-labelings. El-Zanati and Vanden Eynden proved that $K_{m,2} \times Q_n$ has a strong α-labeling and that $K_{m,2} \times P_n$ has an α-labeling for all n. They also prove that if G is a connected bipartite graph with partite sets of odd order such that in each partite set each vertex has the same degree, then $G \times K_2$ does not have a strong α-labeling. As a corollary they have that $K_{m,n} \times K_2$ does not have a strong α-labeling when m and n are odd.

An α-labeling f of a graph G is called *free* by El-Zanati and Vanden Eynden in [888] if the critical number k (in the definition of α-labeling) is greater than 2 and if neither 1 nor $k-1$ is used in the labeling. Their main result is that the union of graphs with free α-labelings has an α-labeling. In particular, they show that $K_{m,n}$, $m > 1$, $n > 2$, has a free α-labeling. They also show that Q_n, $n \geqslant 3$, and $K_{m,2} \times Q_n$, $m > 1$, $n \geqslant 1$, have free α-labelings. El-Zanati [personal communication] has shown that the Heawood graph has a free α-labeling.

Wannasit and El-Zanati [3456] proved that if G is a cubic bipartite graph each of whose components is either a prism, a Möbius ladder, or has order at most 14, then G admits free α-labeling. They conjecture that every bipartite cubic graph admits a free α-labeling.

In [2071] Makadia, Karavadiya, and Kaneria call a vertex v in a graph G with a graceful labeling f a *graceful center* of G if $f(v) = 0$ or $f(v) = |E(G)|$. They say a graph G is a *universal graceful* graph if for every $v \in V(G)$, v is a graceful center for G with respect to some graceful labeling of G. They call G a *universal α-graceful* graph if for every $v \in V(G)$, v is a graceful center for G with respect to some α-graceful labeling of G. They define the *ring sum* of two graphs G_1 and G_2 denoted $G_1 \oplus G_2$, as the graph with vertex set $(V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) - (E(G_1) \cap E(G_2))$. They proved: any graph G that admits α-labeling has at least four graceful centers; if G is a graceful graph, then $G \oplus K_{1,n}$ is graceful; if G is a universal graceful graph, then $G \oplus K_2$ is a graceful; if G_1 is graceful and G_2 has an α-labeling, then the ring sum $G_1 \oplus G_2$ with the graceful center of G_1 and the graceful center of G_2 as a common vertex is a graceful; and if G_1 and G_2 have α labelings, then the ring sum $G_1 \oplus G_2$ with the two graceful centers of G_1 and G_2 as a common vertex has an α labeling.

For connected bipartite graphs Grannell, Griggs, and Holroyd [1148] introduced a labeling that lies between α-labelings and graceful labelings. They call a vertex labeling f of a bipartite graph G with q edges and partite sets D and U *gracious* if f is a bijection from the vertex set of G to $\{0, 1, \dots, q\}$ such that the set of edge labels induced by $f(u) - f(v)$ for every edge uv with $u \in U$ and $v \in D$ is $\{1, 2, \dots, q\}$. Thus a gracious labeling of G with partite sets D and U is a graceful labeling in which every vertex in D has a label lower than every adjacent vertex. They verified by computer that every tree of size up to 20 has a gracious labeling. This led them to conjecture that every tree has a gracious labeling. For any $k > 1$ and any tree T Grannell et al. say that T has a *gracious k-labeling* if the vertices of T can be partitioned into sets D and U in such a way that there is a function f from the verticies of G to the integers modulo k such that the edge labels induced by $f(u) - f(v)$ where $u \in U$ and $v \in D$ have the following properties: the number of edges labeled with 0 is one less than the number of verticies labeled with 0 and for each nonzero integer t the number of edges labeled with t is the same as the number of verticies labeled with t. They prove that every nontrivial tree has a k-gracious labeling for $k = 2, 3, 4$, and 5 and that caterpillars are k-gracious for all $k \geqslant 2$. In [530] Bell and Cummins provided new methods for combining certain families of gracefully labeled graphs to produce new gracefully labeled graphs. If the constituent graphs have a gracious labeling, then the methods presented produce a gracious labeling. They also introduce new infinite families of gracious trees and new classes of graceful trees.

The same labeling that is called gracious by Grannell, Griggs, and Holroyd is called a *near α-labeling* by El-Zanati, Kenig, and Vanden Eynden [885]. The latter prove that if G is a graph with n edges that has a near α-labeling then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x and a cyclic G-decomposition of $K_{n,n}$. They further prove that if G and H have near α-labelings, then so does their weak tensor product (see earlier part of this section) with respect to the corresponding vertex partitions. They conjecture that every tree has a near α-labeling.

In [2052] Mahalingam and Rajendram introduced a new labeling called m-bonacci graceful labeling as follows. A graph G on n edges is *m-bonacci graceful* if the vertices can be labeled with distinct integers from the set of the first n m-bonacci numbers such that the derived edge labels are the first n m-bonacci numbers. They showed that complete graphs, complete bipartite graphs, gear graphs, triangular grid graphs, and wheel graphs are not m-bonacci graceful. They gave m-bonacci graceful labeling for cycles, friendship graphs, polygonal snake graphs, and double polygonal snake graphs and proved that almost all trees are m-bonacci graceful.

For a simple, finite, connected, undirected, non-trivial graph G Sumathi and Raman [3121] introduced the notion of *arithmetic sequential graceful* as an injection $f : V(G) \to \{a, a + d, a + 2d, a + 3d, \dots, 2(a + qd)\}$, where $a \geqslant 0$ and $d \geqslant 1$, with the property that $f^* : E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijection. They proved that stars, double stars, and some star related graphs are arithmetic sequential graceful. [3121] new

Another kind of labelings for trees was introduced by Ringel, Llado, and Serra [2627] in an approach to proving their conjecture $K_{n,n}$ is edge-decomposable into n copies of any given tree with n edges. If T is a tree with n edges and partite sets A and B, they define a labeling f from the set of vertices to $\{1, 2, \ldots, n\}$ to be a *bigraceful* labeling of T if f restricted to A is injective, f restricted to B is injective, and the edge labels given by $f(y) - f(x)$ where yx is an edge with y in B and x in A is the set $\{0, 1, 2, \ldots, n-1\}$. (Notice that this terminology conflicts with that given in Section 2.7 In particular, the Ringel, Llado, and Serra bigraceful does not imply the usual graceful.) Among the graphs that they show are bigraceful are: lobsters, trees of diameter at most 5, stars $S_{k,m}$ with k spokes of paths of length m, and complete d-ary trees for d odd. They also prove that if T is a tree then there is a vertex v and a nonnegative integer m such that the addition of m leaves to v results in a bigraceful tree. They conjecture that all trees are bigraceful.

A *pronic number* is one of the form $n(n+1)$, where n is a positive integer. Porchelvi and Devi [2479] defined a *pronic graceful* labeling of a graph G with $n \geqslant 2$ verticies as a bijection $f : V(G) \to \{0, 2, \ldots, n(n+1)\}$ such that the upon labeling each edge uv with $|f(u) - f(v)|$ the labels are distinct. A graph G is called a *pronic graceful* graph if it admits pronic graceful labeling. They proved that paths, cycles, wheels, stars, twigs, and generalized Peterson graphs are pronic graceful. In [812] they proved that the generalized Peterson graphs $P(6, 2), P(8, 3), (10, 2), P(10, 3)$, and $P(12, 5)$ are pronic graceful. [2479] new [812] new

Table 3 summarizes some of the main results about α-labelings; α indicates that the graphs have an α-labeling.

Table 3: **Summary of Results on α-labelings**

Graph	*α-labeling*
cycles C_n	α iff $n \equiv 0 \pmod 4$ [2648]
caterpillars	α [2648]
n-cube	α [1738]
books $B_{2n},\ B_{4n+1}$	α [2053], [1006]
Möbius ladders M_{2k+1}	α [2352]
$C_m \cup C_n$	α iff m, n are even and $m + n \equiv 0 \pmod 4$[24]
$C_{4m} \cup C_{4m} \cup C_{4m}$ $(m > 1)$	α [1740]
$C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$	α [1740]
$mK_{s,t}$ $(m \geqslant 1, s, t \geqslant 2)$	iff $(m, s, t) \neq (3, 2, 2)$ [1313]
$P_n \times Q_n$	α [2053]
$B_{2n} \times Q_n$	α [2053]
$K_{1,n} \times Q_n$	α [2053]
$K_{m,2} \times Q_n$	α [887]
$K_{m,2} \times P_n$	α [887]
$P_2 \times P_2 \times \cdots \times P_2 \times G$	α when $G = C_{4m},\ P_m,\ K_{3,3},$ $K_{4,4}$ [3011]
$P_2 \times P_2 \times \cdots \times P_2 \times P_m$	α [3011]
$P_2 \times P_2 \times \cdots \times P_2 \times K_{m,m}$	α [3011] when $m = 3$ or 4
$G[\overline{K_n}]$	α when G is α [3012]

3.2 γ-Labelings

In 2004 Chartrand, Erwin, VanderJagt, and Zhang [676] define a γ-labeling of a graph G of size m as a 1-1 function f from the vertices of G to $\{0,1,2,\dots,m\}$ that induces an edge labeling f' defined by $f'(uv) = |f(u)-f(v)|$ for each edge uv. They define the following parameters of a γ-labeling: $\mathrm{val}(f) = \Sigma f'(e)$ over all edges e of G; $\mathrm{val}_{\max}(G) = \max\{\ \mathrm{val}(f) : f$ is a γ-labeling of $G\}$, $\mathrm{val}_{\min}(G) = \min\{\ \mathrm{val}(f) : f$ is a $\gamma -$ labeling of $G\}$. Among their results are the following:
$\mathrm{val}_{\min}(P_n) = \mathrm{val}_{\max}(P_n) = \lfloor (n^2-2)/2 \rfloor$; $\mathrm{val}_{\min}(C_n) = 2(n-1)$; for even $n \geqslant 4$, $\mathrm{val}_{\max}(C_n) = n(n+2)/2$; for odd $n \geqslant 3$, $\mathrm{val}_{\max}(C_n) = (n-1)(n+3)/2$; for odd n, $\mathrm{val}_{\min}(K_n) = \binom{n+1}{3}$; for odd n, $\mathrm{val}_{\max}(K_n) = (n^2-1)(3n^2-5n+6)/24$; for even n, $\mathrm{val}_{\max}(K_n) = n(3n^3-5n^2+6n-4)/24$; for every $n \geqslant 3$, $\mathrm{val}_{\min}(K_{1,n-1}) = \binom{\lfloor \frac{n+1}{2} \rfloor}{2} + \binom{\lceil \frac{n+1}{2} \rceil}{2}$; $\mathrm{val}_{\max}(K_{1,n-1}) = \binom{n}{2}$ for a connected graph of order n and size m, $\mathrm{val}_{\min}(G) = m$ if and only if G is isomorphic to P_n; if G is maximal outerplanar of order $n \geqslant 2$, $\mathrm{val}_{\min}(G) \geqslant 3n-5$ and equality occurs if and only if $G = P_n^2$; if G is a connected r-regular bipartite graph of order n and size m where $r \geqslant 2$, then $\mathrm{val}_{\max}(G) = rn(2m-n+2)/4$.

In another paper on γ-labelings of trees Chartrand, Erwin, VanderJagt, and Zhang [677] prove for $p,q \geqslant 2$, $\mathrm{val}_{\min}(S_{p,q})$ (that is, the graph obtained by joining the centers of $K_{1,p}$ and $K_{1,q}$ by an edge)$= (\lfloor p/2 \rfloor+1)^2+(\lfloor q/2 \rfloor+1)^2-(n_p\lfloor p/2 \rfloor+1)^2+(n_q\lfloor (q+2)/2 \rfloor+1)^2)$, where n_i is 1 if i is even and n_i is 0 if n_i is odd; $\mathrm{val}_{\min}(S_{p,q}) = (p^2+q^2+4pq-3p-3q+2)/2$; for a connected graph G of order n at least 4, $\mathrm{val}_{\min}(G) = n$ if and only if G is a caterpillar with maximum degree 3 and has a unique vertex of degree 3; for a tree T of order n at least 4, maximum degree Δ, and diameter d, $\mathrm{val}_{\min}(T) \geqslant (8n+\Delta^2-6\Delta-4d+\delta_\Delta)/4$ where δ_Δ is 0 if Δ is even and δ_Δ is 0 if Δ is odd. They also give a characterization of all trees of order n at least 5 whose minimum value is $n+1$.

Saduakdee and Khemmani [2695] investigated connected graphs having the unique γ-min labeling. They determined the minimum value of a γ-labeling for some classes of trees and showed that they have no unique γ-min labeling.

In [630] Buratti and Del Fra solved the existence problem for cyclic k-cycle systems of the complete graph K_v with $v \equiv 1 \pmod{2k}$, and the existence problem for cyclic k-cycle systems of the complete m-partite graph $K_{m\times k}$ for m and k odd. As a particular consequence, a cyclic p-cycle system of K_v with p a prime exists for all admissible values of v but $(p,v) \neq (3,9)$. This was previously known only for $p = 3,5,7$.

In [2694] Sanaka determined $\mathrm{val}_{\max}(K_{m,n})$ and $\mathrm{val}_{\min}(K_{m,n})$. In [629] Bunge, Chantasartraaamee, El-Zanati, and Vanden Eynden generalized γ-labelings by introducing two labelings for tripartite graphs. Graphs G that admit either of these labelings guarantee the existence of cyclic G-decompositions of K_{2nx+1} for all positive integers x. They also proved that, except for $C_3 \cup C_3$, the disjoint union of two cycles of odd length admits one of these labelings.

3.3 Graceful-like Labelings

As a means of attacking graph decomposition problems, Rosa [2648] invented another analogue of graceful labelings by permitting the vertices of a graph with q edges to assume labels from the set $\{0, 1, \ldots, q+1\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, \ldots, q-1, q\}$ or $\{1, 2, \ldots, q-1, q+1\}$. He calls these *$\hat{\rho}$-labelings*. Frucht [986] used the term *nearly graceful labeling* instead of *$\hat{\rho}$-labelings*. Frucht [986] has shown that the following graphs have nearly graceful labelings with edge labels from $\{1, 2, \ldots, q-1, q+1\}$: $P_m \cup P_n$; $S_m \cup S_n$; $S_m \cup P_n$; $G \cup K_2$ where G is graceful; and $C_3 \cup K_2 \cup S_m$ where m is even or $m \equiv 3 \pmod{14}$. Seoud and Elsakhawi [2771] have shown that all cycles are nearly graceful. Barrientos [435] proved that C_n is nearly graceful with edge labels $1, 2, \ldots, n-1, n+1$ if and only if $n \equiv 1$ or $2 \pmod 4$. Nurvazly and Sugeng [2315] proved that $\Theta(C_3)^n$ graphs (n copies of C_3 that share an edge) have $\hat{\rho}$ labelings. Gao [1021] shows that a variation of banana trees is odd-graceful and in some cases has a nearly graceful labeling. (A graph G with q edges is *odd-graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$).

For a graph G with p vertices and q directed edges that are assigned distinct vertex labels in $\{0, \ldots, q\}$ and distinct edge labels in $\{1, \ldots, p\}$ so that the label of the directed edge from u to v is $(f(v) - f(u)) \bmod (q+1)$ (this generalizes Rosa's ρ-valuations. Knuth [1707] has observed that there is a nice data structure for storing a graph or digraph in a computer. He calls this a *graceful data structure* labeling.

In 1988 Rosa [2650] conjectured that triangular snakes with $t \equiv 0$ or $1 \pmod 4$ blocks are graceful and those with $t \equiv 2$ or $3 \pmod 4$ blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [2200] proved Rosa's conjecture while introducing the slightly stronger concept of *almost graceful* by permitting the vertex labels to come from $\{0, 1, 2, \ldots, q-1, q+1\}$ while the edge labels are $1, 2, \ldots, q-1, q$, or $1, 2, \ldots, q-1, q+1$. More generally, Rosa [2650] conjectured that all triangular cacti are either graceful or near graceful and suggested the use of Skolem sequences to label some types of triangular cacti. Dyer, Payne, Shalaby, and Wicks [862] verified the conjecture for two families of triangular cacti using Langford sequences to obtain Skolem and hooked Skolem sequences with specific subsequences.

Seoud and Elsakhawi [2771] and [2772] have shown that the following graphs are almost graceful: C_n; $P_n + \overline{K_m}$; $P_n + K_{1,m}$; $K_{m,n}$; $K_{1,m,n}$; $K_{2,2,m}$; $K_{1,1,m,n}$; $P_n \times P_3$ $(n \geqslant 3)$; $K_5 \cup K_{1,n}$; $K_6 \cup K_{1,n}$, and ladders.

For a graph G with p vertices, q edges, and $1 \leqslant k \leqslant q$, Eshghi [899] defines a *holey α-labeling with respect to k* as an injective vertex labeling f for which $f(v) \in \{1, 2, \ldots, q+1\}$ for all v, $\{|f(u) - f(v)| \mid \text{for all edges } uv\} = \{1, 2, \ldots, k-1, k+1, \ldots, q+1\}$, and there exist an integer γ with $0 \leqslant \gamma \leqslant q$ such that $\min\{f(u), f(v)\} \leqslant \gamma \leqslant \max\{f(u), f(v)\}$. He proves the following: P_n has a holey α-labeling with respect to all k; C_n has a holey α-labeling with respect to k if and only if either $n \equiv 2 \pmod 4$, k is even, and $(n, k) \neq (10, 6)$, or $n \equiv 0 \pmod 4$ and k is odd.

Recall from Section 2.2 that a *kC_n-snake* is a connected graph with k blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to C_n. In addition to his results on the graceful kC_n-snakes given in Section 2.2, Barrientos [439] proved that when k is odd the linear kC_6-snake is nearly graceful and that $C_m \cup K_{1,n}$ is nearly graceful when $m = 3, 4, 5$, and 6.

Yet another kind of labeling introduced by Rosa in his 1967 paper [2648] is a ρ-labeling. (Sometimes called a *rosy* labeling). A *ρ-labeling* (or ρ-valuation) of a graph is an injection from the vertices of the graph with q edges to the set $\{0, 1, \ldots, 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, \ldots, a_q$, then $a_i = i$ or $a_i = 2q + 1 - i$. Rosa [2648] proved that a cyclic decomposition of the edge set of the complete graph K_{2q+1} into subgraphs isomorphic to a given graph G with q edges exists if and only if G has a ρ-labeling. (A decomposition of K_n into copies of G is called *cyclic* if the automorphism group of the decomposition itself contains the cyclic group of order n.) It is known that every graph with at most 11 edges has a ρ-labeling and that all lobsters have a ρ-labeling (see [659]).

In [468] Barrientos and Minion proved that a tree admits a ρ–labeling when the deletion of some of its leaves results in a graceful tree. They use this result to prove the existence of ρ-labeling for several families of trees such as lobsters and those of diameter up to seven. Similarly, they showed that if T is any graceful tree of size n and k is an integer such that $2k \geqslant n + 1$, then any tree of size $n + 2k$ obtained attaching a path of length 2 to k distinct vertices of T has a ρ-labeling.

Donovan, El-Zanati, Vanden Eyden, and Sutinuntopas [844] prove that rC_m has a ρ-labeling (or a more restrictive labeling) when $r \leqslant 4$. They conjecture that every 2-regular graph has a ρ-labeling. Gannon and El-Zanati [1015] proved that for any odd $n \geqslant 7, rC_n$ admits ρ-labelings. The cases $n = 3$ and $n = 5$ were done in [840] and [884]. Aguado, El-Zanati, Hake, Stob, and Yayla [69] give a ρ-labeling of $C_r \cup C_s \cup C_t$ for each of the cases where $r \equiv 0,\ s \equiv 1,\ t \equiv 1 \pmod 4$; $r \equiv 0,\ s \equiv 3,\ t \equiv 3 \pmod 4$; and $r \equiv 1$, $s \equiv 1,\ t \equiv 3 \pmod 4$; (iv) $r \equiv 1,\ s \equiv 2,\ t \equiv 3 \pmod 4$; (v) $r \equiv 3,\ s \equiv 3,\ t \equiv 3 \pmod 4$. Caro, Roditty, and Schőnheim [659] provide a construction for the adjacency matrix for every graph that has a ρ-labeling. They ask the following question: If H is a connected graph having a ρ-labeling and q edges and G is a new graph with q edges constructed by breaking H up into disconnected parts, does G also have a ρ-labeling? Kézdy [1689] defines a *stunted tree* as one whose edges can be labeled with $e_1, e_2, \ldots, e_n$ so that e_1 and e_2 are incident and, for all $j = 3, 4, \ldots, n$, edge e_j is incident to at least one edge e_k satisfying $2k \leqslant j - 1$. He uses Alon's "Combinatorial Nullstellensatz" to prove that if $2n + 1$ is prime, then every stunted tree with n edges has a ρ-labeling.

Jeba Jesintha and Ezhilarasi Hilda [1390] introduced a variation of Rosa's ρ-labeling as follows. A *$\rho^{\star}$-labeling* of a graph G is an injection from the vertices of the graph with q edges to the set $\{0, 1, \ldots, 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $e_1, e_2, \ldots, e_q$, then $e_i = i$ or $e_i = 2q - i$. They prove that all paths and shell-butterfly graphs have a $\rho^{\star}$-labeling.

In [459] Barrientos and Minion proved the existence of ρ-labelings for some types of forests that considerably reduce the number of trees that need to be studied to prove Kotzig's Conjecture that states that K_{2n+1} can be cyclically decomposed into $2n + 1$ subgraphs isomorphic to a given tree with n edges. Among their results are the following.

If T_1 and T_2 admit α-labelings such that one of the end-vertices of the edge of weight 1 in T_2 is a leaf, then $T_1 \cup T_2$ admits a ρ-labeling. If $G_1, G_2, \ldots, G_k$ is a collection of graphs that admit α-labelings, where G_k is a caterpillar of size at least $k-2$, then $\bigcup_{i=1}^{k} G_i$ admits a ρ-labeling. Let $\mathcal{R}$ denote the family that consists of all trees G such that G has a branch H, (i.e., $G-H$ is a tree) that is a caterpillar, where the excess of $G-H$ is at most the size of H. They prove that G admits a ρ-labeling when $G \in \mathcal{R}$.

Recall a kayak paddle $KP(k,m,l)$ is the graph obtained by joining C_k and C_m by a path of length l. Fronček and Tollefeson [981], [982] proved that $KP(r,s,l)$ has a ρ-labeling for all cases. As a corollary they have that the complete graph K_{2n+1} is decomposable into kayak paddles with n edges.

In [964] Fronček generalizes the notion of an α-labeling by showing that if a graph G on n edges allows a certain type of ρ-labeling), called α_2-labeling, then for any positive integer k the complete graph K_{2nk+1} can be decomposed into copies of G.

In their investigation of cyclic decompositions of complete graphs El-Zanati, Vanden Eynden, and Punnim [890] introduced two kinds of labelings. They say a bipartite graph G with n edges and partite sets A and B has a *θ-labeling* h if h is a one-to-one function from $V(G)$ to $\{0,1,\ldots,2n\}$ such that $\{|h(b)-h(a)| \ ab \in E(G), a \in A, b \in B\} = \{1,2,\ldots,n\}$. They call h a *ρ^+-labeling* of G if h is a one-to-one function from $V(G)$ to $\{0,1,\ldots,2n\}$ and the integers $h(x)-h(y)$ are distinct modulo $2n+1$ taken over all ordered pairs (x,y) where xy is an edge in G, and $h(b) > h(a)$ whenever $a \in A, b \in B$ and ab is an edge in G. Note that θ-labelings are ρ^+-labelings and ρ^+-labelings are ρ-labelings. They prove that if G is a bipartite graph with n edges and a ρ^+-labeling, then for every positive integer x there is a cyclic G-decomposition of K_{2nx+1}. They prove the following graphs have ρ^+-labelings: trees of diameter at most 5, C_{2n}, lobsters, and comets (that is, graphs obtained from stars by replacing each edge by a path of some fixed length). They also prove that the disjoint union of graphs with α-labelings have a θ-labeling and conjecture that all forests have ρ-labelings.

A *σ-labeling* of $G(V,E)$ is a one-to-one function f from V to $\{0,1,\ldots,2|E|\}$ such that $\{|f(u)-f(v)| \mid uv \in E(G)\} = \{1,2,\ldots,|E|\}$. Such a labeling of G yields cyclic G-decompositions of K_{2n+1} and of $K_{2n+2}-F$, where F is a 1-factor of K_{2n+2}. El-Zanati and Vanden Eynden (see [68]) have conjectured that every 2-regular graph with n edges has a ρ-labeling and, if $n \equiv 0$ or 3 (mod 4), then every 2-regular graph has a σ-labeling. Aguado and El-Zanati [68] have proved that the latter conjecture holds when the graph has at most three components.

Given a bipartite graph G with partite sets X and Y and graphs H_1 with p vertices and H_2 with q vertices, Fronček and Winters [983] define the *bicomposition* of G and H_1 and H_2, $G[H_1,H_2]$, as the graph obtained from G by replacing each vertex of X by a copy of H_1, each vertex of Y by a copy of H_2, and every edge xy by a graph isomorphic to $K_{p,q}$ with the partite sets corresponding to the vertices x and y. They prove that if G is a bipartite graph with n edges and G has a θ-labeling that maps the vertex set $V = X \cup Y$ into a subset of $\{0,1,2,\ldots,2n\}$, then the bicomposition $G[\overline{K_p},\overline{K_q}]$ has a θ-labeling for every $p,q \geqslant 1$. As corollaries they have: if a bipartite graph G with n edges and at most $n+1$ vertices has a gracious labeling (see §3.1), then the bicomposition graph $G[\overline{K_p},\overline{K_q}]$

has a gracious labeling for every $p, q \geqslant 1$, and if a bipartite graph G with n edges has a θ-labeling, then for every $p, q \geqslant 1$, the bicomposition $G[\overline{K_p}, \overline{K_q}]$ decomposes the complete graph K_{2npq+1}.

In a paper published in 2009 [889] El-Zanati and Vanden Eynden survey "Rosa-type" labelings. That is, labelings of a graph G that yield cyclic G-decompositions of K_{2n+1} or K_{2nx+1} for all natural numbers x. The 2009 survey by Fronček [963] includes generalizations of ρ- and α-labelings that have been used for finding decompositions of complete graphs that are not covered in [889].

Blinco, El-Zanati, and Vanden Eynden [572] call a non-bipartite graph *almost-bipartite* if the removal of some edge results in a bipartite graph. For these kinds of graphs G they call a labeling f a γ-*labeling* of G if the following conditions are met: f is a ρ-labeling; G is tripartite with vertex tripartition A, B, C with $C = \{c\}$ and $\bar{b} \in B$ such that $\{\bar{b}, c\}$ is the unique edge joining an element of B to c; if av is an edge of G with $a \in A$, then $f(a) < f(v)$; and $f(c) - f(\bar{b}) = n$. (In § 3.2 the term γ-labeling is used for a different kind of labeling.) They prove that if an almost-bipartite graph G with n edges has a γ-labeling then there is a cyclic G-decomposition of K_{2nx+1} for all x. They prove that all odd cycles with more than 3 vertices have a γ-labeling and that $C_3 \cup C_{4m}$ has a γ-labeling if and only if $m > 1$. In [628] Bunge, El-Zanati, and Vanden Eynden prove that every 2-regular almost bipartite graph other than C_3 and $C_3 \cup C_4$ have a γ-labeling.

In [572] Blinco, El-Zanati, and Vanden Eynden consider a slightly restricted ρ^+-labeling for a bipartite graph with partite sets A and B by requiring that there exists a number λ with the property that $\rho^+(a) \leqslant \lambda$ for all $a \in A$ and $\rho^+(b) > \lambda$ for all $b \in B$. They denote such a labeling by ρ^{++}. They use this kind of labeling to show that if G is a 2-regular graph of order n in which each component has even order then there is a cyclic G-decomposition of K_{2nx+1} for all x. They also conjecture that every bipartite graph has a ρ-labeling and every 2-regular graph has a ρ-labeling.

Dufour [857] and Eldergill [868] have some results on the decomposition of complete graphs using labeling methods. Balakrishnan and Sampathkumar [414] showed that for each positive integer n the graph $\overline{K_n} + 2K_2$ admits a ρ-labeling. Balakrishnan [408] asks if it is true that $\overline{K_n} + mK_2$ admits a ρ-labeling for all n and m. Fronček [962] and Fronček and Kubesa [976] have introduced several kinds of labelings for the purpose of proving the existence ofspecial kinds of decompositions of complete graphs into spanning trees.

For positive integers c and d, let $K_{c \times d}$ denote the complete multipartite graph with c parts, each containing d vertices. Let G with n edges be the union of two vertex-disjoint even cycles. In [3056] Su et al. use Rosa-type graph labelings to show that there exists a cyclic G- decomposition of $K_(2n+1) \times t$, $K_{(n/2+1)\times 4t}, K_{5\times(n/2)t}$, and of K_{2nt} for every positive integer t. If $n \equiv 0 \pmod 4$,then there also exists a cyclic G-decomposition of $K_{n+1} \times 2t$, $K_{(n/4)+1} \times 8t$, $K_9 \times (n/4)t$, and of $K_{3\times nt}$ for every positive integer t.

For (p, q)-graphs with $p = q + 1$, Frucht [986] has introduced a stronger version of almost graceful graphs by permitting as vertex labels $\{0, 1, \ldots, q-1, q+1\}$ and as edge labels $\{1, 2, \ldots, q\}$. He calls such a labeling *pseudograceful*. Frucht proved that P_n $(n \geqslant 3)$, combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star), $C_3 \cup P_n$ $(n \neq 3)$, and $C_4 \cup P_n$ $(n \neq 1)$ are pseudograceful whereas $K_{1,n}$ $(n \geqslant 3)$ is not. Kishore [1704] proved that $C_s \cup P_n$ is pseudograceful when $s \geqslant 5$ and $n \geqslant (s+7)/2$ and that $C_s \cup S_n$ is pseudograceful when $s = 3, s = 4$, and $s \geqslant 7$. Seoud and Youssef [2803] and [2799] extended the definition of pseudograceful to all graphs with $p \leqslant q+1$. They proved that K_m is pseudograceful if and only if $m = 1, 3$, or 4 [2799]; $K_{m,n}$ is pseudograceful when $n \geqslant 2$, and $P_m + \overline{K_n}$ $(m \geqslant 2)$ [2803] is pseudograceful. They also proved that if G is pseudograceful, then $G \cup K_{m,n}$ is graceful for $m \geqslant 2$ and $n \geqslant 2$ and $G \cup K_{m,n}$ is pseudograceful for $m \geqslant 2, n \geqslant 2$ and $(m,n) \neq (2,2)$ [2799]. They ask if $G \cup K_{2,2}$ is pseudograceful whenever G is. Seoud and Youssef [2799] observed that if G is a pseudograceful Eulerian graph with q edges, then $q \equiv 0$ or 3 (mod 4). Youssef [3545] has shown that C_n is pseudograceful if and only if $n \equiv 0$ or 3 (mod 4), and for $n > 8$ and $n \equiv 0$ or 3 (mod 4), $C_n \cup K_{p,q}$ is pseudograceful for all $p, q \geqslant 2$ except $(p,q) = (2,2)$. Youssef [3542] has shown that if H is pseudograceful and G has an α-labeling with k being the smaller vertex label of the edge labeled with 1 and if either $k+2$ or $k-1$ is not a vertex label of G, then $G \cup H$ is graceful. In [3546] Youssef shows that if G is (p,q) pseudograceful graph with $p = q+1$, then $G \cup S_m$ is Skolem-graceful (see Section 3.5 for the definition). As a corollary he obtains that for all $n \geqslant 2$, $P_n \cup S_m$ is Skolem-graceful if and only if $n \geqslant 3$ or $n = 2$ and m is even.

In [3551] Youssef generalizes his results in [3542] and provides new families of disconnected graphs that have α-labelings and pseudo α-labelings. (A *pseudo* α-labeling f is an α-labeling for which there is an integer k_j with the property that for each edge xy of the graph either $f(x) \leqslant k_j < f(y)$ or $f(y) \leqslant k_j < f(x)$.)

For a graph G Ichishima, Muntaner-Batle, and Oshima [1293] defined the *beta-number* of G, $\beta(G)$, to be either the smallest positive integer n for which there exists an injective function f from the vertices of G to $\{1, 2, \ldots, n\}$ such that when each edge uv is labeled $|f(u) - f(v)|$ the resulting set of edge labels is $\{c, c+1, \ldots, c+|E(G)|-1\}$ for some positive integer c or $+\infty$ if there exists no such integer n. They defined the *strong beta-number* of G to be either the smallest positive integer n for which there exists an injective function f from the vertices of G to $\{1, 2, \ldots, n\}$ such that when each edge uv is labeled $|f(u) - f(v)|$ the resulting set of edge labels is $\{1, 2, \ldots, |E(G)|\}$ or $+\infty$ if there exists no such integer n. They gave some necessary conditions for a graph to have a finite (strong) beta-number and some sufficient conditions for a graph to have a finite (strong) beta-number. They also determined formulas for the beta-numbers and strong beta-numbers of C_n, $2C_n, K_n$ $(n \geqslant 2)$, $S_m \cup S_n$, $P_m \cup S_n$, and prove that nontrivial trees and forests without isolated vertices have finite strong beta-numbers. In [1282] Ichishima, López, Muntaner-Batte, and Oshima proved that if G is a bipartite graph and m is odd, then $\beta(mG) \leqslant m|E(G)| + m - 1$. If G has the additional property that G is a graceful nontrivial tree, then $\beta(mG) = m|V(G)| + m - 1$. They also investigate the (strong) beta-number of forests with components that are isomorphic to either paths or stars. They propose new conjectures on the (strong) beta-number of forests. In [1309] Ichishima and Oshima determine a formula for the (strong) beta-number of the linear forests $P_m \cup P_n$. As a corollary they provide a partial formula for the beta-number of the disjoint union of multiple copies of the same linear forest. In [1295] Ichishima, Muntaner-Batle, Oshima provide lower and upper bounds for $\beta(G+nK_1)$ when $\beta(G) = |V(G)|-1$ and formulas for

$\beta(G + nK_1)$ and $\beta_s(G + nK_1)$) when $\beta_s(G) = |V(G)| - 1$. They also determine formulas for $\beta(G + K_{1,n})$ and $\beta_s(G + K_{1,n})$ when $\beta_s(G) = |V(G)| - 1$. They conclude with two problems.

In [1288] Ichishima, Oshima, and Takahashi establish a lower bound for the beta- [1288] new
number of an arbitrary galaxy under certain conditions. They also introduce the notions of odd symmetric and even symmetric galaxies and determine formulas for the beta-number and gamma-number of odd symmetric galaxies. As corollaries, they provide formulas for the beta-number and gamma-number of the disjoint union of multiple copies of the same galaxy when the number of copies is odd. In addition to these, the present an upper bound for the beta-number of even symmetric galaxies and obtain partial formulas for the beta-number and gamma-number of even symmetric galaxies.

For a graph G of order p and size q and every positive integer n Ichishima, Muntaner-Batle, and Oshima [1298] proved if $\beta(G) = p - 1$, then there exists some positive integer c such that $q + np \leqslant \beta(G + nK_1) \leqslant c + q + np - 1$; if $\beta_s(G) = p - 1$, then $\beta(G + nK_1) = \beta_s(G + nK_1) = q + np$ and $G + nK_1$ is graceful; and if $q = p - 1$ and $\beta_s(G) = p - 1$, then $\beta(G + S_n) = \beta_s(G + S_n) = (n + 2)p + n - 1$. In particular, if T is a graceful tree of order p then $\beta(T + nK_1) = \beta_s(T + nK_1) = (n + 1)p - 1$. Moreover, $T + nK_1$ and $T + S_n$ are graceful.

In [1301] Ichishima, Muntaner-Batle, and Oshima establish a lower bound for the strong beta-number of an arbitrary galaxy (that is, a forest whose components are stars) under certain conditions. They also determine formulas for the (strong) beta-number and gracefulness of galaxies with three and four components. As corollaries, they provide formulas for the beta-number and gracefulness of the disjoint union of multiple copies of the same galaxies if the number of copies is odd. They pose some problems and conjecture. In [1290] Ichishima and Muntaner-Batle determined formulas for the (strong) beta-number and gracefulness of galaxies with five components. In [1305] Ichishima, Muntaner-Batle, and Oshima determined formulas for the (strong) beta-number and gamma-number of galaxies with five components. As a corollary of these results, they provide formulas for the beta-number and gamma-number of the disjoint union of multiple copies of the same galaxies if the number of copies is odd.

McTavish [2120] has investigated labelings of graphs with q edges where the vertex and edge labels are from $\{0, \ldots, q, q + 1\}$. She calls these *$\tilde{\rho}$-labelings*. Graphs that have $\tilde{\rho}$-labelings include cycles and the disjoint union of P_n or S_n with any graceful graph.

Frucht [986] has made an observation about graceful labelings that yields nearly graceful analogs of α-labelings and weakly α-labelings in a natural way. Suppose $G(V, E)$ is a graceful graph with the vertex labeling f. For each edge xy in E, let $[f(x), f(y)]$ (where $f(x) \leqslant f(y)$) denote the interval of real numbers r with $f(x) \leqslant r \leqslant f(y)$. Then the intersection $\cap[f(x), f(y)]$ over all edges $xy \in E$ is a unit interval, a single point, or empty. Indeed, if f is an α-labeling of G then the intersection is a unit interval; if f is a weakly α-labeling, but not an α-labeling, then the intersection is a point; and, if f is a graceful but not a weakly α-labeling, then the intersection is empty. For nearly graceful labelings, the intersection also gives three distinct classes.

Let $G(V, E)$ be a graph without isolated vertices and with q edges. The *gracefulness* grac(G) of G is the smallest positive integer k for which there exists an injective function $f : V \to \{0, 1, 2, \ldots, k\}$ such that the edge induced function $g_f : E \to \{1, 2, \ldots, k\}$ defined by $g_f(uv) = |f(u) - f(v)|$ for all edges uv is also injective. Let $c(f) = \max\{i : 1, 2, \ldots, i\}$ are edge labels} and let $m(G) = \max_f\{c(f)\}$ where the maximum is taken over all injective functions f from V to the nonnegative integers such that g_f is also injective. The measure $m(G)$ is called m-gracefulness of G. It determines how close G is to being graceful. Pereira, Singh, Arumugam [2380] prove that there are infinitely many nongraceful graphs with m-gracefulness $q-1$ and give necessary conditions for an Eulerian graph with q edges and K_p with q edges to have m-gracefulness $q-1$ and $q-2$. They prove that K_5 is the only complete graph to have m-gracefulness $q-1$. They also give an upper bound for the highest possible vertex label of K_p if $m(K_p) = q-2$.

A (p, q)-graph G is said to be a *super graceful* graph if there is a a bijective function $f : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, p+q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ labeling. Perumal, Navaneethakrishnan, Nagarajan, Arockiaraj [2385] and [2386] show that the graphs P_n, C_n, $P_m \odot nK_1$, $K_{m,n}$, and $P_n \odot K_1$ minus a pendent edge at an endpoint of P_n are super graceful graphs. Lau, Shiu, and Ng [1810] study the super gracefulness of complete graphs, the disjoint union of certain star graphs, the complete tripartite graphs $K_{(1,1,n)}$, and certain families of trees. They also provide four methods of constructing new super graceful graphs. They prove all trees of order at most 7 are super graceful and conjecture that all trees are super graceful. Amutha and Uma Devi [180] proved the following graphs are super graceful: fans, double fans $DF_n = P_n + \overline{K_2}$ $(n \geqslant 2)$, and for $(m \geqslant 3, n \geqslant 2)$ the graphs obtained by identifying a central vertex of the star S_m with an end vertex of path in $P_n + K_1$.

For $k \geqslant 1$, Lau, Shiu, and Ng [1815] say a bijection $f : V \cup E \to [k, k+p+q-1]$ [1815] new
is a *k-super graceful labeling* if $f(uv) = |f(u) - f(v)|$ for every edge uv in G. A graph G is *k-super graceful* if it admits a k-super graceful labeling. This is a generalization of super graceful labeling defined by Perumal, Navaneethakrishnan, Nagarajan, Arockiaraj in [2386]. It was referred to as a k-sequential labeling by Slater in [3000], in which Slater gave necessary and sufficient conditions for a star to be k-sequential. In [1814] Lau, [1814] new
Shiu, and Ng investigated the existence of k-sequential labelings (which they call k-super graceful labelings) of paths, cycles, caterpillars, complete bipartite and complete tripartite graphs.

In [875] Elsonbaty and Daoud introduce a new version of gracefulness called an edge even graceful labeling of graphs. A bijective function f from the edges of a (p, q)-graph G to $\{2, 4, \ldots, 2q\}$ is said to be an *edge even graceful labeling* of G if the induced function f^* from the vertices to $\{0, 2, \ldots, 2q\}$ defined by $f^*(e)$ is the sum of $f(e) \pmod{\max(p, q)}$ is injective. They prove the following graphs have edge even graceful labelings: P_n if and only if n is odd, C_n if and only if n is odd, $K_{1,n}$ if and only if n is even, wheels, fans, friendship graphs, and double wheels $W_{n,n}$. The polar grid graph $P_{m,n}$ consists of n copies of C_m, a new vertex v_0, and m copies on P_{n+1} that share a endpoint at v_0 The graph is drawn as m concentric circles with a center at a new vertex v_0 and the m vertices of each cycle lie on a line with one endpoint at v_0 and the other endpoint at the outermost cycle in such a way that the n vertices of the copies on P_{n+1} other the v_0 intersect the vertices of cycles. Daoud [781] provided necessary and sufficient conditions for the polar grid graph to be edge even graceful.

Singh and Devaraj [2977] call a graph G with p vertices and q edges *triangular graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, T_q\}$ where T_q is the qth triangular number and the labels induced on each edge uv by $|f(u) - f(v)|$ are the first q triangular numbers. They prove the following graphs are triangular graceful: paths, level 2 rooted trees, olive trees (see § 2.1 for the definition), complete n-ary trees, double stars, caterpillars, C_{4n}, C_{4n} with pendent edges, the one-point union of C_3 and P_n, and unicyclic graphs that have C_3 as the unique cycle. They prove that wheels, helms, flowers (see §2.2 for the definition) and K_n with $n \geqslant 3$ are not triangular graceful. They conjecture that all trees are triangular graceful. In [2843] Sethuraman and Venkatesh introduced a new method for combining graceful trees to obtain trees that have α-labelings.

Van Bussel [3336] considered two kinds of relaxations of graceful labelings as applied to trees. He called a labeling *range-relaxed graceful* it is meets the same conditions as a graceful labeling except the range of possible vertex labels and edge labels are not restricted to the number of edges of the graph (the edges are distinctly labeled but not necessarily labeled 1 to q where q is the number of edges). Similarly, he calls a labeling *vertex-relaxed graceful* if it satisfies the conditions of a graceful labeling while permitting repeated vertex labels. He proves that every tree T with q edges has a range-relaxed graceful labeling with the vertex labels in the range $0, 1, \ldots, 2q-d$ where d is the diameter of T and that every tree on n vertices has a vertex-relaxed graceful labeling such that the number of distinct vertex labels is strictly greater than $n/2$. In 2017 Sethuraman, Ragukumar, and Slater [2830] improved the bound on the range-relaxed graceful labeling given by Van Bussel in [3336] in 2002 for a tree T.

The *range-relaxed graceful game* is a maker-breaker game played in a simple graph G where two players, Alice and Bob, alternately assign an unused label $f(v) \in \{0, \ldots, k\}$ $(k \geqslant |E(G)|)$, to an unlabeled vertex $v \in V(G)$. If both ends of an edge $vw \in E(G)$ are already labeled, then the label of the edge is defined as $|f(v) - f(w)|$. Alice's goal is to end up with a vertex labeling of G where all edges of G have distinct labels, and Bob's goal is to prevent this from happening. When it is required that $k = |E(G)|$, the game is called a *graceful game*. The range-relaxed graceful game and the graceful game were proposed by Tuza in 2017 [3233]. In [2319] Oliveira, Dantas, and Lui, considered a question about the least number of consecutive non-negative integer labels necessary for Alice to win the game on an arbitrary simple graph G and also asked if Alice can win the range-relaxed graceful game on G with the set of labels $\{0, \ldots, k+1\}$ once it is known that she can win with the set $\{0, \ldots, k\}$. They investigated the graceful game in Cartesian and corona products of graphs, and determined that Bob has a winning strategy in all investigated families independently of who starts the game. Additionally, they partially answer Tuza's questions presenting the first results in the range-relaxed graceful game and proving that Alice wins on any simple graph G with order n, size m, and maximum degree Δ, for any set of labels $\{0, \ldots, k\}$ with $k \geqslant (n-1) + 2\Delta(m-\Delta) + (\Delta(\Delta-1))/2$.

In [455], Barrientos and Krop introduce left- and right-layered trees as trees with a specific representation and define the excess of a tree. Applying these ideas, they show a range-relaxed graceful labeling which improves the upper bound for maximum vertex label given by Van Bussel in [3336]. They also improve the bounds given by Rosa and Širáň in [2651] for the α-size and gracesize of lobsters.

Sekar [2736] calls an injective function ϕ from the vertices of a graph with q edges to $\{0, 1, 3, 4, 6, 7, \ldots, 3(q-1), 3q-2\}$ *one modulo three graceful* if the edge labels induced by labeling each edge uv with $|\phi(u) - \phi(v)|$ is $\{1, 4, 7, \ldots, 3q-2\}$. He proves that the following graphs are one modulo three graceful: P_m; C_n if and only if $n \equiv 0 \bmod 4$; $K_{m,n}$; $C_{2n}^{(2)}$ (the one-point union of two copies of C_{2n}); $C_n^{(t)}$ for $n = 4$ or 8 and $t > 2$; $C_6^{(t)}$ and $t \geqslant 4$; caterpillars; stars; lobsters; banana trees; rooted trees of height 2; ladders; the graphs obtained by identifying the endpoints of any number of copies of P_n; the graph obtained by attaching pendent edges to each endpoint of two identical stars and then identifying one endpoint from each of these graphs; the graph obtained by identifying a vertex of C_{4k+2} with an endpoint of a star; n-polygonal snakes (see §2.2) for $n \equiv 0 \pmod 4$; n-polygonal snakes for $n \equiv 2 \pmod 4$ where the number of polygons is even; crowns $C_n \odot K_1$ for n even; $C_{2n} \odot P_m$ (C_{2n} with P_m attached at each vertex of the cycle) for $m \geqslant 3$; chains of cycles (see §2.2) of the form $C_{4,m}, C_{6,2m}$, and $C_{8,m}$. He conjectures that every one modulo three graceful graph is graceful.

A *subdivided shell graph* is obtained by subdividing the edges in the path of the shell graph. Jeba Jesintha and Ezhilarasi Hilda [1385] proved that the subdivided uniform shell bow graphs (that is, double shells in which each shell has the same order) are one modulo three graceful. Jeba Jesintha and Ezhilarasi Hilda [1384] proved the disjoint union of two subdivided shell graphs are one modulo three graceful.

In [2569] Ramachandran and Sekar introduced the notion of one modulo N graceful as follows. For a positive integer N a graph G with q edges is said to be *one modulo N graceful* if there is an injective function ϕ from the vertex set of G to $\{0, 1, N, N+1, 2N, 2N+1, \ldots, (q-1)N, (q-1)N+1\}$ such that ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, (q-1)N+1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$. They proved the following graph are one modulo N graceful for all positive integers N: paths, caterpillars, and stars [2569]; n-polygonal snakes, $C_n^{(t)}$, $P_{a,b}$ [2583]; the splitting graphs $S'(P_{2n})$, $S'(P_{2n+1})$, $S'(K_{1,n})$, all subdivision graphs of double triangular snakes, and all subdivision graphs of $2m$-triangular snakes [2570]; the graph $L_n \otimes S_m$ obtained from the ladder L_n ($P_n \times P_2$) by identifying one vertex of L_n with any vertex of the star S_m other than the center of S_m [2572]; arbitrary supersubdivisions of paths, disconnected paths, cycles, and stars [2571]; and regular bamboo trees and coconut trees [2573]. Ramachandran and Sekar [2574] proved the supersubdivisions of ladders are one modulo N graceful for all positive integers N. In [2575] Ramachandran and Sekar proved that the crowns, armed crowns, and chain of even cycles are one modulo N graceful for all positive integers N.

In 1983 Bange and Barkauskas z [418] introduced the notion of Fibonacci graceful graphs as follows. A function $f : V(G) \to \{0, 1, 2, \ldots, F_q\}$, where G has q edges and F_q is the qth Fibonacci number, is a *Fibonacci graceful labeling* if the induced edge labeling $\overline{f}(uv) = |f(u) - f(v)|$ is a bijection to the set of the first q Fibonacci numbers. Such a graph is called *Fibonacci graceful*. They derived a number of properties of Fibonacci [418] new

graceful graphs and provided some forbidden subgraphs of Fibonacci graceful graphs. Other results include: C_n is Fibonacci graceful if and only if $n = 0$ or 2 mod 3, trees with at least 7 vertices are Fibonacci graceful, and a maximal outerplanar graph with at least four vertices is Fibonacci graceful if and only if it has exactly two vertices of degree 2. Kathiresan and Amutha in [1667] prove the following: K_n is Fibonacci graceful if and [1667] new
only if $n \leqslant 3$; if an Eulerian graph with q edges is Fibonacci graceful, then $q \equiv 0$ (mod 3); paths are Fibonacci graceful; fans $P_n \odot K_1$ are Fibonacci graceful; squares of paths P_n^2 are Fibonacci graceful; and caterpillars are Fibonacci graceful. Dharman and Shanmuga Sundaram [815] proved that balloon trees, barycentric subdivisions of bistars, umbrella [815] new
graphs, and double comb graphs are Fibonacci graceful graphs.

Kalyan and Kempepatil [1568] proved that trees and the graphs obtained by joining a [1568] new
vertex of C_{3m} and a vertex of C_{3n} by P_2 or P_3 admit Fibonacci graceful labelings. They define a function $f : V(G) \to \{0, F_1, F_2, \ldots, F_q\}$, where F_i is the ith Fibonacci number, to be a *super Fibonacci graceful* labeling if the induced labeling $\overline{f}(uv) = |f(u) - f(v)|$ is onto the set $\{F_1, F_2, \ldots, F_q\}$. They show that bistars $B_{n,n}$ are Fibonacci graceful but not super Fibonacci graceful for $n \geqslant 5$; cycles C_n are super Fibonacci graceful if and only if $n \equiv 0$ (mod 3); if G is Fibonacci or super Fibonacci graceful, then $G \odot K_1$ is Fibonacci graceful; if G_1 and G_2 are super Fibonacci graceful graphs in which no two adjacent vertices have the labels 1 and 2, then $G_1 \cup G_2$ is Fibonacci graceful; and if $G_1, G_2, \ldots, G_n$ are super Fibonacci graceful graphs in which no two adjacent vertices are labeled with 1 and 2, then the amalgamation of $G_1, G_2, \ldots, G_n$ obtained by identifying the vertices having labels 0 is also a super Fibonacci graceful. Karthikeyan, Arthi, Abinaya, Swathi, Madhumathi [1657] proved that friendship graphs $C_3^{(t)}$ and the graphs obtained by the one-point union of copies of K_4 with an edge deleted are super Fibonacci graceful.

Vaidya and Prajapati [3296] proved: the graphs obtained joining a vertex of C_{3m} and a vertex of C_{3n} by a path P_k are Fibonacci graceful; the graphs obtained by starting with any number of copies of C_{3m} and joining each copy with a copy of the next by identifying the end points of a path with a vertex of each successive pair of C_{3m} (the paths need not be the same length) are Fibonacci graceful; the one point union of C_{3m} and C_{3n} is Fibonacci graceful; the one point union of k cycles C_{3m} is super Fibonacci graceful; every cycle C_n with $n \equiv 0$ (mod 3) or $n \equiv 1$ (mod 3) is an induced subgraph of a super Fibonacci graceful graph; and every cycle C_n with $n \equiv 2$ (mod 3) can be embedded as a subgraph of a Fibonacci graceful graph. [3041] Sridevi, Navaneethakrishnan, and Nagarajan proved [3041] new
the following graphs are super Fibonacci graceful: the graphs obtained by identifying the apex of a fan with the end point of a path, the graphs obtained by identifying the apex of a fan with the vertex of maximum degree of $K_{1,n} \odot P_2$, the graphs obtained by identifying a vextex of C_{3n} with the end point of a path, the graphs obtained by identifying a vextex of C_{3n} with the center of a star, and the graphs obtained by identifying each endpoint a star with the center of $K_{1,2}$.

For a graph G with q edges an injective function f from the vertices of G to $\{0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$, where F_i is the ith Fibonacci number (as defined by Kathiresan and Amuth above), is said to be *almost super Fibonacci graceful* if the induced edge labeling $f*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$ or

$\{0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$. Sridevi, Navaneethakrishnan, and Nagarajan [3041] proved that paths, combs, graphs obtained by subdividing each edge of a star, and some special types of extension of cycle related graphs are almost super Fibonacci graceful labeling. Sridevi, Navaneethakrishnan, and Nagarajan [3040] showed that paths, combs, and some [3040] new special types of extension of cycle-related graphs are almost super Fibonacci graceful.

For a graph G and a vertex v of G, a *vertex switching* G_v is the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. Vaidya and Vihol [3321] prove the following: trees are Fibonacci graceful; the graph obtained by switching of a vertex in cycle is Fibonacci graceful; wheels and helms are not Fibonacci graceful; the graph obtained by switching of a vertex in a cycle is super Fibonacci graceful except $n \geqslant 6$; the graph obtained by switching of a vertex in cycle C_n for $n \geqslant 6$ can be embedded as an induced subgraph of a super Fibonacci graceful graph; and the graph obtained by joining two copies of a fixed fan with an edge is Fibonacci graceful.

The *Perrin sequence* of numbers P_n is defined by the linear recurrence relation satisfying the conditions: $P_1 = 3, P_2 = 0, P_3 = 2$, and $P_n = P_{n-2} + P_{n-3}$, if $n \geqslant 4$. Letting P_i be the i^{th} term of the Perrin sequence and $P_0 = 0$, Sugumaran and Rajesh [3118] introduced the notion of Perrin graceful labeling as follows: A function f is called a *Perrin graceful* labeling of a graph G, if $f : V(G) \to \{P_0, P_1, P_2, ..., P_q\}$ is injective and the induced function $f^{(*)} : E(G) \to \{P_1, P_2, ..., P_q\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is bijective. A graph that admits Perrin graceful labeling is called a *Perrin graceful* graph. In [3118] Sugumaran and Rajesh proved that the following graphs are Perrin graceful graphs: $K_{1,n}$, $B_{n,n}$, $P_n \odot K_1$, $C_n \odot K_1$, and $< K_{1,n}; 4 >$.

For $n \geqslant 1$ the Pell numbers are defined as $p_0 = 0,\ p_1 = 1$, and $p_{n+1} = 2p_n + p_{n-1}$. For a graph G with q edges Muthuramakrishnan and Sutha [2236] introduced the concept of *Pell graceful* labeling as an injective function f from $V(G)$ to the Pell numbers $\{0, 1, 2, \ldots, p_q\}$ such that the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the Pell numbers $\{p_1, p_2, \ldots, p_q\}$ They prove that paths, combs $P_n \odot K_1$ $(n \geqslant 3)$, and the graphs obtained by the one point union of paths of lengths $1, 2, \ldots, n$ $(n \geqslant 3)$ are Pell graceful.

In [613] Brešar and Klavžar define a natural extension of graceful labelings of certain tree subgraphs of hypercubes. A subgraph H of a graph G is called *isometric* if for every two vertices u, v of H, there exists a shortest u-v path that lies in H. The isometric subgraphs of hypercubes are called *partial cubes*. Two edges xy, uv of G are in Θ-relation if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. A Θ-relation is an equivalence relation that partitions $E(G)$ into Θ-classes. A Θ-*graceful labeling* of a partial cube G on n vertices is a bijection $f \colon V(G) \to \{0, 1, \ldots, n-1\}$ such that, under the induced edge labeling, all edges in each Θ-class of G have the same label and distinct Θ-classes get distinct labels. They prove that several classes of partial cubes are Θ-graceful and the Cartesian product of Θ-graceful partial cubes is Θ-graceful. They also show that if there exists a class of partial cubes that contains all trees and every member of the class admits a Θ-graceful labeling then all trees are graceful.

Cohen and Kovše showed that the graph obtained by merging two vertices of two 4-cycles is not a Θ-graceful partial cube, thereby answering in the negative a question of

Brešar and Klavžar [613] who asked whether every partial cube is Θ-graceful.

Table 4 provides a summary results about graceful-like labelings adapted from [611]. "Y" indicates that all graphs in that class have the labeling; "N" indicates that not all graphs in that class have the labeling; "?" means unknown; "C" means conjectured.

Table 4: **Summary of Results on Graceful-like labelings**

Graph	α-labeling	β-labeling	σ-labeling	ρ-labeling
Cycle $C_n,\ n \equiv 0 \bmod 4$	Y [2648]	Y	Y	Y
Cycle $C_n,\ n \equiv 3 \bmod 4$	N [2648]	Y [2648]	Y	Y
Wheels	N	Y [984], [1255]	Y	Y
Trees				
Yes, if order $\leqslant$	5	35 [914]	54	
Paths	Y [2648]	Y	Y	Y
Caterpillars	Y [2648]	Y	Y	Y
Firecrackers	Y [692]	Y	Y	Y
Lobsters	N[573]	?C [544]	Y	Y [659]
Bananas	?	Y [2822], [2821]	Y	Y
Symmetrical trees	N [573]	Y [548]	Y	Y
Olive trees	?	Y [2355], [13]	Y	Y
Diameter < 8	N [3447]	Y	Y	Y
< 5 end vertices	N [573]	Y [2648]	Y	Y
Max degree 3	N [2651]	C	C	C
Max degree 3 and perfect matching	C [608]	C	C	C

3.4 *k*-graceful Labelings

A natural generalization of graceful graphs is the notion of k-graceful graphs introduced independently by Slater [3001] in 1982 and by Maheo and Thuillier [2054] in 1982. A graph G with q edges is *k-graceful* if there is labeling f from the vertices of G to $\{0, 1, 2, \ldots, q+k-1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k+1, \ldots, q+k-1\}$. Obviously, 1-graceful is graceful and it is readily shown that any graph that has an α-labeling is k-graceful for all k. Graphs that are k-graceful for all k are sometimes called *arbitrarily graceful*. The result of Barrientos and Minion [456] that all snake polyominoes are α-graphs partially answers a question of Acharya [30] and supports his conjecture that if the length of every cycle of a graph is a multiple of 4, then the graph is arbitrarily graceful. In [2772] Seoud and Elsakhawi show that $P_2 \oplus \overline{K_2}$ $(n \geqslant 2)$ is arbitrarily graceful. Ng [2282] has shown that there are graphs that are k-graceful for all k but do not have an α-labeling.

Results of Maheo and Thuillier [2054] together with those of Slater [3001] show that: C_n is k-graceful if and only if either $n \equiv 0$ or $1 \pmod 4$ with k even and $k \leqslant (n-1)/2$, or $n \equiv 3 \pmod 4$ with k odd and $k \leqslant (n^2-1)/2$. Maheo and Thuillier [2054] also proved that the wheel W_{2k+1} is k-graceful and conjectured that W_{2k} is k-graceful when $k \neq 3$ or $k \neq 4$. This conjecture was proved by Liang, Sun, and Xu [1942]. Kang [1631] proved that $P_m \times C_{4n}$ is k-graceful for all k. Lee and Wang [1906] showed that the graphs obtained from a nontrivial path of even length by joining every other vertex to one isolated vertex (a *lotus*), the graphs obtained from a nontrivial path of even length by joining every other vertex to two isolated vertices (a *diamond*), and the graphs obtained by arranging vertices into a finite number of rows with i vertices in the ith row and in every row the jth vertex in that row is joined to the jth vertex and $j+1$st vertex of the next row (a *pyramid*) are k-graceful. Liang and Liu [1930] have shown that $K_{m,n}$ is k-graceful. Bu, Gao, and Zhang [622] have proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are k-graceful for all k. Acharya (see [30]) has shown that a k-graceful Eulerian graph with q edges must satisfy one of the following conditions: $q \equiv 0 \pmod 4$, $q \equiv 1 \pmod 4$ if k is even, or $q \equiv 3 \pmod 4$ if k is odd. Bu, Zhang, and He [627] have shown that an even cycle with a fixed number of pendent edges adjoined to each vertex is k-graceful. Lu, Pan, and Li [2041] have proved that $K_{1,m} \cup K_{p,q}$ is k-graceful when $k > 1$, and p and q are at least 2. Jirimutu, Bao, and Kong [1552] have shown that the graphs obtained from $K_{2,n}$ ($n \geqslant 2$) and $K_{3,n}$ ($n \geqslant 3$) by attaching $r \geqslant 2$ edges at each vertex is k-graceful for all $k \geqslant 2$. Seoud and Elsakhawi [2772] proved: paths and ladders are arbitrarily graceful; and for $n \geqslant 3$, K_n is k-graceful if and only if $k = 1$ and $n = 3$ or 4. Li, Li, and Yan [1927] proved that $K_{m,n}$ is k-graceful graph. Pradhan and Kamesh [2481] showed that the hairy cycle $C_n \cdot rK_1$ ($n \equiv 3$ (mod 4), the graph obtained by adding a pendent edge to each pendent vertex of hairy cycle $C_n \cdot K_1$; $n \equiv 0 \pmod 4$), double graphs of path P_n, and double graphs of combs $P_n \cdot K_1$ are k-graceful.

Yao, Cheng, Zhongfu, and Yao [3530] have shown: a tree of order p with maximum degree at least $p/2$ is k-graceful for some k; if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$ and $d_{T_2}(u_2) \geqslant |T_2|/2$, then T is k-graceful for some positive k; if a tree T has two edges u_1u_2 and u_2u_3 such that the three components T_1, T_2, and T_3 of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$, $d_{T_2}(u_2) \geqslant |T_2|/2$, and $d_{T_3}(u_3) \geqslant |T_3|/2$, then T is k-graceful for some $k > 1$; and every Skolem-graceful (see 3.5 for the definition) tree is k-graceful for all $k \geqslant 1$. They conjecture that every tree is k-graceful for some $k > 1$.

Several authors have investigated the k-gracefulness of various classes of subgraphs of grid graphs. Acharya [28] proved that all 2-dimensional polyminoes that are convex and Eulerian are k-graceful for all k; Lee [1836] showed that Mongolian tents and Mongolian villages are k-graceful for all k (see §2.3 for the definitions); Lee and K. C. Ng [1860] proved that all Young tableaus (see §2.3 for the definitions) are k-graceful for all k. (A special case of this is $P_n \times P_2$.) Lee and H. K. Ng [1860] subsequently generalized these results on Young tableaus to a wider class of planar graphs.

In [465] Barrientos and Minion say that two caterpillars Γ and Ω of size n are *analogous* if the stable sets of Γ have the same cardinalities as the stable sets of Ω. They prove that if Ω is an induced subgraph of a gracefully labeled graph G, such that the induced labeling is a bipartite k-labeling shifted c units, then the graph G' obtained by replacing Ω with any other caterpillar Γ analogous to Ω, is a graceful graph. This result is used to generalize several existing results that use k-graceful labelings of paths such as the subdivision of graceful trees [633], the α-labeling of the ith attachment tree [2843], the α-labelings of path-like trees [438], the α-labelings of the graphs obtained by identifying the end-vertices of b paths of length a with two new vertices, as well as the graceful labelings of the armed crowns [2736].

Duan and Qi [856] use $G_t(m_1, n_1; m_2, n_2; \ldots; m_s, n_s)$ to denote the graph composed of the s complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_s,n_s}$ that have only t $(1 \leqslant t \leqslant \min\{m_1, m_2, \ldots, m_s\})$ common vertices but no common edge and $G(m_1, n_1; m_2, n_2)$ to denote the graph composed of the complete bipartite graphs K_{m_1,n_1}, K_{m_2,n_2} with exactly one common edge. They prove that these graphs are k-graceful graphs for all k.

Let $c, m, p_1, p_2, \ldots, p_m$ be positive integers. For $i = 1, 2, \ldots, m$, let S_i be a set of $p_i + 1$ integers and let D_i be the set of positive differences of the pairs of elements of S_i. If all these differences are distinct then the system $D_1, D_2, \ldots, D_m$ is called a *perfect system of difference sets starting at* c if the union of all the sets D_i is $c, c+1, \ldots, c-1+\sum_{i=1}^{m} \binom{p_i+1}{2}$. There is a relationship between k-graceful graphs and perfect systems of difference sets. A perfect system of difference sets starting with c describes a c-graceful labeling of a graph that is decomposable into complete subgraphs. A survey of perfect systems of difference sets is given in [18].

Acharya and Hegde [44] generalized k-graceful labelings to (k, d)*-graceful labelings* by permitting the vertex labels to belong to $\{0, 1, 2, \ldots, k + (q-1)d\}$ and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be $\{k, k+d, k+2d, \ldots, k+(q-1)d\}$. They also introduce an analog of α-labelings in the obvious way. Notice that a (1,1)-graceful labeling is a graceful labeling and a $(k, 1)$-graceful labeling is a k-graceful labeling. Bu and Zhang [626] have shown: $K_{m,n}$ is (k, d)-graceful for all k and d; for $n > 2, K_n$ is (k, d)-graceful if and only if $k = d$ and $n \leqslant 4$; if $m_i, n_i \geqslant 2$ and $\max\{m_i, n_i\} \geqslant 3$, then $K_{m_1,n_1} \cup K_{m_2,n_2} \cup \cdots \cup K_{m_r,n_r}$ is (k, d)-graceful for all k, d, and r; if G has an α-labeling, then G is (k, d)-graceful for all k and d; a k-graceful graph is a (kd, d)-graceful graph; a (kd, d)-graceful connected graph is k-graceful; and a (k, d)-graceful graph with q edges that is not bipartite must have $k \leqslant (q-2)d$.

Let T be a tree with adjacent vertices u_0 and v_0 and pendent vertices u and v such that the length of the path $u_0 - u$ is the same as the length of the path $v_0 - v$. Hegde and Shetty [1231] call the graph obtained from T by deleting u_0v_0 and joining u and v an *elementary parallel transformation* of T. They say that a tree T is a T_p*-tree* if it can be transformed into a path by a sequence of elementary parallel transformations. They prove that every T_p-tree is (k, d)-graceful for all k and d and every graph obtained from a T_p-tree by subdividing each edge of the tree is (k, d)-graceful for all k and d.

Yao, Cheng, Zhongfu, and Yao [3530] have shown: a tree of order p with maximum degree at least $p/2$ is (k, d)-graceful for some k and d; if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$ and T_2 is a caterpillar, then T is Skolem-graceful (see 3.5 for the definition); if a tree T

has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$ and $d_{T_2}(u_2) \geqslant |T_2|/2$, then T is (k, d)-graceful for some $k > 1$ and $d > 1$; if a tree T has two edges u_1u_2 and u_2u_3 such that the three components T_1, T_2, and T_3 of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$, $d_{T_2}(u_2) \geqslant |T_2|/2$, and $d_{T_3}(u_3) \geqslant |T_3|/2$, then T is (k, d)-graceful for some $k > 1$ and $d > 1$; and every Skolem-graceful tree is (k, d)-graceful for $k \geqslant 1$ and $d > 0$. They conjecture that every tree is (k, d)-graceful for some $k > 1$ and $d > 1$.

Hegde [1215] has proved the following: if a graph is (k, d)-graceful for odd k and even d, then the graph is bipartite; if a graph is (k, d)-graceful and contains C_{2j+1} as a subgraph, then $k \leqslant jd(q - j - 1)$; K_n is (k, d)-graceful if and only if $n \leqslant 4$; C_{4t} is (k, d)-graceful for all k and d; C_{4t+1} is $(2t, 1)$-graceful; C_{4t+2} is $(2t - 1, 2)$-graceful; and C_{4t+3} is $(2t + 1, 1)$-graceful.

A *semismooth graceful graph* is a bipartite graph G with the property that for some fixed positive integer $t \leqslant q$ and all positive integers l there is an injective map $g : V \longrightarrow \{0, 1, \ldots, t - l, t + l + 1, \ldots, q + l\}$ such that the induced edge labeling map $g^\star : E \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined by $g^\star(e) = |g(u) - g(v)|$ is a bijection. Kaneria, Gohil, and Makadia [1585] prove every semismooth graceful graph is a (k, d)-graceful; graphs obtained by joining two semismooth graceful graphs with an arbitrary path is a semismooth graceful graph; and the notions of graceful labeling and odd-even graceful labelings are equivalent. (A graph G with q edges is *odd-even graceful* if there is an injection f from the vertices of G to $\{1, 3, 5, \ldots, 2q + 1\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $\{2, 4, 6, \ldots, 2q\}$). Kaneria, Meghpara and Khoda [1592] prove: a smooth graceful labeling for a graph is also an α-labeling for the graph; a graph that has an α-labeling is a semismooth graceful graph; graphs that admit an α-labeling are semismooth graceful graphs; if m is even and H has an α-labeling, then the path union $P(m \cdot H)$ is a smooth graceful graph; and the path union $P(m \cdot H)$ has an α-labeling.

Nurvazly, Chasanah, and Wiranto [2316] showed that the corona of the ladder L_2 and $\overline{K_n}$ and of L_3 and $\overline{K_n}$, which they call *millipedes*, admits graceful, $\hat{\rho}$, and odd-even graceful labelings. In [3074] Sudha and Kanniga proved that tensor product of a star and P_2 is odd-even graceful. (The *tensor product* $G \otimes H$ of graphs G and H, has the vertex set $V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in $G \otimes H$ if and only if u' is adjacent with v' and u is adjacent with v.) In [3372] Venkatesh, Mahalakshmi, and Amirthavahini use $C_{n,k}$ to denote the dragon obtained by joining an end point of P_k with a vertex of C_n and $C^t_{n,k}$ to denote the graph obtained by taking one-point union of t copies of $C_{n,k}$ at the common vertex v. They proved that the graph $C^t_{n,k}$ admits a graceful labeling, an odd graceful labeling, and odd-even graceful labeling for all values of t with $n = 4, k = 1$, and that $C^t_{n,1}$ admits vertex cordial labeling for all values of n and t, except $n \equiv 2 \bmod 4$ (see Section 3.7). Nurvazly and Sugeng [2315] proved that $\Theta(C_3)^n$ graphs (n copies of C_3 that share an edge) have odd-even graceful labelings. Anitha, Selvam, and Thirusangu [215] provide k-graceful and odd-even graceful labelings for the extended duplicate graph of the kite graph. [2316] new

For a graph G let $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$ be $n \geqslant 2$ copies of G. The graph obtained by joining vertices u, v of $G^{(i)}$ with same vertices of the graph $G^{(i+1)}$ by two edges, for all $i = 1, 2, \ldots, n-1$ is called the *double path union* of n copies of the graph G. Such graphs can obtained in $\frac{p(p-1)}{2}$ different ways, where $p = |V(G)|$ and are denoted by $D(n \cdot G)$. Kaneria, Teraiya and Meghpara [1620] prove the double path unions of C_{4m}, $K_{m,n}$, and P_{2m} have α-labelings.

Hegde [1213] calls a (k, d)-graceful graph (k, d)-*balanced* if it has a (k, d)-graceful labeling f with the property that there is some integer m such that for every edge uv either $f(u) \leqslant m$ and $f(v) > m$, or $f(u) > m$ and $f(v) \leqslant m$. He proves that if a graph is $(1, 1)$-balanced then it is (k, d)-graceful for all k and d and that a graph is $(1, 1)$-balanced graph if and only if it is (k, k)-balanced for all k. He conjectures that all trees are (k, d)-balanced for some values of k and d.

Slater [3004] has extended the definition of k-graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices, and the countably infinite Dutch windmill is k-graceful for all k.

In [1237] Hegde and Shivarajkumar extend the idea of k-graceful labeling of undirected graphs to directed graphs as follows. A simple directed graph D with n vertices and e edges is labeled by assigning each vertex a distinct element from the set Z_{e+k} and assigning the edge xy from vertex x to vertex y the label $\theta(x, y) = \theta(y) - \theta(x) \bmod(e + k)$, where $\theta(y)$ and $\theta(x)$ are the values assigned to the vertices y and x respectively. A labeling is a *k-graceful* labeling if all $\theta(x, y)$ are distinct and belong to $\{k, k + 1, \ldots, k + e - 1\}$. If a digraph D admits a k-graceful labeling then D is called a *k-graceful digraph*. They provide some values of k for which the unidirectional cycles admit a k-graceful labeling; give a necessary and sufficient condition for the outspoken unicyclic wheel to be k-graceful; and prove that to provide a list of values of k for which the unicyclic wheel is k-graceful is NP-complete.

More specialized results on k-graceful labelings can be found in [1836], [1860], [1864], [3001], [621], [623], and [622].

Graceful-type labelings methods have been used for cryptographical password construction for network data [3424], [3423], [3425], and [3132].

3.5 Skolem-Graceful Labelings

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see [1890]) calls a graph G with p vertices and q edges *Skolem-graceful* if there is an injection from the set of vertices of G to $\{1, 2, \ldots, p\}$ such that the edge labels induced by $|f(x) - f(y)|$ for each edge xy are $1, 2, \ldots, q$. A necessary condition for a graph to be Skolem-graceful is that $p \geqslant q+1$. Lee and Wui [1920] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree. Yao, Cheng, Zhongfu, and Yao [3530] have shown that a tree of order p with maximum degree at least $p/2$ is Skolem-graceful. Although the disjoint union of trees cannot be graceful, they can be Skolem-graceful. Lee and Wui [1920] prove that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has even size. In [720] Choudum and Kishore show that the disjoint union of k copies of the star $K_{1,2p}$ is Skolem graceful if

$k \leqslant 4p+1$ and the disjoint union of any number of copies of $K_{1,2}$ is Skolem graceful. For $k \geqslant 2$, let $St(n_1, n_2, \ldots, n_k)$ denote the disjoint union of k stars with $n_1, n_2, \ldots, n_k$ edges. Lee, Wang, and Wui [1913] showed that the 4-star $St(n_1, n_2, n_3, n_4)$ is Skolem-graceful for some special cases and conjectured that all 4-stars are Skolem-graceful. Denham, Leu, and, Liu [801] proved this conjecture. Kishore [1704] has shown that a necessary condition for $St(n_1, n_2, \ldots, n_k)$ to be Skolem graceful is that some n_i is even or ($k \equiv 0$) or 1 (mod 4) (see also [3564] . He conjectures that each one of these conditions is sufficient. Yue, Yuan-sheng, and Xin-hong [3564] show that for k at most 5, a k-star is Skolem-graceful if at one star has even size or $k \equiv 0$ or 1 (mod 4). Choudum and Kishore [718] proved that all 5-stars are Skolem graceful.

Lee, Quach, and Wang [1876] showed that the disjoint union of the path P_n and the star of size m is Skolem-graceful if and only if $n = 2$ and m is even or $n \geqslant 3$ and $m \geqslant 1$. It follows from the work of Skolem [2993] that nP_2, the disjoint union of n copies of P_2, is Skolem-graceful if and only if $n \equiv 0$ or 1 (mod 4). Harary and Hsu [1186] studied Skolem-graceful graphs under the name *node-graceful*. Frucht [986] has shown that $P_m \cup P_n$ is Skolem-graceful when $m + n \geqslant 5$. Bhat-Nayak and Deshmukh [559] have shown that $P_{n_1} \cup P_{n_2} \cup P_{n_3}$ is Skolem-graceful when $n_1 < n_2 \leqslant n_3,\ n_2 = t(n_1+2)+1$ and n_1 is even and when $n_1 < n_2 \leqslant n_3,\ n_2 = t(n_1+3)+1$ and n_1 is odd. They also prove that the graphs of the form $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_i}$ where $i \geqslant 4$ are Skolem-graceful under certain conditions. In [806] Deshmukh states the following results: the sum of all the edges on any cycle in a Skolem graceful graph is even; $C_5 \cup K_{1,n}$ if and only if $n = 1$ or 2; $C_6 \cup K_{1,n}$ if and only if $n = 2$ or 4.

Youssef [3542] proved that if G is Skolem-graceful, then $G + \overline{K_n}$ is graceful. In [3546] Youssef shows that that for all $n \geqslant 2,\ P_n \cup S_m$ is Skolem-graceful if and only if $n \geqslant 3$ or $n = 2$ and m is even. Yao, Cheng, Zhongfu, and Yao [3530] have shown that if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$ and T_2 is a caterpillar or have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$ and $d_{T_2}(u_2) \geqslant |T_2|/2$, then T is Skolem-graceful.

A graph $G = (V, E)$ is said to be (k, d)-*Skolem graceful* if there exists a bijection f from $V(G)$ to $\{12, \ldots, |V|\}$ such that the induced edge labeling g_f defined by $g_f(uv) = |f(u) - f(v)|$ is a bijection from E to $\{k, k+d, \ldots, k+(q-1)d\}$ where k and d are positive integers. Such a labeling is called a (k, d)-*Skolem graceful labeling* of G. In [2381] Pereira, Singh, and Arumugam present a few basic results on (k, d)-Skolem graceful graphs and prove that nK_2 is $(2, 1)$-Skolem graceful if and only if $n \equiv 0$ or 3 (mod 4), which produces the Langford sequence $L(2, n)$.

Mendelsohn and Shalaby [2134] defined a *Skolem labeled* graph $G(V, E)$ as one for which there is a positive integer d and a function $L\colon V \to \{d, d+1, \ldots, d+m\}$, satisfying (a) there are exactly two vertices in V such that $L(v) = d+i$, $0 \leqslant i \leqslant m$; (b) the distance in G between any two vertices with the same label is the value of the label; and (c) if G' is a proper spanning subgraph of G, then L restricted to G' is not a Skolem labeled graph. Note that this definition is different from the Skolem-graceful labeling of Lee, Quach, and Wang. A *hooked Skolem sequence* of order n is a sequence $s_1, s_2, \ldots, s_{2n+1}$ such that $s_{2n} = 0$ and for each $j \in \{1, 2, \ldots, n\}$, there exists a unique $i \in \{1, 2, \ldots, 2n-1, 2n+1\}$

such that $s_i = s_{i+j} = j$. Mendelsohn [2133] established the following: any tree can be embedded in a Skolem labeled tree with $O(v)$ vertices; any graph can be embedded as an induced subgraph in a Skolem labeled graph on $O(v^3)$ vertices; for $d = 1$, there is a Skolem labeling or the minimum hooked Skolem (with as few unlabeled vertices as possible) labeling for paths and cycles; for $d = 1$, there is a minimum Skolem labeled graph containing a path or a cycle of length n as induced subgraph. In [2133] Mendelsohn and Shalaby prove that the necessary conditions in [2134] are sufficient for a Skolem or minimum hooked Skolem labeling of all trees consisting of edge-disjoint paths of the same length from some fixed vertex. Graham, Pike, and Shalaby [1146] obtained various Skolem labeling results for grid graphs. Among them are $P_1 \times P_n$ and $P_2 \times P_n$ have Skolem labelings if and only if $n \equiv 0$ or 1 mod 4; and $P_m \times P_n$ has a Skolem labeling for all m and n at least 3.

In [2395] Pike, Sanaei, and Shalaby introduce pseudo-Skolem sequences, which are similar to Skolem-type sequences in their structures and applications. They use known Skolem-type sequences to constructions of such sequences and discuss applications of these sequences to Skolem labelingsre graphs such that H is bipartite, and give formulas for the gamma-number of rail-siding graphs and caterpillars.

In [748] Clark and Sanaei present (hooked) vertex Skolem labelings for generalized Dutch windmills whenever such labelings exist. They present a novel technique for showing that generalized Dutch windmills with more than two cycles cannot be Skolem labelled and that those composed of two cycles of lengths m and n, $n \geqslant m$ cannot be Skolem labelled if and only if $n - m \equiv 3$ or 5 (mod 8) and m is odd.

3.6 Odd-Graceful Labelings

Gnanajothi [1104, p. 182] defined a graph G with q edges to be *odd-graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. She proved that the class of odd-graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs by showing that every graph with an α-labeling has an odd-graceful labeling and every graph with an odd cycle is not odd-graceful. She also proved the following graphs are odd-graceful: P_n; C_n if and only if n is even; $K_{m,n}$; combs $P_n \odot K_1$ (graphs obtained by joining a single pendent edge to each vertex of P_n); books; crowns $C_n \odot K_1$ (graphs obtained by joining a single pendent edge to each vertex of C_n) if and only if n is even; the disjoint union of copies of C_4; the one-point union of copies of C_4; $C_n \times K_2$ if and only if n is even; caterpillars; rooted trees of height 2; the graphs obtained from P_n ($n \geqslant 3$) by adding exactly two leaves at each vertex of degree 2 of P_n; the graphs obtained from $P_n \times P_2$ by deleting an edge that joins to end points of the P_n paths; the graphs obtained from a star by adjoining to each end vertex the path P_3 or by adjoining to each end vertex the path P_4. She conjectures that all trees are odd-graceful and proves the conjecture for all trees with order up to 10. Barrientos [442] has extended this to trees of order up to 12. Eldergill [868] generalized Gnanajothi's result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd-graceful. He also proved that the one-point union of any number of copies of C_6 is odd-graceful. Kathiresan [1665] has shown that ladders and graphs obtained from them by subdividing each step exactly once are odd-graceful. Barrientos [445] and [442] has proved the following graphs are odd-graceful: every forest whose components are caterpillars; every tree with diameter at most five is odd-graceful; and all disjoint unions of caterpillars. He conjectures that every bipartite graph is odd-graceful. In [2268] Neela and Selvaraj partially resolved a Barrientos's conjecture by showing that the following graphs are odd-graceful: finite unions of paths, stars, and caterpillars; finite unions of ladders; finite unions of paths, bistars and caterpillars; finite unions of graphs obtained by the one end point union of an odd number of paths of uniform length; and the coronas $K_{m;n} \odot rK_l$. Gao, Zhang, and Xu [1031] proved that $P_n \times P_m$ ($m = 2$, 3 or 4), generalized crown graphs $C_n \odot K_{1,t}$, and gears are odd graceful.

Seoud, Diab, and Elsakhawi [2769] have shown that a connected complete r-partite graph is odd-graceful if and only if $r = 2$ and that the join of any two connected graphs is not odd-graceful. Yan [3511] proved that $P_m \times P_n$ is odd-graceful labeling. Vaidya and Shah [3304] prove that the splitting graph and the shadow graph of bistar are odd-graceful. (The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking 2 copies G_1 and G_2 of G and joining each vertex u in G_1 to the neighbors of the corresponding vertex v in G_2. Li, Li, and Yan [1927] proved that $K_{m,n}$ is odd-graceful. Liu, Wang, and Lu [1972] that proved that a class of bicyclic graphs with a common edge is odd-graceful. Moussa and Badr [2199] proved tha ladders and subdivisions of ladders with pendent edges are odd-graceful.

Sekar [2736] has shown the following graphs are odd-graceful: the graph obtained by identifying an end point of P_n with every vertex of C_m where $n \geqslant 3$ and m is even; $P_{a,b}$ when $a \geqslant 2$ and b is odd (see §2.7); $P_{2,b}$ and $b \geqslant 2$; $P_{4,b}$ and $b \geqslant 2$; $P_{a,b}$ when a and b are even and $a \geqslant 4$ and $b \geqslant 4$; $P_{4r+1,4r+2}$; $P_{4r-1,4r}$; all n-polygonal snakes with n even; $C_n^{(t)}$ (see §2.2 for the definition); graphs obtained by beginning with C_6 and repeatedly forming the one-point union with additional copies of C_6 in succession; graphs obtained by beginning with C_8 and repeatedly forming the one-point union with additional copies of C_8 in succession; graphs obtained from even cycles by identifying a vertex of the cycle with the endpoint of a star; $C_{6,n}$ and $C_{8,n}$ (see §2.7); the splitting graph of P_n (see §2.7) the splitting graph of C_n, n even; lobsters, banana trees, and regular bamboo trees (see §2.1).

Yao, Cheng, Zhongfu, and Yao [3530] have shown the following: if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$ and T_2 is a caterpillar, then T is odd-graceful; and if a tree T has a vertex of degree at least $|T|/2$, then T is odd-graceful. They conjecture that for trees the properties of being Skolem-graceful and odd-graceful are equivalent. Recall a banana tree is a graph obtained by starting with any number os stars and connecting one end-vertex from each to a new vertex. Zhenbin [3593] has shown that graphs obtained by starting with any number of stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are odd-graceful.

Solairaju and Chithra [3018] defined a graph G with q edges to be *edge-odd graceful* if there is an bijection f from the edges of the graph to $\{1, 3, 5, \ldots, 2q-1\}$ such that, when each vertex is assigned the sum of all the edges incident to it mod $2q$, the resulting vertex labels are distinct. They prove they following graphs are odd-graceful: paths with at least 3 vertices; odd cycles; ladders $P_n \times P_2$ $(n \geqslant 3)$; stars with an even number of edges; and crowns $C_n \odot K_1$. In [3019] they prove the following graphs have edge-odd graceful labelings: P_n $(n > 1)$ with a pendent edge attached to each vertex (combs); the graph obtained by appending $2n+1$ pendent edges to each endpoint of P_2 or P_3; and the graph obtained by subdividing each edge of the star $K_{1,2n}$.

For a graph G, Kulli and Muddebihai [1772] define the *lict graph* of G as the graph whose vertex set is the union of the edges of G and the set of cutpoints of G where two vertices are adjacent if and only if the corresonding members of G are adjacent or the corresponding members of G are incident. They define the *litact graph* of G as the graph whose vertex set is the union of the edges of G and the set of cutpoints of G where two vertices are adjacent if and only if the corresonding members of G are adjacent or the corresponding members of G are adjacent or incident. Mirajkar and Sthavarmath [2153] provided edge-odd graceful labeling for the lict graph of P_n for $n > 1$ and odd, the lict graph of the one-point of C_n and P_2, and the litact graph of P_n for $n \geqslant 4$.

A *subdivided shell graph* is obtained by subdividing the edges in the path of the shell graph. Let $G_1, G_2, \ldots, G_n$ be n subdivided shell graphs of any order. The graph SSG(n) is obtained by adding an edge to apexes of G_i and $G_{i+1}, i = 1, 2, \ldots, n-1$. Jeba Jesintha and Ezhilarasi Hilda [1392] that SSG(2) is odd graceful. In [1386] and [1391] Jeba Jesintha and Ezhilarasi Hilda proved that the subdivided uniform shell bow graphs (that is, double shells in which each shell has the same order) are odd graceful and shell butterfly graphs are edge-odd graceful. Daoud [780] provided necessary and sufficient conditions for $C_m \times P_n$ and $C_m \times C_n$ to be edge-odd graceful.

Gao [1024] has proved the following graphs are odd-graceful: the union of any number of paths; the union of any number of stars; the union of any number of stars and paths; $C_m \cup P_n$; $C_m \cup C_n$; and the union of any number of cycles each of which has order divisible by 4.

If f is an odd-graceful labeling of a bipartite graph G with bipartition (V_1, V_2) such that $\max\{f(u) : u \in V_1\} < \min\{f(v) : v \in V_2\}$, Zhou, Yao, Chen, and Tao [3601] say that f is a *set-ordered odd-graceful labeling* of G. They proved that every lobster is odd-graceful and adding leaves to a connected set-ordered odd-graceful graph is an odd-graceful graph.

In [2758] Seoud and Abdel-Aal determined all odd-graceful graphs of order at most 6 and proved that if G is odd-graceful then $G \cup K_{m,n}$ is odd-graceful. In [2777] Seoud and Helmi proved: if G has an odd-graceful labeling f with bipartition (V_1, V_2) such that $\max\{f(x) : f(x) \text{ is even}, x \in V_1\} < \min\{f(x) : f(x) \text{ is odd}, x \in V_2\}$, then G has an α-labeling; if G has an α-labeling, then $G \odot \overline{K_n}$ is odd-graceful; and if G_1 has an α-labeling and G_2 is odd-graceful, then $G_1 \cup G_2$ is odd-graceful. They also proved the following graphs have odd-graceful labelings: dragons obtained from an even cycle; graphs obtained from a gear graph by attaching a fixed number of pendent edges to each vertex of degree 2 on rim of the wheel of the graph; $C_{2m} \odot \overline{K_n}$; graphs obtained from an even cycle by attaching a fixed number of pendent edges to every other vertex; graphs obtained by identifying an endpoint of a star S_n $(n \geqslant 3)$ with a vertex of an even cycle; the graphs consisting of two even cycles of the same order that share a common vertex with any number of pendent edges attached at the common vertex; and the graphs obtained by joining two even cycles of the same order by an edge. Seoud, El Sonbaty, and Abd El Rehim [2770] proved that the conjunction $P_m \wedge P_n$ for all $n, m \geqslant 2$ and the conjunction $K_2 \wedge F_n$ for n even are odd-graceful. Jeba Jesintha and Ezhilarasi Hilda [1384] proved the disjoint union of two subdivided shell graphs is odd-graceful and the one vertex union of three subdivided shells are odd-graceful.

In [2196] and [2197] Moussa proved that $C_m \cup P_n$ is odd-graceful in some cases and gave algorithms to prove that for all $m \geqslant 2$ the graphs $P_{4r-1;m}$, $r = 1, 2, 3$ and $P_{4r+1;m}$, $r = 1, 2$ are odd-graceful. ($P_{n;m}$ is the graph obtained by identifying the endpoints of m paths each of length n). He also presented an algorithm that showed that closed spider graphs and the graphs obtained by joining one or two copies of P_m to each vertex of the path P_n are odd-graceful. Moussa and Badr [2195] proved that $C_m \odot P_n$ is odd-graceful if and only if m is even (see also [272]). Badr, Moussa, and Kathiresan [272] proved ladders are odd graceful.

Moussa [2198] defines the *tensor product,* $P_m \wedge P_n$, of P_m and P_n as the graph with vertices v_i^j, $i = 1, \ldots, n$; $j = 1, \ldots, m$ and edges $v_1^j v_2^{j+1}, v_2^{j+1} v_3^j, \ldots, v_{n-1}^j v_n^{j+1}$ for j odd and $v_1^j v_2^{j-1}, v_2^{j-1} v_3^j, \ldots, v_{n-1}^j v_n^{j-1}$ for j even. He proves that $P_m \wedge P_m$ is odd-graceful.

In [3] Abdel-Aal generalized the notions of shadow graphs and splitting graphs are follows. The m-shadow graph $D_m(G)$ of a connected graph G is constructed by taking m copies of $G_1, G_2, \ldots, G_m$ of G , and joining each vertex u in G_i to the neighbors of the corresponding vertex v in G_j for $1 \leqslant i, j \leqslant m$. The m-splitting graph $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G m new vertices, $v^1, v^2, \ldots, v^m$, such that v^i, $1 \leqslant i \leqslant m$ is adjacent to every vertex that is adjacent to v in G_j. Thus the 2-shadow graph is the shadow graph $D_2(G)$ and the 1-splitting graph of G is the splitting graph of G. Abdel-Aal proved the following graphs are odd-graceful: $D_m(P_n), D_m(P_n \oplus \overline{K_2})$ (the symmetric product of P_n and $\overline{K_2}), D_m(K_{r,s})$, $Spl_m(P_n)$, $Spl_m(K_{1,n})$, and $Spl_m(P_n \oplus \overline{K_2})$.

Vaidya and Bijukumar [3261] proved the following are odd-graceful: graphs obtained by joining two copies of C_n by a path; graphs that are two copies of an even cycle that share a common edge; graphs that are the splitting graph of a star; and graphs that are the tensor product of a star and P_2. Jeba Jesintha, Jaya Glory, and Elakiya Solai [1393] proved that the path unions of caterpillars are odd graceful. [1393] new

Acharya, Germina, Princy, and Rao [40] proved that every bipartite graph G can be embedded in an odd-graceful graph H. The construction is done in such a way that if G is planar and odd-graceful, then so is H. Varkey and Sunoj [3344] investigate some new families of odd graceful graphs generated from various graph operations on the given graph.

In [688] Chawathe and Krishna extend the definition of odd-gracefulness to countably infinite graphs and show that all countably infinite bipartite graphs that are connected and locally finite have odd-graceful labelings.

Solairaju and Chithra [3018] defined a graph G with q edges to be *edge-odd graceful* if there is an bijection f from the edges of the graph to $\{1, 3, 5, \dots, 2q-1\}$ such that, when each vertex is assigned the sum of all the edges incident to it mod $2q$, the resulting vertex labels are distinct. They prove they following graphs are odd-graceful: paths with at least 3 vertices; odd cycles; ladders $P_n \times P_2$ ($n \geqslant 3$); stars with an even number of edges; and crowns $C_n \odot K_1$. In [3019] they prove the following graphs have edge-odd graceful labelings: P_n ($n > 1$) with a pendent edge attached to each vertex (combs); the graph obtained by appending $2n+1$ pendent edges to each endpoint of P_2 or P_3; and the graph obtained by subdividing each edge of the star $K_{1,2n}$.

Singhun [2981] proved the following graphs have edge-odd graceful labelings: W_{2n}; $W_n \odot K_1$; and $W_n \odot K_m$, when n is odd, m is even, and n divides m. Seoud and Salim [2792] present edge-odd graceful labelings for the following families of graphs: W_n for $n \equiv 1, 2$ and 3 (mod 4); $C_n \odot \overline{K_{2m-1}}$; even helms; $P_n \odot \overline{K_{2m}}$; and $K_{2,s}$. They also provide two theorems about non edge-odd graceful graphs. Susanti, Ernanto1, and Surodjo [3150] found edge-odd graceful labelings for some classes of prism related graphs.

In [3042] Sridevi, Navaeethakrishnan, Nagarajan, and Nagarajan call a graph G with q edges *odd-even graceful* if there is an injection f from the vertices of G to $\{1, 3, 5, \dots, 2q+1\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $\{2, 4, 6, \dots, 2q\}$. They proved that P_n, combs $P_n \odot K_1$, stars $K_{1,n}, K_{1,2,n}, K_{m,n}$, and bistars $B_{m,n}$ are odd-even graceful.

Sudha and Babu [3071] say a graph G with q edges is *even-even graceful* if there is an injection f from the edges of G to $\{2, 4, 6, \dots, 2q\}$ such that, the induced map f^+ from $V(G)$ to $\{0, 2, \dots, 2k-2\}$ defined by $f^*(x) = \Sigma(f(xy)$ (mod $2k$) where $k = \max(p, q)$ is injective and each value is $f^*(x)$ is even. They proved that dumbbells, stars, $C_n \times P_2$, and $K_1 + C_n$ are even-even graceful.

Behera, Mishra, and Nayak [527] proved the following: bistars $B_{r,r}$ are even-even graceful, combs are even-even graceful, the trees obtained by joining and even number of pendent edges to the endpoint of a path are even-even graceful, the graphs obtained by identifying the center of a star and a vertex of C_3 are odd-even graceful, the graphs obtained by identifying the center of a star and a vertex of C_3 and two pendent edges at the other two verticies are odd-even graceful, and the graphs obtained by identifying the center of a star with a vertex of C_n and the endpoints of the star with the opposite vertices of C_n is odd-even graceful.

In [782] Daoud zele introduced vertex odd graceful labelings as follows. Let G be a graph with q edges. A function f is called a *vertex odd graceful*abeling of G if f $E(G) \to \{1, 2, 3, \dots, 2q\}$ is an injection and the induced function f^* $V(G) \to \{1, 3, \dots, 2q-1\}$ defined as $f^*(u) = \sum_{uv \in E(G)} f(uv)$ (mod $2q$) is also an injection. A graph which admits a vertex odd graceful labeling is called a *vertex odd graceful* graph. Necessary and sufficient conditions for prisme, tori, wheels, fans and books to be vertex odd graceful are given.

3.7 Cordial Labelings

Cahit [640] has introduced a variation of both graceful and harmonious labelings. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Call f a *cordial labeling* of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Cahit [641] proved the following: every tree is cordial; K_n is cordial if and only if $n \leqslant 3$; $K_{m,n}$ is cordial for all m and n; the friendship graph $C_3^{(t)}$ (i.e., the one-point union of t 3-cycles) is cordial if and only if $t \not\equiv 2 \pmod 4$; all fans are cordial; the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$ (see also [853]); maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang, and Kwong [1780] determine all m and n for which mK_n is cordial. Youssef [3546] proved that every Skolem-graceful graph (see 3.5 for the definition) is cordial. Liu and Zhu [1981] proved that a 3-regular graph of order n is cordial if and only if $n \not\equiv 4 \pmod 8$. In [1319] Imran, Cancan, Ali, Nadeem, Mushtaq, Aslam, Riaz, proved various comb related graphs are cordial. Kastrati, Myrvold, Panjer, and Williams [1661] proved that a forest is cordial if and only if it does not have $4k+2$ components and every vertex has odd-degree. [1319] new [1661] new

A *k-angular cactus* is a connected graph all of whose blocks are cycles with k vertices. In [641] Cahit proved that a k-angular cactus with t cycles is cordial if and only if $kt \not\equiv 2 \pmod 4$. This was improved by Kirchherr [1702] who showed any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to 2 (mod 4). Kirchherr [1703] also gave a characterization of cordial graphs in terms of their adjacency matrices. Ho, Lee, and Shee [1254] proved: $P_n \times C_{4m}$ is cordial for all m and all odd n; the composition G and H is cordial if G is cordial and H is cordial and has odd order and even size (see §2.3 for definition of composition); for $n \geqslant 4$ the composition $C_n[K_2]$ is cordial if and only if $n \not\equiv 2 \pmod 4$; the Cartesian product of two cordial graphs of even size is cordial. Ho, Lee, and Shee [1253] showed that a unicyclic graph is cordial unless it is C_{4k+2} and that the generalized Petersen graph (see §2.7 for the definition) $P(n,k)$ is cordial if and only if $n \not\equiv 2 \pmod 4$. Khan [1684] proved that a graph that consisting of a finite number of cycles of finite length joined at a common cut vertex is cordial if and only if the number of edges is not congruent to 2 mod 4.

Du [853] determines the maximal number of edges in a cordial graph of order n and gives a necessary condition for a k-regular graph to be cordial. Riskin [2628] proved that Möbius ladders M_n (see §2.3 for the definition) are cordial if and only if $n \geqslant 3$ and $n \not\equiv 2 \pmod 4$. (See also [2772].) Diab and Nada [836] show that $P_n \odot P_m$ is cordial; except for n and m both equal to 2 (mod 4), $C_n \odot C_m$ is cordial; and when $n \equiv 2 \pmod 4$ and m is odd, $C_n \odot C_m$ is not cordial. In [2686] Salehi, Mukhin, and Saputro showed that Q_n is cordial for all $n > 1$.

Seoud and Abdel Maqusoud [2760] proved that if G is a graph with n vertices and m edges and every vertex has odd degree, then G is not cordial when $m + n \equiv 2 \pmod 4$. They also prove the following: for $m \geqslant 2$, $C_n \times P_m$ is cordial except for the case $C_{4k+2} \times P_2$; P_n^2 is cordial for all n; P_n^3 is cordial if and only if $n \neq 4$; and P_n^4 is cordial if and only if $n \neq 4, 5$, or 6. Seoud, Diab, and Elsakhawi [2769] have proved the following graphs are cordial: $P_n + P_m$ for all m and n except $(m,n) = (2,2)$; $C_m + C_n$ if $m \not\equiv 0 \pmod 4$ and $n \neq 2 \pmod 4$; $C_n + K_{1,m}$ for $n \not\equiv 3 \pmod 4$ and odd m except $(n,m) = (3,1)$; $C_n + \overline{K_m}$ when n is odd, and when n is even and m is odd; $K_{1,m,n}$; $K_{2,2,m}$; the n-cube; books B_n if and only if $n \not\equiv 3 \pmod 4$; $B(3,2,m)$ for all m; $B(4,3,m)$ if and only if m is even; and

$B(5,3,m)$ if and only if $m \not\equiv 1 \pmod 4$ (see §2.4 for the notation $B(n,r,m)$). In [3014] Solairaju and Arockiasamy prove that various families of subgraphs of grids $P_m \times P_n$ are cordial.

Diab [829], [830], and [832] proved the following graphs are cordial: $C_m + P_n$ if and only if $(m,n) \neq (3,3), (3,2)$, or (3,1); $P_m + K_{1,n}$ if and only if $(m,n) \neq (1,2)$; $P_m \cup K_{1,n}$ if and only if $(m,n) \neq (1,2)$; $C_m \cup K_{1,n}$; $C_m + \overline{K_n}$ for all m and n except $m \equiv 3 \pmod 4$ and n odd, and $m \equiv 2 \pmod 4$ and n even; $C_m \cup \overline{K_n}$ for all m and n except $m \equiv 2 \pmod 4$; $P_m + \overline{K_n}$; $P_m \cup \overline{K_n}$; $P_m^2 \cup P_n^2$ except for $(m,n) = (2,2)$ or (3,3); $P_n^2 + P_m$ except for $(m,n) = (3,1), (3,2), (2,2), (3,3)$, and (4,2); $P_n^2 \cup P_m$ except for $(n,m) = (2,2), (3,3)$, and (4,2); $P_n^2 + C_m$ if and only if $(n,m) \neq (1,3), (2,3)$, and $(3,3)$.$P_n + \overline{K_m}$; $C_n + K_{1,m}$ for all $n > 3$ and all m except $n \equiv 3 \pmod 4$; $C_n + K_{1,m}$ for $n \equiv 3 \pmod 4$ $(n \neq 3)$ and even $m \geqslant 2$; and $C_m \times C_n$ if and only if $2mn$ is not congruent to 2 (mod 4).

In [831] Diab proved the graphs $W_n + W_m$ are cordial if and only if one of the following conditions is not satisfied: (i) $(n,m) = (3,3)$, (ii) $n = 3$ and $m \equiv 1 \pmod 4$, (iii) $n \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$; the graphs $W_n \cup W_m$ are cordial if and only if one of the following conditions is not satisfied: (i) $n = 3$ and $m \equiv 1 \pmod 4$, (ii) $n \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$; the graphs $W_n + P_m$ are cordial if and only if one of the following conditions is not satisfied: (i) $(n,m) = (3,1), (3,2)$ and $(3,3)$, (ii) $n \equiv 3 \pmod 4$ and $m = 1$. They also prove that $W_n \cup P_m$ and $W_n \cup C_m$ are cordial for all m and n and $W_n + C_m$ is cordial if and only if $(m,n) \neq (3,3)$ and $(3,4)$. In [833] Diab showed that the second power of C_n is cordial if and only if $n = 3$ or n is even and greater than 4. He also investigated the cordiality of the join and union of pairs of second power of cycles and graphs consisting of one second power of cycle with one cycle and one path.

In [2239] Nada, Diab, Elrokh, and Sabra proved that $P_n \odot C_m$ is cordial if and only if $\gcd(n,m) \neq 1$ or 3 (mod 4); in [2238] they proved $C_n \odot P_m$ is cordial for all $n \geqslant 3$ and $m \geqslant 1$. Nada, Elrokh, and Elshafey [2241] provided necessary and sufficient conditions for $F_n^2 = K_1 + P_n^2$, $F_n^2 + F_m^2$, and $F_n^2 + F_m^2$ to be cordial.

The generalized *Jahangir graph* $J_{m,n}$ $m > 3$, $n > 1$ is a graph on $mn + 1$ vertices, consisting of a cycle C_{mn} with one additional vertex that is adjacent to n vertices of C_{mn} at distance m to each other on C_{mn}. Gajjar and Des [1000] proved $J_{m,n}$ is cordial for all $m > 3$ and $n > 1$, except for $J_{1,4n-1}$.

Youssef [3548] has proved the following: If G and H are cordial and one has even size, then $G \cup H$ is cordial; if G and H are cordial and both have even size, then $G + H$ is cordial; if G and H are cordial and one has even size and either one has even order, then $G + H$ is cordial; $C_m \cup C_n$ is cordial if and only if $m + n \not\equiv 2 \pmod 4$; mC_n is cordial if and only if $mn \not\equiv 2 \pmod 4$; $C_m + C_n$ is cordial if and only if $(m,n) \neq (3,3)$ and $\{m \pmod 4, n \pmod 4\} \neq \{0,2\}$; and if P_n^k is cordial, then $n \geqslant k + 1 + \sqrt{k-2}$. He conjectures that this latter condition is also sufficient. He confirms the conjecture for $k = 5, 6, 7, 8$, and 9. Elirokh and Rabie [874] proved $P_n^4 + P_m^4$ and $P_n^4 \cup P_m^4$ are cordial for all $n, m \geqslant 7$, and $C_n^4 + C_m^4$, and $C_n^4 \cup C_m^4$ are cordial for all n, m except $(n,m) = (7,7)$.

Lee and Liu [1854] have shown that the complete n-partite graph is cordial if and only if at most three of its partite sets have odd cardinality (see also [853]). Lee, Lee, and Chang [1829] prove the following graphs are cordial: the Cartesian product of an arbitrary number of paths; the Cartesian product of two cycles if and only if at least one of them is even; and the Cartesian product of an arbitrary number of cycles if at least one of them has length a multiple of 4 or at least two of them are even. Ali Al-Shamiri, Elrokh, El-Mashtawye, and Tallah [160] showed that the Cartesian product of a path and a cycle is cordial under some conditions and that the Cartesian product of two paths is cordial. Elrokh, Elmshtaye, and Abd El-hay [818] provided necessary and sufficient conditions for cone and lemniscate graphs to be cordial. [818] new

Shee and Ho [2875] have investigated the cordiality of the one-point union of n copies of various graphs. For $C_m^{(n)}$, the one-point union of n copies of C_m, they prove:

(i) If $m \equiv 0 \pmod 4$, then $C_m^{(n)}$ is cordial for all n;

(ii) If $m \equiv 1$ or $3 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(iii) If $m \equiv 2 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if n is even.

For $K_m^{(n)}$, the one-point union of n copies of K_m, Shee and Ho [2875] prove:

(i) If $m \equiv 0 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 3 \pmod 4$;

(ii) If $m \equiv 4 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 1 \pmod 4$;

(iii) If $m \equiv 5 \pmod 8$, then $K_m^{(n)}$ is not cordial for all odd n;

(iv) $K_4^{(n)}$ is cordial if and only if $n \not\equiv 1 \pmod 4$;

(v) $K_5^{(n)}$ is cordial if and only if n is even;

(vi) $K_6^{(n)}$ is cordial if and only if $n > 2$;

(vii) $K_7^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(viii) $K_n^{(2)}$ is cordial if and only if n has the form p^2 or $p^2 + 1$.

For $W_m^{(n)}$, the one-point union of n copies of the wheel W_m with the common vertex being the center, Shee and Ho [2875] show:

(i) If $m \equiv 0$ or $2 \pmod 4$, then $W_m^{(n)}$ is cordial for all n;

(ii) If $m \equiv 3 \pmod 4$, then $W_m^{(n)}$ is cordial if $n \not\equiv 1 \pmod 4$;

(iii) If $m \equiv 1 \pmod 4$, then $W_m^{(n)}$ is cordial if $n \not\equiv 3 \pmod 4$. For all n and all $m > 1$ Shee and Ho [2875] prove $F_m^{(n)}$, the one-point union of n copies of the fan $F_m = P_m + K_1$ with the common point of the fans being the center, is cordial (see also [1950]). The *flag* Fl_m is obtained by joining one vertex of C_m to an extra vertex called the *root*. Shee and Ho [2875] show all $Fl_m^{(n)}$, the one-point union of n copies of Fl_m with the common point being the root, are cordial. In his 2001 Ph. D. thesis Selvaraju [2739] proves that the one-point union of any number of copies of a complete bipartite graph is cordial. Benson and Lee [536] have investigated the regular windmill graphs $K_m^{(n)}$ and determined precisely which ones are cordial for $m < 14$.

Diab and Mohammedm [835] proved the following: the join of two fans $F_n + F_m$ is cordial if and only if $n, m \geqslant 4$; $F_n \cup F_m$ is cordial if and only if $(n, m) \neq$ (1,1) or (2,2); $F_n + P_m$ is cordial if and only if $(n, m) \neq$ (1,2), (2,1), (2,2), (2,3), or (3,2); $F_n \cup P_m$ is cordial if and only if $(n, m) \neq$ (1,2); $F_n + C_m$ is cordial if and only if $(n, m) \neq$ (1,3), (2,3) or (3,3); and $F_n \cup C_m$ is cordial if and only if $(n, m) \neq (2, 3)$. Hefnawy, Elsid, and Euat Tallah [1208] gave necessary and sufficient conditions for a cordial labeling of thesum of the second power of the path $P_n^2 + K_{1,m}$ and $P_n^2 \cup K_{1,m}$.

Andar, Boxwala, and Limaye [187], [188], and [191] have proved the following graphs are cordial: helms; closed helms; generalized helms obtained by taking a web (see 2.2 for the definitions) and attaching pendent vertices to all the vertices of the outermost cycle in the case that the number cycles is even; flowers (graphs obtained by joining the vertices of degree one of a helm to the central vertex); sunflower graphs (that is, graphs obtained by taking a wheel with the central vertex v_0 and the n-cycle $v_1, v_2, \ldots, v_n$ and additional vertices $w_1, w_2, \ldots, w_n$ where w_i is joined by edges to v_i, v_{i+1}, where $i+1$ is taken modulo n); multiple shells (see §2.2); and the one point unions of helms, closed helms, flowers, gears, and sunflower graphs, where in each case the central vertex is the common vertex.

In [2487], [2488], [2489], [2494], and [2490] Prajapati and Gajjar provided results about the existence of cordial labelings of graphs obtained from paths, cycles, flower graphs, sunflower graphs, flower snarks, lotus inside a circle graphs, helms, closed helms, armed helms ($W_n \oplus P_2$), and webs by the duplication of vertices and edges. In [872] Elrokh and Elkom proved that certain classes any four-leaved rose graphs (the one-point union of four cycles of the same length) are cordial. [872] new

Du [854] proved that the disjoint union of $n \geqslant 2$ wheels is cordial if and only if n is even or n is odd and the number of vertices of in each cycle is not 0 (mod 4) or n is odd and the number of vertices of in each cycle is not 3 (mod 4). Prajapati and Gajjar [2486] prove $\overline{W_n}$ is not cordial if $n \not\equiv 4, 7 \pmod 8$ and $\overline{C_n}$ is not cordial if $n \not\equiv 4, 7 \pmod 8$.

Let $\mathcal{O}$ be the family of all cordial graphs of odd order and odd size for which there is no cordial labeling g such that $e_g(0) - e_g(1) = 1$. Barrientos and Minion [473] proved that if G is a cordial graph such that $G \notin \mathcal{O}$, then the corona $K_1 \odot G$ is cordial. They use this result to prove that $H \odot G$ is cordial when G and H are cordial and G has even order and even size or $G \notin \mathcal{O}$. In addition, $H \odot G$ is cordial when G is a cordial graph of odd order and even size and H is any graph of order m and size $n \in \{m-1, m, m+1\}$. If H is bipartite such that the difference of the cardinalities of its partite sets is at most one, and G is a cordial graph of even order and odd size that admits a cordial labeling g such that $e_g(0) - e_g(1) = 1$, then the corona $H \odot G$ is cordial. Barrientos and Minion proved the cordiality of certain circulant graphs; they also proved that for every positive integer k, the k-splitting of a cordial graph of even size, results in a cordial graph. They provide sufficient conditions to prove that any super subdivision of a graph G is cordial. They study the cordiality of the join of two cordial graphs, proving that $G + H$ is cordial when G and H have even order and even size, or both have odd order and even size, or both graphs have odd order, odd size, and the dominating weight in both graphs is not 1, or G has even order, odd size, and the dominating weight on both graphs is not the same, or both G and H have odd order, but only one has odd size, and the dominating weight is 0. They also prove that when G is a cordial graph of odd order and even size, the one-point union of t copies of G is cordial.

In [473] Barrientos and Minion provide necessary conditions for the cordiality of coronas of cordial graphs, prove the cordiality of a family of circulant graphs, prove that any splitting graph of a cordial graph of even order and even size is cordial, determine a condition that a graph must satisfy in order that any super subdivision of it is cordial, prove the cordiality of the joint of two cordial graphs, and determine when a one-point union of a cordial graph is cordial.

For positive integers m and n divisible by 4 Venkatesh [3370] constructs graphs obtained by appending a copy of C_n to each vertex of C_m by identifying one vertex of C_n with each vertex of C_m and iterating by appending a copy of C_n to each vertex of degree 2 in the previous step. He proves that the graphs obtained by successive iterations are cordial.

Elumalai and Sethurman [879] proved: cycles with parallel cords are cordial and n-cycles with parallel P_k-chords (see §2.2 for the definition) are cordial for any odd positive integer k at least 3 and any $n \not\equiv 2 \pmod 4$ of length at least 4. They call a graph H an *even-multiple subdivision graph* of a graph G if it is obtained from G by replacing every edge uv of G by a pair of paths of even length starting at u and ending at v. They prove that every even-multiple subdivision graph is cordial and that every graph is a subgraph of a cordial graph. In [3464] Wen proves that generalized wheels $C_n + mK_1$ are cordial when m is even and $n \not\equiv 2 \pmod 4$ and when m is odd and $n \not\equiv 3 \pmod 4$. Kuppusamy and Guruswamy [1781] show that the subdivision graph of $K_{2,n}$ is graceful for $n \geqslant 1$ and the subdivision graph of the shell graph $C(n, n-3)$ is graceful for $n \geqslant 4$.

Vaidya, Ghodasara, Srivastav, and Kaneria investigated graphs obtained by joining two identical graphs by a path. They prove: graphs obtained by joining two copies of the same cycle by a path are cordial [3272]; graphs obtained by joining two copies of the same cycle that has two chords with a common vertex with opposite ends of the chords joining two consecutive vertices of the cycle by a path are cordial [3272]; graphs obtained by joining two rim verticies of two copies of the same wheel by a path are cordial [3274]; and graphs obtained by joining two copies of the same Petersen graph by a path are cordial [3274]. They also prove that graphs obtained by replacing one vertex of a star by a fixed wheel or by replacing each vertex of a star by a fixed Petersen graph are cordial [3274]. In [3313] Vaidya, Ghodasara, Srivastav, and Kaneria investigated graphs obtained by joining two identical cycles that have a chord are cordial and the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed cycle with a chord that forms a triangle with two consecutive edges of the cycle and joining each G_i to G_{i+1} $(i = 1, 2, \ldots, n-1)$ by an edge that is incident with the endpoints of the chords in G_i and G_{i+1} are cordial. Vaidya, Dani, Kanani, and Vihol [3267] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of G_i to the center of G_{i+1} $(i = 1, 2, \ldots, n-1)$ by an edge are cordial.

Ghodasara, Rokad, and Jadav [1086] prove that the path union of $P_n \times P_n$ is cordial. They also prove that the graph obtained by joining two copies of $P_n \times P_n$ by a path is cordial. Ghodasara and Jadav [1079] prove: the graph obtained by joining a finite number of copies of $P_n \times P_n$ by path is cordial; the star of $P_n \times P_n$ is cordial; and the path union of the star of $P_n \times P_n$ is cordial. Rokad and Patadiya [2645] proved that the shadow graph, splitting graph, and the degree splitting graph of a star are cordial graphs. They also showed that the jewel graph and the jellyfish graph are cordial.

Ghodasara and Rokad prove [1087] the star of $K_{n,n}$ $(n \geqslant 2)$ is cordial, the path union of $K_{n,n}$ $(n \geqslant 2)$ is cordial, and the graph obtained by joining two copies of $K_{n,n}$ $(n \geqslant 2)$ by a path is cordial [1087]. In [1088] the same authors prove that a vertex switching of any non-apex vertex of a wheel graph, a vertex switching of any internal vertex of a flower graph, a vertex switching of any non-apex vertex of a gear graph, and a vertex switching of any non-apex vertex of a shell graph are cordial graphs. In [1089] they proved that a barycentric subdivision of a shell graph, a barycentric subdivision of $K_{n,n}$, and a barycentric subdivision of a wheel are cordial. Ghodasara and Sonchhatra [1090] prove that the graph obtained by joining two copies of the same fan by a path is cordial. They also prove that the star of a fan is cordial and the graph obtained by joining two copies of the star of the same fan by a path is cordial [1090]. Elrokh, Nada, and El-Shafey [873] showed that $P_k \odot F_m^2$ (F_m is the fan graph with $m+1$ vertices) is cordial for all $k \geqslant 1$ and $m \geqslant 4$.

Vaidya, Kanani, Srivastav, and Ghodasara [3282] proved: graphs obtained by subdividing every edge of a cycle with exactly two extra edges that are chords with a common endpoint and whose other end points are joined by an edge of the cycle are cordial; graphs obtained by subdividing every edge of the graph obtained by starting with C_n and adding exactly three chords that result in two 3-cycles and a cycle of length $n-3$ are cordial; graphs obtained by subdividing every edge of a Petersen graph are cordial. Sankar and Sethuramam zske [2709] showed that the subdivision graph $S(K_2, n)$ is graceful and cordial, for $n \geqslant 1$ and the shell graph $S(C(n, n-3))$ is graceful and cordial for $n \geqslant 4$.

Recall the shell $C(n, n-3)$ is the cycle C_n with $n-3$ cords sharing a common endpoint. Vaidya, Dani, Kanani, and Vihol [3268] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed shell and joining common endpoint of the chords of G_i to the common endpoint of the chords of G_{i+1} ($i = 1, 2, \ldots, n-1$) by an edge are cordial. Vaidya, Dani, Kanani, and Vihol [3283] define $C_n(C_n)$ as the graph obtained by subdividing each edge of C_n and connecting the new n vertices to form a copy of C_n inscribed the original C_n. They prove that $C_n(C_n)$ is cordial if $n \not\equiv 2 \pmod 4$; the graphs obtained by starting with copies $G_1, G_2, \ldots, G_k$ of $C_n(C_n)$ the graph obtained by joining a vertex of degree 2 in G_i to a vertex of degree 2 in G_{i+1} ($i = 1, 2, \ldots, n-1$) by an edge are cordial; and the graphs obtained by joining vertex of degree 2 from one copy of $C_n(C_n)$ to a vertex of degree 2 to another copy of $C_n(C_n)$ by any finite path are cordial. Vaidya and Shah [3309] and [3310] proved that following graphs are cordial: the shadow graph of the bistar $B_{n,n}$, the splitting graph of $B_{n,n}$, the degree splitting graph of $B_{n,n}$, alternate triangular snakes, alternate quadrilateral snakes, double alternate triangular snakes, and double alternate quadrilateral snakes. In [3312] Vaidya and Shah give cordial labelings of the degree splitting graph of paths, shells, helms, and gears.

A graph $C(2n, n-2)$ is called an *alternate shell* if $C(2n, n-2)$ is obtained from the cycle C_{2n} $(v_0, v_1, v_2, \ldots, v_{2n-1})$ by adding $n-2$ chords between the vertex v_0 and the vertices v_{2i+1}, for $1 \leqslant i \leqslant n-2$. Sethuraman and Sankar [2831] proved that some graphs obtained by merging alternate shells and joining certain vertices by a path have α-labelings.

Vaidya, Srivastav, Kaneria, and Ghodasara [3314] proved that a cycle with two chords that share a common vertex and the opposite ends of which join two consecutive vertices of the cycle is cordial. For a graph G Vaidya, Ghodasara, Srivastav, and Kaneria [3273] introduced the graph G^* called the *star of G* as the graph obtained by replacing each vertex of the star $K_{1,n}$ by a copy of G and prove that $C_n^{\,*}$ admits cordial labeling. Vaidya and Dani [3263] proved that the graphs obtained by starting with n copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of G_i to the center of G_{i+1} by an edge as well as each of the centers to a new vertex x_i $(1 \leqslant i \leqslant n-1)$ by an edge admit cordial labelings. An *arbitrary supersubdivison* H of a graph G is the graph obtained from G by replacing every edge of G by $K_{2,m}$, where m may vary for each edge arbitrarily. Vaidya and Kanani [3275] proved that arbitrary supersubdivisions of paths and stars admit cordial labelings. Vaidya and Dani [3264] prove that arbitrary supersubdivisions of trees, $K_{m,n}$, and $P_m \times P_n$ are cordial. They also prove that an arbitrary supersubdivision of the graph obtained by identifying an end vertex of a path with every vertex of a cycle C_n is cordial except when n is odd, m_i $(1 \leqslant i \leqslant n)$ are odd, and m_i $(n+1 \leqslant i \leqslant mn)$ of the K_{2,m_i} are even. Recall for a graph G and a vertex v of G Vaidya, Srivastav, Kaneria, and Kanani [3315] define a *vertex switching* G_v as the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. They proved that the graphs obtained by the switching of a vertex in C_n admit cordial labelings. They also show that the graphs obtained by the switching of any arbitrary vertex of cycle C_n with one chord that forms a triangle with two consecutive edges of the cycle are cordial. Moreover they prove that the graphs obtained by the switching of any arbitrary vertex in cycle with two chords that share a common vertex the opposite ends of which join two consecutive vertices of the cycle are cordial.

The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it. Vaidya and Vihol [3317] prove that the middle graph $M(G)$ of an Eulerian graph is Eulerian with $|E(M(G))| = \sum_{i=1}^{n}(d(v_i)^2 + 2e)/2$. They prove that middle graphs of paths, crowns $C_n \odot K_1$, stars, and tadpoles (that is, graphs obtained by appending a path to a cycle) admit cordial labelings.

Vaidya and Dani [3266] define the *duplication of an edge $e = uv$ of a graph G by a new vertex w* as the graph G' obtained from G by adding a new vertex w and the edges wv and wu. They prove that the graphs obtained by duplication of an arbitrary edge of a cycle and a wheel admit a cordial labeling. Starting with k copies of fixed wheel W_n, $W_n^{(1)}, W_n^{(2)}, \ldots,$ $W_n^{(k)}$, Vaidya, Dani, Kanani, and Vihol [3270] define $G =< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ as the graph obtained by joining the center vertices of each of $W_n^{(i)}$ and $W_n^{(i+1)}$ to a new vertex x_i where $1 \leqslant i \leqslant k-1$. They prove that $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ are cordial graphs. Kaneria and Vaidya [1623] define the *index of cordiality* of G as n if the disjoint union of n copies of G is cordial but the disjoint union of fewer than n copies of G is not cordial. They obtain several results on index of cordiality of K_n. In the same paper they investigate cordial labelings of graphs obtained by replacing each vertex of $K_{1,n}$ by a graph G. Kaneria, Jariya, and Karavadiya [1586] proved that the index of cordiality for K_n is at most 6 for n at most 105; the index of cordiality for K_n is at most 4, when n can be expressed as sum of square of two integers; and it is at most 8 when a particular different condition on the edge labels are met. See also [1354].

In [191] Andar et al. define a *t-ply graph* $P_t(u, v)$ as a graph consisting of t internally disjoint paths joining vertices u and v. They prove that $P_t(u, v)$ is cordial except when it is Eulerian and the number of edges is congruent to 2 (mod 4). In [192] Andar, Boxwala, and Limaye prove that the one-point union of any number of plys with an endpoint as the common vertex is cordial if and only if it is not Eulerian and the number of edges is congruent to 2 (mod 4). They further prove that the path union of shells obtained by joining any point of one shell to any point of the next shell is cordial; graphs obtained by attaching a pendent edge to the common vertex of the cords of a shell are cordial; and cycles with one pendent edge are cordial.

For a graph G and a positive integer t, Andar, Boxwala, and Limaye [189] define the *t-uniform homeomorph* $P_t(G)$ of G as the graph obtained from G by replacing every edge of G by vertex disjoint paths of length t. They prove that if G is cordial and t is odd, then $P_t(G)$ is cordial; if $t \equiv 2 \pmod 4$ a cordial labeling of G can be extended to a cordial labeling of $P_t(G)$ if and only if the number of edges labeled 0 in G is even; and when $t \equiv 0 \pmod 4$ a cordial labeling of G can be extended to a cordial labeling of $P_t(G)$ if and only if the number of edges labeled 1 in G is even. In [190] Ander et al. prove that $P_t(K_{2n})$ is cordial for all $t \geqslant 2$ and that $P_t(K_{2n+1})$ is cordial if and only if $t \equiv 0 \pmod 4$ or t is odd and $n \not\equiv 2 \pmod 4$, or $t \equiv 2 \pmod 4$ and n is even.

In [192] Andar, Boxwala, and Limaya show that a cordial labeling of G can be extended to a cordial labeling of the graph obtained from G by attaching $2m$ pendent edges at each vertex of G. For a binary labeling g of the vertices of a graph G and the induced edge labels given by $g(e) = |g(u) - g(v)|$ let $v_g(j)$ denote the number of vertices labeled with j and $e_g(j)$ denote the number edges labeled with j. Let $i(G) = \min\{|e_g(0) - e_g(1)|\}$ taken over all binary labelings g of G with $|v_g(0) - v_g(1)| \leqslant 1$. Andar et al. also prove that a cordial labeling g of a graph G with p vertices can be extended to a cordial labeling of the graph obtained from G by attaching $2m + 1$ pendent edges at each vertex of G if and only if G does not satisfy either of the conditions: (1) G has an even number of edges and $p \equiv 2 \pmod 4$; (2) G has an odd number of edges and either $p \equiv 1 \pmod 4$ with $e_g(1) = e_g(0) + i(G)$ or $n \equiv 3 \pmod 4$ and $e_g(0) = e_g(1) + i(G)$. Andar, Boxwala, and Limaye [193] also prove: if g is a binary labeling of the n vertices of graph G with induced edge labels given by $g(e) = |g(u) - g(v)|$ then g can be extended to a cordial labeling of $G \odot \overline{K_{2m}}$ if and only if n is odd and $i(G) \equiv 2 \pmod 4$; $K_n \odot \overline{K_{2m}}$ is cordial if and only if $n \neq 4 \pmod 8$; $K_n \odot \overline{K_{2m+1}}$ is cordial if and only if $n \neq 7 \pmod 8$; if g is a binary labeling of the n vertices of graph G with induced edge labels given by $g(e) = |g(u) - g(v)|$ then g can be extended to a cordial labeling of $G \odot C_t$ if $t \neq 3$ mod 4, n is odd and $e_g(0) = e_g(1)$. For any binary labeling g of a graph G with induced edge labels given by $g(e) = |g(u) - g(v)|$ they also characterize in terms of $i(G)$ when g can be extended to graphs of the form $G \odot \overline{K_{2m+1}}$.

For graphs $G_1, G_2, \ldots, G_n$ ($n \geqslant 2$) that are all copies of a fixed graph G, Shee and Ho [2876] call a graph obtained by adding an edge from G_i to G_{i+1} for $i = 1, \ldots, n-1$ a *path union* of G (the resulting graph may depend on how the edges are chosen). Among their results they show the following graphs are cordial: path-unions of cycles; path-unions of any number of copies of K_m when $m = 4, 6$, or 7; path-unions of three or more copies of K_5; and path-unions of two copies of K_m if and only if $m - 2, m$, or $m + 2$ is a perfect square. They also show that there exist cordial path-unions of wheels, fans, unicyclic graphs, Petersen graphs, trees, and various compositions.

Lee and Liu [1854] give the following general construction for the forming of cordial graphs from smaller cordial graphs. Let H be a graph with an even number of edges and a cordial labeling such that the vertices of H can be divided into t parts $H_1, H_2, \ldots, H_t$ each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let G be any graph and $G_1, G_2, \ldots, G_t$ be any t subsets of the vertices of G. Let (G, H) be the graph that is the disjoint union of G and H augmented by edges joining every vertex in G_i to every vertex in H_i for all i. Then G is cordial if and only if (G, H) is. From this it follows that: all generalized fans $F_{m,n} = \overline{K_m} + P_n$ are cordial; the generalized bundle $B_{m,n}$ is cordial if and only if m is even or $n \not\equiv 2 \pmod 4$ ($B_{m,n}$ consists of $2n$ vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ with an edge from v_i to u_i and $2m$ vertices $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ with x_i joined to v_i and y_i joined to u_i); if m is odd the generalized wheel $W_{m,n} = \overline{K_m} + C_n$ is cordial if and only if $n \not\equiv 3 \pmod 4$. If m is even, $W_{m,n}$ is cordial if and only if $n \not\equiv 2 \pmod 4$; a complete k-partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.

Sethuraman and Selvaraju [2840] have shown that certain cases of the union of any number of copies of K_4 with one or more edges deleted and one edge in common are cordial. Youssef [3552] has shown that the kth power of C_n is cordial for all n when $k \equiv 2 \pmod 4$ and for all even n when $k \equiv 0 \pmod 4$. Ramanjaneyulu, Venkaiah, and Kothapalli [2576] give cordial labelings for a family of planar graphs for which each face is a 3-cycle and a family for which each face is a 4-cycle. Acharya, Germina, Princy, and Rao [40] prove that every graph G can be embedded in a cordial graph H. The construction is done in such a way that if G is planar or connected, then so is H.

Recall from §2.7 that a graph H is a *supersubdivision* of a graph G, if every edge uv of G is replaced by $K_{2,m}$ (m may vary for each edge) by identifying u and v with the two vertices in $K_{2,m}$ that form the partite set with exactly two members. Vaidya and Kanani [3275] prove that supersubdivisions of paths and stars are cordial. They also prove that supersubdivisions of C_n are cordial provided that n and the various values for m are odd.

Raj and Koilraj [2552] proved that the splitting graphs of $P_n, C_n, K_{m,n}, W_n, nK_2$, and the graphs obtained by starting with k copies of stars $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(k)}$ and joining the central vertex of $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ to a new vertex x_{p-1} for each $2 \leqslant p \leqslant k$ are cordial.

Seoud, El Sonbaty, and Abd El Rehim [2770] proved the following graphs are cordial: $K_{1,l,m,n}$ when mn is even; $P_m + K_{1,n}$ if n is even or n is odd and ($m \neq 2$); the conjunction graph $P_4 \wedge C_n$ is cordial if n is even; and the join of the one-point union of two copies of C_n and K_1.

Recall $< K_{1,n_1}, \ldots, K_{1,n_t} >$ is the graph obtained by starting with the stars $K_{1,n_1}, \ldots, K_{1,n_t}$ and joining the center vertices of K_{1,n_i} and $K_{1,n_{i+1}}$ to a new vertex v_i where $1 \leqslant i \leqslant k-1$. Kaneria, Jariya, and Meghpara [1590] proved that $< K_{1,n_1}, \ldots, K_{1,n_t} >$ is cordial and every graceful graph with $|v_f(\text{odd}) - v_f(\text{even})| \leqslant 1$ is cordial. Kaneria, Meghpara, and Makadia [1618] proved that the cycle of complete graphs $C(t \cdot K_{m,n})$ and the cycle of wheels $C(t \cdot W_n)$ are cordial. Kaneria, Makadia, and Meghpara [1605] proved that the cycle of cycles $C(t \cdot C_n)$ is cordial for $t \geqslant 3$. Kaneria, Makadia, and Meghpara [1606] proved that a star of K_n and a cycle of n copies of K_n are cordial. Kaneria, Viradia, Jariya, and Makadia [1625] proved that the cycle of paths $C(t \cdot P_n)$ is cordial, product cordial (see Section 7.6), and total edge product cordial.

Jeba Jesintha and Subashini proved the following graphs are cordial: the cycle of vertex switching of cycles [1395]; the path union of vertex switching of wheels in increasing order [1404]; the path union of jelly fish graphs is cordial and cycle of jelly fish graphs [1405]; the star of fixed trees of diameter three tree [1397]; and the path union of vertex switching of cycles in increasing order [1399]. In [1414] Jeba Jesintha, Vinodhini, and Lakshmi, proved that a star glued with subdivided shell graph and super subdivision of circular ladders $(C_n \times K_2)$ admit cordial labelings. [1414] new

Cordial labelings and variations of them for fractal graphs are given in [2721] and [2720]. [2721] new [2720] new

Cahit [646] calls a graph *H-cordial* if it is possible to label the edges with the numbers from the set $\{1, -1\}$ in such a way that, for some k, at each vertex v the sum of the labels on the edges incident with v is either k or $-k$ and the inequalities $|v(k) - v(-k)| \leqslant 1$ and $|e(1) - e(-1)| \leqslant 1$ are also satisfied, where $v(i)$ and $e(j)$ are, respectively, the number of vertices labeled with i and the number of edges labeled with j. He calls a graph H_n-*cordial* if it is possible to label the edges with the numbers from the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ in such a way that, at each vertex v the sum of the labels on the edges incident with v is in the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ and the inequalities $|v(i) - v(-i)| \leqslant 1$ and $|e(i) - e(-i)| \leqslant 1$ are also satisfied for each i with $1 \leqslant i \leqslant n$. Among Cahit's results are: $K_{n,n}$ is H-cordial if and only if $n > 2$ and n is even; and $K_{m,n}, m \neq n$, is H-cordial if and only if $n \equiv 0 \pmod 4$, m is even and $m > 2, n > 2$. Unfortunately, Ghebleh and Khoeilar [1078] have shown that other statements in Cahit's paper are incorrect. In particular, Cahit states that K_n is H-cordial if and only if $n \equiv 0 \pmod 4$; W_n is H-cordial if and only if $n \equiv 1 \pmod 4$; and K_n is H_2-cordial if and only if $n \equiv 0 \pmod 4$ whereas Ghebleh and Khoeilar instead prove that K_n is H-cordial if and only if $n \equiv 0$ or $3 \pmod 4$ and $n \neq 3; W_n$ is H-cordial if and only if n is odd; K_n is H_2-cordial if $n \equiv 0$ or $3 \pmod 4$; and K_n is not H_2-cordial if $n \equiv 1 \pmod 4$. Ghebleh and Khoeilar also prove every wheel has an H_2-cordial labeling. In [954] Freeda and Chellathurai prove that the following graphs are H_2-cordial: the join of two paths, the join of two cycles, ladders, and the tensor product $P_n \otimes P_2$. They also prove that the join of W_n and W_m where $n + m \equiv 0 \pmod 4$ is H-cordial. Cahit generalizes the notion of H-cordial labelings in [646].

A graph $G(V, E)$ is called H_k-*cordial* if it has an H-cordial labeling f such that for each edge e and each vertex v of G have the label $1 \leqslant |f(e)| \leqslant k$, $1 \leqslant |f(v)| \leqslant k$ and $|v_f(i) - v_f(-i))| \leqslant 1$, $|e_f(i)) - e_f(-i)| \leqslant 1$ for each i with $1 \leqslant i \leqslant k$. Ratilal and Parmar [2606] investigated H_k-cordial labelings of triangular snakes, double triangular snakes, triple triangular snakes, alternate triangular snakes, double alternate triangular snakes, irregular triangular snakes, quadrilateral snakes, double quadrilateral snakes, alternate quadrilateral snakes, and irregular quadrilateral snakes. Joshi and Parmar [1557] investigated the H-, H_2- and H_3-cordiality of the following snakes: triangular, double triangular, triple trianglar, quadrilateral, double quadrilateral, alternate trigular, double alternate trigular, irregular triangular, quadrilateral, double quadrilateral, alternate quadrilateral, and irregular quadrilateral. Joshi and Pamar [1557] investigated H_k-cordial labeling of p-triangular, m-polygonal snakes, double m-polygonal snakes, alternate m-polygonal snakes, double alternate m-polygonal snakes, irregular m-polygonal snakes, and double irregular m-polygonal snakes.

Cahit and Yilmaz [650] call a graph E_k-*cordial* if it is possible to label the edges with the numbers from the set $\{0, 1, 2, \ldots, k-1\}$ in such a way that, at each vertex v, the sum of the labels on the edges incident with v modulo k satisfies the inequalities $|v(i) - v(j)| \leqslant 1$ and $|e(i) - e(j)| \leqslant 1$, where $v(s)$ and $e(t)$ are, respectively, the number of vertices labeled with s and the number of edges labeled with t. Cahit and Yilmaz prove the following graphs are E_3-cordial: P_n $(n \geqslant 3)$; stars S_n if and only if $n \not\equiv 1 \pmod 3$; K_n $(n \geqslant 3)$; C_n $(n \geqslant 3)$; friendship graphs; and fans F_n $(n \geqslant 3)$. They also prove that S_n $(n \geqslant 2)$ is E_k-cordial if and only if $n \not\equiv 1 \pmod k$ when k is odd or $n \not\equiv 1 \pmod{2k}$ when k is even and $k \neq 2$. Ni, Liu, and Lu [2271] demonstrate the E_3-cordiality of W_n, $P_m \times P_n$, $K_{m,n}$, and trees.

Bapat and Limaye [424] provide E_3-cordial labelings for: K_n $(n \geqslant 3)$; snakes whose blocks are all isomorphic to K_n where $n \equiv 0$ or $2 \pmod 3$; the one-point union of any number of copies of K_n where $n \equiv 0$ or $2 \pmod 3$; graphs obtained by attaching a copy of K_n where $n \equiv 0$ or $3 \pmod 3$ at each vertex of a path; and $K_m \odot K_n$. Sridharan and Umarani [3045] proved: for odd $n > 1$ and $k \geqslant 2$, $P_n \odot K_1$ is E_k-cordial; for n even and $n \neq k/2$, $P_n \odot K_1$ is E_k-cordial; and certain cases of fans are E_k-cordial. Youssef [3549] gives a necessary condition for a graph to be E_k-cordial for certain k. He also gives some new families of E_k-cordial graphs and proves Lee's [1886] conjecture about the edge-gracefulness of the disjoint union of two cycles. Venkatesh, Salah, and Sethuraman [3375] proved that C_{2n+1} snakes and C_{2n+1}^{2t} are E_2-cordial. Liu, Liu, and Wu [1980] provide two necessary conditions for a graph G to be E_k-cordial and prove that every P_n $(n \geqslant 3)$ is E_p-cordial if p is odd. They also discuss the E_2-cordiality of a graph G under the condition that some subgraph of G has a 1-factor. Liu and Liu [1979] proved that a graph with no isolated vertex is E_2-cordial if and only if it does not have order $4n + 2$. Bapat and Limaye [425] prove that helms, one point unions of helms, and path unions of helms are E_3-cordial. Jinnah and Beena [1547] prove the graphs P_n $(n \geqslant 3), C_n$ where $n \neq 4$ mod 8, and K_n $(n \geqslant 3)$ are E_4-cordial graphs. They also prove that every graph of order at least 3 is a subgraph of an E_4-cordial graph.

Hovey [1258] introduced a simultaneous generalization of harmonious and cordial labelings. For any Abelian group A (under addition) and graph $G(V, E)$ he defines G to be A-*cordial* if there is a labeling of V with elements of A such that for all a and b in A when the edge ab is labeled with $f(a) + f(b)$, the number of vertices labeled with a and the number of vertices labeled b differ by at most one and the number of edges labeled with a and the number labeled with b differ by at most one. In the case where A is the cyclic group of order k, the labeling is called k-*cordial*. With this definition we have: if $G(V, E)$ is a graph with $|E| \geqslant |V| - 1$ then $G(V, E)$ is harmonious if and only if G is $|E|$-cordial; G is cordial if and only if G is 2-cordial.

Hovey obtained the following: caterpillars are k-cordial for all k; all trees are k-cordial for $k = 3, 4$, and 5; odd cycles with pendent edges attached are k-cordial for all k; cycles are k-cordial for all odd k; for k even, C_{2mk+j} is k-cordial when $0 \leqslant j \leqslant \frac{k}{2} + 2$ and when

$k < j < 2k$; $C_{(2m+1)k}$ is not k-cordial; K_m is 3-cordial; and, for k even, K_{mk} is k-cordial if and only if $m = 1$.

Hovey advances the following conjectures: all trees are k-cordial for all k; all connected graphs are 3-cordial; and C_{2mk+j} is k-cordial if and only if $j \neq k$, where k and j are even and $0 \leqslant j < 2k$. The last conjecture was verified by Tao [3189]. Tao's result combined with those of Hovey show that for all positive integers k the n-cycle is k-cordial with the exception that k is even and $n = 2mk + k$. Tao also proved that the crown with $2mk + j$ vertices is k-cordial unless $j = k$ is even, and for $4 \leqslant n \leqslant k$ the wheel W_n is k-cordial unless $k \equiv 5 \pmod 8$ and $n = (k + 1)/2$. In [3231] Tuczyński, Wenus, and Wesek proved a conjecture of Cichacz, Görlich, and Tuza [745] that all hypertrees are 2-cordial. They also proved that all hypertree are 3-cordial.

In [2372] Patrias and Pechenik initiated the study of classes of finite Abelian groups A for which particular graphs are A-cordial. Their results include: P_{2m} and P_{2m+1} are not ${Z_2}^m$-cordial, all paths are A-cordial when A is an Abelian group of odd order, if A is an Abelian group of order n and P_n is A-cordial, then all paths are A-cordial, and P_{2n} is A-cordial when $A = Z_2 \times Z_k$ and $n = |A|$.

They conjecture that for a finite abelian group A, all paths are A-cordial if and only if A has an element of order greater than 2. Cichacz [737] proved that all cycle graphs are A-cordial for any Abelian group A of odd order. In [1654] and [1655] Chidambaram, Athisayanathan, and Ponraj proved that hypercubes, books, $C_n \times K_2$, and $P_n \times K_3$ and the splitting graphs of paths, cycles, and wheels are $\{1, -1, i, -i\}$-cordial. Since the group $\{1, -1, i, -i\}$ is cyclic, this is same as 4-cordial. In [2538] Radha, Venkatesan, Vitaldas, and Perumal prove that triangular ladders, alternate triangular snakes, alternate quadrilateral snakes, and double triangular snakes admit $\{1, -1, i, -i\}$ cordial labelings. [2538] new

Erickson [895]et al. showed that the friendship graph F_n is Z_{3m}-cordial and conjectured that F_n is Z_m-cordial except when n is even and not divisible by 4 and $m = 3n/d$, where d is odd.

In [3555] Youssef and Al-Kuleab proved the following: if G is a (p_1, q_1) k-cordial graph and G is a (p_2, q_2) k-cordial graph with p_1 or $p_2 \equiv 0 \pmod k$ and q_1 or $q_2 \equiv 0 \pmod k$, then $G + H$ is k-cordial; if G is a (p_1, q_1) 4-cordial graph and G is a (p_2, q_2) 4-cordial graph with p_1 or $p_2 \not\equiv 2 \pmod 4$ and q_1 or $q_2 \equiv 0 \pmod k$, then $G + H$ is 4-cordial; and $K_{m,n,p}$ is 4-cordial if and only if $(m, n, p) \mod 4 \not\equiv (0, 2, 2)$ or $(2, 2, 2)$.

In [819] ELrokh, Ismail, El-hay, and Elmshtaye define a *cubic roots cordial* labeling f of the vertices of a graph G with $1, \omega$, and ω^2, where $\omega^3 = 1$, with induced edge labeling $f^* : E(G)$ to $\{1, \omega, \omega^2\}$ defined by $f^*(uv) = f(u)f(v)$ if both the number of vertices and the number edges labeled with x and the number of vertices and the number edges labeled with y differ by at most 1. Since $\{1, \omega, \omega^2\}$ is isomorphic the group Z_3, cubic roots cordial is the same as 3-cordial. They prove the all nontrivial cases of paths, cycles, fans, and $G \cup H$ where G and H paths or cycles admit a cubic roots cordial. They also prove that wheels W_n are cubic roots cordial except when $n = 2 \bmod 3$ and n is even. [819] new

In [3547] Youssef obtained the following results: C_{2k} with one pendent edge is not $(2k + 1)$-cordial for $k > 1$; K_n is 4-cordial if and only if $n \leqslant 6$; C_n^2 is 4-cordial if and only if $n \not\equiv 2 \pmod 4$; and $K_{m,n}$ is 4-cordial if and only if $n \not\equiv 2 \pmod 4$; He also provides some necessary conditions for a graph to be k-cordial. Driscol zele [849] proved that all trees are 7-cordial.

Modha and Kanani [2178] prove that following graphs have a 5-cordial labeling: the shadow graph of a path and a cycle, graphs obtained by one point duplication and duplication of an edge by a vertex in cycle, and the graph obtained by the barycentric subdivision of wheel. In [2171] Modha and Kanani proved prisms, webs, flowers, and closed helms admit 5-cordial labelings. In [2172] they proved that fans are k-cordial for all k and double fans are k-cordial for all odd k and $n = (k+1)/2$. In [2174] they proved that the following graphs are k-cordial: W_n for odd k, $n = mk + j, m \geqslant 0$, $1 \leqslant j \leqslant k-1$ except for $j = (k-1)/2$; the total graphs of paths (recall $T(P_n)$ has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in P_n); the square C_n^2 for odd $k \leqslant n$; the path union of n copies of C_k where k is odd; and C_n with one pendent edge for odd $k \leqslant n$. Rathod and Kanani [2602] proved P_n^2 is k-cordial for all k and cycles with a single pendent edge are k-cordial for all even k. In [2599] Rathod and Kanani proved the middle graph, total graph, and splitting graph of a path are 4-cordial and P_n^2 and triangular snakes are 4-cordial. Modha and Kanani [2175] proved: W_n is k-cordial for all odd k and for all $n = mk + j,\ m \geqslant 0,\ 1 \leqslant j \leqslant k-1$ except for $j = k-1$; the path union of copies of C_k is k-cordial for odd k; the total graph of P_n is k-cordial for all k; the square C_n^2 is k-cordial for odd k odd and $n \geqslant k$; and the graphs obtained by appending an edge to C_n is k-cordial for odd k and $n \geqslant k$. Modha and Kanani [2177] prove the following graphs are k-cordial: $P_m \times C_k,\ P_m \times C_{k+1},\ P_m \times C_{k+3}$ for all odd k and $m \geqslant 2$, and $P_m \times C_{2k-1}$ for all odd $k, m \geqslant 2$ and $m \neq tk$. Rathod and Kanani [2602] [2604] prove that following graphs are 4-cordial: the splitting graph of $K_{1,n}$; triangular books; and the one point union any number of copies of the fan f_3; braid graphs; triangular ladders; and irregular quadrilateral snakes obtained from the path P_n with consecutive vertices $u_1, u_2, \ldots, u_n$ and new vertices $v_1, v_2, \ldots, v_{n-2}, w_1, w_2$, and edges $u_iv_i,\ w_iu_{i+2},\ v_iw_i$ for all $1 \leqslant i \leqslant n-2$. Rathod and Kanani [2603] prove wheels, fans, friendship graphs, double fans, and helms are 5-cordial. Driscoll, Krop, and Nguyen [842] proved that all trees are 6-cordial. In [1577], [1578], and [2173] Kanani and Modha prove that fans, friendship graphs, ladders, double fans, double wheels, wheels, helms, closed helms, and webs are 7-cordial graphs and wheels, fans and friendship graphs, gears, double fans, and helms are 4-cordial graphs. In [2724] Sathish Narayanan and Vijayaragavan obtained 3-divisor cordial labelings for graphs derived from paths.

Cichacz, Görlich and Tuza [745] extended the definition of k-cordial labeling for hypergraphs. They presented various sufficient conditions on a hypertree H (a connected hypergraph without cycles) to be k-cordial. From their theorems it follows that every k-uniform hypertree is k-cordial, and every hypertree with odd order or size is 2-cordial. Modha and Kanani [2176] prove the following graphs are k-cordial for all k: bistars, restricted square graphs $B_{n,n}^2$, the one-point union of C_3 and $K_{1,n}$, and $P_n \odot K_1$.

In [2836] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph G and successively form supersubdivisions (see §2.7 for the definition) that are cordial in the case that every edge in G is replaced by $K_{2,m}$ where m is even. Sethuraman and Selvaraju [2835] also show that the one-vertex union of any number of copies of $K_{m,n}$ is cordial and that the one-edge union of k copies of shell graphs $C(n, n-3)$ (see §2.2) is cordial for all $n \geqslant 4$ and all k. They conjectured that the one-point union of any number of copies of graphs of the form $C(n_i, n_i - 3)$ for various $n_i \geqslant 4$ is cordial. This was proved by Yue, Yuansheng, and Liping in [3567]. Riskin [2630] claimed that K_n is $(Z_2 \times Z_2)$-cordial if and only if n is at most 3 and $K_{m,n}$ is $(Z_2 \times Z_2)$-cordial if and only if $(m, n) \neq (2, 2)$. (Many authors use V_4 to denote $Z_2 \times Z_2$.) However, Pechenik and Wise [2375] report that the correct statement for $K_{m,n}$ is $K_{m,n}$ is $(Z_2 \times Z_2)$-cordial if and only if m and n are not both congruent to 2 mod 4. Seoud and Salim [2788] gave an upper bound on the number of edges of a graph that admits a $(Z_2 \oplus Z_2)$-cordial labeling in terms the number of vertices. Rathod and Kanani [2601] prove the following graphs are $(Z_2 \times Z_2)$-cordial for all n and m: $C_n \odot mK_1$, $C_n \odot K_2$, and graphs obtained by appending a single edge to one vertex of C_n. In Rathod and Kanani [2605] and [2600] proved the following graphs are $(Z_2 \times Z_2)$-cordial: alternate triangular snakes, alternate double triangular snakes, alternate triple triangular snakes, quadrilateral snakes, alternate quadrilateral snakes, double quadrilateral snakes, and double alternate quadrilateral snakes.

In [2375] Pechenik and Wise investigate $Z_2 \times Z_2$-cordiality of complete bipartite graphs, paths, cycles, ladders, prisms, and hypercubes. They proved that all complete bipartite graphs are $Z_2 \times Z_2$-cordial except $K_{m,n}$ where $m, n \equiv 2 \bmod 4$; all paths are $Z_2 \times Z_2$-cordial except P_4 and P_5; all cycles are $Z_2 \times Z_2$-cordial except C_4, C_5, C_k, where $k \equiv 2 \bmod 4$; and all ladders $P_2 \times P_k$ are $Z_2 \times Z_2$-cordial except C_4. They also introduce a generalization of A-cordiality involving digraphs and quasigroups, and show that there are infinitely many Q-cordial digraphs for every quasigroup Q. Jinnah and Nair [1548] proved that all trees except P_4 and P_5 are $Z_2 \times Z_2$-cordial and the graphs obtained by subdividing the pendent edges of $C_n \odot K_1$ are $Z_2 \times Z_2$ -cordial for all n.

Cairnie and Edwards [653] have determined the computational complexity of cordial and k-cordial labelings. They prove the conjecture of Kirchherr [1703] that deciding whether a graph admits a cordial labeling is NP-complete. As a corollary, this result implies that the same problem for k-cordial labelings is NP-complete. They remark that even the restricted problem of deciding whether connected graphs of diameter 2 have a cordial labeling is also NP-complete.

For a (p, q) graph G and a bijection f from $V(G)$ to $\{1, 2, \ldots, p\}$ Ponraj, Annathurai, and Kala [2414] introduced a new graph labeling as follows. For each edge uv assign the remainder when $f(u)$ is divided by $f(v)$ or when $f(v)$ is divided by $f(u)$ depending on whether $f(u) \geqslant f(v)$ or $f(v) \geqslant f(u)$. The function f is called a *remainder cordial* labeling of G if $|\eta_e - \eta_o| \leqslant 1$ where η_e and η_o respectively denote the number of edges labeled with even integers and the number of edges labeled with odd integers. A graph G with a remainder cordial labeling is called a *remainder cordial* graph. In [2414] and [2419] they proved that the following graphs are remainder cordial: paths, cycles, stars, bistars, crowns, combs, $K_{2,n}$, $S(K_{1,n})$, $S(B_{n,n})$, P_n^2, wheels, subdivisions of wheels, $K_{2,2n}$, and the graph obtained by subdividing the pendent edges of the bistar $B_{n,n}$. They also proved the following star related graphs are remainder cordial: $K_{1,n} \cup B_{n,n}$, $P_n \cup K_{1,n}$, $P_n \cup B_{n,n}$, $K_{1,n} \cup S(K_{1,n})$, $K_{1,n} \cup S(B_{n,n})$, $P_n^2 \cup K_{1,n}$, ${P_n}^2 \cup B_{n,n}$, and $S(K_{1,n}) \cup S(B_{n,n})$.

They conjecture that K_n is remainder cordial if and only if $n \leqslant 3$. Ponraj, Annathurai, and Kala [2415] generalize remainder cordial labelings as follows. Let f be a function from $V(G)$ to $\{1, 2, \ldots, k\}$ where $2 < k \leqslant |V(G)|$. For each edge uv assign the remainder when $f(u)$ is divided by $f(v)$ or when $f(v)$ is divided by $f(u)$ depending on whether $f(u) \geqslant f(v)$ or $f(v) \geqslant f(u)$. The function f is called a *k-remainder cordial* labeling of G if $|v_f(i) - v_f(j)| \leqslant 1$, for $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labeled with x and $|\eta_e - \eta_o| \leqslant 1$ where η_e and η_o respectively denote the number of edges labeled with even integers and the number of edges labeled with odd integers. A graph that admits a k-remainder cordial labeling is called a *k-remainder cordial* graph. In [2415], [216], [217], and [2420] they proved the following. Every graph is a subgraph of a connected k-remainder cordial graph for $k \geqslant 4$. Note that when $k = 2$, the number of edges with label 0 is q so there does not exists a 2-remainder cordial labeling. They further investigate the 3-remainder cordial labeling behavior of paths, cycles, stars, combs, crowns, wheels, fans, squares of paths, subdivisions of wheels, subdivisions of stars, subdivisions of combs, armed crowns, and $K_{1,n} \odot K_2$. They further proved that W_n is 3-remainder cordial if and only if $n \equiv 1 \pmod 3$, $K_{1,n}$ is 3-remainder cordial if and only if $n \in \{1, 2, 3, 4, 5, 6, 7, 9\}$, and K_n is 3-remainder cordial if and only if $n \leqslant 3$. In [2416], [2417], and [2418] Ponraj, Annathurai, and Kala proved the following graphs are 4-remainder cordial: complete graphs, paths, cycles, crowns, stars, bistars, books, subdivisions of stars, subdivisions of bistars, subdivisions of jelly fish, flowers, sunflowers, lotuses inside a circle, friendship graphs, webs, triangular snakes, durer graphs, planar grids, mongolian tents, prisms, dragon graphs $C_m@P_n$ (the graph obtained by identifying an endpoint of P_n with one vertex of C_m), crossed prisms CP_{2n}, and $K_2 + mK_1$ ($m \equiv 0, 1, 3 \pmod 4$). They also investigate the 4-remainder cordial labeling of $L_n \odot mK_1$, $L_n \odot K_2$, $L_n \odot mK_1$, $P_n \odot K_1$, $P_n \odot 2K_1$, $C_n \odot K_1$, and $S(P_n \odot K_1)$.

In Bapat [283] introduces the following new labeling. A graph $G(V, E)$ has a L-*cordial* labeling if there is a bijection f from $E(G)$ to $\{1, 2, \ldots, |E|\}$ that assigns 0 to a vertex v if the largest label on the edges incident to v is even and assigns 1 to v otherwise and this assignment satisfies the condition that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1. A graph that admits an L-cordial labeling is called as L-*cordial* graph. He shows that stars, path, cycles, and triangular snakes are L-cordial graphs

In [683] Chartrand, Lee, and Zhang introduced the notion of uniform cordiality as follows. Let f be a labeling from $V(G)$ to $\{0, 1\}$ and for each edge xy define $f^*(xy) = |f(x) - f(y)|$. For $i = 0$ and 1, let $v_i(f)$ denote the number of vertices v with $f(v) = i$ and $e_i(f)$ denote the number of edges e with $f^*(e) = i$. They call a such a labeling f *friendly* if $|v_0(f) - v_1(f)| \leqslant 1$. A graph G for which every friendly labeling is cordial is called *uniformly cordial.* They prove that a connected graph of order $n \geqslant 2$ is uniformly cordial if and only if $n = 3$ and $G = K_3$, or n is even and $G = K_{1,n-1}$.

In [2628] Riskin introduced two measures of the noncordiality of a graph. He defines the *cordial edge deficiency* of a graph G as the minimum number of edges, taken over all friendly labelings of G, needed to be added to G such that the resulting graph is cordial. If a graph G has a vertex labeling f using 0 and 1 such that the edge labeling f_e given by $f_e(xy) = |f(x) - f(y)|$ has the property that the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1, the *cordial vertex deficiency* defined as ∞. Riskin proved: the cordial edge deficiency of K_n $(n > 1)$ is $\lfloor \frac{n}{2} \rfloor - 1$; the cordial vertex deficiency of K_n is $j - 1$ if $n = j^2 + \delta$, when δ is $-2, 0$ or 2, and ∞ otherwise. In [2628] Riskin determines the cordial edge deficiency and cordial vertex deficiency for the cases when the Möbius ladders and wheels are not cordial. In [2629] Riskin determines the cordial edge deficiencies for complete multipartite graphs that are not cordial and obtains a upper bound for their cordial vertex deficiencies.

Recall a graph G the graph G^*, called the *star of* G, is the graph obtained by replacing each vertex G with the star $K_{1,n}$. In [1619] Kaneria, Patadiya and Teraiya introduced a balanced cordial labeling for a graph by saying that a cordial labeling f is a *vertex balanced cordial* if it satisfies the condition$v_f(0) = v_f(1)$; f is a *balanced cordial* if it satisfies the conditions $e_f(0) = e_f(1)$ and $v_f(0) = v_f(1)$. Kaneria, Teraiya, and Patadiya [1622] proved the path union $P(t \cdot C_{4n})$ is a balanced cordial if t is odd and it is vertex balanced cordial if t is even; $C(t \cdot C_{4n})$ is a balanced cordial if $t \equiv 0 \pmod 4$ and it is a vertex balanced cordial if $t \equiv 1, 3 \pmod 4$; and $C^{\star}_{4n}$ is balanced cordial. They proved $P_n \times C_{4t}$ is balanced cordial; $C_{2n} \times C_{4t}$ is balanced cordial; and $G_1 \odot G_2$ is cordial when G_1 is cordial and G_2 is a balanced cordial. Kaneria and Teraiya [1621] prove if G is a balanced cordial, then so is G^*; if G is a balanced cordial, then so is $P_{2n+1} \times G$; and if G is a balanced cordial, then so is $\overline{G}^*$.

An *integer cordial labeling* of a graph $G^*(p, q)$ is an injective map $g : V \to [\frac{-p}{2}, \ldots, \frac{p}{2}]^*$ or $[-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor]$ as p is even or odd, which induces an edge labeling $g : E \to \{0, 1\}$ defined by $g(uv) = 1$ if $g(u) + g(v) \geqslant 0$ and 0 otherwise such that the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1. If a graph has integer cordial labeling it is called an *integer cordial graph*. In [1128] Gondalia and Rokad investigated the existence of integer cordial labelings of star and bistar related graphs.

For a a planar graph G with an with integer cordial labeling g and the face labeling $g**$ from the faces of G defined by $g**(f) = 1$ if $g(v_1) + g(v_2) + \ldots + g(v_n) = 0$ and $g**(f) = 0$ otherwise, where $v_1, v_2, \ldots, v_n$ are the vertices of face f. Such a labeling is called a *face integer cordial* labeling of graph G if the number of faces labeled with 0 and the number of faces labeled with 1 differ by at most 1. Parameswari, Saradha Pritha, and Rajeswari [2341] proved that the lilly graph, $2K_{1,n} + 2P_n$ $(n \geqslant 2)$, admits an integer cordial lableling and the vanessa graph, $2F_n + K_{1,n}$ $(n \geqslant 2)$, admits an integer cordial labeling and a face integer cordial lableling. Sheriff, Abbas, and Raj [2883] proved that wheels, fans, friendship graphs, triangular snakes, double triangular snakes, the star of cycles, the degree splitting graph of bistars are face integer cordial graphs. [2341] new [2883] new

For a simple connected graph $G(V, E)$, Sahaya, Maya, and Nicholas [2672] introduced the concept of *product integer cordial* labeling of as an injective mapping $f : V \to \{1, 2, \ldots, |V|\}$ such that the induced edge labeling f^* on E defined $f^*(uv) = 1$ or 0 according as $f(u)f(v)$ even or odd respectively, has the property that the number of edges labeled with 1 and the the number of edges labeled with 0 differ by at most 1. They proved that the following graphs admit product integer cordial labelings: paths, friendship graphs, C_n if and only if n is odd, and $K_{m,n}$ if and only if one of m and n is [2672] new

1. In [2846] Shah and Parmar proved that nontrivial triangular snakes, double triangular snakes, triple triangular snakes, and alternate triangular snakes graph admits integer cordial labelings. In [2847] Shah and Parmar proved that m-triangular snakes, quadrilateral snakes, double quadrilateral snakes, m-quadrilateral snakes, pentagonal snakes, double pentagonal snakes, and m-pentagonal snakes are integer cordial graphs. They also proved that triangular snake graph, double triangular snakes, alternate triangular snakes, m-triangular snakes, quadrilateral snakes, double quadrilateral snakes, m-quadrilateral snakes, pentagonal snakes, double pentagonal snakes, and m-pentagonal snakes graph are integer cordial graphs. In [2848] Shah and Parmar proved that alternate m-triangular snakes, quadrilateral snakes, alternate m-quadrilateral snakes, pentagonal snakes, alternate m-pentagonal snakes, irregular triangular snakes, irregular quadrilateral snakes, and irregular pentagonal snakes are integer cordial graphs. [2846] new [2847] new [2848] new

If f is a binary vertex labeling of a graph G Lee, Liu, and Tan [1855] defined a partial edge labeling of the edges of G by $f^*(uv) = 0$ if $f(u) = f(v) = 0$ and $f^*(uv) = 1$ if $f(u) = f(v) = 1$. They let $e_0(G)$ denote the number of edges uv for which $f^*(uv) = 0$ and $e_1(G)$ denote the number of edges uv for which $f^*(uv) = 1$. They say G is *balanced* if it has a friendly labeling f such that if $|e_0(f) - e_1(f)| \leqslant 1$. In the case that the number of vertices labeled 0 and the number of vertices labeled 1 are equal and the number of edges labeled 0 and the number of edges labeled 1 are equal they say the labeling is *strongly balanced*. They prove: P_n is balanced for all n and is strongly balanced if n is even; $K_{m,n}$ is balanced if and only if m and n are even, m and n are odd and differ by at most 2, or exactly one of m or n is even (say $n = 2t$) and $t \equiv -1, 0, 1 \pmod{|m-n|}$; a k-regular graph with p vertices is strongly balanced if and only if p is even and is balanced if and only if p is odd and $k = 2$; and if G is any graph and H is strongly balanced, the composition $G[H]$ (see §2.3 for the definition) is strongly balanced. In [1735] Kong, Lee, Seah, and Tang show: $C_m \times P_n$ is balanced if m and n are odd and is strongly balanced if either m or n is even; and $C_m \odot K_1$ is balanced for all $m \geqslant 3$ and strongly balanced if m is even. They also provide necessary and sufficient conditions for a graph to be balanced or strongly balanced. Lee, Lee, and Ng [1826] show that stars are balanced if and only if the number of edges of the star is at most 4. Kwong, Lee, Lo, and Wang [1787] define a graph G to be *uniformly balanced* if $|e_0(f) - e_1(f)| \leqslant 1$ for every vertex labeling f that satisfies if $|v_0(f) - v_1(f)| \leqslant 1$. They present several ways to construct families of uniformly balanced graphs. Kim, Lee, and Ng [1697] prove the following: for any graph G, mG is balanced for all m; for any graph G, mG is strongly balanced for all even m; if G is strongly balanced and H is balanced, then $G \cup H$ is balanced; mK_n is balanced for all m and strongly balanced if and only if $n = 3$ or mn is even; if H is balanced and G is any graph, the $G \times H$ is strongly balanced; if one of m or n is even, then $P_m[P_n]$ is balanced; if both m and n are even, then $P_m[P_n]$ is balanced; and if G is any graph and H is strongly balanced, then the tensor product $G \otimes H$ is strongly balanced. (The *tensor product* $G \otimes H$ of graphs G and H, has the vertex set $V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in $G \otimes H$ if and only if u' is adjacent with v' and u is adjacent with v.)

A graph G is *k-balanced* if there is a function f from the vertices of G to $\{0, 1, 2, \ldots, k-$

$1\}$ such that for the induced function f^* from the edges of G to $\{0, 1, 2, \ldots, k-1\}$ defined by $f^*(uv) = |f(u) - f(v)|$ the number of vertices labeled i and the number of edges labeled j differ by at most 1 for each i and j. Seoud, El Sonbaty, and Abd El Rehim [2770] proved the following: if $|E| \geqslant 2k+1$ and $|V| \leqslant k$ then $G(V, E)$ is not k-balanced; if $|E| \geqslant 3k+1$, $(k \geqslant 2)$ and $3k-1 \geqslant |V| \geqslant 2k+1$ then $G(V, E)$ is not k-balanced; r-regular graphs with $3 \leqslant r \leqslant n-1$ are not r-balanced; if G_1 has m vertices and G_2 has n vertices then $G_1 + G_2$ is not $(m+n)$-balanced for $m, n \geqslant 5$; $P_3 \times P_n$ with edge set E is $3n$-balanced and $|E|$-balanced; $L_n \times P_2$ $(L_n = P_n \times P_2)$ with vertex set V and edge set E is $|V|$-balanced and k-balanced for $k \geqslant |E|$ but not n-balanced for $n \geqslant 2$; the one-point union of two copies of $K_{2,n}$ is $2n$-balanced, $|V|$-balanced, and $|E|$-balanced not is 3-balanced when $n \geqslant 4$. They also proved that the composition graph $P_n[P_2]$ is not n-balanced for $n \geqslant 3$, is not $2n$-balanced for $n \geqslant 5$, and is not $|E|$-balanced.

A graph whose edges are labeled with 0 and 1 so that the absolute difference in the number of edges labeled 1 and 0 is no more than one is called *edge-friendly*. We say an edge-friendly labeling induces a *partial vertex labeling* if vertices which are incident to more edges labeled 1 than 0, are labeled 1, and vertices which are incident to more edges labeled 0 than 1, are labeled 0. Vertices that are incident to an equal number of edges of both labels are called *unlabeled*. Call a procedure on a labeled graph a label switching algorithm if it consists of pairwise switches of labels. Krop, Lee, and Raridan [1762] prove that given an *edge-friendly labeling* of K_n, we show a label switching algorithm producing an edge-friendly relabeling of K_n such that all the vertices are labeled.

In 2017 [286] Bapat introduced a new labeling as follows. A function f from the vertices of a graph $G(E, V)$ to $\{0, 1, 2, \ldots, |V|-1\}$ is called an *extended vertex edge additive cordial* labeling if the induced function f^* from the edges of G to $\{0, 1\}$ defined by $f^*(uv) = f(u) + f(v) \pmod 2$ for all edges uv of G has the property that the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. Bapat [286] proved paths, stars, $K_{2,n}, K_{3,n}, K_{4,n}$, $P_n \odot C_3$, and $P_n \odot C_4$ admit extended vertex edge additive cordial labeling.

Let $G(p, q)$ a simple finite connected graph. Given a bijective function f from $E(G)$ to $\{0, 1, \ldots, q-1\}$ Bapat [287] calls a bijective function f^* from $E(G)$ to $\{0, 1, 2, \ldots, q-1\}$ an *extended edge vertex cordial (eevc) labeling* if the induced function f^* from $V(G)$ to $\{0, 1\}$ defined by $f^*(u) = \Sigma f(uv)$ mod 2 where the sum is taken over all edges incident to u has the property that the number of vertices labeled with 0 differs from the number labeled with 1 by at most 1. He shows that P_n $(n \neq 2 \bmod 4)$, C_n $(n \neq 2 \bmod 4)$, $K_{1,n}$ $(n \neq 1 \bmod 4)$, graphs obtained by joining the centers of two copies of $K_{1,2n+1}$ by an edge, and triangular snakes have eevc labelings.

Murali, Thirusangu, Madura Meenakshi [2208] say a graph $G = (V, E)$ is *biconditional cordial* if there is a function $f : V \to \{0, 1\}$ such that the induced edge function $f^* : E \to \{0, 1\}$ defined by $f^*(uv) = 1$ if $(u) = f(v)$ and 0 if $f(u) \neq f(v)$ and the number of vertices labeled with 0 and the number labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number labeled with 1 differ by at most 1. Kalaimathi and Balamurugan [1564] prove the existence of the biconditional cordial labeling for complete bipartite graphs, books with triangular pages, sunflower graphs and web graphs. Nedumaran, Thirusangu, and Celin Mary [2266] proved that the graph consisting of k copies of a double star admits a biconditional cordial labeling. Kalaimathi, Balamurugan, and Rao [1565] proved the existence of the biconditional cordial labelings for super subdivisions of ladders and grids $P_m \times P_n$.

A *total cordial* labeling of a graph G is a cordial labeling of vertex set and edge set such that the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. Elrokh, Al-Shamiri, Nada, and El-hay [870] provided necessary and sufficient conditions for the existence of cordial and total cordial labelings for the corona product of paths and the one-point union C_m^2 and C_n^2.

3.8 The Friendly Index–Balance Index

Recall a function f from $V(G)$ to $\{0,1\}$ where for each edge xy, $f^*(xy) = |f(x) - f(y)|$, $v_i(f)$ is the number of vertices v with $f(v) = i$, and $e_i(f)$ is the number of edges e with $f^*(e) = i$ is called *friendly* if $|v_0(f) - v_1(f)| \leqslant 1$. Lee and Ng [1863] define the *friendly index set* of a graph G as FI(G)= $\{|e_0(f) - e_1(f)|$ where f runs over all friendly labelings f of $G\}$. They proved: for any graph G with q edges FI(G) $\subseteq \{0, 2, 4, \ldots, q\}$ if q is even and FI(G)$\subseteq \{1, 3, \ldots, q\}$ if q is odd; for $1 \leqslant m \leqslant n$, FI($K_{m,n}$)= $\{(m - 2i)^2 |\ 0 \leqslant i \leqslant \lfloor m/2 \rfloor\}$ if $m + n$ is even; and FI($K_{m,n}$)= $\{i(i + 1)|\ 0 \leqslant i \leqslant m\}$ if $m + n$ is odd. In [1866] Lee and Ng prove the following: FI(C_{2n}) = $\{0, 4, 8, \ldots, 2n\}$ when n is even; FI(C_{2n}) = $\{2, 6, 10, \ldots, 2n\}$ when n is odd; and FI(C_{2n+1}) = $\{1, 3, 5, \ldots, 2n - 1\}$. Elumalai [878] defines a *cycle with a full set of chords* as the graph PC_n obtained from $C_n = v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the cords $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}, v_{(n+2)/2}$ when n is even and $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-3)/2}, v_{(n+3)/2}$ when n is odd. Lee and Ng [1865] prove: FI(PC_{2m+1}) = $\{3m - 2, 3m - 4, 3m - 6, \ldots, 0\}$ when m is even and FI(PC_{2m+1}) = $\{3m - 2, 3m - 4, 3m - 6, \ldots, 1\}$ when m is odd; FI(PC_4) = $\{1, 3\}$; for $m \geqslant 3$, FI(PC_{2m}) = $\{3m - 5, 3m - 7, 3m - 9, \ldots, 1\}$ when m is even; FI(PC_{2m}) = $\{3m - 5, 3m - 7, 3m - 9, \ldots, 0\}$ when m is odd.

Salehi and Lee [2681] determined the friendly index for various classes of trees. Among their results are: for a tree with q edges that has a perfect matching, the friendly index is the odd integers from 1 to q and for $n \geqslant 2$, FI(P_n)= $\{n - 1 - 2i|\ 0 \leqslant i\lfloor (n - 1)/2 \rfloor$. Law [1821] determined the full friendly index sets of spiders and disproved a conjecture by Salehi and Lee [2681] that the friendly index set of a tree forms an arithmetic progression. In [1869] Lee, Ng, and Lau determine the friendly index sets of several classes of spiders. Gao, Sun, and Lee [1029] determined the full friendly index of $P_m \times P_n$ with the extra $mn + 1 - m - n$ edges $u_{ij} - u_{(i+1)(j+1)}$. Sun, Gao, and Lee [3131] determined the full friendly index and friendly index for the twisted product of Mőbius ladders. Sinha and Kaur [2967] determined the full edge friendly index of stars, wheels, 2-regular graphs, and mP_n. In [2896] Shiu determined the full edge-friendly index sets of complete bipartite graphs. Salehi and McGinn [2684] obtained partial results about the friendly index set of Q_n and strenghten a conjecture about the friendly index set of Q_n made in [2686]. Teffilia1 and Devaraj [3195] found the friendly index set of the graphs obtained by identifying the central vertex of a fan with the endpoint of a path (*umbrella*), the graphs obtained by identifying the central vertex of a star with the endpoint of a path, the graphs obtained by identifying the endpoints of copies of P_2 (*globe*), the splitting graph of a star, and $P_2 + mK_1$. Lee, Low, Ng, and Wang [1857] determined the friendly index sets for various classes of disjoint unions of stars. Gao, Ruo-Yuan, Lee, Ren, and Sun [1028] determined $FFI(G)$, $FI(G)$ and $FPCI(G)$ for a class of cubic graphs G.

Lee and Ng [1865] define $PC(n, p)$ as the graph obtained from the cycle C_n with consecutive vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the p cords joining v_i to v_{n-i} for $1 \leqslant p\lfloor n/2 \rfloor - 1$. They prove $\mathrm{FI}(PC(2m+1, p)) = \{2m+p-1, 2m+p-3, 2m+p-5, \ldots, 1\}$ if p is even and $\mathrm{FI}(PC(2m+1, p)) = \{2m+p-1, 2m+p-3, 2m+p-5, \ldots, 0\}$ if p is odd; $\mathrm{FI}(PC(2m, 1)) = \{2m-1, 2m-3, 2m-5, \ldots, 1\}$; for $m \geqslant 3$, and $p \geqslant 2$, $\mathrm{FI}(PC(2m, p)) = \{2m+p-4, 2m+p-6, 2m+p-8, \ldots, 0\}$ when p is even, and $\mathrm{FI}(PC(2m, p)) = \{2m+p-4, 2m+p-6, 2m+p-8, \ldots, 1\}$ when p is odd. More generally, they show that the integers in the friendly index of a cycle with an arbitrary nonempty set of parallel chords form an arithmetic progression with a common difference 2. Shiu and Kwong [2901] determine the friendly index of the grids $P_n \times P_2$. The maximum and minimum friendly indices for $C_m \times P_n$ were given by Shiu and Wong in [2932].

In [1867] Lee and Ng prove: for $n \geqslant 2$, $\mathrm{FI}(C_{2n} \times P_2) = \{0, 4, 8, \ldots, 6n-8, 6n\}$ if n is even and $\mathrm{FI}(C_{2n} \times P_2) = \{2, 6, 10, \ldots, 6n-8, 6n\}$ if n is odd; $\mathrm{FI}(C_3 \times P_2) = \{1, 3, 5\}$; for $n \geqslant 2$, $\mathrm{FI}(C_{2m+1} \times P_2) = \{6n-1\} \cup \{6n-5-2k |$ where $k \geqslant 0$ and $6n-5-2k \geqslant 0\}$; $\mathrm{FI}(M_{4n})$ (here M_{4n} is the Möbius ladder with $4n$ steps) $= \{6n-4-4k |$ where $k \geqslant 0$ and $6n-4-4k \geqslant 0\}$; $\mathrm{FI}(M_{4n+2}) = \{6n+3\} \cup \{6n-5-2k |$ where $k \geqslant 0$ and $6n-5-2k > 0\}$. In [1788] Kwong, Lee, and Ng completely determine the friendly index of all 2-regular graphs. As a corollary, they show that $C_m \cup C_n$ is cordial if and only if $m + n = 0, 1$ or $3 \pmod 4$. Ho, Lee, and Ng [1251] determine the friendly index sets of stars and various regular windmills. In [3464] Wen determines the friendly index of generalized wheels $C_n + mK_1$ for all $m > 1$. In [2680] Salehi and De determine the friendly index sets of certain caterpillars of diameter 4 and disprove a conjecture of Lee and Ng [1866] that the friendly index sets of trees form an arithmetic progression. The maximum and minimum friendly indices for for $C_m \times P_n$ were given by Shiu and Wong in [2932]. Salehi and Bayot [2677] have determined the friendly index set of $P_m \times P_n$. In [1867] Lee and Ng determine the friendly index sets for two classes of cubic graphs, prisms d Möbius ladders. Sinha and Kaur [2967] investigate the full region index sets of friendly labelings of cycles, wheels fans, and $P_2 \times P_n$.

For positive integers $a \leqslant b \leqslant c$, Lee, Ng, and Tong [1872] define the *broken wheel* $W(a, b, c)$ with three spokes as the graph obtained from K_4 with vertices u_1, u_2, u_3, c by inserting vertices $x_{1,1}, x_{1,2}, \ldots, x_{1,a-1}$ along the edge u_1u_2, $x_{2,1}, x_{2,2}, \ldots, x_{2,b-1}$ along the edge u_2u_3, $x_{3,1}, x_{3,2}, \ldots, x_{3,c-1}$ along the edge u_3u_1. They determine the friendly index set for broken wheels with three spokes.

Lee and Ng [1865] define a *parallel chord* of C_n as an edge of the form v_iv_{n-i} ($i < n-1$) that is not an edge of C_n. For $n \geqslant 6$, they call the cycle C_n with consecutive vertices $v_1, v_2, \ldots, v_n$ and the edges $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}v_{(n+2)/2}$ for n even and $v_2v_{n-1}, v_3v_{n-2}, \ldots, v_{(n-1)/2}v_{(n+3)/2}$ for n odd, C_n with a *full set of parallel chords*. They determine the friendly index of these graphs and show that for any cycle with an arbitrary non-empty set of parallel chords the numbers in its friendly index set form an arithmetic progression with common difference 2.

For a graph $G(V, E)$ and a graph H rooted at one of its vertices v, Ho, Lee, and Ng [1250] define a *root-union of* (H, v) *by* G as the graph obtained from G by replacing each vertex of G with a copy of the root vertex v of H to which is appended the rest of the structure of H. They investigate the friendly index set of the root-union of stars by cycles.

For a graph $G(V, E)$, the *total graph* $T(G)$ of G, is the graph with vertex set $V \cup E$ and edge set $E \cup \{(v, uv)|\ v \in V, uv \in E\}$. Note that the total graph of the n-star is the friendship graph and the total graph of P_n is a triangular snake. Lee and Ng [1862] use $SP(1^n, m)$ to denote the spider with one central vertex joining n isolated vertices and a path of length m. They show: $\mathrm{FI}(K_1 + 2nK_2)$ (friendship graph with $2n$ triangles) $= \{2n, 2n-4, 2n-8, \ldots, 0\}$ if n is even; $\{2n, 2n-4, 2n-8, \ldots, 2\}$ if n is odd; $\mathrm{FI}(K_1+(2n+1)K_2) = \{2n+1, 2n-1, 2n-3, \ldots, 1\}$; for n odd, $\mathrm{FI}(T(P_n)) = \{3n-7, 3n-11, 3n-15, \ldots, z\}$ where $z = 0$ if $n \equiv 1 \pmod 4$ and $z = 2$ if $n \equiv 3 \pmod 4$; for n even, $\mathrm{FI}(T(P_n)) = \{3n-7, 3n-11, 3n-15, \ldots, n+1\} \cup \{n-1, n-3, n-5, \ldots, 1\}$; for $m \leqslant n-1$ and $m+n$ even, $\mathrm{FI}(T(SP(1^n, m))) = \{3(m+n)-4, 3(m+n)-8, 3(m+n)-12, \ldots, (m+n) \pmod 4\}$; for $m+n$ odd, $\mathrm{FI}(T(SP(1^n, m))) = \{3(m+n)-4, 3(m+n)-8, 3(m+n)-12, \ldots, m+n+2\} \cup \{m+n, m+n-2, m+n-4, \ldots, 1\}$; for $n \geqslant m$ and $m+n$ even, $\mathrm{FI}(T(SP(1^n, m))) = \{|4k-3(m+n)|\ |(n-m+2)/2 \leqslant k \leqslant m+n\}$; for $n \geqslant m$ and $m+n$ odd, $\mathrm{FI}(T(SP(1^n, m))) = \{|4k-3(m+n)|\ |(n-m+3)/2 \leqslant k \leqslant m+n\}$.

Kwong and Lee [1784] determine the friendly index any number of copies of C_3 that share an edge in common and the friendly index any number of copies of C_4 that share an edge in common. Lau, Gao, Lee, and Sun determine the friendly index sets and the cordiality of the edge-gluing of a complete graph K_n and n copies of cycles C_3.

For a planar graph $G(V, E)$ Sinha and Kaur [2986] extended the notion of an index set of a friendly labeling to regions of a planar graph and determined the full region index sets of friendly labeling of cycles, wheels fans, and grids $P_n \times P_2$.

An edge-friendly labeling f of a graph G induces a function f^* from $V(G)$ to $\{0, 1\}$ defined as the sum of all edge labels mod 2. The *edge-friendly index set*, $I_f(G)$, of f is the number of vertices of f labeled 1 minus the number of vertices labeled 0. The *edge-friendly index set* of a graph G, EFI(G), is $\{|I_f(G)|\}$ taken over all edge-friendly labelings f of G. The *full edge-friendly index set* of a graph G, FEFI(G), is $\{I_f(G)\}$ taken over all edge-friendly labelings f of G. Sinha and Kaur [2985] determined the full edge-friendly index sets of stars, 2-regular graphs, wheels, and mP_n. In [2987] Sinha and Kaur extended the notion of index set of an edge-friendly labeling to regions of a planar graph and determined the full region index set of edge-friendly labelings of cycles, wheels, fans $P_n + K_1$, double fans $P_n + \overline{K_2}$, and grids $P_m \times P_n$ $(m \geqslant 2, n \geqslant 3)$. Sinha and Kaur [2966] investigate the full edge-friendly index sets of double stars, fans generalized fans, and $P_n \times P_2$. In [2895] Shiu determined the extreme values of edge-friendly indices of complete bipartite graphs.

In [1698] Kim, Lee, and Ng define the *balance index set* of a graph G as $\{|e_0(f)-e_1(f)|\}$ where f runs over all friendly labelings f of G. Zhang, Lee, and Wen [1826] investigate the balance index sets for the disjoint union of up to four stars and Zhang, Ho, Lee, and Wen [3576] investigate the balance index sets for trees with diameter at most four. Kwong, Lee, and Sarvate [1792] determine the balance index sets for cycles with one pendent edge, flowers, and regular windmills. Lee, Ng, and Tong [1871] determine the balance index set of certain graphs obtained by starting with copies of a given cycle and successively identifying one particular vertex of one copy with a particular vertex of the next. For graphs G and H and a bijection π from G to H, Lee and Su [1892] define $\mathrm{Perm}(G, \pi, H)$ as the graph obtaining from the disjoint union of G and H by joining each v in G to $\pi(v)$ with an edge. They determine the balanced index sets of the disjoint union of cycles and the balanced index sets for graphs of the form $\mathrm{Perm}(G, \pi, H)$ where G and H are regular graphs, stars, paths, and cycles with a chord. They conjecture that the balanced index set for every graph of the form $\mathrm{Perm}(G, \pi, H)$ is an arithmetic progression. Lee, Ho, and Su [1842] investigated the balance index sets of k-level wheel graphs.

Wen [3463] determines the balance index set of the graph that is constructed by identifying the center of a star with one vertex from each of two copies of C_n and provides a necessary and sufficient for such graphs to be balanced. In [1895] Lee, Su, and Wang determine the balance index sets of the disjoint union of a variety of regular graphs of the same order. Kwong [1782] determines the balanced index sets of rooted trees of height at most 2, thereby settling the problem for trees with diameter at most 4. His method can be used to determine the balance index set of any tree. The *homeomorph* $Hom(G, p)$ of a graph G is the collection of graphs obtained from G by adding p ($p \geqslant 0$) additional degree 2 vertices to its edges. For any regular graph G, Kong, Lee, and Lee [1728] studied the changes of the balance index sets of $Hom(G, p)$ with respect to the parameter p. They derived explicit formulas for their balance index sets provided new examples of uniformly balanced graphs. In [601] Bouchard, Clark, Lee, Lo, and Su investigate the balance index sets of generalized books and ear expansion graphs. In [2652] Rose and Su provided an algorithm to calculate the balance index sets of a graph. Hua and Raridan [1267] determine the balanced index sets of all complete bipartite graphs with a larger part of odd cardinality and a smaller part of even cardinality.

In[2902] Shiu and Kwong made a major advance by introducing an easier approach to find the balance index sets of a large number of families of graphs in a unified and uniform manner. They use this method to determine the balance index sets for r-regular graphs, amalgamations of r-regular graphs, complete bipartite graphs, wheels, one point unions of regular graphs, sun graphs, generalized theta graphs, m-ary trees, spiders, grids $P_m \times P_n$, and cylinders $C_m \times P_n$. They provide a formula that enables one to determine the balance index sets of many biregular graphs (that is, graphs with the property that there exist two distinct positive integers r and s such that every vertex has degree r or s).

A labeling f from the vertices of a graph G to $\{0, 1\}$ is said to be *vertex-friendly* if the number of vertices labeled with 0 and the number labeled with 1 differ by at most 1. The *vertex balance index set* of G is $|e_0(f) - e_1(f)|$ taken over all vertex-friendly labelings f. Adiga, Subbaraya, Shrikanth and Sriraj [60] completely determined the vertex balance index set of K_n, $K_{m,n}$, $C_n \times P_2$, and complete binary trees.

Manico and Pedrano [2075] prove that if the number of edges in a vertex-friendly of a graph G is even, then the vertex balance index of G contains only even numbers, and if the number of edges in a vertex-friendly graph G is odd, then the the vertex balance index of G contains only odd numbers. Furthermore, they provide the vertex balance index set of triangular snakes, quadrilateral snakes, double triangular snakes), and double quadrilateral snakes.

In [2901] Shiu and Kwong define the *full friendly index set* of a graph G as $\{e_0(f) - e_1(f)\}$ where f runs over all friendly labelings of G. The full friendly index for $P_2 \times P_n$ is given by Shiu and Kwong in [2901]. The full friendly index of $C_m \times C_n$ is given by Shiu and Ling in [2918]. In [2983] and [2984] Sinha and Kaur investigated the full friendly index sets complete graphs, cycles, fans, double fans, wheels, double stars, $P_3 \times P_n$, and the tensor product of P_2 and P_n. Shiu and Ho [2898] investigated the full friendly index sets of cylinder graphs $C_m \times P_2$ $(m \geqslant 3), C_m \times P_3$ $(m \geqslant 4)$, and $C_3 \times P_n$ $(n \geqslant 4)$. These results, together with previously proven ones, completely determine the full friendly index of all cylinder graphs. Shiu and Ho [2899] study the full friendly index set and the full product-cordial index set of odd twisted cylinders and two permutation Petersen graphs. Gao [1017] determined the full friendly index set of $P_m \times P_n$, but he used the terms "edge difference set" instead of "full friendly index set" and "direct product" instead of "Cartesian product." The *twisted cylinder* graph is the permutation graph on $4n$ $(n \geqslant 2)$ vertices, $P(2n; \sigma)$, where $\sigma = (1, 2)(3, 4) \cdots (2n-1, 2n)$ (the product of n transpositions). Shiu and Lee [2916] determined the full friendly index sets of twisted cylinders.

In [714] and [1785] Chopra, Lee, and Su and Kwong and Lee introduce a dual of balance index sets as follows. For an edge labeling f using 0 and 1 they define a partial vertex labeling f^* by assigning 0 or 1 to $f^*(v)$ depending on whether there are more 0-edges or 1-edges incident to v and leaving $f^*(v)$ undefined otherwise. For $i = 0$ or 1 and a graph $G(V, E)$, let $e_f(i) = |\{uv \in E : f(uv) = i\}|$ and $v_f(i) = |\{v \in V : f^*(v) = i\}|$. They define the *edge-balance index* of G as $\mathrm{EBI}(G) = \{|v_f(0) - v_f(1)| :$ the edge labeling f satisfies $|e_f(0) - e_f(1)| \leqslant 1\}$. Among the graphs whose edge-balance index sets have been investigated by Lee and his colleagues are: fans and wheels [714]; generalized theta graphs [1785]; flower graphs [1786] and [1786]; stars, paths, spiders, and double stars [1903]; $(p, p+1)$-graphs [1897]; prisms and Möbius ladders [3437]; 2-regular graphs, complete graphs [3436]; and the envelope graphs of stars, paths, and cycles [724]. (The *envelope graph* of $G(V, E)$ is the graph with vertex set $V(G) \cup E(G)$ and set $E(G) \cup \{(u, (u, v)) : U \in V, (u, v) \in E)\}$).

Lee, Kong, Wang, and Lee [1729] found the $\mathrm{EBI}(K_{m,n})$ for $m = 1, 2, 3, 4, 5$ and $m = n$. Krop, Minion, Patel, and Raridan [1764] did the case for complete bipartite graphs with both parts of odd cardinality. Dao, Hua, Ngo, and Raridan [779] determined the edge-balanced index sets for complete even bipartite graphs. Krop and Sikes [1766] determined $\mathrm{EBI}(K_{m,m-2a})$ for $1 \leqslant a \leqslant (m-3)/4$ and m odd.

For a graph G and a connected graph H with a distinguished vertex s, the L-product of G and (H, s), $G \times_L (H, s)$, is the graph obtained by taking $|V(G)|$ copies of (H, s) and identifying each vertex of G with s of a single copy of H. In [716] and [605] Chou, Galiardi, Kong, Lee, Perry, Bouchard, Clark, and Su investigated the edge-balance index sets of L-product of cycles with stars. Bouchard, Clark, and Su [604] gave the exact values of the edge-balance index sets of L-product of cycles with cycles.

Chopra, Lee, and Su [717] prove that the edge-balance index of the fan $P_3 + K_1$

is $\{0, 1, 2\}$ and edge-balance index of the fan $P_n + K_1$, $n \geqslant 4$, is $\{0, 1, 2, \ldots, n-2\}$. They define the broken fan graphs $BF(a, b)$ as the graph with $V(BF(a, b)) = \{c\} \cup \{v_1, \ldots, v_a\} \cup \{u_1, \ldots, u_b\}$ and $E(BF(a, b)) = \{(c, v_i) | \; i = 1, \ldots, a\} \cup \{(c, u_i) | \; 1, \ldots, b\} \cup E(P_a) \cup E(P_b)$ ($a \geqslant 2$ and $b \geqslant 2$). They prove the edge-balance index set of $BF(a, b)$ is $\{0, 1, 2, \ldots, a+b-4\}$. In [1893] Lee, Su, and Todt give the edge-balance index sets of broken wheels. See also [3057] and [3219]. In [1827] Lee, Lee, and Su present a technique that determines the balance index sets of a graph from its degree sequence. In addition, they give an explicit formula giving the exact values of the balance indices of generalized friendship graphs, envelope graphs of cycles, and envelope graphs of cubic trees.

3.9 *k*-equitable Labelings

In 1990 Cahit [642] proposed the idea of distributing the vertex and edge labels among $\{0, 1, \ldots, k-1\}$ as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V, E)$ and any positive integer k, assign vertex labels from $\{0, 1, \ldots, k-1\}$ so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. Cahit has called a graph with such an assignment of labels *k-equitable*. Note that $G(V, E)$ is graceful if and only if it is $|E| + 1$-equitable and $G(V, E)$ is cordial if and only if it is 2-equitable. Cahit [641] has shown the following: C_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$; the triangular snake with n blocks is 3-equitable if and only if n is even; the friendship graph $C_3^{(n)}$ is 3-equitable if and only if n is even; an Eulerian graph with $q \equiv 3 \pmod 6$ edges is not 3-equitable; and all caterpillars are 3-equitable [641]. Cahit [641] claimed to prove that W_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$ but Youssef [3544] proved that W_n is 3-equitable for all $n \geqslant 4$. Youssef [3542] also proved that if G is a k-equitable Eulerian graph with q edges and $k \equiv 2$ or $3 \pmod 4$ then $q \not\equiv k \pmod{2k}$. Cahit conjectures [641] that a triangular cactus with n blocks is 3-equitable if and only if n is even. In [642] Cahit proves that every tree with fewer than five end vertices has a 3-equitable labeling. He conjectures that all trees are k-equitable [643]. In 1999 Speyer and Szaniszló [3039] proved Cahit's conjecture for $k = 3$. Coles, Huszar, Miller, and Szaniszlo [751] proved caterpillars, symmetric generalized n-stars (or symmetric spiders), and complete n-ary trees are 4-equitable. Vaidya and Shah [3303] proved that the splitting graphs of $K_{1,n}$ and the bistar $B_{n,n}$ and the shadow graph of $B_{n,n}$ are 3-equitable. Rokad [2643] found 3-equitable labelings of the ring sum of different graphs.

Vaidya, Dani, Kanani, and Vihol [3267] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of G_i to the center of G_{i+1} ($i = 1, 2, \ldots, n-1$) by an edge are 3-equitable. Recall the shell $C(n, n-3)$ is the cycle C_n with $n-3$ cords sharing a common endpoint called the *apex*. Vaidya, Dani, Kanani, and Vihol [3268] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed shell and joining each apex of G_i to the apex of G_{i+1} ($i = 1, 2, \ldots, n-1$) by an edge are 3-equitable. For a graph G and vertex v of G, Vaidya, Dani, Kanani, and Vihol

[3269] prove that the graphs obtained from the wheel W_n, $n \geqslant 5$, by duplicating (see 3.7 for the definition) any rim vertex is 3-equitable and the graphs obtained from the wheel W_n by duplicating the center is 3-equitable when n is even and not 3-equitable when n is odd and at least 5. They also show that the graphs obtained from the wheel W_n, $n \neq 5$, by duplicating every vertex is 3-equitable.

Vaidya, Srivastav, Kaneria, and Ghodasara [3314] prove that cycle with two chords that share a common vertex with opposite ends that are incident to two consecutive vertices of the cycle is 3-equitable. Vaidya, Ghodasara, Srivastav, and Kaneria [3273] prove that star of cycle ${C_n}^*$ is 3-equitable for all n. Vaidya and Dani [3263] proved that the graphs obtained by starting with n copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining the center of G_i to the center of G_{i+1} by an edge and each center to a new vertex x_i $(1 \leqslant i \leqslant n-1)$ by an edge have 3-equitable labeling. Vaidya and Dani [3266] prove that the graphs obtained by duplication of an arbitrary edge of a cycle or a wheel have 3-equitable labelings.

The *Mycielski graph* of a graph G is obtained from G by adding to each vertex v a new vertex u that is adjacent to the neighbors of v and a adding a new vertex w that is adjacent to every u. In [2707] Sangeeta, Parthiban, Selvaraju proved the non-existence of 3-equitable labelings for non-trivial cases of the total graphs of fans, the middle graph of ladders, the degree splitting graphs of friendship graphs, and the Mycielskian graphs of paths. [2707] new

Recall $G =< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ 1s the graph obtained by joining the center vertices of each of $W_n^{(i)}$ and $W_n^{(i+1)}$ to a new vertex x_i where $1 \leqslant i \leqslant k-1$. Vaidya, Dani, Kanani, and Vihol [3270] prove that $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ is 3-equitable. Vaidya and Vihol [3318] prove that any graph G can be embedded as an induced subgraph of a 3-equitable graph thereby ruling out any possibility of obtaining any forbidden subgraph characterization for 3-equitable graphs.

The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, G' and G'' and joining each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''. Vaidya, Vihol, and Barasara [433] prove that the shadow graph of C_n is 3-equitable except for $n = 3$ and 5 while the shadow graph of P_n is 3-equitable except for $n = 3$. They also prove that the middle graph of P_n is 3-equitable and the middle graph of C_n is 3-equitable for n even and not 3-equitable for n odd.

Bhut-Nayak and Telang have shown that crowns $C_n \odot K_1$, are k-equitable for $k = n, \ldots, 2n-1$ [564] and $C_n \odot K_1$ is k-equitable for all n when $k = 2, 3, 4, 5$, and 6 [565].

In [2759] Seoud and Abdel Maqsoud prove: a graph with n vertices and q edges in which every vertex has odd degree is not 3-equitable if $n \equiv 0 \pmod 3$ and $q \equiv 3 \pmod 6$; all fans except $P_2 + \overline{K_1}$ are 3-equitable; all double fans $P_n + \overline{K_2}$ except $P_4 + \overline{K_2}$ are 3-equitable; P_n^2 is 3-equitable for all n except 3; $K_{1,1,n}$ is 3-equitable if and only if $n \equiv 0$ or $2 \pmod 3$; $K_{1,2,n}$, $n \geqslant 2$, is 3-equitable if and only if $n \equiv 2 \pmod 3$; $K_{m,n}$, $3 \leqslant m \leqslant n$, is 3-equitable if and only if $(m,n) = (4,4)$; and $K_{1,m,n}$, $3 \leqslant m \leqslant n$, is 3-equitable if and only if $(m,n) = (3,4)$. They conjectured that C_n^2 is not 3-equitable for all $n \geqslant 3$. However, Youssef [3550] proved that C_n^2 is 3-equitable if and only if n is at least 8. Youssef [3550] also proved that $C_n + \overline{K_2}$ is 3-equitable if and only if n is even and at least 6 and determined the maximum number of edges in a 3-equitable graph as a function of the number of its vertices. For a graph with n vertices to admit a k-equitable labeling, Seoud and Salim [2788] proved that the number of edges is at most $k\lceil(n/k)\rfloor^2 + k - 1$.

Bapat and Limaye [422] have shown the following graphs are 3-equitable: helms H_n, $n \geqslant 4$; flowers (see §2.2 for the definition); the one-point union of any number of helms; the one-point union of any number of copies of K_4; K_4-snakes (see §2.2 for the definition); C_t-snakes where $t = 4$ or 6; C_5-snakes where the number of blocks is not congruent to 3 modulo 6. A *multiple shell* $\mathrm{MS}\{n_1^{t_1}, \ldots, n_r^{t_r}\}$ is a graph formed by t_i shells each of order n_i, $1 \leqslant i \leqslant r$, that have a common apex. Bapat and Limaye [423] show that every multiple shell is 3-equitable and Chitre and Limaye [706] show that every multiple shell is 5-equitable. In [707] Chitre and Limaye define the *H-union* of a family of graphs $G_1, G_2, \ldots, G_t$, each having a graph H as an induced subgraph, as the graph obtained by starting with $G_1 \cup G_2 \cup \cdots \cup G_t$ and identifying all the corresponding vertices and edges of H in each of $G_1, \ldots, G_t$. In [707] and [708] they proved that the $\overline{K_n}$-union of gears and helms H_n ($n \geqslant 6$) are edge-3-equitable.

Szaniszló [3180] has proved the following: P_n is k-equitable for all k; K_n is 2-equitable if and only if $n = 1, 2$, or 3; K_n is not k-equitable for $3 \leqslant k < n$; S_n is k-equitable for all k; $K_{2,n}$ is k-equitable if and only if $n \equiv k - 1 \pmod{k}$, or $n \equiv 0, 1, 2, \ldots, \lfloor k/2 \rfloor - 1 \pmod{k}$, or $n = \lfloor k/2 \rfloor$ and k is odd. She also proves that C_n is k-equitable if and only if k meets all of the following conditions: $n \neq k$; if $k \equiv 2, 3 \pmod{4}$, then $n \neq k - 1$ and $n \not\equiv k \pmod{2k}$. Coles, Huszar, Miller, and Szaniszló [751] proved that all caterpillars, symmetric generalized n-stars (or symmetric spiders), and complete n-ary trees for all are 4-equitable.

Vickrey [3360] has determined the k-equitability of complete multipartite graphs. He shows that for $m \geqslant 3$ and $k \geqslant 3$, $K_{m,n}$ is k-equitable if and only if $K_{m,n}$ is one of the following graphs: $K_{4,4}$ for $k = 3$; $K_{3,k-1}$ for all k; or $K_{m,n}$ for $k > mn$. He also shows that when k is less than or equal to the number of edges in the graph and at least 3, the only complete multipartite graphs that are k-equitable are $K_{kn+k-1,2,1}$ and $K_{kn+k-1,1,1}$. Partial results on the k-equitability of $K_{m,n}$ were obtained by Krussel [1767].

In [3557] Youssef and Al-Kuleab proved the following: C_n^3 is 3-equitable if and only if n is even and $n \geqslant 12$; gear graphs are k-equitable for $k = 3, 4, 5, 6$; ladders $P_n \times P_2$ are 3-equitable for all $n \geqslant 2$; $C_n \times P_2$ is 3-equitable if and only if $n \not\equiv \pmod{6}$; Möbius ladders M_n are 3-equitable if and only if $n \not\equiv (\mathrm{mod} 6)$; and the graphs obtained from $P_n \times P_2$ ($n \geqslant 2$) where by adding the edges $u_i v_{i+1}$ ($1 \leqslant i \leqslant n - 1$) to the path vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$.

In [1993] López, Muntaner-Batle, and Rius-Font prove that if n is an odd integer and F is optimal k-equitable for all proper divisors k of $|E(F)|$, then nF is optimal k-equitable for all proper divisors k of $|E(F)|$. They also prove that if $m - 1$ and n are odd, then then nC_m is optimal k-equitable for all proper divisors k of $|E(F)|$.

As a corollary of the result of Cairnie and Edwards [653] on the computational complexity of cordially labeling graphs it follows that the problem of finding k-equitable labelings of graphs is NP-complete as well.

Seoud and Abdel Maqsoud [2760] call a graph *k-balanced* if the vertices can be labeled from $\{0, 1, \ldots, k-1\}$ so that the number of edges labeled i and the number of edges labeled j induced by the absolute value of the differences of the vertex labels differ by at most 1. They prove that P_n^2 is 3-balanced if and only if $n = 2, 3, 4$, or 6; for $k \geqslant 4$, P_n^2 is not k-balanced if $k \leqslant n-2$ or $n+1 \leqslant k \leqslant 2n-3$; for $k \geqslant 4$, P_n^2 is k-balanced if $k \geqslant 2n-2$; for $k, m, n \geqslant 3$, $K_{m,n}$ is k-balanced if and only if $k \geqslant mn$; for $m \leqslant n$, $K_{1,m,n}$ is k-balanced if and only if (i) $m = 1$, $n = 1$ or 2, and $k = 3$; (ii) $m = 1$ and $k = n+1$ or $n+2$; or (iii) $k \geqslant (m+1)(n+1)$.

In [3550] Youssef gave some necessary conditions for a graph to be k-balanced and some relations between k-equitable labelings and k-balanced labelings. Among his results are: C_n is 3-balanced for all $n \geqslant 3$; K_n is 3-balanced if and only if $n \leqslant 3$; and all trees are 2-balanced and 3-balanced. He conjectures that all trees are k-balanced ($k \geqslant 2$).

Bloom has used the term k-equitable to describe another kind of labeling (see [3482] and [3483]). He calls a graph *k-equitable* if the edge labels induced by the absolute value of the difference of the vertex labels have the property that every edge label occurs exactly k times. Bloom calls a graph of order n *minimally k-equitable* if the vertex labels are 1, 2,…, n and it is k-equitable. Both Bloom and Wojciechowski [3482], [3483] proved that C_n is minimally k-equitable if and only if k is a proper divisor of n. Barrientos and Hevia [453] proved that if G is k-equitable of size $q = kw$ (in the sense of Bloom), then $\delta(G) \leqslant w$ and $\Delta(G) \leqslant 2w$. Barrientos, Dejter, and Hevia [452] have shown that forests of even size are 2-equitable. They also prove that for $k = 3$ or $k = 4$ a forest of size kw is k-equitable if and only if its maximum degree is at most $2w$ and that if 3 divides $mn+1$, then the double star $S_{m,n}$ is 3-equitable if and only if $q/3 \leqslant m \leqslant \lfloor (q-1)/2 \rfloor$. ($S_{m,n}$ is P_2 with m pendent edges attached at one end and n pendent edges attached at the other end.) They discuss the k-equitability of forests for $k \geqslant 5$ and characterize all caterpillars of diameter 2 that are k-equitable for all possible values of k. Acharya and Bhat-Nayak [50] have shown that coronas of the form $C_{2n} \odot K_1$ are minimally 4-equitable. In [434] Barrientos proves that the one-point union of a cycle and a path (dragon) and the disjoint union of a cycle and a path are k-equitable for all k that divide the size of the graph. Barrientos and Havia [453] have shown the following: $C_n \times K_2$ is 2-equitable when n is even; books B_n ($n \geqslant 3$) are 2-equitable when n is odd; the vertex union of k-equitable graphs is k-equitable; and wheels W_n are 2-equitable when $n \not\equiv 3 \pmod 4$. They conjecture that W_n is 2-equitable when $n \equiv 3 \pmod 4$ except when $n = 3$. Their 2-equitable labelings of $C_n \times K_2$ and the n-cube utilized graceful labelings of those graphs.

M. Acharya and Bhat-Nayak [51] have proved the following: the crowns $C_{2n} \odot K_1$ are minimally 2-equitable, minimally $2n$-equitable, minimally 4-equitable, and minimally n-equitable; the crowns $C_{3n} \odot K_1$ are minimally 3-equitable, minimally $3n$-equitable, minimally n-equitable, and minimally 6-equitable; the crowns $C_{5n} \odot K_1$ are minimally 5-equitable, minimally $5n$-equitable, minimally n-equitable, and minimally 10-equitable; the crowns $C_{2n+1} \odot K_1$ are minimally $(2n+1)$-equitable; and the graphs P_{kn+1} are k-equitable.

In [436] Barrientos calls a k-equitable labeling *optimal* if the vertex labels are consecutive integers and *complete* if the induced edge labels are $1, 2, \ldots, w$ where w is the number of distinct edge labels. Note that a graceful labeling is a complete 1-equitable labeling. Barrientos proves that $C_m \odot nK_1$ (that is, an m-cycle with n pendent edges attached at each vertex) is optimal 2-equitable when m is even; $C_3 \odot nK_1$ is complete 2-equitable when n is odd; and that $C_3 \odot nK_1$ is complete 3-equitable for all n. He also shows that $C_n \odot K_1$ is k-equitable for every proper divisor k of the size $2n$. Barrientos and Havia [453] have shown that the n-cube ($n \geqslant 2$) has a complete 2-equitable labeling and that $K_{m,n}$ has a complete 2-equitable labeling when m or n is even. They conjecture that every tree of even size has an optimal 2-equitable labeling.

3.10 Hamming-graceful Labelings

Mollard, Payan, and Shixin [2184] introduced a generalization of graceful graphs called Hamming-graceful. A graph $G = (V, E)$ is called *Hamming-graceful* if there exists an injective labeling g from V to the set of binary $|E|$-tuples such that $\{d(g(v), g(u))|\ uv \in E\} = \{1, 2, \dots, |E|\}$ where d is the Hamming distance. Shixin and Yu [2939] have shown that all graceful graphs are Hamming-graceful; all trees are Hamming-graceful; C_n is Hamming-graceful if and only if $n \equiv 0$ or 3 (mod 4); if K_n is Hamming-graceful, then n has the form k^2 or $k^2 + 2$; and K_n is Hamming-graceful for $n = 2, 3, 4, 6, 9, 11, 16,$ and 18. They conjecture that K_n is Hamming-graceful for n of the forms k^2 and $k^2 + 2$ for $k \geqslant 5$.

4 Variations of Harmonious Labelings

4.1 Sequential and Strongly c-harmonious Labelings

Chang, Hsu, and Rogers [674] and Grace [1143], [1144] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling f of a graph G with q vertices to be *strongly c-harmonious* if the vertex labels are from $\{0, 1, \ldots, q-1\}$ and the edge labels induced by $f(x) + f(y)$ for each edge xy are $c, \ldots, c+q-1$. Strongly 1-harmoinious labelings are more simply called *strongly harmonious*. Grace called such a labeling *sequential*. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices whereas Grace allows the vertex labels to range from 0 to q with no vertex label being used twice. For graphs other than trees, we use the term c-sequential labelings interchangeably with strongly c-harmonious labelings. By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [1144] proved that caterpillars, caterpillars with a pendent edge, odd cycles with zero or more pendent edges, trees with α-labelings, wheels W_{2n+1}, and P_n^2 are sequential. Liu and Zhang [1963] finished off the crowns $C_{2n} \odot K_1$. (The case $C_{2n+1} \odot K_1$ was a special case of Grace's results. Liu [1976] proved crowns are harmonious.)

Bača and Youssef [390] investigated the existence of harmonious labelings for the corona graphs of a cycle and a graph G. They proved that if $G+K_1$ is strongly harmonious with the 0 label on the vertex of K_1, then $C_n \odot G$ is harmonious for all odd $n \geqslant 3$. By combining this with existing results they have as corollaries that the following graphs are harmonious: $C_n \odot C_m$ for odd $n \geqslant 3$ and $m \not\equiv 2 \pmod 3$; $C_n \odot K_{s,t}$ for odd $n \geqslant 3$; and $C_n \odot K_{1,s,t}$ for odd $n \geqslant 3$.

Bu [618] also proved that crowns are sequential as are all even cycles with m pendent edges attached at each vertex. Figueroa-Centeno, Ichishima, and Muntaner-Batle [934] proved that all cycles with m pendent edges attached at each vertex are sequential. Wu [3488] has shown that caterpillars with m pendent edges attached at each vertex are sequential. exactly one path of fixed length to each vertex of some path is sequential.

Singh has proved the following: $C_n \odot K_2$ is sequential for all odd $n > 1$ [2972]; $C_n \odot P_3$ is sequential for all odd n [2973]; $K_2 \odot C_n$ (each vertex of the cycle is joined by edges to the end points of a copy of K_2) is sequential for all odd n [2973]; helms H_n are sequential when n is even [2973]; and $K_{1,n} + K_2$, $K_{1,n} + \overline{K}_2$, and ladders are sequential [2975]. Santhosh [2710] has shown that $C_n \odot P_4$ is sequential for all odd $n \geqslant 3$. Both Grace [1143] and Reid (see [1006]) have found sequential labelings for the books B_{2n}. Jungreis and Reid [1561] have shown the following graphs are sequential: $P_m \times P_n$ $(m, n) \neq (2, 2)$; $C_{4m} \times P_n$ $(m, n) \neq (1, 2)$; $C_{4m+2} \times P_{2n}$; $C_{2m+1} \times P_n$; and $C_4 \times C_{2n}$ $(n > 1)$. The graphs $C_{4m+2} \times C_{2n+1}$ and $C_{2m+1} \times C_{2n+1}$ fail to satisfy a necessary parity condition given by Graham and Sloane [1147]. The remaining cases of $C_m \times P_n$ and $C_m \times C_n$ are open. Gallian, Prout, and Winters [1007] proved that all graphs $C_n \times P_2$ with a vertex or an edge deleted are sequential. Zhu and Liu [3602] give necessary and sufficient conditions for sequential graphs, provide a characterization of non-tree sequential graphs by way of by vertex closure, and obtain characterizations of sequential trees.

Gnanajothi [1104] [pp. 68-78] has shown the following graphs are sequential: $K_{1,m,n}$; mC_n, the disjoint union of m copies of C_n if and only if m and n are odd; books with triangular pages or pentagonal pages; and books of the form B_{4n+1}, thereby answering a question and proving a conjecture of Gallian and Jungreis [1006]. Sun [3127] has also proved that B_n is sequential if and only if $n \not\equiv 3 \pmod 4$. Ichishima and Oshima [1313] pose determining whether or not $mK_{s,t}$ is sequential as a problem.

Yuan and Zhu [3565] have shown that mC_n is sequential when m and n are odd. Although Graham and Sloane [1147] proved that the Möbius ladder M_3 is not harmonious, Gallian [1001] established that all other Möbius ladders are sequential (see §2.3 for the definition of Möbius ladder). Chung, Hsu, and Rogers [674] have shown that $K_{m,n} + K_1$, which includes $S_m + K_1$, is sequential. Seoud and Youssef [2798] proved that if G is sequential and has the same number of edges as vertices, then $G + \overline{K_n}$ is sequential for all n. Recall that $\Theta(C_m)^n$ denotes the book with n m-polygonal pages. Lu [2039] proved that $\Theta(C_{2m+1})^{2n}$ is $2mn$-sequential for all n and $m = 1, 2, 3, 4$, and $\Theta(C_m)^2$ is $(m-2)$-sequential if $m \geqslant 3$ and $m \equiv 2, 3, 4, 7 \pmod 8$.

Zhou and Yuan [3599] have shown that for every c-sequential graph G with p vertices and q edges and any positive integer m the graph $(G + \overline{K_m}) + \overline{K_n}$ is also c-sequential when $q - p + 1 \leqslant m \leqslant q - p + c$. Zhou [3598] has shown that the analogous results hold for strongly c-harmonious graphs. Zhou and Yuan [3599] have shown that for every c-sequential graph G with p vertices and q edges and any positive integer m the graph $(G + \overline{K_m}) + \overline{K_n}$ is c-sequential when $q - p + 1 \leqslant m \leqslant q - p + c$.

Shee [1881] proved that every graph is a subgraph of a sequential graph. Acharya, Germina, Princy, and Rao [40] prove that every connected graph can be embedded in a strongly c-harmonious graph for some c. Miao and Liang [2135] use $C_n(d; i, j; P_k)$ to denote a cycle C_n with path P_k joining two nonconsecutive vertices x_i and x_j of the cycle, where d is the distance between x_i and x_j on C_n. They proved that the graph $C_n(d; i, j; P_k)$ is strongly c-harmonious when $k = 2, 3$ and integer $n \geqslant 6$. Lu [2038] provides three techniques for constructing larger sequential graphs from some smaller one: an attaching construction, an adjoining construction, and the join of two graphs. Using these, he obtains various families of sequential or strongly c-indexable graphs.

For $1 \leqslant s \leqslant n_3$, let $C_n(i : i_1, i_2, \ldots, i_s)$ denote an n-cycle with consecutive vertices $x_1, x_2, \ldots, x_n$ to which the s chords $x_i x_{i_1}, x_i x_{i_2}, \ldots, x_i x_{i_s}$ have been added. Liang [1938] proved a variety of graphs of the form $C_n(i : i_1, i_2, \ldots, i_s)$ are strongly c-harmonious.

Youssef [3547] observed that a strongly c-harmonious graph with q edges is c-cordial for all $c \geqslant q$ and a strongly k-indexable graph is k-cordial for every k. The converse of this latter result is not true.

In [1310] Ichishima and Oshima show that the hypercube Q_n ($n \geqslant 2$) is sequential if and only if $n \geqslant 4$. They also introduce a special kind of sequential labeling of a graph G with size $2t + s$ by defining a sequential labeling f to be a *partitional labeling* if G is bipartite with partite sets X and Y of the same cardinality s such that $f(x) \leqslant t + s - 1$ for all $x \in X$ and $f(y) \geqslant t - s$ for all $y \in Y$, and there is a positive integer m such that the induced edge labels are partitioned into three sets $[m, m+t-1], [m+t, m+t+s-1]$, and $[m+t+s, m+2t+s-1]$ with the properties that there is an involution π, which is an automorphism of G such that π exchanges X and Y, $x\pi(x) \in E(G)$ for all $x \in X$, and $\{f(x)+f(\pi(x))|\ x \in X\} = [m+t, m+t+s-1]$. They prove if G has a partitional labeling, then $G \times Q_n$ has a partitional labeling for every nonnegative integer n. Using this together with existing results and the fact that every graph that has a partitional labeling is sequential, harmonious, and felicitous (see §4.5) they show that the following graphs are partitional, sequential, harmonious, and felicitous: for $n \geqslant 4$, hypercubes Q_n; generalized books $S_{2m} \times Q_n$; and generalized ladders $P_{2m+1} \times Q_n$.

In [1311] Ichishma and Oshima proved the following: if G is a partitional graph, then $G \times K_2$ is partitional, sequential, harmonious and felicitous; if G is a connected bipartite graph with partite sets of distinct odd order such that in each partite set each vertex has the same degree, then $G \times K_2$ is not partitional; for every positive integer m, the book B_m is partitional if and only if m is even; the graph $B_{2m} \times Q_n$ is partitional if and only if $(m, n) \neq (1, 1)$; the graph $K_{m,2} \times Q_n$ is partitional if and only if $(m, n) \neq (2, 1)$; for every positive integer n, the graph $K_{m,3} \times Q_n$ is partitional when $m = 4, 8, 12$, or 16. As open problems they ask which m and n is $K_{m,n} \times K_2$ partitional and for which l, m and n is $K_{l,m} \times Q_n$ partitional?

Ichishma and Oshima [1311] also investigated the relationship between partitional graphs and strongly graceful graphs (see §3.1 for the definition) and partitional graphs and strongly felicitous graphs (see §4.5) for the definition). They proved the following. If G is a partitional graph, then $G \times K_2$ is partitional, sequential, harmonious and felicitous. Assume that G is a partitional graph of size $2t+s$ with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that $\lambda_1 = \max\{f(x) : x \in X\}$ and $\lambda_2 = \max\{f(y) : y \in Y\}$. If $\lambda_1 + 1 = m + 2t + s - \lambda_2$, where $m = \min\{f(x) + f(y) : xy \in E(G)\} = \min\{f(y) : y \in Y\}$, then G has a strong α-valuation. Assume that G is a partitional graph of size $2t+s$ with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that $\lambda_1 = \max\{f(x) : x \in X\}$ and $\lambda_2 = \max\{f(y) : y \in Y\}$. If $\lambda_1 + 1 = m + 2t + s - \lambda_2$, where $m = \min\{f(x) + f(y) : xy \in E(G)\} = \min\{f(y) : y \in Y\}$, then G is strongly felicitous. Assume that G is a partitional graph of size $2t+s$ with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that $\mu_1 = f(x_1) = \min\{f(x) : x \in X\}$ and $\mu_2 = f(y_1) = \min\{f(y) : y \in Y\}$. If $t + s = m + 1$ and $\mu_1 + \mu_2 = m$, where $m = \min\{f(x) + f(y) : xy \in E(G)\}$ and $x_1y_1 \in E(G)$, then G has a strong α-valuation and strongly felicitous labeling.

Vaidya and Lekha [3292] proved the following graphs are odd sequential: P_n, C_n for $n \equiv 0 \pmod 4$, crowns $C_n \bigodot K_1$ for even n, the graph obtained by duplication of arbitrary vertex in even cycles, path unions of stars, arbitrary super subdivisions in P_n, and shadows of stars. They also introduced the concept of a *bi-odd sequential* labeling of a graph G as one for which both G and its line graph $L(G)$ admit odd sequential labeling. They proved P_n and C_n for $n \equiv \pmod 4$ are bi-odd sequential graphs and trees are bi-odd sequential if and only if they are paths. They also prove that P_4 is the only graph with the property that it and its complement are odd sequential.

Arockiaraj, Mahalakshmi, and Namasivayam [230] proved that the subdivision graphs of the following graphs have odd sequential labelings (they call them *odd sum* labelings): triangular snakes; quadrilateral snakes; slanting ladders SL_n $(n > 1)$ (the graphs obtained from two paths $u_1u_2\dots u_n$ and $v_1v_2\dots v_n$ by joining each u_i with v_{i+1}); $C_p \odot K_1$, $H_n \odot K_1$, $C_m@C_n$ (the graph obtained by attaching paths P_n to C_m by identifying the endpoints of the paths with each successive pairs of vertices of C_m); $P_m \times P_n$; and graphs obtained by the duplication of a vertex of a path and the duplication of a vertex of a cycle. Arockiaraj, Mahalakshmi, and Namasivayam [232] investigate the odd sum labeling behavior of paths, combs, cycles, crowns, and ladders under duplication of an edge. In [233] they investigated the odd sum property of shadow graphs, edge duplication graphs and vertex identification graphs. In [1132] Gopi proved the following graphs are odd sum graphs: graphs H_n obtained from two copies of P_n $(n \geqslant 3)$ with vertices $v_1, v_2, \dots, v_n$ and $u_1, u_2, \dots, u_n$ by joining $v_{(n+1)/2}$ and $u_{(n+1)/2}$ if n is odd and $v_{n/2}$ and $u_{(n+2)/2}$ if n is even; graphs obtained from H_n by attaching a fixed number of pendent edges at each vertex, graphs obtained from P_n $(n \geqslant 4)$ by attaching a two pendent edges at each interior vertex; and graphs obtained from P_m $(m \geqslant 4)$ by identifying an endpoint of the star S_n $(n \geqslant 2)$ with each vertex of P_m. In [1136] Gopi and Irudaya Mary proved that slanting ladders, shadow graphs of stars and bistars and mirror graphs and duplicate vertex graphs of paths with at least four verticies are odd sum graphs. In [1131] Gopi proved that alternative quadrilateral snakes $A(D(Q_n))$ $(n \geqslant 4)$ are odd sum graphs.

Arockiaraj and Mahalakshmi [229] proved the following graphs have odd sequential labelings (odd sum labelings): P_n $(n > 1)$, C_n if and only if $n \equiv 0 \pmod 4$; $C_{2n} \odot K_1$; $P_n \times P_2$ $(n > 1)$; $P_m \odot K_1$ if m is even or m is odd and $n = 1$ or 2; the balloon graph $P_m(C_n)$ obtained by identifying an end point of P_m with a vertex of C_n if either $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$ and $m \not\equiv 1 \pmod 3$; quadrilateral snakes Q_n; $P_m \odot C_n$ if $m > 1$ and $n \equiv 0 \pmod 4$; $P_m \odot Q_3$; bistars; $C_{2n} \times P_2$; the trees T_p^n obtained from n copies of T_p by joining an edge uu' between every pair of consecutive paths where u is a vertex in ith copy of the path and u' is the corresponding vertex in the $(i+1)$th copy of the path; H_n-*graphs* obtained by starting with two copies of P_n with vertices $v_1, v_2, \dots, v_n$ and $u_1, u_2, \dots, u_n$ and joining the vertices $v_{(n+1)/2}$ and $u_{(n+1)/2}$ if n is odd and the vertices $v_{n/2+1}$ and $u_{n/2}$ if n; and $H_n \odot mK_1$.

Arockiaraj and Mahalakshmi [231] proved the splitting graphs of following graphs have odd sequential labelings (odd sum labelings): P_n; C_n if and only if $n \equiv 0 \pmod 4$; $P_n \odot K_1$; $C_{2n} \odot K_1$; $K_{1,n}$ if and only if $n \leqslant 2$; $P_n \times P_2$ $(n > 1)$; slanting ladders SL_n $(n > 1)$; the quadrilateral snake Q_n; and H_n-graphs.

Among the strongly 1-harmonious (also called *strongly harmonious*) graphs are: fans F_n with $n \geqslant 2$ [674]; wheels W_n with $n \not\equiv 2 \pmod 3$ [674]; $K_{m,n} + K_1$ [674]; French windmills $K_4^{(t)}$ [1263], [1634]; the friendship graphs $C_3^{(n)}$ if and only if $n \equiv 0$ or $1 \pmod 4$ [1263], [1634], [3509]; $C_{4k}^{(t)}$ [3128]; and helms [2555].

Seoud, Diab, and Elsakhawi [2769] have shown that the following graphs are strongly harmonious: $K_{m,n}$ with an edge joining two vertices in the same partite set; $K_{1,m,n}$; the composition $P_n[P_2]$ (see §2.3 for the definition); $B(3, 2, m)$ and $B(4, 3, m)$ for all m (see §2.4 for the notation); P_n^2 $(n \geqslant 3)$; and P_n^3 $(n \geqslant 3)$. Seoud et al. [2769] have also proved: B_{2n} is strongly $2n$-harmonious; P_n is strongly $\lfloor n/2 \rfloor$-harmonious; ladders L_{2k+1} are strongly $(k+1)$-harmonious; and that if G is strongly c-harmonious and has an equal number of vertices and edges, then $G + \overline{K_n}$ is also strongly c-harmonious.

Bača and Youssef [390] investigated the existence of harmonious labelings for the corona graphs of a cycle and a graph G, and for the corona graph of K_2 and a tree. They prove: if join of a graph G of order p and K_1, $G + K_1$, is strongly harmonious with the 0 label on the vertex of K_1, then the corona of C_n with G, $C_n \odot G$, is harmonious for all odd $n \geqslant 3$; if T is a strongly c-harmonious tree of odd size q and $c = \frac{q+1}{2}$ then the corona of K_2 with T, $K_2 \odot T$, is also strongly c-harmonious; if a unicyclic graph G of odd size q is a strongly c-harmonious and $c = \frac{q-1}{2}$ then the corona of K_2 with G, $K_2 \odot G$, is also strongly c-harmonious.

Seenivasan and Lourdusamy [2734] define an *absolutely harmonious labeling* f as an injection from the vertex set of a graph G with q edges to the set $\{0, 1, 2, \ldots, q-1\}$, if when each edge uv is assigned $f(u) + f(v)$, the resulting edge labels can be arranged as $a_0, a_1, a_2, \ldots, a_{q-1}$ where $a_i = q - i$ or $q + i$ for $0 \leqslant i \leqslant q - 1$. When G is a tree one of the vertex labels may be assigned to exactly two vertices. A graph that admits absolutely harmonious labeling is called an *absolutely harmonious* graph. Observe that a strongly harmonious graph is an absolutely harmonious graph. They prove the following graphs are absolutely harmnious: P_n $(n \geqslant 3)$, $P_n \odot \overline{K_m}$, $C_n \odot \overline{K_m}$, the banana tree obtained by joining a vertex of degree 1 of each of any number of copies of $K_{1,n}$ to an isolated vertex, ladders, triangular snakes, quadrilateral snakes, mK_4, K_n if and only if $n = 3$ or 4. They also prove that if G is an absolutely harmonious graph, then there exists a partition (V_1, V_2) of the vertex set $V(G)$, such that the number of edges connecting the vertices of V_1 to the vertices of V_2 is exactly $\lceil q/2 \rceil$ snd that if every vertex of an absolutely harmonious graph with q edges is even then $q \equiv 1$ or 2. As corollaries of the latter condition, they have that C_n when $n \equiv 1$ or 2 (mod 4), $C_m \times C_n$ when m and n are odd, and mK_3, $m \geqslant 2$ are not absolutely harmonious.

Sethuraman and Selvaraju [2839] have proved that the graph obtained by joining two complete bipartite graphs at one edge is graceful and strongly harmonious. They ask whether these results extend to any number of complete bipartite graphs.

For a graph $G(V, E)$ Gayathri and Hemalatha [1060] define an *even sequential harmonious labeling* f of G as an injection from V to $\{0, 1, 2, \ldots, 2|E|\}$ with the property that the induced mapping f^+ from E to $\{2, 4, 6, \ldots, 2|E|\}$ defined by $f^+(uv) = f(u) + f(v)$ when $f(u)+f(v)$ is even, and $f^+(uv) = f(u)+f(v)+1$ when $f(u)+f(v)$ is odd, is an injection. They prove the following have even sequential harmonious labelings (all cases are the nontrivial ones): $P_n, P_n^+, C_n(\ n \geqslant 3)$, triangular snakes, quadrilateral snakes, Möbius ladders, $P_m \times P_n$ $(m \geqslant 2, n \geqslant 2)$, $K_{m,n}$; crowns $C_m \odot K_1$, graphs obtained by joining the centers of two copies of $K_{1,n}$ by a path; banana trees (see §2.1), P_n^2, closed helms (see §2.2), $C_3 \odot nK_1$ $(n \geqslant 2)$; $D \odot K_{1,n}$ where D is a dragon (see §2.2); $\langle K_{1,n} : m \rangle$ $(m, n \geqslant 2)$ (see §4.5); the wreath product $P_n * \overline{K_2}$ $(n \geqslant 2)$ (see §4.5); combs $P_n \odot K_1$; the one-point union of the end point of a path to a vertex of a cycle (tadpole); the one-point union of the end point of a tadpole and the center of a star; the graphs PC_n obtained from $C_n = v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the cords $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}, v_{(n+2)/2}$ when n

is even and $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-3)/2}, v_{(n+3)/2}$ when n is odd (that is, cycles with a full set of cords); $P_m \odot nK_1$; the one-point union of a vertex of a cycle and the center of a star; graphs obtained by joining the centers of two stars with an edge; graphs obtained by joining two disjoint cycles with an edge (dumbbells); graphs consisting of two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex (butterflies).

In [1296] Ichishima, Muntaner-Batle, and Oshima define the *harmonious number*, $\eta(G)$, of a graph G with q edges as the smallest positive integer n for which there exists an injective function f from $V(G)$ to Z_{n+1} such that each uv of G is labeled $f(u)+f(v)$ (mod q) and the resulting edge labels are distinct, or $+\infty$ if there exists no such integer n. If such functions exist, they are called *harmonious numberings*. The *strong harmonious number*, $\eta_s(G)$, of a graph G is defined to be either the smallest positive integer n such that $n = \eta(G)$ with the additional property that there exists an integer λ such that $\min\{f(u), f(v)\} \leqslant \lambda \leqslant \max\{f(u), f(v)\}$ for each edge in G or $+\infty$ if there exists no such integer n. They provide a necessary condition for a graph to have a finite harmonious number and sufficient conditions for a graph to have an infinite (strong) harmonious number. In addition, they examine the relations between harmonious numbers, gamma-numbers, alpha-numbers, and super edge magic deficiencies (see §5.2). They determine the formulas for the (strong) harmonious numbers of some 2-regular graphs and all complete bipartite graphs.

In her PhD thesis [2235] (see also [1061]) Muthuramakrishnan defined a labeling f of a graph $G(V,E)$ to be *k-even sequential harmonious* if f is an injection from V to $\{k-1, k, k+1, \ldots, k+2q-1\}$ such that the induced mapping f^+ from E to $\{2k, 2k+2, 2k+4, \ldots, 2k+2q-2\}$ defined by $f^+(uv) = f(u)+f(v)$ if $f(u)+f(v)$ is even and $f^+(uv) = f(u)+f(v)+1$ if $f(u)+f(v)$ is odd are distinct. A graph G is called a *k-even sequential harmonious* graph if it admits a k-even sequential harmonious labeling. Among the numerous graphs that she proved to be k-even sequential harmonious are: paths, cycles, $K_{m,n}$, P_n^2 ($n \geqslant 3$), crowns $C_m \odot K_1$, $C_m@P_n$ (the graph obtained by identifying an endpoint of P_n with one vertex of C_m), double triangular snakes, double quadrilateral snakes, bistars, grids $P_m \times P_n$ ($m, n \geqslant 2$), $P_n[P_2]$, $C_3 \odot nK_1$ ($n \geqslant 2$), flags Fl_m (the cycle C_m with one pendent edge), dumbbell graphs (two disjoint cycles joined by an edge) butterfly graphs B_n (two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex), $K_2 + nK_1$, $\overline{K_n} + 2K_2$, banana trees, sparklers $P_m@K_{1,n}$ ($m, n \geqslant 2$), (graphs obtained by identifying an endpoint of P_m with the center of a star), twigs (graphs obtained from P_n ($n \geqslant 3$) by attaching exactly two pendent edges at each internal vertex of P_n), festoon graphs $P_m \odot nK_1$ ($m \geqslant 2$), the graphs $T_{m,n,t}$ obtained from a path P_t by appending m edges at one endpoint of P_t and n edges at the other endpoint of P_t, $L_n \odot K_1$ (L_n is the ladder $P_n \times P_2$), shadow graphs of paths, stars and bistars, and split graphs of paths and stars. Muthuramakrishnan also defines k-odd sequential harmonious labeling of graphs in the natural way and obtains a handful of results.

In [521] Beatress and Sarasija introduced a new harmonious-like labeling as follows. A graph $G(V,E)$ with n vertices and m edges is said to be a *square harmonius* graph if there exists an injection f from $V(G)$ to $\{1, 2, \ldots, m^2+1\}$ such that the induced mapping f^* from $E(G)$ to $\{1, 4, 9, \ldots, m^2\}$ defined by $f^*(uv) = (f(u) + f(v)) \bmod (m^2 + 1)$ is a bijection. Such a function f is called a *square harmonious labeling* of G. They prove that P_n $(n \geqslant 3), K_{1,n}$ $(n \geqslant 2)$, bistars, combs, $P_n \odot pK_1$ $(n \geqslant 2)$, and $C_3@pK_1$ $(p \geqslant 2)$ are square harmonious graphs. Lawas and Lim [1822] proved that stars have a square harmonious labelings.

In [475] Barrientos and Youssef generalize the concepts of harmonious and (k, d)-arithmetic graphs by relaxing the injectivity constraint of the corresponding labelings. They call these labelings *semi harmonious* and *semi (k, d)-arithmetic*. They showed the existence of a semi harmonious labeling for several types of cycle-related graphs and characterized the cycles that admit semi harmonius labelings. They proved that if G is semi (k, d)-arithmetic, then it is also semi (rk, rd)-arithmetic for every $r \geqslant 1$ and that nG is both semi (k, d)-arithmetic and semi $(k + d, d)$-arithmetic. They further showed that any graph whose components are complete bipartite graphs is semi (k, d)-arithmetic for any ordered pair (k, d) of positive integers, and if G_i is a semi (k_i, d)-arithmetic graph of size q_i for each $i = 1, 2$, then $G_1 \cup G_2$ is semi (k_i, d)-arithmetic. [475] new

4.2 (k, d)-arithmetic Labelings

Acharya and Hegde [44] have generalized sequential labelings as follows. Let G be a graph with q edges and let k and d be positive integers. A labeling f of G is said to be (k, d)-*arithmetic* if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x) + f(y)$ for each edge xy are $k, k + d, k + 2d, \ldots, k + (q - 1)d$. They obtained a number of necessary conditions for various kinds of graphs to have a (k, d)-arithmetic labeling. The case where $k = 1$ and $d = 1$ was called *additively graceful* by Hegde [1209]. Hegde [1209] showed: K_n is additively graceful if and only if $n = 2, 3$, or 4; every additively graceful graph except K_2 or $K_{1,2}$ contains a triangle; and a unicyclic graph is additively graceful if and only if it is a 3-cycle or a 3-cycle with a single pendent edge attached. Jinnah and Singh [1549] noted that P_n^2 is additively graceful. Hegde [1210] proved that if G is strongly k-indexable, then G and $G + \overline{K_n}$ are (kd, d)-arithmetic. Acharya and Hegde [46] proved that K_n is (k, d)-arithmetic if and only if $n \geqslant 5$ (see also [624]). They also proved that a graph with an α-labeling is a (k, d)-arithmetic for all k and d. Bu and Shi [624] proved that $K_{m,n}$ is (k, d)-arithmetic when k is not of the form id for $1 \leqslant i \leqslant n - 1$. For all $d \geqslant 1$ and all $r \geqslant 0$, Acharya and Hegde [44] showed the following: $K_{m,n,1}$ is $(d + 2r, d)$-arithmetic; C_{4t+1} is $(2dt + 2r, d)$-arithmetic; C_{4t+2} is not (k, d)-arithmetic for any values of k and d; C_{4t+3} is $((2t + 1)d + 2r, d)$-arithmetic; W_{4t+2} is $(2dt + 2r, d)$-arithmetic; and W_{4t} is $((2t + 1)d + 2r, d)$-arithmetic. They conjecture that C_{4t+1} is $(2dt + 2r, d)$-arithmetic for some r and that C_{4t+3} is $(2dt + d + 2r, d)$-arithmetic for some r. Hegde and Shetty [1229] proved the following: the generalized web $W(t, n)$ (see §2.2 for the definition) is $((n - 1)d/2, d)$-arithmetic and $((3n - 1)d/2, d)$-arithmetic for odd n; the join of the generalized web $W(t, n)$ with the center removed and $\overline{K_p}$ where n is odd is $((n - 1)d/2, d)$-arithmetic; every T_p-tree (see §3.2 for the definition) with q edges and every tree obtained by subdividing every edge of a T_p-tree exactly once is $(k + (q - 1)d, d)$-arithmetic for all k and d. Lu, Pan, and Li [2041] proved that $K_{1,m} \cup K_{p,q}$

is (k, d)-arithmetic when $k > (q-1)d + 1$ and $d > 1$.

Yu [3560] proved that a necessary condition for C_{4t+1} to be (k, d)-arithmetic is that $k = 2dt + r$ for some $r \geqslant 0$ and a necessary condition for C_{4t+3} to be (k, d)-arithmetic is that $k = (2t + 1)d + 2r$ for some $r \geqslant 0$. These conditions were conjectured by Acharya and Hegde [44]. Singh proved that the graph obtained by subdividing every edge of the ladder L_n is $(5, 2)$-arithmetic [2971] and that the ladder L_n is $(n, 1)$-arithmetic [2974]. He also proves that $P_m \times C_n$ is $((n-1)/2, 1)$-arithmetic when n is odd [2974]. Acharya, Germina, and Anandavally [38] proved that the subdivision graph of the ladder L_n is (k, d)-arithmetic if either d does not divide k or $k = rd$ for some $r \geqslant 2n$ and that $P_m \times P_n$ and the subdivision graph of the ladder L_n are (k, k)-arithmetic if and only if k is at least 3. Lu, Pan, and Li [2041] proved that $S_m \cup K_{p,q}$ is (k, d)-arithmetic when $k > (q-1)d+1$ and $d > 1$.

A graph is called *arithmetic* if it is (k, d)-arithmetic for some k and d. Singh and Vilfred [2979] showed that various classes of trees are arithmetic. Singh [2974] has proved that the union of an arithmetic graph and an arithmetic bipartite graph is arithmetic. He conjectures that the union of arithmetic graphs is arithmetic. He provides an example to show that the converse is not true.

Germina and Anandavally [1072] investigated embedding of graphs in arithmetic graphs. They proved: every graph can be embedded as an induced subgraph of an arithmetic graph; every bipartite graph can be embedded in a (k, d)-arithmetic graph for all k and d such that d does not divide k; and any graph containing an odd cycle cannot be embedded as an induced subgraph of a connected (k, d)-arithmetic with $k < d$.

In [3531] Yao, Liu, and Yao give necessary and sufficient conditions for a tree to have the following mutually equivalent labelings: set-ordered odd-graceful, (k, d)-graceful, super edge-magic total, odd-elegant (see §4.4), harmonious, (k, d)-arithmetic, and edge-antimagic (see §6.1).

4.3 (k, d)-indexable Labelings

Acharya and Hegde [44] call a graph with p vertices and q edges (k, d)-*indexable* if there is an injective function from V to $\{0, 1, 2, \ldots, p-1\}$ such that the set of edge labels induced by adding the vertex labels is a subset of $\{k, k+d, k+2d, \ldots, k+q(d-1)\}$. When the set of edges is $\{k, k+d, k+2d, \ldots, k+q(d-1)\}$ the graph is said to be *strongly* (k, d)-*indexable*. A $(k, 1)$-graph is more simply called k-*indexable* and strongly 1-indexable graphs are simply called *strongly indexable*. Notice that strongly indexable graphs are a stronger form of sequential graphs and for trees and unicyclic graphs the notions of sequential labelings and strongly k-indexable labelings coincide. Hegde and Shetty [1234] have shown that the notions of $(1, 1)$-strongly indexable graphs and super edge-magic total labelings (see §5.2) are equivalent.

Zhou [3598] has shown that for every k-indexable graph G with p vertices and q edges the graph $(G + \overline{K_{q-p+k}}) + \overline{K_1}$ is strongly k-indexable. Acharaya and Hegde prove that the only nontrivial regular graphs that are strongly indexable are K_2, K_3, and $K_2 \times K_3$, and that every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. Acharya and Hegde [44] call a graph with p vertices *indexable* if there is an injective labeling of the vertices with labels from $\{0, 1, 2, \ldots, p-1\}$ such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [242] who also proved that all trees are indexable. Bu and Shi [625] also proved that all trees are indexable and that all unicyclic graphs with the cycle C_3 are indexable. Hegde [1210] has shown the following: every graph can be embedded as an induced subgraph of an indexable graph; if a connected graph with p vertices and q edges ($q \geqslant 2$) is (k, d)-indexable, then $d \leqslant 2$; $P_m \times P_n$ is indexable for all m and n; if G is a connected $(1, 2)$-indexable graph, then G is a tree; the minimum degree of any $(k, 1)$-indexable graph with at least two vertices is at most 3; a caterpillar with partite sets of orders a and b is strongly $(1, 2)$-indexable if and only if $|a - b| \leqslant 1$; in a connected strongly k-indexable graph with p vertices and q edges, $k \leqslant p-1$; and if a graph with p vertices and q edges is (k, d)-indexable, then $q \leqslant (2p - 3 - k + d)/d$. As a corollary of the latter, it follows that K_n ($n \geqslant 4$) and wheels are not (k, d)-indexable.

Lee and Lee [1825] provide a way to construct a (k, d)-strongly indexable graph from two given (k, d)-strongly indexable graphs. Lee and Lo [1856] show that every given (1,2)-strongly indexable spider can extend to an (1,2)-strongly indexable spider with arbitrarily many legs.

Seoud, Abd El Hamid, and Abo Shady [2757] proved the following graphs are indexable: $P_m \times P_n$ ($m, n \geqslant 2$); the graphs obtained from $P_n + K_1$ by inserting one vertex between every two consecutive vertices of P_n; the one-point union of any number of copies of $K_{2,n}$; and the graphs obtained by identifying a vertex of a cycle with the center of a star. They showed P_n is strongly $\lceil n/2 \rceil$-indexable; odd cycles C_n are strongly $\lceil n/2 \rceil$-indexable; $K_{(m,n)}$ ($m, n > 2$) is indexable if and only if m or n is at most 2. For a simple indexable graph $G(V, E)$ they proved $|E| \leqslant 2|V| - 3$. Also, they determine all indexable graphs of order at most 6.

Hegde and Shetty [1233] also prove that if G is strongly k-indexable Eulerian graph with q edges then $q \equiv 0, 3 \pmod 4$ if k is even and $q \equiv 0, 1 \pmod 4$ if k is odd. They further showed how strongly k-indexable graphs can be used to construct polygons of equal internal angles with sides of different lengths.

Germina [1069] has proved the following: fans $P_n + K_1$ are strongly indexable if and only if $n = 1, 2, 3, 4, 5, 6$; $P_n + K_2$ is strongly indexable if and only if $n \leqslant 2$; the only strongly indexable complete m-partite graphs are $K_{1,n}$ and $K_{1,1,n}$; ladders $P_n \times P_2$ are $\lceil \frac{n}{2} \rceil$-strongly indexable, if n is odd; $K_n \times P_k$ is a strongly indexable if and only if $n = 3$; $C_m \times P_n$ is 2-strongly indexable if m is odd and $n \geqslant 2$; $K_{1,n} + K_i$ is not strongly indexable for $n \geqslant 2$; for $G_i \cong K_{1,n}$, $1 \leqslant i \leqslant n$, the sequential join $G \cong (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$ is strongly indexable if and only if, either $i = n = 1$ or $i = 2$ and $n = 1$ or $i = 1, n = 3$; $P_1 \cup P_n$ is strongly indexable if and only if $n \leqslant 3$; $P_2 \cup P_n$ is not strongly indexable; $P_2 \cup P_n$ is $\lceil \frac{n+3}{2} \rceil$-strongly indexable; mC_n is k-strongly indexable if and only if m and n are odd; $K_{1,n} \cup K_{1,n+1}$ is strongly indexable; and $mK_{1,n}$ is $\lceil \frac{3m-1}{2} \rceil$-strongly indexable when m is odd.

Acharya and Germina [33] proved that every graph can be embedded in a strongly indexable graph and gave an algorithmic characterization of strongly indexable unicyclic graphs. In [35] they provide necessary conditions for an Eulerian graph to be strongly k-indexable and investigate strongly indexable (p, q)-graphs for which $q = 2p - 3$.

Hegde and Shetty [1229] proved that for n odd the generalized web graph $W(t, n)$ with the center removed is strongly $(n-1)/2$-indexable. Hegde and Shetty [1234] define a *level joined planar grid* as follows. Let u be a vertex of $P_m \times P_n$ of degree 2. For every pair of distinct vertices v and w that do not have degree 4, introduce an edge between v and w provided that the distance from u to v equals the distance from u to w. They prove that every level joined planar grid is strongly indexable. For any sequence of positive integers $(a_1, a_2, \ldots, a_n)$ Lee and Lee [1824] show how to associate a strongly indexable $(1, 1)$-graph. As a corollary, they obtain the aforementioned result Hegde and Shetty on level joined planar grids.

Section 5.2 of this survey includes a discussion of a labeling method called super edge-magic. In 2002 Hegde and Shetty [1234] showed that a graph has a strongly k-indexable labeling if and only if it has a super edge-magic labeling.

4.4 Elegant Labelings

In 1981 Chang, Hsu, and Rogers [674] defined an *elegant labeling* f of a graph G with q edges as an injective function from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $f(x) + f(y) \pmod{(q+1)}$ the resulting edge labels are distinct and nonzero. An injective labeling f of a graph G with q vertices is called *strongly k-elegant* if the vertex labels are from $\{0, 1, \ldots, q\}$ and the edge labels induced by $f(x) + f(y) \pmod{(q+1)}$ for each edge xy are $k, \ldots, k + q - 1$. Note that in contrast to the definition of a harmonious labeling, for an elegant labeling it is not necessary to make an exception for trees.

Whereas the cycle C_n is harmonious if and only if n is odd, Chang et al. [674] proved that C_n is elegant when $n \equiv 0$ or $3 \pmod 4$ and not elegant when $n \equiv 1 \pmod 4$. Chang et al. further showed that all fans are elegant and the paths P_n are elegant for $n \not\equiv 0 \pmod 4$. Cahit [639] then showed that P_4 is the only path that is not elegant. Balakrishnan, Selvam, and Yegnanarayanan [416] have proved numerous graphs are elegant. Among them are $K_{m,n}$ and the mth-subdivision graph of $K_{1,2n}$ for all m. They prove that the bistar $B_{n,n}$ (K_2 with n pendent edges at each endpoint) is elegant if and only if n is even. They also prove that every simple graph is a subgraph of an elegant graph and that several families of graphs are not elegant. Deb and Limaye [792] have shown that triangular snakes (see §2.2 for the definition) are elegant if and only if the number of triangles is not equal to $3 \pmod 4$. In the case where the number of triangles is $3 \pmod 4$ they show the triangular snakes satisfy a weaker condition they call *semi-elegant* whereby the edge label 0 is permitted. In [793] Deb and Limaye define a graph G with q edges to be *near-elegant* if there is an injective function f from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $f(x) + f(y) \pmod{(q+1)}$ the resulting edge labels are distinct and not equal to q. Thus, in a near-elegant labeling, instead of 0 being the missing value in the edge labels, q is the missing value. Deb and Limaye show that triangular snakes where the number of triangles is $3 \pmod 4$ are near-elegant. For any positive integers $\alpha \leqslant \beta \leqslant \gamma$ where β is at least 2, the *theta graph* $\theta_{\alpha,\beta,\gamma}$ consists of three edge disjoint paths of lengths α, β, and γ having the same end points. Deb and Limaye [793] provide elegant and near-elegant labelings for some theta graphs where $\alpha = 1, 2$, or 3. Seoud and Elsakhawi [2771] have proved that the following graphs are elegant: $K_{1,m,n}$; $K_{1,1,m,n}$; $K_2 + \overline{K_m}$; $K_3 + \overline{K_m}$; and $K_{m,n}$ with an edge joining two vertices of the same partite set. Elumalai and Sethuraman [881] proved ${P_2}^n$, $P_m^2 + \overline{K_n}$, $S_m + S_n$, $S_m + \overline{K_m}$, $C_3 \times P_m$, and even cycles C_{2n} with vertices $a_0, a_1, \ldots, a_{2n-1}, a_0$ and $2n - 3$ chords $a_0a_2, a_0a_3, \ldots, a_0a_{2n-2}$ $(n \geqslant 2)$ are elegant. Zhou [3598] has shown that for every strongly k-elegant graph G with p vertices and q edges and any positive integer m the graph $(G + \overline{K_m}) + \overline{K_n}$ is also strongly k-elegant when $q - p + 1 \leqslant m \leqslant q - p + k$.

If f is a strongly k-elegant labeling of a bipartite graph G with partite sets V_1 and V_2 and max $f(u) <$ min $f(v)$ for all u in V_1 and v in V_2, f is said to be a *set-ordered strongly k-elegant* labeling of G. Su, Wang, and Yao [3058] proved that if a connected (p, q)-graph admits a strongly k-elegant labeling, then $q \leqslant 2p-3$ and if a graph is a set-ordered strongly k-elegant, then $q = p - 1$. They constructed several classes of large-scale trees that have strongly k-elegant labelings through graph operations that connect edges between two vertices or identify two vertices to form a new graph and proved that caterpillars with p vertices admits a set-ordered strongly k-elegant labelings. They further showed that a graph admits a strongly k-elegant labeling if and only if it has a super edge-magic total labeling (SEMT)–see Section 5.2.

Sethuraman and Elumalai [2814] proved that every graph is a vertex induced subgraph of a elegant graph and present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a strong form of elegant labeling. Acharya, Germina, Princy, and Rao [40] prove that every (p, q)-graph G can be embedded in a connected elegant graph H. The construction is done in such a way that if G is planar and elegant (harmonious), then so is H.

In [2813] Sethuraman and Elumalai define a graph H to be a $K_{1,m}$-*star extension* of a graph G with p vertices and q edges at a vertex v of G where $m > p - 1 - deg(v)$ if H is obtained from G by merging the center of the star $K_{1,m}$ with v and merging $p-1-deg(v)$ pendent vertices of $K_{1,m}$ with the $p - 1 - deg(v)$ nonadjacent vertices of v in G. They prove that for every graph G with p vertices and q edges and for every vertex v of G and every $m \geqslant 2^{p-1} - 1 - q$, there is a $K_{1,m}$-star extension of G that is both graceful and harmonious. In the case where $m \geqslant 2^{p-1} - q$, they show that G has a $K_{1,m}$-star extension that is elegant. Sethuraman and Selvaraju [2840] have shown that certain cases of the union of any number of copies of K_4 with one or more edges deleted and one edge in common are elegant.

In [893] Ephremnath and Elumlai say a graph G is a *cycle with a chord Hamiltonian path* if G is obtained from the cycle $v_0, v_1, \ldots, v_{n-1}, v_0$ $(n \geqslant 6)$ by adding the chords $v_1v_{n-1}, v_vv_{n-2}, \ldots, v_\alpha v_\beta$ where $\alpha = \beta = (n - 2)/2$ if n is even and $\alpha = (n + 3)/2$, $\beta = (n - 1)/2$ if n is odd. They proved that C_n $(n \geqslant 6)$ with a chord Hamiltonian path is harmonious and elegant.

Gallian extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let G be a graph with q edges and H a finite Abelian group (under addition) of order q. Define G to be *H-harmonious* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x)+f(y)$ the resulting edge labels are distinct. When G is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley, and Jungreis [516] have shown that if H is a finite Abelian group of order $n > 1$ then C_n is H-harmonious if and only if H has a non-cyclic or trivial Sylow 2-subgroup and H is not of the form $Z_2 \times Z_2 \times \cdots \times Z_2$. Thus, for example, C_{12} is not Z_{12}-harmonious but is $(Z_2 \times Z_2 \times Z_3)$-harmonious. In [867] Ehard, Glock, and Joos apply rainbow colorings to graph decompositions and harmonious labeling of graphs.

Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let G be a graph with q edges and H a finite Abelian group (under addition) with $q+1$ elements. We say G is *H-elegant* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x)+f(y)$ the resulting set of edge labels is the non-identity elements of H. Beals et al. [516] proved that if H is a finite Abelian group of order n with $n \neq 1$ and $n \neq 3$, then C_{n-1} is H-elegant using only the non-identity elements of H as vertex labels if and only if H has either a non-cyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang, Hsu, and Rogers [674] by showing that C_n is elegant when $n \equiv 2 \pmod 4$. Mollard and Payan [2183] also proved that C_n is elegant when $n \equiv 2 \pmod 4$ and gave another proof that P_n is elegant when $n \neq 4$. In 2014 Ollis [2320] used harmonious labelings for Z_m given by Beals, Gallian, Headley, and Jungreis in [516] to construct new Latin squares of odd order.

A function f is said to be an *odd-elegant* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q-1$ such that the induced mapping $f^*(uv) = f(u)+f(v) \pmod{2q}$ from the edges of G to the odd integers between 1 to $2q-1$ is a bijection. Zhou, Yao, and Chen [3600] proved that every lobster is odd-elegant. In [3422] Wang, Xu, Ma, and Zhang gave a new type of graphical passwords based on odd-elegant labeled graphs. See also [3423] and [3583].

For a graph $G(V,E)$ and an Abelian group H Valentin [3335] defines a *polychrome* labeling of G by H to be a bijection f from V to H such that the edge labels induced by $f(uv) = f(v)+f(u)$ are distinct. Valentin investigates the existence of polychrome labelings for paths and cycles for various Abelian groups.

4.5 Felicitous Labelings

Another generalization of harmonious labelings are felicitous labelings. An injective function f from the vertices of a graph G with q edges to the set $\{0,1,\ldots,q\}$ is called *felicitous* if the edge labels induced by $f(x)+f(y) \pmod q$ for each edge xy are distinct. (Recall a harmonious labeling only allows the vertex labels $0,1,\ldots,q-1$.) This definition first appeared in a paper by Lee, Schmeichel, and Shee in [1881] and is attributed to E. Choo. labeling of the graph. Balakrishnan and Kumar [413] proved the conjecture of Lee, Schmeichel, and Shee [1881] that every graph is a subgraph of a felicitous graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitous are: C_n except when $n \equiv 2 \pmod 4$ [1881]; $K_{m,n}$ when $m,n > 1$ [1881]; $P_2 \cup C_{2n+1}$ [1881]; $P_2 \cup C_{2n}$ [3199]; $P_3 \cup C_{2n+1}$ [1881]; $S_m \cup C_{2n+1}$ [1881]; K_n if and only if $n \leqslant 4$ [2813]; $P_n + \overline{K_m}$ [2813]; the friendship graph $C_3^{(n)}$ for n odd [1881]; $P_n \cup C_3$ [2877]; $P_n \cup C_{n+3}$ [3199]; and the one-point union of an odd cycle and a caterpillar [2877]. Shee [2873] conjectured that $P_m \cup C_n$ is felicitous when $n > 2$ and $m > 3$. Lee, Schmeichel, and Shee [1881] ask for which m and n is the one-point union of n copies of C_m felicitous. They showed that in the case where mn is twice an odd integer the graph is not felicitous. In contrast to the situation for felicitous labelings, we remark that C_{4k} and $K_{m,n}$ where $m, n > 1$ are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious. Lee, Schmeichel, and Shee [1881] conjectured that the n-cube is felicitous. This conjecture was proved by Figueroa-Centeno and Ichishima in 2001 [929].

Balakrishnan, Selvam, and Yegnanarayanan [415] obtained numerous results on felicitous labelings. The *wreath product*, $G * H$, of graphs G and H has vertex set $V(G) \times V(H)$ and (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. They define $H_{n,n}$ as the graph with vertex set $\{u_1, \ldots, u_n; v_1, \ldots, v_n\}$ and edge set $\{u_iv_j | \ 1 \leqslant i \leqslant j \leqslant n\}$. They let $\langle K_{1,n} : m\rangle$ denote the graph obtained by taking m disjoint copies of $K_{1,n}$, and joining a new vertex to the centers of the m copies of $K_{1,n}$. They prove the following are felicitous: $H_{n,n}$; $P_n * \overline{K_2}$; $\langle K_{1,m} : m\rangle$; $\langle K_{1,2} : m\rangle$ when $m \not\equiv 0 \pmod 3$, or $m \equiv 3 \pmod 6$, or $m \equiv 6 \pmod{12}$; $\langle K_{1,2n} : m\rangle$ for all m and $n \geqslant 2$; $\langle K_{1,2t+1} : 2n+1\rangle$ when $n \geqslant t$; P_n^k when $k = n-1$ and $n \not\equiv 2 \pmod 4$, or $k = 2t$ and $n \geqslant 3$ and $k < n-1$; the join of a star and $\overline{K_n}$; and graphs obtained by joining two end vertices or two central vertices of stars with an edge. Yegnanarayanan [3533] conjectures that the graphs obtained from an even cycle by attaching n new vertices to each vertex of the cycle is felicitous. This conjecture was verified by Figueroa-Centeno, Ichishima, and Muntaner-Batle in [934]. In [2836] Sethuraman and Selvaraju [2840] have shown that certain cases of the union of any number of copies of K_4 with 3 edges deleted and one edge in common are felicitous. Sethuraman and Selvaraju [2836] present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a felicitous labeling. Krisha and Dulawat [1757] give algorithms for finding graceful, harmonious, sequential, felicitous, and antimagic (see §5.7) labelings of paths. A *linear cactus* $P_m(K_n)$ is a connected graph in which all the blocks are isomorphic to a complete graph K_n and block-cutpoint is a path P_{2m-1}. Gomathi [1125] proved the follow graphs are felicitous: $P_m(K_4)$, splitting graphs of $(B_{n,n)})$, planar graphs $\mathrm{Pl}_{m,n}$, and $C_{2k+1} \odot S_m$. Gomathi and Nagarajan [1120] proved the following graphs are felicitous: a vertex switching of C_n ($n \geqslant 4$), a vertex switching of C_n ($n \geqslant 4$) with one chord, a vertex duplication of C_n, and the square of the book $B_{n,n}$ ($n \geqslant 2$). Ezhilarasi Hilda and Jeba Jesintha [1247] proved that all shell flower graphs are felicitous.

In [3078] Sudhakar, Ranjani, Swathy, and Balaji provided a technique for coding a secret messages by applying an even felicitous labeling for the union of two star graphs using a GMJ (Graph Message Jumbled) code. They include two illustrations for converting plain text into cipher text (picture coding) and a method for a felicitous labeling of a graph. In [3077] Manshath, Hariprabakaran, Veerasamy, and Balaji used a GMJ code to create a confidential message by applying an even felicitous labeling to a bistar. [3078] new [3077] new

Figueroa-Centeno, Ichishima, and Muntaner-Batle [935] define a felicitous graph to be *strongly felicitous* if there exists an integer k so that for every edge uv, $\min\{f(u), f(v)\} \leqslant k < \max\{f(u), f(v)\}$. For a graph with p vertices and q edges with $q \geqslant p-1$ they show that G is strongly felicitous if and only if G has an α-labeling (see §3.1). They also show that for graphs G_1 and G_2 with strongly felicitous labelings f_1 and f_2 the graph obtained from G_1 and G_2 by identifying the vertices u and v such that $f_1(u) = 0 = f_2(v)$ is strongly felicitous and that the one-point union of two copies of C_m where $m \geqslant 4$ and m is even is strongly felicitous. As a corollary they have that the one-point union of n copies of C_m where m is even and at least 4 and $n \equiv 2 \pmod 4$ is felicitous. They conjecture that the one-point union of n copies of C_m is felicitous if and only if $mn \equiv 0, 1$, or 3 (mod 4). In [939] Figueroa-Centeno, Ichishima, and Muntaner-Batle prove that $2C_n$ is strongly felicitous if and only if n is even and at least 4. They conjecture [939] that mC_n is felicitous if and only if $mn \not\equiv 2 \pmod 4$ and that $C_m \cup C_n$ is felicitous if and only if $m + n \not\equiv 2 \pmod 4$.

As consequences of their results about super edge-magic labelings (see §5.2) Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [939] have the following corollaries: if m and n are odd with $m \geqslant 1$ and $n \geqslant 3$, then mC_n is felicitous; $3C_n$ is felicitous if and only if $n \not\equiv 2 \pmod 4$; and $C_5 \cup P_n$ is felicitous for all n.

For a graph G with q edges Shainy and Balaji [2879] call a one-to-one function f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ a *even felicitous* if the edge labels generated by $(f(r) + f(s)) \operatorname{mod}(2q-1)$ for each edge are even and distinct. They proved that stars, bistars, the union two stars, and the union of three stars are even felicitous graph.

In [2074] Manickam, Marudai, and Kala prove the following graphs are felicitous: the one-point union of m copies of C_n if $mn \equiv 1, 3$ mod 4; the one-point union of m copies of C_4; mC_n if $mn \equiv 1, 3 \pmod 4$; and mC_4. These results partially answer questions raised by Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima in [935] and [939].

Chang, Hsu, and Rogers [674] have given a sequential counterpart to felicitous labelings. They call a graph with q edges *strongly c-elegant* if the vertex labels are from $\{0, 1, \ldots, q\}$ and the edge labels induced by addition are $\{c, c+1, \ldots, c+q-1\}$. (A strongly 1-elegant labeling has also been called a *consecutive* labeling.) Notice that every strongly c-elegant graph is felicitous and that strongly c-elegant is the same as $(c, 1)$-arithmetic in the case where the vertex labels are from $\{0, 1, \ldots, q\}$. Chang et al. [674] have shown: K_n is strongly 1-elegant if and only if $n = 2, 3, 4$; C_n is strongly 1-elegant if and only if $n = 3$; and a bipartite graph is strongly 1-elegant if and only if it is a star. Shee [2874] has proved that $K_{m,n}$ is strongly c-elegant for a particular value of c and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

Seoud and Elsakhawi [2773] have shown: $K_{m,n}$ $(m \leqslant n)$ with an edge joining two vertices of the same partite set is strongly c-elegant for $c = 1, 3, 5, \ldots, 2n+2$; $K_{1,m,n}$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 2m$ when $m = n$, and for $c = 1, 3, 5, \ldots, m+n+1$ when $m \neq n$; $K_{1,1,m,m}$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 2m+1$; $P_n + \overline{K_m}$ is strongly $\lfloor n/2 \rfloor$-elegant; $C_m + \overline{K_n}$ is strongly c-elegant for odd m and all n for $c = (m-1)/2, (m-1)/2+2, \ldots, 2m$ when $(m-1)/2$ is even and for $c = (m-1)/2, (m-1)/2+2, \ldots, 2m-(m-1)/2$ when $(m-1)/2$ is odd; ladders L_{2k+1} $(k > 1)$ are strongly $(k+1)$-elegant; and $B(3, 2, m)$ and $B(4, 3, m)$ (see §2.4 for notation) are strongly 1-elegant and strongly 3-

elegant for all m; the composition $P_n[P_2]$ (see §2.3 for the definition) is strongly c-elegant for $c = 1, 3, 5, \ldots, 5n-6$ when n is odd and for $c = 1, 3, 5, \ldots, 5n-5$ when n is even; P_n is strongly $\lfloor n/2 \rfloor$-elegant; P_n^2 is strongly c-elegant for $c = 1, 3, 5, \ldots, q$ where q is the number of edges of P_n^2; and P_n^3 $(n > 3)$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 6k-1$ when $n = 4k$; $c = 1, 3, 5, \ldots, 6k+1$ when $n = 4k+1$; $c = 1, 3, 5, \ldots, 6k+3$ when $n = 4k+2$; $c = 1, 3, 5, \ldots, 6k+5$ when $n = 4k+3$.

In [464] Barrientos and Minion study a technique to transform an α-labeling of some snakes whose cells are squares into a felicitous labeling and the felicitous labeling into a harmonious labeling. They prove that all quadrilateral snakes, all snake polyominoes, and all hybrid quadrilateral snakes are both, felicitous (see §4.5) and harmonious. A *hybrid quadrilateral snake* is a snake obtained with n copies of C_4 where the ith copy of C_4 is attached to the $(i+1)$th copy via vertex amalgamation or edge amalgamation. Barrientos and Minion [464] prove that all hybrid quadrilateral snakes admit α-labelings.

4.6 Odd Harmonious and Even Harmonious Labelings

Liang and Bai [1940] introduced odd harmonious labelings by defining a function f to be an *odd harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q-1$ such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2q-1$ is a bijection. A function f is said to be a *strongly odd harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to q such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2q-1$ is a bijection. Liang and Bai [1940] have shown the following: odd harmonious graphs are bipartite; if a (p, q)-graph is odd harmonious, then $2\sqrt{q} \leqslant p \leqslant 2q-1$; if a (p, q)-graph with degree sequence $(d_1, d_2, \ldots, d_p)$ is odd harmonious, then $\gcd(d_1, d_2, \ldots, d_p)$ divides q^2; P_n $(n > 1)$ is odd harmonious and strongly odd harmonious; C_n is odd harmonious if and only if $n \equiv 0 \bmod 4$; K_n is odd harmonious if and only if $n = 2$; $K_{n_1,n_2,\ldots,n_k}$ is odd harmonious if and only if $k = 2$; K_n^t is odd harmonious if and only if $n = 2$; $P_m \times P_n$ is odd harmonious; the tadpole graph obtained by identifying the endpoint of a path with a vertex of an n-cycle is odd harmonious if $n \equiv 0 \bmod 4$; the graph obtained by appending two or more pendent edges to each vertex of C_{4n} is odd harmonious; the graph obtained by subdividing every edge of the cycle of a wheel (gear graphs) is odd harmonious; the graph obtained by appending an edge to each vertex of a strongly odd harmonious graph is odd harmonious; and caterpillars and lobsters are odd harmonious. They conjecture that every tree is odd harmonious.

Liang and Bai [1940] also showned that the kC_4-snake graph is an odd harmonious graph. Abdel-Aal [4] generalize this result by showing that the kC_n-snake with string $1, 1, \ldots, 1$ for $n \equiv 0 \pmod 4$ are odd harmonious. He also showed that the kC_4 snake with m pendent edges is odd harmonius and that all subdivisions of $2m$-triangular snakes are odd harmonious. Alyani, Firmansah, Giyarti, and Sugeng [171] constructed odd [171] new
harmonious labelings for kC_n-snakes for $n = 4$ and $n = 8$ and gave odd harmonious labelings for a variation of kC_n-snakes. An n *hair-*kC_4 *snake* is a graph obtained by attaching n leaves to vertices of degree two in a kC_4-snake graph. Mumtaz and Silaban

[2204] proved that n hair-kC_4-snakes are odd harmonious. A $k(G)$ *snake graph* is a graph obtained from a path on k edges by replacing each edge by a graph isomorphic to G. Asumpta, Purwanto, and Chandra [262] showed that the snake graph $k(G)$ is odd harmonious when G is a gear graph based on W_3 and when G is $K_{2,m}$ $(m \geqslant 2)$. [2204] new [262] new

Renuka and Palanivelu [2615] proved that the one vertex union of some classes of complete bipartite graphs, the one vertex union of paths, and extended bistars and their subdivisions are odd harmonious. They further proved that the collection of paths passing through another path at a selected common mid vertex and each vertex of ladder appended by an edge are strongly odd harmonious and that the subdivision of the Cartesian product of two paths is strongly odd harmonious. [2615] new

Abdel-Aal also proved that a necessary condition for odd harmonious Eulerian graphs with q edges is $q \equiv 0 \pmod 4$ and that the following graphs are odd harmonious: $C_m \times P_n$ $(n \geqslant 2, m \equiv 0 \pmod 4)$; $C_{4m} \odot C_4$; $S_n \odot \overline{K_m}$; two copies of an even n-cycle sharing a common edge is an odd harmonious graph when $n \equiv 0 \pmod 4$; two copies of an even n-cycle sharing a common vertex is odd harmonious when $n \equiv 0 \pmod 4$; and graphs obtained from $K_{2,n}$ $(n \geqslant 2)$ by adding r pendent edges to one of the two vertices of degree n and s pendent edges to the other vertex of degree n. In [2203] Mumtaz, John, and Silaban proved that the grid-like graph of order $2mn+mm+n$ and size $4mn$ obtained by arranging m-rows connected rows of nC_4-snake graphs is odd harmonious.

Vaidya and Shah [3300] prove that the shadow graphs (see §3.8 for the definition) of path P_n and star K_{1n} are odd harmonious. They also show that the splitting graphs (see §2.7 for the definition)) of path P_n and star $K_{1,n}$ are odd harmonious. In [3301] Vaidya and Shah proved the following graphs are odd harmonious: the shadow graph and the splitting graph of bistar $B_{n,n}$; the arbitrary supersubdivision of paths; graphs obtained by joining two copies of cycle C_n for $n \equiv 0 (\text{mod } 4)$ by an edge; and the graphs $H_{n,n}$, where $V(H_{n,n}) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, v_n\}$ and $E(H_{n,n}) = \{v_i u_j : 1 \leqslant i \leqslant n,\ n-i+1 \leqslant j \leqslant n\}$. In [3511] Yan proves that $P_m \times P_n$ is strongly odd harmonious. Koppendrayer [1730] has proved that every graph with an α-labeling is odd harmonious. Li, Li, and Yan [1927] proved that $K_{m,n}$ is odd strongly harmonious. Saputri, Sugeng, and Fronček [2716] proved that the graph obtained by joining C_n to C_k by an edge (dumbbell graph $D_{n,k,2}$) is odd harmonious for $n \equiv k \equiv 0 \pmod 4$ and $n \equiv k \equiv 2 \pmod 4$, and $C_n \times P_m$ is odd harmonious if and only if $n \equiv 0 \pmod 4$. They also observe that $C_n \odot K_1$ with $n \equiv 0 \pmod 4$ is odd harmonious. Pujiwatia, Halikinb, and Wijayac [2530] proved double stars and even cycles odd harmonious. Sugeng, Surip, and Rismayati [3092] proved that the m-shadow of C_{4n} and nontivial gears are odd harmonious graphs.

Jeyanthi [1484] proved that the shadow and splitting graphs of $K_{2,n}$, C_{4n}, the double quadrilateral snakes $DQ(n)$ $(n \geqslant 2)$, and the graph $H_{n,n}$ with vertex set $V(H_{n,n}) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and the edge set $E(H_{n,n}) = \{v_i u_j : 1 \leqslant i \leqslant n, n-i+1 \leqslant j \leqslant n\}$ are odd harmonious. Jeyanthi and Philo [1484] proved that the shadow graphs $D_2(K_{2,n})$ and $D_2(H_{n,n})$ are odd harmonious and the splitting of graphs of $K_{2,n}$ and $H_{n,n}$ are odd harmonious. They also showed that the shadow graph $D_2(C_n)$ is odd harmonious if $n \equiv 0 \pmod 4$, the splitting of C_n is odd harmonious if $n \equiv 0 \pmod 4$, and the double quadrilateral snake $DQ(n)$ is odd harmonious for $n \geqslant 2$. In [1488] Jeyanthi and Philo prove that super subdivision of cycles, ladders, $C_{4n} \oplus K_{1,m}$, and uniform fire crackers are odd harmonious graphs. Jeyanthi and Philo [1494] proved that the graph $P_{n-1}(1, 2, 3, \dots, n)$ obtained from a path of n vertices $v_1, v_2, \dots, v_{n-1}$ by appending a path of length $n - i$ at each v_i and certain one point unions of the end points of paths are odd harmonious.

Philo, Jeyanthi, and Davazz [2869] proved that the following graphs are odd harmonious: the path union of r copies of $K_{m,n}$; the path union of copies of complete bipartite graphs; the graphs obtained from t copies of $K_{m,n}$ by joining each one to the next one with an islolated vertex (the *join sum*) ; the one point union of t copies of complete bipartite graphs (not necessarily the same); the graphs obtained by replacing each vertex of $K_{1,t}$, except the apex vertex, with the graph $K_{m,n}$; and the one point union of isomorphic path graphs of $K_{m,n}$. Let $u_{i,1}, u_{i,2}, \dots, u_{i,m}$ and $v_{i,1}, v_{i,2}, \dots, v_{i,n}$ be the vertices of r copies of $K_{m,n}$. The *circle formation* [2869] of r copies of $K_{m,n}$ is the graph obtained by joining the ith copy of $K_{m,n}$ $(1 \leqslant i \leqslant r)$ by joining the vertices $u_{i,m}$ to $u_{i+1,1}$ for $1 \leqslant i \leqslant r - 1$ and joining the vertex $u_{r,m}$ to $u_{1,1}$. Philo, Jeyanthi, and Davazz [2869] proved that he circle formation of r copies of $K_{m,n}$ when $r = 0 (\bmod 4)$ is odd harmonious. [2869] new [2869] new [2869] new

Suptri, Sugeng, and Fronček [2716] proved that that dumbbell graphs $D_{n,k,2}$ are odd harmonious if and only if $n, k = 2(\bmod 4)$. In [1490] Jeyanthi and Philo proved that the graphs obtained by attaching m pendant vertices to each vertex of paths of odd length, the splitting graph of combs, slanting ladders, graphs obtained from m copies of $\langle K_{1,n} : K_{1,n} \rangle$ by joining one leaf of ith copy of $\langle K_{1,n} : K_{1,n} \rangle$ with the center of $(i + 1)$th copy of $\langle K_{1,n} : K_{1,n} \rangle$ where $1 \leqslant i \leqslant m - 1$, and H-super subdivisions of P_n and C_{4n} admit odd harmonious labeling. Moveover, they observe that all strongly odd harmonious graphs admit a mean labeling, an odd mean labeling, an odd sequential labeling, all odd sequential graphs are odd harmonious, and all odd harmonious graphs are even sequential harmonious. They also proved the n-splitting graphs for paths, stars, and symmetric product between paths and null graphs are odd harmonious graphs for all n. Selvaraju, Balaganesan, and Renuka [2740] proved that quadrilateral snakes and k-regular caterpillars are odd harmonious. Abdel-Aal1 and Seoud [5] proved that the m-shadow graphs for paths and complete bipartite graphs are odd harmonious graphs for all m. [2716] new [1490] new [2740] new [5] new

For a T_p-tree T with vertices $v_1, v_2, \dots, v_n$, the graph $T\hat{o}P_m$ is obtained from T and n copies of P_m by identifying a pendant vertex of ith copy of P_m with vertex v_i of T. For C_n with consecutive vertices $v_1, v_2, \dots, v_n$, the graph $C_n\hat{o}P_m$ is obtained from C_n and n copies of P_m by identifying a pendant vertex of ith copy of P_m with vertex v_i of C_n. Jeyanthi and Philo [1496] proved that T_p-trees, $T\hat{o}P_m$, $T\hat{o}2P_m$, regular bamboo trees, $C_n\hat{o}P_m$, $C_n\hat{o}2P_m$, and subdivided grid graphs are odd harmonious.

Recall a *subdivided shell graph* is obtained by subdividing the edges in the path of the shell graph. Let $G_1, G_2, \dots, G_n$ be n subdivided shell graphs of any order. The graph SSG(n) is obtained by adding an edge to apexes of G_i and $G_{i+1}, i = 1, 2, \dots, (n-1)$. Jeba Jesintha and Ezhilarasi Hilda [1392] proved that the subdivided shell graph and SSG(2) are odd harmonious.

The following definitions are taken from [1495]. The *m-shadow graph* $D_m(G)$ of a connected graph G is constructed by taking m-copies of G, $G_1, G_2, G_3, \dots, G_m$, and joining each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leqslant j \leqslant m$. The *m-splitting graph* $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G m new vertices, $v^1, v^2, \ldots, v^m$ such that v^i, $1 \leqslant i \leqslant m$, is adjacent to every vertex that is adjacent to v in G. Note that the 2-shadow graph is the shadow graph $D_2(G)$ and the 1-splitting graph is splitting graph. The *m-mirror graph* $M_m(G)$ is defined as the disjoint union of m copies of G, $G_1, G_2, \ldots, G_m$, together with additional edges joining each vertex of G_i to its corresponding vertex in G_{i+1}, $1 \leqslant i \leqslant m-1$. The graph $\overline{W_{m,n}}$ is obtained from the gear graph arising from the wheel W_n as follows: Join the vertices v_i and v_{i+2} with the new vertices v_{i+1}^j for $1 \leqslant j \leqslant m$ and $2 \leqslant i \leqslant n-2$ and join v_n and v_2 with v_{2i-1}. The graph $K_{2,n}(r,s)$ is obtained from $K_{2,n}$ $(n \geqslant 2)$ by adding r and s pendent edges to the two vertices of degree n. The graph $G = \langle C_n : K_{2,m} : C_r \rangle$ is obtained from $K_{2,m}$ with the partite set $\{u, v\}$ by identifying the vertex u with a vertex of C_n and the vertex v with a vertex of C_r. Let P_n be a path on n vertices denoted by $(1,1), (1,2), \ldots, (1,n)$ and with $n-1$ edges denoted by $e_1, e_2, \ldots, e_{n-1}$ where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$. The *step ladder* graph $S(T_n)$ has $(n^2+3n-2)/2$ vertices denoted by $(1,1), (1,2), \ldots, (1,n), (2,1), (2,2), \ldots, (2,n), (3,1), (3,2), \ldots, (3,n-1), (4,1), \ldots, (4,n-2), \ldots, (n,1), (n,2)$ and n^2+n+2 edges. In any ordered pair (i,j), i denotes the row (counted from bottom to top) and j denotes the column(from left to right) in which the vertex occurs.

The *cocktail party* graph, $H_{m,n}$ $(m, n \geqslant 2)$, is the graph with a vertex set $V = \{v_1, v_2, \ldots, v_{mn}\}$ partitioned into n independent sets $V = \{I_1, I_2, \ldots, I_n\}$ each of size m such that $v_iv_j \in E$ for all $i, j \in \{1, 2, \ldots, mn\}$ where $i \in I_p$, $j \in I_q$, $p \neq q$.

Jeyanthi and Philo [1487] proved that following graphs are odd harmonious: $D_m(P_n)$ for all $m, n \geqslant 2$; $Spl_m(P_n)$ for $m, n \geqslant 2$; $D_m(H_{n,n})$ for all $m \geqslant 2$ and $n \geqslant 1$; $Spl_m(H_{n,n})$ for all $m \geqslant 2$ and $n \geqslant 1$; $D_m(K_{r,s})$ for all $r, s \geqslant 1$; $Spl_m(K_{r,s})$ for all $m \geqslant 2$ and $r, s \geqslant 1$; $D_m(P_n \oplus \overline{K_2})$ for all $m, n \geqslant 2$; $Spl_m(P_n \oplus \overline{K_2})$, $m, n \geqslant 2$; and $Spl_m(C_n)$ if and only if $n \equiv 0 \pmod 4$.

Jeyanthi and Philo [1495] proved that following graphs are odd harmonious: $\overline{W_{m,n}}$ for $n \equiv 0 \pmod 4$, $m \geqslant 1$; $D_m(P_n \odot K_1)$ (the authors use the notion C_{bn} for the comb $P_n \odot K_1$) for all $m \geqslant 2$ and $n \geqslant 1$; $Spl_m(K_{2,n}(r,s))$; $\langle C_n : K_{2,m} : C_r \rangle$ for $n, r \equiv 0 \pmod 4$ and $m \geqslant 2$; and the graphs obtained by arranging vertices into a finite number of rows (at least 2) with i vertices in the ith row and in every row the jth vertex in that row is joined to the jth vertex and $j+1$st vertex of the next row (a *pyramid*) for $n \geqslant 2$. They also prove that if G is a strongly odd harmonious tree, then $M_m(G)$ is odd harmonious.

Let P_{2n} be a path of length $2n-1$ with $2n$ vertices, denoted by $(1,1), (1,2), \ldots, (1,2n)$ and with $2n-1$ edges, denoted by $e_1, e_2, \ldots, e_{2n-1}$ where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$. On each edge e_i for $i = n+1, n+2, \ldots, 2n-1$, we erect a ladder with $2n+1-i$ steps including the edge e_i. The double sided step ladder graph $2S(T_{2\times n})$ has vertices denoted by $(1,1), (1,2), \ldots, (1,2n)$,$(2,1), (2,2), \ldots, (2,2n)$,$(3,2), (3,3), \ldots, (3,2n-1)$,$(4,3), (4,4), \ldots, (4,2n-2), \ldots, (n+1,n), (n+1,n+1)$. In any ordered pair (i,j), i denotes the row (counted from bottom to top) and j denotes the column (from left to right) in which the vertex occurs. Jeyanthi and Philo [1493] proved that the path union of t copies of $S(T_n)$, the double sided step ladder $2S(T_{2\times n})$, the path union of

t copies of $2S(T_{2\times n})$, $S(t.C_{bn})$, $S(t.C_4)$, ${C_4}^t$, ${C_6}^t$, and ${C_8}^t$ are odd harmonious graphs. Jeyanthi and Philo [1489] proved that path union of r copies of $K_{m,n}$, the path union of r copies of K_{m_i,n_i}, $1 \leqslant i \leqslant r$, $K^t_{m,n}$, $K^t_{(m_1,n_1),(m_2,n_2),\ldots,(m_t,n_t)}$, the join sum of graph $\langle K_{m,n}; K_{m,n}; \ldots, K_{m,n}$ (t copies $\rangle$, $\langle K_{m_1,n_1}; K_{m_2,n_2}; \ldots, K_{m_t,n_t}\rangle$, the circle formation of r copies of $K_{m,n}$ when $r \equiv 0 \pmod 4$, $S(t.K_{m,n})$ and $P^t_n(t.n.K_{p,q})$ are odd harmonious graphs. Jeyanthi and Philo [1492] proved that the subdivided shell graphs, disjoint union of two subdivided shell graphs, subdivided shell flower graphs, and subdivided uniform shell bow graphs are odd harmonious. Jeyanthi, Philo, and Youssef [1497] proved that the path union of t copies of $P_m \times P_n$, the path union of t copies of $P_{m_i} \times P_{n_i}$ where $1 \leqslant i \leqslant t$, the vertex union of t copies of $P_m \times P_n$, the vertex union of t different copies of $P_{m_i} \times P_{n_i}$ where $1 \leqslant i \leqslant t$, the one point union of path of $P^t_n(t.n.P_m \times P_m)$, and the super subdivision of grid graph $P_m \times P_n$ are odd harmonious graphs.

Recall from Section 2.7 that for even $n > 2$ a *plus graph* of size n (denoted by Pl_n) is the graph obtained by starting with paths $P_2, P_4, \ldots, P_{n-2}, P_n, P_n, P_{n-2}, \ldots, P_4, P_2$ arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. Jeyanthi [1486] proved that following graphs are odd harmonious: Pl_n where $n \equiv 0 \pmod 2$, $n \neq 2$; path unions of finitely many copies of Pl_n where $n \equiv 0 \pmod 2$, $n \neq 2$; open stars of plus graphs $S(t.Pl_n)$ where $n \equiv 0 \pmod 2$, $n \neq 2$ and t odd; graphs obtained by joining C_m, $m \equiv 0 \pmod 4$ and a plus graph Pl_n, $n \equiv 0 \pmod 2$, $n \neq 2$ with a path of arbitrary length; the graph obtained by replacing all vertices of $K_{1,t}$, except the apex vertex, by the path union of n copies of the graph Pl_m.

In [1488] Jeyanthi and Philo prove that super subdivision of cycles, ladders, $C_{4n} \oplus K_{1,m}$, and uniform fire crackers are odd harmonious graphs. They also proved the (m, n)-firecracker graph obtained by the concatenation of m n-stars by linking one leaf from each is odd harmonius; the arbitrary super subdivision of cycles C_m are odd harmonious; and the super subdivision of ladders are odd harmonious. Jeyanthi and Philo [1495] proved that the m-mirror graph $M_m(G)$ $(m \geqslant 2)$, m-splitting graph of $K_{2,n}(r, s)$ (obtained from $K_{2,n}$, $(n > 2)$ by adding r and s $(r, s > 1)$ pendent edges to the two vertices of degree n), $\overline{W_{(m,4n)}}$ obtained from the gear graph of W_n by joining the vertices v_i and v_{i+2} with the new vertices v^j_{i+1} for $1 \leqslant j \leqslant m$ and $2 \leqslant i \leqslant n-2$ and joining v_n and v_2 with v^j_1 for $1 \leqslant j \leqslant m$, $\langle C_{4n} : K_{2,m} : C_{4r}\rangle$ obtained from $K_{2,m}$ with one partite set $V_1 = \{u, v\}$ and C_r by identifying the vertex u of V_1 with a vertex of C_n and the other vertex v of V_1 with a vertex of C_r, and the pyramid graph $PY_n(n \geqslant 2)$ are odd harmonious graphs. They also proved that G is a strongly odd harmonious tree, then $M_m(G)$ is an odd harmonious.

The *edge comb product* of two graphs G and H is the graph formed by taking one copy of G and $|E(G)|$ copies of H, then attaching the i-th copy of H at the edge e to the i-th edge of G. In [949] Sarasvati, Halikin, and Wijaya showed that graphs constructed by the edge comb product of P_n and C_4 and the shadow of C_4 are odd harmonious. Firmansah and Tasari [950] gave odd harmonious labelings for a line amalgamation of double quadrilateral graphs and the graphs obtained by connecting two copies of double quadrilateral graph by an edge. Firmansah and Giyarti [946] proved that graphs obtaned by the edge amalgamation of double quadrilateral graphs are odd harmonious. Febriana and Sugeng [947] proved that squid graphs (obtained from C_n by add pendant edges to

[949] new
[950] new
[946] new
[947] new

one vertex of C_n) are odd harmonious if and only if n is even and double squid graphs
(obtained from two copies of C_n that share one common vertex and adding pendant edges
to the common vertex) are odd harmonious if and only $n \geqslant 4$ is even. In [943], [942], [943] new
[945] Firmansah proved that multiple net snake graphs and a variation of the double [942] new
quadrilateral windmill graphs, layered graphs, and some classes of string graphs are odd
harmonious graphs. Firmansah, Tasari, and Yuwono [948] proved that the zinnia flower [945] new
graph and its variations are odd harmonious graphs. [948] new

In [1488] Jeyanthi and Philo modified the notion of odd harmonious by defining an odd harmonious labelings as a function f to be an *odd harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q-1$ such that the induced mapping $f^*(uv) = f(u) + f(v) \bmod (2q)$ from the edges of G to the odd integers between 1 to $2q-1$ is a bijection. Using this definition they proved that an m-cycle and an n-cycle sharing a common vertex is an odd harmonious if and only if either both $m,\ n \equiv 0 \pmod 4$ or both $m,\ n \equiv 2 \pmod 4$ and the same holds for an m-cycle and an n-cycle sharing a common edge. They also proved that any two even cycles sharing a common vertex and a common edge are odd harmonious graphs.

Sarasija and Binthiya [2717] say a function f is an *even harmonious* labeling of a graph G with q edges if $f : V \to \{0, 1, \ldots, 2q\}$ is injective and the induced function $f^* : E \to \{0, 2, \ldots, 2(q-1)\}$ defined as $f^*(uv) = f(u)+f(v) \pmod{2q}$ is bijective. Notice that for an even harmonious labeling of a connected graph all the vertex labels must have the same parity. Moveover, in the case of even harmonious labelings for connected graphs there is no loss of generality to assume that all the vertex labels are even integers and the duplicate vertex is 0. They proved the following graphs are even harmonious: nontrivial paths; complete bipartite graphs; odd cycles; bistars $B_{m,n}$; $K_2 + \overline{K_n}$; P_n^2; and the friendship graphs F_{2n+1}. López, Muntaner-Batle and Rious-Font [1992] proved that every super edge-magic graph (see Section 5.2 for the definition of super edge-magic) with p vertices and q edges where $q \geqslant p-1$ has an even harmonious labeling. In [3554] Youssef provided a necessary condition for some regular graphs to be even harmonious, showed that the disjoint union of two k-sequential graphs is even $2k$-sequential under some conditions, and showed that in some cases G is k-sequential implies mG is even $2k$-sequential for all positive integer m.

Because 0 and $2q$ are equal modulo $2q$ the following retricted form of even harmonious labelings is of interest. A function f is said to be a *properly even harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q-1$ and the induced function f^* from the edges of G to $\{0, 2, \ldots, 2q-2\}$ defined by $f^*(xy) = f(x)+f(y) \pmod{2q}$ is bijective. In their definition of properly even harmonious in [1008] Gallian and Schoenhard incorrectly required that the vertex labels should be the even integers from 0 to $2q-2$. For connected graphs the two definitions are equivalent but for disconnected graph they are not. They used vertex labels from 0 to $2q-1$ for their results on disconnected graphs.

A graph with a properly even harmonious labeling is said to be *properly even harmonious.* Gallian and Schoenhard [1008] say a properly even harmonious labeling of a graph with q edges is *strongly even harmonious* if it satisfies the additional condition that for any two adjacent vertices with labels u and v, $0 < u + v \leqslant 2q$.

Jared Bass [513] has observed that for connected graphs any harmonious labeling of a graph with q edges yields an even harmonious labeling by simply multiplying each vertex label by 2 and adding the vertex labels modulo $2q$. Thus we know that every connected harmonious graph is an even harmonious graph and every connected graph that is not a tree that has a harmonious labeling also has a properly even harmonious labeling. Conversely, a properly even harmonious labeling of a connected graph with q edges (assuming that the vertex labels are even) yields a harmonious labeling of the graph by dividing each vertex label by 2 and adding the vertex labels modulo q.

Gallian and Schoenhard [1008] proved the following: wheels W_n and helms H_n are properly even harmonious when n is odd; nP_2 is even harmonious for n odd; nP_2 is properly even harmonious if and only if n is even; K_n is even harmonious if and only if $n \leqslant 4$; C_{2n} is not even harmonious when n is odd; $C_n \cup P_3$ is properly even harmonious when odd $n \geqslant 3$; $C_4 \cup P_n$ is even harmonious when $n \geqslant 2$; $C_4 \cup F_n$ is even harmonious when $n \geqslant 2$; $S_m \cup P_n$ is even harmonious when $n \geqslant 2$; $K_4 \cup S_n$ is properly even harmonious; $P_m \cup P_n$ is properly even harmonious for all $m \geqslant 2$ and $n \geqslant 2$; $C_3 \cup P_n^2$ is even harmonious when $n \geqslant 2$; $C_4 \cup P_n^2$ is even harmonious when $n \geqslant 2$; the disjoint union of two or three stars where each star has at least two edges and one has at least three edges is properly even harmonious; $P_m^2 \cup P_n$ is even harmonious for $m \geqslant 2$ and $2 \leqslant n < 4m - 5$; the one-point union of two complete graphs each with at least 3 vertices is not even harmonious; $S_m \cup P_n$ is strongly even harmonious if $n \geqslant 2$; and $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is strongly even harmonious for $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t$ and $t < \frac{n_1}{2} + 2$. They conjecture that $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is strongly even harmonious if at least one star has more than 2 edges. They also note that $C_4, C_8, C_{12}, C_{16}, C_{20}, C_{24}$ are even harmonious and conjecture that C_{4n} is even harmonious for all n. This conjecture was proved by Youssef [3552] who also proved that if a connected even harmonious graph with q edges where q is even and each vertex has degree divisible by 2^k $(k \geqslant 1)$, then q is divisible by 2^{k+1}. As a corollary of the latter he gets that C_{4n+2}^2 is not even harmonious. Hall, Hillesheim, Kocina, and Schmit [1172] proved that nC_{2m+1} is properly even harmonious for all n and m.

In [1009] and [1010] Gallian and Stewart investigated properly even harmonious labelings of unions of graphs. They use ${P_m}^{+t}$ to denote the graph obtained from the path P_m by appending t edges to an endpoint; ${Cat_m}^{+t}$ to denote a caterpillar of path length m with t pendent edges; and ${C_m}^{+t}$ to denote an m-cycle with t pendent edges. They proved the following graphs are properly even harmonious: nP_m if n is even and $m \geqslant 2$; $P_n \cup K_{m,2}$ for n odd and $n > 1$, $m > 1$; $P_n \cup S_{m_1} \cup S_{m_2}$ for $n > 2$ and $m_1 + m_2$ is odd; $C_n \cup S_{m_1} \cup S_{m_2}$ for n odd and m_1, $m_2 > 3$; ${P_m}^{+t} \cup {P_n}^{+s}$; the union of any number of caterpillars; $C_m \cup {Cat_n}^{+t}$ for $m > 1$ odd, $n > 1$; $C_4 \cup {Cat_m}^{+t}$; the union of C_4 and a hairy cycle; $K_4 \cup {C_m}^{+n}$ for some cases; $W_4 \cup {C_m}^{+n}$ for some cases; $C_4 \cup (P_n + \overline{K_2})$ for $n > 1$; $K_4 \cup (P_n + \overline{K_m})$ for $n \equiv 1, 2 \pmod 4$; $C_3 \cup (P_n + \overline{K_m})$ for $n \equiv 1, 2 \pmod 4$; $W_4 \cup (P_n + \overline{K_m})$ for $n \equiv 1, 2 \pmod 4$; $W_4 \cup P_n$ for $n \equiv 1, 2 \pmod 4$; $K_4 \cup P_n$ for $n > 1$ and $n \equiv 1, 2 \pmod 4$; $K_4 \cup P_{m_1}^2 \cup P_{m_2}^2 \cup \cdots \cup P_{m_n}^2$ for $m_i > 2$, $n \geqslant 1$; $W_4 \cup P_{m_1}^2 \cup P_{m_2}^2 \cup \cdots \cup P_{m_n}^2$ for $m_i > 2$, $n \geqslant 1$; $C_m \cup P_n^2$ for $m \equiv 3 \pmod 4$ and $n > 1$; and $2P_m \cup 2P_n$. They also prove that nP_3 is even harmonious if $n > 1$ is odd and $P_{m_1}^2 \cup P_{m_2}^2 \cup \cdots \cup P_{m_n}^2$ is strongly even harmonious for $m > 2,\ n \geqslant 1$.

Gallian and Stewart [1011] call an injective labeling f of a graph G with q edges *even $2a$-sequential* if the vertex labels are from $\{0, 1, \ldots, 2q-1\}$ and the edge labels induced by $f(u) + f(v)$ for each edge uv are $2a, 2a+2, \ldots, 2a+2q-2$. When G is a tree, the allowable vertex labels are $0, 1, \ldots, 2q$. For connected a-sequential graphs, a connected $2a$-sequential graph can be obtained by multiplying all the vertex labels by 2. Notice that the vertex labels in resulting graph belong to $\{0, 2, \ldots, 2q-2\}$ (or $\{0, 2, \ldots, 2q\}$ for trees) and the edges labels are from $2a$ to $2a+2q-2$. Moreover, a connected a-sequential graph can be obtained from a connected even $2a$-sequential graph with even vertex labels by dividing all the vertex labels by 2. Likewise, a $2a$-sequential labeling of a connected graph with odd vertex labels induces an a-sequential labeling of the graph by subtracting 1 from each vertex label and dividing by 2. Thus for connected graphs, a-sequential is equivalent to $2a$-sequential. They prove that if G is even $2a$-sequential the following graphs are properly even harmonious: $G \cup P_m^2$ for $m > 2$, $G \cup P_n$ for $n > 1$, $n \equiv 1, 2 \pmod 4$, $G \cup {C_m}^{+t}$ for some cases, $G \cup {Cat_m}^{+n}$ for $m > 1$, and $G \cup W_{2n+1}$.

For n and k odd and $m, n, k, t > 1$, Mbianda and Gallian (see [2111]) proved the following graphs have properly even harmonious labelings: mP_3 for even m; $2P_m \cup 2P_n \cup S_t$; $2P_m \cup 2P_n \cup P_k$; $2P_m \cup 2P_n \cup C_k$; $2P_m \cup 2P_n \cup C_4$; $2P_m \cup 2P_n \cup 2K_4$; $2P_m \cup 2P_n \cup 2W_4$; $2P_m \cup 2P_n \cup 2C_k$; $F_n \cup K_4$ ($F_n = P_n + K_1$ is the fan); $F_n \cup 2K_4$; $F_n \cup W_4$; $F_n \cup 2W_4$; $W_n \cup K_4$; $W_n \cup 2K_4$; $W_n \cup W_4$; $W_n \cup 2W_4$; $(C_n + K_1) \cup K_4$; $(C_n + K_1) \cup W_4$; $(C_n + K_1) \cup 2K_4$; $(C_n + K_1) \cup 2W_4$; and $(C_n + \overline{K_2}) \cup K_4$ $((C_n + \overline{K_2})$ is the double cone). Gallian [1005] proved the following graphs have properly even harmonious labelings (in all cases $m, n > 1$): mP_n for m even; $2P_m \cup 2P_n \cup 2C_3$; $2P_m \cup 2P_n \cup 2C_4$; $2P_m \cup 2P_n \cup C_3 \cup C_4$; $F_n \cup P_4$; $F_n \cup 2P_4$; $F_n \cup C_4$; and $F_n \cup 2C_4$.

Binthiya and Sarasija [571] prove the following graphs are even harmonious: $C_n \odot mK_1$ (n odd), $P_n \odot mK_1$ ($n > 1$ odd), $C_{2n}@K_2$, P_n (n even) with $n-1$ copies of $m\overline{K_1}$, the shadow graph $D_2(K_{1,n})$, the splitting graph $spl(K_{1,n})$, and the graph obtained from the P_n (n even) with $n-1$ copies of $\overline{K_m}$ incident with first $n-1$ vertices of P_n. Vargheese and Arun [3342] prove that the triangular books, the disjoint union of two triangular book graphs, total graphs $T(P_n)$, the disjoint union of $T(P_n)$ and a triangular book, and the graph obtained by joining the centers of two disjoint copies of $K_{1,n}$ to an isolated vertex are even harmonious.

Lasim, Halikin, and Wijaya [1799] showed how to build new harmonious, odd harmonious, even harmonious labelings based on the existing such labelings. [1799] new

In [520] Beatress and Sarasija introduced the notion of even-odd harmonious graphs as follows. Let G be a graph with m vertices and n edges. An injective mapping from the vertices of G to $\{1, 3, 5, \ldots, 2m-1\}$ is called an *even-odd harmonious* labeling of G if the induced edge mapping f^* from the edges of G to $\{0, 2, 4, \ldots, 2(n-1)\}$ is a bijection and $f^*(uv) = (f(u)^* + f^*(v)) \bmod 2n$ for all edges uv. They proves that the bistars, cycles with one pendent edge, crowns, $K_{1,m,n}$, ${P_n}^2$ ($n \geqslant 4$), and nP_2 are even-odd harmonious graphs. Kalaimathi and Balamurugan [1563] proved caterpillars, lobsters, coconut trees, spider trees, and star graphs admit even-odd harmonious labelings. Zala, Chotaliya, and Chaurasiya [3570] proved H-graphs, combs, and bistars graph have even-odd harmonious labelings. Ganeshwari and Sudhana [1013] proved that paths, cycles, stars, bistars, combs, ${P_n}^2$, $C_3@pK_1$ and $C_{2r-1}@K_1$ are even-odd average harmonious graphs.

5 Magic-type Labelings

5.1 Magic Labelings

Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedláček [2732] in 1963.[2] Responding to a problem raised by Sedláček, Stewart [3053] and [3054] studied various ways to label the edges of a graph in the mid 1960s. Stewart calls a connected graph *semi-magic* if there is a labeling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. (Berge [538] used the term "regularisable" for this notion.) A semi-magic labeling where the edges are labeled with distinct positive integers is called a *magic* labeling. Stewart calls a magic labeling *supermagic* if the set of edge labels consists of consecutive positive integers. The classic concept of an $n \times n$ magic square in number theory corresponds to a supermagic labeling of $K_{n,n}$. Stewart [3053] proved the following: K_n is magic for $n = 2$ and all $n \geqslant 5$; $K_{n,n}$ is magic for all $n \geqslant 3$; fans F_n are magic if and only if n is odd and $n \geqslant 3$; wheels W_n are magic for $n \geqslant 4$; and W_n with one spoke deleted is magic for $n = 4$ and for $n \geqslant 6$. Stewart [3053] also proved that $K_{m,n}$ is semi-magic if and only if $m = n$. In [3054] Stewart proved that K_n is supermagic for $n \geqslant 5$ if and only if $n > 5$ and $n \not\equiv 0 \pmod 4$. Sedláček [2733] showed that Möbius ladders M_n (see §2.3 for the definition) are supermagic when $n \geqslant 3$ and n is odd and that $C_n \times P_2$ is magic, but not supermagic, when $n \geqslant 4$ and n is even. Shiu, Lam, and Lee [2908] have proved: the composition of C_m and $\overline{K_n}$ (see §2.3 for the definition) is supermagic when $m \geqslant 3$ and $n \geqslant 2$; the complete m-partite graph $K_{n,n,\ldots,n}$ is supermagic when $n \geqslant 3$, $m > 5$ and $m \not\equiv 0 \pmod 4$; and if G is an r-regular supermagic graph, then so is the composition of G and $\overline{K_n}$ for $n \geqslant 3$. Ho and Lee [1249] showed that the composition of K_m and $\overline{K_n}$ is supermagic for $m = 3$ or 5 and $n = 2$ or n odd. Bača, Holländer, and Lih [335] have found two families of 4-regular supermagic graphs. Shiu, Lam, and Cheng [2905] proved that for $n \geqslant 2$, $mK_{n,n}$ is supermagic if and only if n is even or both m and n are odd. Ivančo [1338] gave a characterization of all supermagic regular complete multipartite graphs. He proved that Q_n is supermagic if and only if $n = 1$ or n is even and greater than 2 and that $C_n \times C_n$ and $C_{2m} \times C_{2n}$ are supermagic. He conjectures that $C_m \times C_n$ is supermagic for all m and n. Trenklér [3224] has proved that a connected magic graph with p vertices and q edges other than P_2 exits if and only if $5p/4 < q \leqslant p(p-1)/2$. In [3129] Sun, Guan, and Lee give an efficient algorithm for finding a magic labeling of a graph. In [3467] Wen, Lee, and Sun show how to construct a supermagic multigraph from a given graph G by adding extra edges to G.

Sudarsana, Suryanto, Lusianti, and Putri [3067] show how magic graph labelings can be used to increase the security level of encrypted text on social media. Angel Sherin and Maheswari [196] and Ali Ahmed and Baskar Babujee [121] used magic labeling of graph to devise encryption and decryption schemes. In [1118] Lakshmi, Sudhakar, and Sudhakar provided a cryptographic technique for data encryption and decryption using

[196] new
[121] new
[1118] new

[2] A comprehensive expository treatment of magic labelings is given by Bača, Miller, Ryan, and Semaničová-Feňovčíkováin [370].

magic labelings of graphs.

In [1748] Kovář provides a general technique for constructing supermagic labelings of copies of certain kinds of regular supermagic graphs. In particular, he proves: if G is a supermagic r-regular graph ($r \geqslant 3$) with a proper edge r coloring, then nG is supermagic when r is even and supermagic when r and n are odd; if G is a supermagic r-regular graph with m vertices and has a proper edge r coloring and H is a supermagic s-regular graph with n vertices and has a proper edge s coloring, then $G \times H$ is supermagic when r is even or n is odd and is supermagic when s or m is odd.

Kovář, Kravčenko, Krbeček, and Silber [1750] affirmatively answered a question by Madaras about existence of supermagic graphs with arbitrarily many different degrees. Their construction provided graphs with all degrees even. They asked if there exists a supermagic graph with d different odd degrees for any positive integer d. This question was answered affirmatively by Fronček and Qiu with a construction based on the use of 3-dimensional magic rectangles.

In [848] Drajnová, Ivančo, and Semaničová proved that the maximal number of edges in a supermagic graph of order n is 8 for $n = 5$ and $\frac{n(n-1)}{2}$ for $6 \leqslant n \not\equiv 0 \pmod 4$, and $\frac{n(n-1)}{2} - 1$ for $8 \leqslant n \equiv 0 \pmod 4$. They also establish some bounds for the minimal number of edges in a supermagic graph of order n. Ivančo, and Semaničová [1348] proved that every 3-regular triangle-free supermagic graph has an edge such that the graph obtained by contracting that edge is also supermagic and the graph obtained by contracting one of the edges joining the two n-cycles of $C_n \times K_2$ ($n \geqslant 3$) is supermagic.

Ivančo [1340] proved: the complement of a d-regular bipartite graph of order $8k$ is supermagic if and only if d is odd; the complement of a d-regular bipartite graph of order $2n$ where n is odd and d is even is supermagic if and only if $(n, d) \neq (3, 2)$; if G_1 and G_2 are disjoint d-regular Hamiltonian graphs of odd order and $d \geqslant 4$ and even, then the join $G_1 \oplus G_2$ is supermagic; and if G_1 is d-regular Hamiltonian graph of odd order n, G_2 is $d-2$-regular Hamiltonian graph of order n and $4 \leqslant d \equiv 0 \pmod 4$, then the join $G_1 \oplus G_2$ is supermagic.

For $k \geqslant 2$ and graphs G and H, the graph $G \odot^k H$ defined as $(G \odot^{k-1} H) \odot H$ (where $G \odot^1 H = G \odot H$) is called the *k-multilevel corona* of G with H. Marbun and Salman [2078] proved $(W_n \odot^{k-1}) \odot C_n$ is W_n-edge magic.

In [552] Bezegová and Ivančo [554] extended the notion of supermagic regular graphs by defining a graph to be *degree-magic* if the edges can be labeled with $\{1, 2, \ldots, |E(G)|\}$ such that the sum of the labels of the edges incident with any vertex v is equal to $(1 + |E(G))/\deg(v)$. They used this notion to give some constructions of supermagic graphs and proved that for any graph G there is a supermagic regular graph which contains an induced subgraph isomorphic to G. In [554] they gave a characterization of complete tripartite degree-magic graphs and in [555] they provided some bounds on the number of edges in degree-magic graphs. They say a graph G is *conservative* if it admits an orientation and a labeling of the edges by $\{1, 2, \ldots, |E(G)|\}$ such that at each vertex the sum of the labels on the incoming edges is equal to the sum of the labels on the outgoing edges. In [553] Bezegová and Ivančo introduced some constructions of degree-magic labelings for a large family of graphs using conservative graphs. Using a connection between degree-magic labelings and supermagic labelings they also constructed supermagic labelings for the disjoint union of some regular non-isomorphic graphs. Among their results are: If G is a δ-regular graph where δ is even and at least 6, and each component of G is a complete multipartite graph of even size, then G is a supermagic graph; for any δ-regular supermagic graph H, the union of disjoint graphs H and G is supermagic; if G is a δ-regular graph with $\delta \equiv 0 \pmod 8$ and each component is a circulant graph, then G is a supermagic graph; for any δ-regular supermagic graph H, the union of disjoint graphs H and G is a supermagic graph; and that the complement of the union of disjoint cycles $C_{n_1}, \ldots, C_{n_k}$ is supermagic when $k \equiv 1 \pmod 4$ and $11 \leqslant n_i \equiv 3 \pmod 8$ for all $i = 1, \ldots, k$. In [1333] Inpoonjal gave necessary and suficient conditions for the existence of degree-magic labelings of graphs obtained by taking the join and composition of complete tripartite graphs.

A graph G is said to be (F, H)-sim-(super)magic if there exists a bijection f' that is simultaneously F-(super) magic and H-(super) magic. In [258], Ashari, Salman, and Simanjuntak consider (K_2, H)-sim-(super) magic graphs where H is isomorphic to three classes of graphs with varied symmetry: a cycle which is symmetric (both vertex-transitive and edge-transitive), a star which is edge-transitive but not vertex-transitive, and a path which is neither vertex-transitive nor edge-transitive. They provide forbidden subgraphs for the existence of (K_2, H)-sim-(super) magic graphs and classify classes of (K_2, H)-sim-(super) magic graphs. They also derive sufficient conditions for edge-(super) magic graphs to be (K_2, H)-sim-(super) magic and utilize such conditions to characterize some (K_2, H)-sim-(super) magic graphs.

Let G' be a copy of a simple graph G and for each vertex v_i of G let u_i be the vertex of G' corresponding with v_i. The *double graph* has vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G') \cup \{u_i v_j \mid u_i \in V(G);\ v_j \in V(G') \text{ and } u_i u_j \in E(G)\}$. Ivančo [1341] establishes sufficient conditions for generalized double graphs to be degree-magic and constructs supermagic labelings of some graphs generalizing double graphs.

Sedláček [2733] proved that graphs obtained from an odd cycle with consecutive vertices $u_1, u_2, \ldots, u_m, u_{m+1}, v_m, \ldots, v_1$ $(m \geqslant 2)$ by joining each u_i to v_i and v_{i+1} and u_1 to v_{m+1}, u_m to v_1 and v_1 to v_{m+1} are magic. Trenklér and Vetchý [3227] have shown that if G has order at least 5, then G^n is magic for all $n \geqslant 3$ and G^2 is magic if and only if G is not P_5 and G does not have a 1-factor whose every edge is incident with an end-vertex of G. Avadayappan, Jeyanthi, and Vasuki [263] have shown that k-sequential trees are magic (see §4.1 for the definition).

Seoud and Abdel Maqsoud [2759] proved that $K_{1,m,n}$ is magic for all m and n and that P_n^2 is magic for all n. However, Serverino has reported that P_n^2 is not magic for $n = 2, 3$, and 5 [1076]. Jeurissan [1378] characterized magic connected bipartite graphs. Ivančo [1339] proved that bipartite graphs with $p \geqslant 8$ vertices, equal sized partite sets, and minimum degree greater than p are magic. Bača [296] characterizes the structure of magic graphs that are formed by adding edges to a bipartite graph and proves that a regular connected magic graph of degree at least 3 remains magic if an arbitrary edge is deleted. In [3014] Solairaju and Arockiasamy prove that various families of subgraphs of grids $P_m \times P_n$ are magic. Dayanand and Ahmed [787] investigate super magic properties of several classes of connected and disconnected graphs. They show that there can be arbitrarily large gaps among the possible valences for certain super magic graphs. They also prove that the disjoint union of multiple copies of a super magic linear forest is super magic if the number of copies is odd and that the super magic labeling is complementary edge antimagic as well. The *broom* $B_{n,t}$ is a graph obtained by attaching $n-t$ pendent edges to an end point vertex of the path P_t. Marimuthu, Raja, and Raja Durga [2089] prove that $B_{n,n-1}$ is E-super vertex magic if and only if $n \geqslant 3$ is odd and $B_{n,t}$ is not E-super vertex magic for $n \geqslant 4$ and $t \geqslant 3$. In [2270] Nemani and Joshi defined a new class of graph called the *cartoon flower* and showed that a E-super vertex magic labeling does not exist for the class of cartoon flower graphs. They also define the wounded cartoon flower graphs and establish some sufficient conditions for the graph not to be E-super vertex magic. They give examples of some wounded cartoon flowers that admit an E-super vertex magic labeling and some others that do not. In [1775] Kumar and Marimuthu proved that semi-regular bipartite graphs are not E-super vertex-magic and gave an upper bound for the maximum degree of an E-super vertex-magic graph. They also ontained upper and lower bounds of any vertex degree d of a E-super vertex-magic graph. [1775] new

A vertex magic total labeling is said to be a V-*super vertex magic* labeling if $f(V(G)) = \{1, 2, 3, \ldots, |V|\}$. A graph G is called V-*super vertex magic* if it admits a V-super vertex magic labeling. In [3401] Vimal Kumar and Vijayalakshmi establish the V-super vertex magic labelings of some classes of parity graphs (that is, for every two induced paths between the same two vertices both paths have odd length, or both have even length).

A Γ-supermagic labeling of a graph $G(V, E)$ with $|E| = k$ is a bijection from E to an Abelian group Γ of order k such that the sum of labels of all incident edges of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. An existence of a Z_{2nm}-supermagic labeling of Cartesian product of two cycles, $C_n \times C_m$ for n odd was proved recently. This along with an earlier result by Ivanč [1338] proved the existence of a Z_{2nm}-supermagic labeling of $C_m \times C_n$ for every $m, n \geqslant 3$ and conjectured that such labeling is possible for all $C_m \times C_n$. In [977] Fronček and McKeown present a simple unified labeling method for Fronček [972] proved this conjecture for all m, n odd that not relatively prime. [972] new

Ponnappan, Nagaraj, and Prabakaran [2403] say a vertex magic labeling f of a graph $G(V, E)$ is an *odd vertex magic* if f maps V to $\{1, 3, 5, \ldots, 2|V|-1\}$ and E to $\{1, 2, 3, 4, \ldots, |V|+|E|\} - \{1, 3, 5, \ldots, 2|V|-1\}$ if $|E| \geqslant |V|-1$) and otherwise f maps E to $\{2, 4, 6, \ldots, 2|E|\}$ and V to $\{1, 2, 3, 4, \ldots, |V|+|E|\} - \{2, 4, 6, \ldots, 2|E|\}$. They prove that P_n ($n \geqslant 3$), C_n and mC_3 are odd vertex magic if and only if n is odd, $(3, t)$-kites are vertex magic if and only if t is even, and $C_n \odot K_1$ are not odd vertex for all n.

A triplet $[H, \phi, t]$ is called a *supermagic frame* of G if ϕ is a homomorphism of H onto G and $t : E(H) \to \{1, 2, \ldots, |E(H)|\}$ is an injective mapping such that the sum of $t(uw)$ over all $u \in \phi^{-1}(v)$ is independent of the vertex $v \in V(G)$. In 2000, Ivančo proved that if there is a supermagic frame of a graph G, then G is supermagic. Singhun, Boonklurb, and Charnsamorn [2982] construct a supermagic frame of $m \geqslant 2$ copies of the Cartesian product of cycles and show that m copies of the Cartesian product of cycles is supermagic.

A *prime-magic labeling* is a magic labeling for which every label is a prime. Sedláček

[2733] proved that the smallest magic constant for prime-magic labeling of $K_{3,3}$ is 53 while Bača and Holländer [331] showed that the smallest magic constant for a prime-magic labeling of $K_{4,4}$ is 114. Letting σ_n be the smallest natural number such that $n\sigma_n$ is equal to the sum of n^2 distinct prime numbers we have that the smallest magic constant for a prime-magic labeling of $K_{n,n}$ is σ_n. Bača and Hollaänder [331] conjecture that for $n \geqslant 5$, $K_{n,n}$ has a prime-magic labeling with magic constant σ_n. They proved the conjecture for $5 \leqslant n \leqslant 17$ and confirmed the conjecture for $n = 5$, 6 and 7.

Characterizations of regular magic graphs were given by Doob [847] and necessary and sufficient conditions for a graph to be magic were given in [1378], [1543], and [805]. Some sufficient conditions for a graph to be magic are given in [845], [3223], and [2202]. Bertault, Miller, Pé-Rosés, Feria-Puron, and Vaezpour [549] provided a heuristic algorithm for finding magic labelings for specific families of graphs. The notion of magic graphs was generalized in [846] and [2693].

Let $m, n, a_1, a_2, \ldots, a_m$ be positive integers where $1 \leqslant a_i \leqslant \lfloor n/2 \rfloor$ and the a_i are distinct. The *circulant graph* $C_n(a_1, a_2, \ldots, a_m)$ is the graph with vertex set $\{v_1, v_2, \ldots, v_m\}$ and edge set $\{v_i v_{i+a_j} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ where addition of indices is done modulo n. In [2748] Semaničová characterizes magic circulant graphs and 3-regular supermagic circulant graphs. In particular, if $G = C_n(a_1, a_2, \ldots, a_m)$ has degree r at least 3 and $d = \gcd(a_1, n/2)$ then G is magic if and only if $r = 3$ and $n/d \equiv 2 \pmod 4$, $a_1/d \equiv 1 \pmod 2$, or $r \geqslant 4$ (a necessary condition for $C_n(a_1, a_2, \ldots, a_m)$ to be 3-regular is that n is even). In the 3-regular case, $C_n(a_1, n/2)$ is supermagic if and only $n/d \equiv 2 \pmod 4$, $a_1/d \equiv 1 \pmod 2$ and $d \equiv 1 \pmod 2$. Semaničová also notes that a bipartite graph that is decomposable into an even number of Hamilton cycles is supermagic. As a corollary she obtains that $C_n(a_1, a_2, \ldots, a_{2k})$ is supermagic in the case that n is even, every a_i is odd, and $\gcd(a_{2j-1}, a_{2j}, n) = 1$ for $i = 1, 2, \ldots, 2k$ and $j = 1, 2, \ldots, k$.

Ivančo, Kovář, and Semaničová-Feňovčková [1344] characterize all pairs n and r for which an r-regular supermagic graph of order n exists. They prove that for positive integers r and n with $n \geqslant r + 1$ there exists an r-regular supermagic graph of order n if and only if one of the following statements holds: $r = 1$ and $n = 2$; $3 \leqslant r \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 4$; and $4 \leqslant r \equiv 0 \pmod 2$ and $n > 5$. The proof of the main result is based on finding supermagic labelings of circulant graphs. The authors construct supermagic labelings of several circulant graphs.

In [1338] Ivančo completely determines the supermagic graphs that are the disjoint unions of complete k-partite graphs where every partite set has the same order.

Trenklér [3225] extended the definition of supermagic graphs to include hypergraphs and proved that the complete k-uniform n-partite hypergraph is supermagic if $n \neq 2$ or 6 and $k \geqslant 2$ (see also [3226]). In [3094] Sugiyama gave a generalized definition of magic graphs, for which any number of digits can be used to label a vertex and edge, and described the construction of such magic graphs and their properties. He determined the minimum and maximum magic sums for regular graphs, including polygons and polyhedrons, and provided techniques for transforming and synthesizing magic graphs using an affine transform.

For connected graphs of size at least 5, Ivančo, Lastivkova, and Semaničová [1345] provide a forbidden subgraph characterization of the line graphs that can be magic. As a corollary they obtain that the line graph of every connected graph with minimum degree at least 3 is magic. They also prove that the line graph of every bipartite regular graph of degree at least 3 is supermagic.

For any non-trivial abelian group A under addition, a graph G is said to be *strong A-magic* if there exists a labeling f of the edges of G with non-zero elements of A such that the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant, and the constant is same for all possible values of $|V(G)|$. Stella Arputha Mary, Navaneethakrishnan, and Nagarajan [3052] provide strong Z_4-magic labelings for various graphs and strong Z_{4p}-magic labelings for those graphs.

In [2608] Razzaq, Rizvi, and Ali introduce the concept of an *H-group magic total* labeling of a graph G over a finite Abelian group A as a bijection $\lambda : V(G) \cup E(G) \to A$ such that for any subgraph $H'(V', E')$ of G isomorphic to H, the sum $\sum_{v \in V'} \lambda(v) + \sum_{e \in E'} \lambda(e)$ is equal to magic constant k'. A graph is called *H-group magic* if it admits an H-group magic total labeling. They determine the H-group magic total labelings of fan graphs over the finite Abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_t$, where $t \geqslant 3$ and show that disjoint union of isomorphic as well as non-isomorphic copies of fan graphs are H-group magic over $A \cong \mathbb{Z}_3 \times \mathbb{Z}_t$.

For a graph $G(V, E)$ and a set of positive integers S with $|S| = |V|$, Godinho, Singh, and Arumugam [1112] say G is *S-magic* if there exists a bijection $\phi : V \to S$ such that [1112] new
$\sum \phi(u)$ over all $u \in N(v)$ is a constant k for all $v \in V$. They proved that if G is S-magic, then the corresponding magic constant is unique. They proved that several families of graphs are S-magic and that several families are not S-magic. They also determined the set of all S-magic constants for certain classes of graphs for different label sets S.

For a natural number h, Salehi [2675] defines a graph G to be h-magic if there is a labeling α from the edges of G to the nonzero integers in Z_h such that for each vertex v in G the sum of all α values of edges incident to v is a constant (called the *magic sum index*) that is independent of the choice of v. If the constant is 0, G is called a *zero-sum h-magic graph*. The *null set* of graph G is the set of all natural numbers h for which G admits a zero-sum h-magic labeling. In [2675] Salehi determines the null sets for K_n, $K_{m,n}$, C_n, books, and cycles with a P_k chord. Lin and Wang [1951] determine the null sets of generalized wheels and generalized fans, and construct infinitely many examples of Z_h-magic graphs with magic sum zero and present some open problems.

In 2020, Kamatchi, Paramasivam, Prajeesh, Sabeel, and Arumugam [1569] introduced [1569] new
the notion group vertex magic graphs as follows. For a simple undirected graph G and an additive Abelian group A with identity 0, a mapping f from the vertices of G to the nonzero elements of A is said to be an *A-vertex magic* labeling of G if there exists a μ in A such that $w(v) = \sum f(u)$ taken over all $u \in N(v)$ is μ for all vertices v of G. If G admits such a labeling it is called an *A-vertex magic graph*. If G is an A-vertex magic for every non-trivial Abelian group A, G is called a *group vertex magic* graph. They obtained some necessary conditions for a graph to be group vertex magic and gave a characterization of trees with diameter at most 4 that are $Z_2 \times Z_2$-vertex magic. In [1656] Karthik, Sabeel, [1656] new
and Paramasivama provided some necessary conditions for the group vertex magicness

of graphs with at least one pendant edge and for the group vertex magicness of product graphs. They also characterized group vertex magicness of trees of diameter up to 5 for all infinite Abelian groups with finitely many elements of finite order. Sabeel, Paramasivam, Prajeesh, Kamatchi, and Arumugam [2665] characterized the $Z_2 \times Z_2$-vertex magicness [2665] new of any tree with diameter 5. They further characterized A-vertex magic trees of diameter at most 5 for any finite abelian group A and proved that A-vertex magic graphs do not possess any forbidden structures. They gave a method for constructing larger A-vertex magic graphs from the existing ones. Balamoorthy, Bharanedhar, and Kamatchi [277] [277] new obtained various results about the A-vertex magicness for graphs formed using joins, tensor products, and lexicographic products of graphs. Kavitha and Stella Arputha Mary [1678] say a vertex magic labeling of V_4 $(Z_2 \times Z_2)$ is a *hefty* V_4-magic if its magic number [1678] new is 1. They proved regular graphs, K_n $(n \geqslant 3)$. vertex transitive graphs, and $C_m \times C_n$ are hefty V_4-magic graphs and connected acyclic graphs are not V_4-magic graphs. Stella Arputha Mary, Navaneetha Krishnan, and Nagarajan [2108] proved that triangular snakes, [2108] new books, the one-point union of the apexes of t fans, and the splitting graph of paths, are Z_{4p}-magic graphs. In [1945] Liao and and Liu provided characterizations of unicyclic [1945] new graphs with diameter at most 4 that are A-vertex magic and a characterization of bicyclic graphs of diameter 3 thatb are group vertex magic.

In 1976 Sedláček [2733] defined a connected graph with at least two edges to be *pseudo-magic* if there exists a real-valued function on the edges with the property that distinct edges have distinct values and the sum of the values assigned to all the edges incident to any vertex is the same for all vertices. Sedláček proved that when $n \geqslant 4$ and n is even, the Möbius ladder M_n is not pseudo-magic and when $m \geqslant 3$ and m is odd, $C_m \times P_2$ is not pseudo-magic.

A *vertex magic total* labeling of a graph with p vertices and q edges is a bijection from the union of the vertex set and edge set to the consecutive integers $1, 2, \ldots, p+q$ with the property that for every vertex u, the sum of the label of u and the labels of the edges incident with u is a constant k. A vertex magic total labeling is said to be *a-vertex multiple magic* if the set of the labels of the vertices is $\{a, 2a, \ldots, na\}$ and is *b-edge multiple magic b-edge multiple magic* if the set of labels of the edges is $\{b, 2b, \ldots, mb\}$. Nagaraj, Ponnappan, and Prabakaran [2251] provide properties of a-vertex multiple magic graphs and b-edge multiple magic graphs. In [3579] Zhang and Wang verify the existence of E-super vertex magic total labeling for odd regular graphs containing a particular 3-factor. Listiana, Darmaji, and Slamin [1959] investigated the existence of vertex magic [1959] new total labelings of directed sun graphs $S_n = C_n \odot K_1$ and mS_n.

Kong, Lee, and Sun [1736] used the term "magic labeling" for a labeling of the edges with nonnegative integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. In particular, the edge labels need not be distinct. They let $M(G)$ denote the set of all such labelings of G. For any L in $M(G)$, they let $s(L) = \max\{L(e) \mid e \in E\}$ and define the *magic strength* of G as $m(G) = \min\{s(L) \mid L \in M(G)\}$. To distinguish these notions from others with the same names and notation, which we will introduced in the next section for labelings from the set of vertices and edges, we call the Kong, Lee, and Sun version the *edge magic strength* and use $em(G)$

for min$\{s(L)\colon L$ in $M(G)\}$ instead of $m(G)$. Kong, Lee, and Sun [1736] use $DS(k)$ to denote the graph obtained by taking two copies of $K_{1,k}$ and connecting the k pairs of corresponding leafs. They show: for $k > 1$, $em(DS(k)) = k - 1$; $em(P_k + K_1) = 1$ for $k = 1$ or 2, $em(P_k + K_1) = k$ if k is even and greater than 2, and 0 if k is odd and greater than 1; for $k \geqslant 3$, $em(W(k)) = k/2$ if k is even and $em(W(k)) = (k-1)/2$ if k is odd; $em(P_2 \times P_2) = 1$, $em(P_2 \times P_n) = 2$ if $n > 3$, $em(P_m \times P_n) = 3$ if m or n is even and greater than 2; $em(C_3^{(n)}) = 1$ if $n = 1$ (Dutch windmill, – see §2.4), and $em(C_3^{(n)}) = 2n - 1$ if $n > 1$. They also prove that if G and H are magic graphs then $G \times H$ is magic and $em(G \times H) = \max\{em(G), em(H)\}$ and that every connected graph is an induced subgraph of a magic graph (see also [891] and [932]). They conjecture that almost all connected graphs are not magic.

In [1284] Ichishima, López, Muntaner-Batle, and Takahashi introduce the parameter [1284] new
$l(n)$ as the minimum size of a graph G of order n for which all graphs of order n and size at least $l(n)$ have $\mu_s(G) = +\infty$, and provide lower and upper bounds for $l(G)$. Imran, Baig, and Feňovčiková, [1318] established that for $n = 0 \pmod 4$, $\mu_s(C_n \times K_2) \leqslant 3n/2-1$. [1318] new
Ichishima, López, Muntaner-Batle, and Takahashi, improve this bound by showing that $\mu_s(n)+1$ when $\geqslant 4$ is even. Enomoto, Lladó, Nakamigawa, and Ringel [891] posed the conjecture that every nontrivial tree is super edge-magic. They propose a new approach to attack this conjecture. They believe that their approach may also help to resolve the conjecture by Graham and Sloane that every nontrivial tree is harmonious [1147]. Huang, Hanif, Siddiqui, and Nadeem, [1273] showed the super edge-magicness of certain types of [1273] new
generalized combs and disjoint unions of generalized combs and stars.

Recall a *lexicographic product* of two graphs G_1 and G_2 is a graph that arises from G_1 by replacing each vertex of G_1 by a copy of the G_2 and each edge of G_1 with $K_{n,n}$ where n is the order of G_2. Sun and Lee [3130] show that the Cartesian, conjunctive, normal, lexicographic, and disjunctive products of two magic graphs are magic and the sum of two magic graphs is magic. They also determine the edge magic strengths of the products and sums in terms of the edge magic strengths of the components graphs. In [1878] Lee, Saba, and Sun show that the edge magic strength of P_n^k is 0 when k and n are both odd.

In [136] Akka and Warad define the *super magic strength* of a graph G, sms(G) as the minimum of all magic constants $c(f)$ where the minimum is taken over all super magic labeling f of G if there exist at least one such super magic labeling. They determine the super magic strength of paths, cycles, wheels, stars, bistars, P_n^2, $< K_{1,n} : 2 >$ (the graph obtained by joining the centers of two copies of $K_{1,n}$ by a path of length 2), and $(2n+1)P_2$.

For a simple graph $G(V,E)$ a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V|+|E|\}$ is said to be *edge-magic total labeling* of G, if there exists an integer k such that $f(u)+f(uv)+f(v) = k$ for every edge $uv \in E$. If, in addition, $f(V) = \{1, 2, \ldots, |V|\}$, f is said to be an *super edge-magic total labeling*. Deeothi [795] investigated the super edge-magic total strength [795] new
of the family of unicyclic graphs having an odd cycle a varying number of pendant vertices adjacent to each vertex.

A *Halin graph* ia a planar 3-connected graphs that consist of a tree and a cycle connecting the end vertices of the tree. Let G be a (p,q)-graph in which the edges are labeled $k, k+1, \ldots, k+q-1$, where $k \geqslant 0$. In [1896] Lee, Su, and Wang define a graph with p vertices to be *k-edge-magic* for every vertex v the sum of the labels of the incident edges at v are constant modulo p. They investigate some classes of Halin graphs that are k-edge-magic. Lee, Su, and Wang [1898] investigated some classes of cubic graphs that are k-edge-magic andb provided a counterexample to a conjecture that any cubic graph of order $p \equiv 2 \pmod 4$ is k-edge-magic for all k. Shiu and Lau [2912] gave some necessary conditions for families of wheels with certain spokes missing to admit k-edge-magic labelings.

Lau, Alikhani, Lee, and Kocay [1802] (see also [159]) show that maximal outerplanar graphs of orders $p = 4, 5, 7$ are k-edge magic if and only if $k \equiv 2 \pmod p$ and determined all maximal outerplanar graphs that are k-edge magic for $k = 3$ and 4. They also characterize all $(p, p-h)$-graphs that are k-edge magic for $h \geqslant 0$ and conjecture that a maximal outerplanar graph of prime order p is k-edge magic if and only if $k \equiv 2 \pmod p$.

S. M. Lee and colleagues [1917] and [1849] call a graph G *k-magic* if there is a labeling from the edges of G to the set $\{1, 2, \ldots, k-1\}$ such that for each vertex v of G the sum of all edges incident with v is a constant independent of v. The set of all k for which G is k-magic is denoted by $\mathrm{IM}(G)$ and called the *integer-magic spectrum* of G. In [1917] Lee and Wong investigate the integer-magic spectrum of powers of paths. They prove: $\mathrm{IM}(P_4^2)$ is $\{4, 6, 8, 10, \ldots\}$; for $n > 5$, $\mathrm{IM}(P_n^2)$ is the set of all positive integers except 2; for all odd $d > 1$, $\mathrm{IM}(P_{2d}^d)$ is the set of all positive integers except 1; $\mathrm{IM}(P_4^3)$ is the set of all positive integers; for all odd $n \geqslant 5$, $\mathrm{IM}(P_n^3)$ is the set of all positive integers except 1 and 2; and for all even $n \geqslant 6$, $\mathrm{IM}(P_n^3)$ is the set of all positive integers except 2. For $k > 3$ they conjecture: $\mathrm{IM}(P_n^k)$ is the set of all positive integers when $n = k+1$; the set of all positive integers except 1 and 2 when n and k are odd and $n \geqslant k$; the set of all positive integers except 1 and 2 when n and k are even and $k \geqslant n/2$; the set of all positive integers except 2 when n is even and k is odd and $n \geqslant k$; and the set of all positive integers except 2 when n and k are even and $k \leqslant n/2$. In [1894] Lee, Su, and Wang showed that besides the natural numbers there are two types of the integer-magic spectra of honeycomb graphs. Fu, Jhuang and Lin [989] determine the integer-magic spectra of graphs obtained from attaching a path of length at least 2 to the end vertices of each edge of a cycle.

In [1849] Lee, Lee, Sun, and Wen investigated the integer-magic spectrum of various graphs such as stars, double stars (trees obtained by joining the centers of two disjoint stars $K_{1,m}$ and $K_{1,n}$ with an edge), wheels, and fans. In [2678] Salehi and Bennett report that a number of the results of Lee et al. are incorrect and provide a detailed accounting of these errors as well as determine the integer-magic spectra of caterpillars. Shiu and Low [2929] determined the integer-magic spectra and null sets of the Cartesian product of two trees.

Lee, Lee, Sun, and Wen [1849] use the notation $C_m@C_n$ to denote the graph obtained by starting with C_m and attaching paths P_n to C_m by identifying the endpoints of the paths with each successive pairs of vertices of C_m. They prove that $\mathrm{IM}(C_m@C_n)$ is the set of all positive integers if m or n is even and $\mathrm{IM}(C_m@C_n)$ is the set of all even positive integers if m and n are odd.

Lee, Valdés, and Ho [1905] investigate the integer magic spectrum for special kinds of trees. For a given tree T they define the *double tree* DT of T as the graph obtained by creating a second copy T^* of T and joining each end vertex of T to its corresponding vertex in T^*. They prove that for any tree T, IM(DT) contains every positive integer with the possible exception of 2 and IM(DT) contains all positive integers if and only if the degree of every vertex that is not an end vertex is even. For a given tree T they define ADT, the *abbreviated double tree of* T*,* as the the graph obtained from DT by identifying the end vertices of T and T^*. They prove that for every tree T, IM(ADT) contains every positive integer with the possible exceptions of 1 and 2 and IM(ADT) contains all positive integers if and only if T is a path.

Lee, Salehi, and Sun [1880] have investigated the integer-magic spectra of trees with diameter at most four. Among their findings are: if $n \geqslant 3$ and the prime power factorization of $n-1 = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, then $\mathrm{IM}(K_{1,n}) = p_1\mathbb{N} \cup p_2\mathbb{N} \cup \cdots \cup p_k\mathbb{N}$ (here $p_i\mathbb{N}$ means all positive integer multiples of p_i); for $m, n \geqslant 3$, the double star $\mathrm{IM}(DS(m,m))$ (that is, stars $K_{m,1}$ and $K_{n,1}$ that have an edge in common) is the set of all natural numbers excluding all divisors of $m-2$ greater than 1; if the prime power factorization of $m-n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ and the prime power factorization of $n-2 = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, (the exponents are permitted to be 0) then $\mathrm{IM}(DS(m,n)) = A_1 \cup A_2 \cup \cdots \cup A_k$ where $A_i = p_i^{1+s_i}\mathbb{N}$ if $r_i > s_i \geqslant 0$ and $A_i = \emptyset$ if $s_i \geqslant r_i \geqslant 0$; for $m, n \geqslant 3$, $\mathrm{IM}(DS(m,n)) = \emptyset$ if and only if $m-n$ divides $n-2$; if $m, n \geqslant 3$ and $|m-n| = 1$, then $DS(m,n)$ is not magic. Lee and Salehi [1879] give formulas for the integer-magic spectra of trees of diameter four but they are too complicated to include here.

For a graph $G(V,E)$ and a function f from the V to the positive integers, Salehi and Lee [2682] define the *functional extension of* G *by* f, as the graph H with $V(H) = \cup\{u_i |\ u \in V(G) \text{ and } i = 1,2,\ldots,f(u)\}$ and $E(H) = \cup\{u_iu_j |\ uv \in E(G), i = 1,2,\ldots,f(u); j = 1,2,\ldots,f(v)\}$. They determine the integer-magic spectra for P_2, P_3, and P_4.

A *reverse edge magic* (REM) labeling of a graph $G(V,E)$ with p vertices and q edges is a bijection $f : V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ such that $k = f(uv) - (f(u)+f(v))$ is a constant k for any edge $uv \in E(G)$. A REM labeling f is called *reverse super edge magic* (RSEM) labeling if $f(V(G)) = \{1,2,3,\ldots,p\}$ and $f(E(G)) = \{p+1, p+2, p+3, \ldots, p+q\}$. Reddy and Sharief Basha [2609] find some new classes of RSEM labeling and investigate the connection between the RSEM labeling and different classes of labelings.

More specialized results about the integer-magic spectra of amalgamations of stars and cycles are given by Lee and Salehi in [1879].

Table 5 summarizes the state of knowledge about magic-type labelings. In the table, **SM** means semi-magic, **M** means magic, and **SPM** means supermagic. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Table 5: **Summary of Magic Labelings**

Graph	*Types*	*Notes*
K_n	M SPM	if $n = 2,\ n \geqslant 5$ [3053] for $n \geqslant 5$ iff $n > 5$ $n \not\equiv 0 \pmod 4$ [3054]
$K_{m,n}$	SM	if $n \geqslant 3$ [3053]
$K_{n,n}$	M	if $n \geqslant 3$ [3053]
fans f_n	M not SM	iff n is odd, $n \geqslant 3$ [3053] if $n \geqslant 2$ [1076]
wheels W_n	M SM	if $n \geqslant 4$ [3053] if $n = 5$ or 6 [1076]
wheels with one spoke deleted	M	if $n = 4$, $n \geqslant 6$ [3053]
null graph with n vertices		
Möbius ladders M_n	SPM	if $n \geqslant 3$, n is odd [2733]
$C_n \times P_2$	not SPM	for $n \geqslant 4$, n even [2733]
$C_m[\overline{K}_n]$	SPM	if $m \geqslant 3$, $n \geqslant 2$ [2908]
$K_{\underbrace{n,n,\ldots,n}_{p}}$	SPM	$n \geqslant 3$, $p > 5$ and $p \not\equiv 0 \pmod 4$ [2908]
composition of r-regular SPM graph and $\overline{K}_n$	SPM	if $n \geqslant 3$ [2908]
$K_k[\overline{K}_n]$	SPM	if $k = 3$ or 5, $n = 2$ or n odd [1249]
$mK_{n,n}$	SPM	for $n \geqslant 2$ iff n is even or both n and m are odd [2905]
Q_n	SPM	iff $n = 1$ or $n > 2$ even [1338]

Continued on next page

Table 5 – *Continued from previous page*

Graph	*Types*	*Notes*
$C_m \times C_n$	SPM	$m = n$ or m and n are even [1338]
$C_m \times C_n$	SPM?	for all m and n [1338]
connected (p, q)-graph other than P_2	M	iff $5p/4 < q \leqslant p(p-1)/2$ [3224]
G^i	M	$\|G\| \geqslant 5$, $i \geqslant 3$ [3227]
G^2	M	$G \neq P_5$ and G does not have a 1-factor whose every edge is incident with an end-vertex of G [3227]
$K_{1,m,n}$	M	for all m, n [2759]
P_n^2	M	for all n except 2, 3, 5 [2759], [1076]
$G \times H$	M	iff G and H are magic [1736]

5.2 Edge-magic Total and Super Edge-magic Total Labelings

In 1970 Kotzig and Rosa [1743] defined a *magic valuation* of a graph $G(V, E)$ as a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V \cup E|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant (called the *magic constant*). This notion was rediscovered by Ringel and Lladó [2626] in 1996 who called this labeling *edge-magic*. To distinguish between this usage from that of other kinds of labelings that use the word magic we will use the term *edge-magic total* labeling as introduced by Wallis [3407] in 2001. (We note that for 2-regular graphs a vertex-magic total labeling is an edge-magic total labeling and vice versa.) Kotzig and Rosa proved: $K_{m,n}$ has an edge-magic total labeling for all m and n; C_n has an edge-magic total labeling for all $n \geqslant 3$ (see also [1108], [2638], [542], and [891]); and the disjoint union of n copies of P_2 has an edge-magic total labeling if and only if n is odd. They further state that K_n has an edge-magic total labeling if and only if $n = 1, 2, 3, 5$, or 6 (see [1744], [755], and [891]) and ask whether all trees have edge-magic total labelings. Wallis, Baskoro, Miller, and Slamin [3411] enumerate every edge-magic total labeling of complete graphs. They also prove that the following graphs are edge-magic total: paths, crowns, complete bipartite graphs, and cycles with a single edge attached to one vertex. Enomoto, Llado, Nakamigana, and Ringel [891] prove that all complete bipartite graphs are edge-magic total. They also show that wheels W_n are not edge-magic total when $n \equiv 3 \pmod 4$ and conjectured that all other wheels are edge-magic total. This conjecture was proved when $n \equiv 0, 1 \pmod 4$ by Phillips, Rees, and Wallis [2393] and when $n \equiv 6$ (mod 8) by Slamin, Bača, Lin, Miller, and Simanjuntak [2996]. Fukuchi [995] verified all cases of the conjecture independently of the work of others. Slamin et al. further show that all fans are edge-magic total. In 2002 Lee and Kong [1845] conjectured that odd star forests are super edge magic. In 2019 Cerioli, Fernandes, Lee, Lintzmayer, Mota, and da Silva [657] proved this conjecture for odd symmetric star forests and proved that odd uniform forests of catterpillars are edge-magic. Afzal, Ather, Baig, and Maheshwari [63] analyzed pyramidion ladders and C_n-books for their super edge-magicness and gave some methods for finding new super edge-magic graphs from existing ones. In [785] Darmaji [785] new and Saputro use $G(+)P_m(+)H$ $(m \geqslant 2)$ to denote the graph obtained by taking one copy of the graphs G, H, and P_m, then connecting one endpoint of P_m to all vertices of G and the other endpoint of P_m to all vertices of H. For any super edge-magic total graphs G, they provide some graphs H such that $G(+)P_m(+)H$ is also super edge-magic total. They further show how to construct a super edge-magic total graph from a super edge-magic total graph by considering a super edge-magic labeling of the original graph. In [1291] [1291] new Ichishima and Muntaner-Batle study the super edge-magicness of graphs of order n with degree sequences: $4, 2, 2, \ldots, 2$. They also investigate the super edge-magic properties of certain families of graphs.

In [1650] Kanwal, Riasat, Imtiaz, Iftikhar, Javed, and Ashraf define a fork as the graph obtained by starting with three paths of length t with vertices $x_{1,j}, x_{2,j}, x_{3,j}, 1 \leqslant j \leqslant t$, a single new edge $x_{2,0}$ adjacent to $x_{2,1}$, an edge joining $x_{1,1}$ and $x_{2,1}$ and an edge joining $x_{2,1}$ and $x_{3,1}$. They gave super edge-magic total labelings and deficiencies of forks, the disjoint union of a fork with a star, a bistar, and a path, and of trees obtained by starting with two copies of P_{2t+1} and adding an edge joining the middle vertex of each path. The super edge-magic total labeling strengths of forks and the trees are also determined. Girija and Karthikeyan [1103] proved that 3 copies of the jelly fish graphs are super edge magic graphs.

Inspired by Kotzig-Rosa's notion, Enomoto, Lladó, Nakamigawa, and Ringel [891] called a graph $G(V, E)$ with an edge-magic total labeling that has the additional property that the vertex labels are 1 to $|V|$ a *super edge-magic total* labeling (SEMT). Kanwal and Kanwal [1649] determined super edge-magic total labelings and deficiencies for forests formed by two sided generalized combs, stars, combs, and banana trees. A two-sided generalized comb $Cb^2_{a,b}$, where b is odd, is obtained from a path P_{a+1} by attaching two paths $P_{(b+1)/2}$ to each of the vertices of degree two and one vertex of degree one of P_{a+1}. In [1645] Kanwal, Azam, and Iftikhar investigate the SEMT strength of generalized comb and the SEMT labeling and deficiency of forests composed of two components, where one of the components for each forest is a generalized comb and other component is star, bistar, comb, or path. In [1647] Kanwal, Imtiaz, Iftikhar, Ashraf, Arshad, Irfan, and Sumbal z studied the super edge-magic deficiency of paths, caterpillars, and the disjoint union of a 2-sided generalized comb with a bistar. They also provide the super edge-magic total strength for a 2-sided generalized comb. Javed, Riasat, and Kanwal [1366] study super edge-magic total labeling and deficiencies of forests consisting of combs, generalized combs, and stars. Their results provide the evidence to support a conjecture proposed by Figueroa-Centeno, Ichishima, and Muntaner-Bartle [937]. Cerioli, Fernandes, Lee, Lintzmayer [656] proved certain forests of stars admit a super edge-magic labeling and that certain forests of caterpillars admit an edge-magic labeling.

Baskoro, Sudarsana, and Cholily [512] provided some constructions of new super edge-magic graphs from some old ones by attaching 1, 2, or 3 pendent vertices and edges. In [1700] Kim introduces a new construction of new super edge-magic graphs by attaching any number pendent vertices and edges under some conditions. In [785] Darmaji and Saputro define a graph $G(+)P_m(+)H$ where $m \geqslant 2$ as a graph obtained by taking one copy of the graphs G and H and P_m, then connect an end point of P_m to all vertices of G and the other end P_m to all vertices of H. For any super edge-magic total graphs G, they provide some graphs H such that $G(+)P_m(+)H$ is also super edge-magic total. They further show how to construct a super edge-magic total graph from a super edge-magic total graph by considering a super edge-magic labeling of the origin graph. One such instance is $(P_{2n} \cup mK_1) + 2K_1$. [785] new

Ringel and Llado [2626] prove that a graph with p vertices and q edges is not edge-magic total if q is even and $p + q \equiv 2 \pmod 4$ and each vertex has odd degree. Ringel and Llado conjecture that trees are edge-magic total. In [502] Baskar Babujee and Rao show that the path with n vertices has an edge-magic total labeling with magic constant $(5n + 2)/2$ when n is even and $(5n + 1)/2$ when n is odd. For stars with n vertices they provide an edge-magic total labeling with magic constant $3n$. In [904] Eshghi and Azimi discuss a zero-one integer programming model for finding edge-magic total labelings of large graphs.

Santhosh [2713] proved that for n odd and at least 3, the crown $C_n \odot P_2$ has an edge-magic total labeling with magic constant $(27n + 3)/2$ and for n odd and at least 3, $C_n \odot P_3$ has an edge-magic total labeling with magic constant $(39n + 3)/2$. Ngurah and Adiwijaya [2286] investigated whether various classes of chain graphs formed from ladders, triangular ladders, diagonal ladders, C_4, and K_4 have an edge-magic or super edge-magic labelings. Baig and Afzal [274] investigated the super edge-magicness of special classes of graphs having maximum magic constant $k = 3p$.

In [955] Freyberg introduced a generalization of edge-magic total labeling that allows multiple labels on the vertices or edges of a graph. He used this new labeling as a tool to construct face-magic labelings of some infinite families of graphs. He then considered the question "Given a graph G, for which $a, b, c \in \{0, 1\}$ does G admit a face-magic labeling of type (a, b, c)? He completely answered this question for two families of chained cycles, ladders and subdivided ladders, fans and subdivided fans, and wheels and subdivided wheels. See [956] for some corrections for these results. Freyberg [957] provided $(1, 1, 1)$-face-magic labelings for square tilings, hexagon tilings on a torus, and a special class of triangle tilings on a cylinder. [956] new [957] new

Ahmad, Baig, and Imran [100] define a *zig-zag triangle* as the graph obtained from the path $x_1, x_2, \ldots, x_n$ by adding n new vertices $y_1, y_2, \ldots, y_n$ and new edges y_1x_1, y_nx_{n-1}; x_iy_i for $1 \leqslant i \leqslant n$; $y_ix_{i-1}y_ix_{i+1}$ for $2 \leqslant i \leqslant n - 1$. They define a graph Cb_n as one obtained from the path $x_1, x_2, \ldots, x_n$ adding $n - 1$ new vertices $y_1, y_2, \ldots, y_{n-1}$ and new edges y_ix_{i+1} for $1 \leqslant i \leqslant n - 1$. The graph Cb^*_n is obtained from the Cb_n by joining a new edge x_1y_1. They prove that zig-zag triangles, graphs that are the disjoint union of a star and a banana tree, certain disjoint unions of stars, and for $n \geqslant 4$, $Cb_n^* \cup Cb_{n-1}$ are super edge-magic total. Baig, Afzal, Imran, and Javaid [275] investigate the existence of super edge-magic labeling of volvox and pancyclic graphs.

The *super edge-magic deficiency* of a graph G, denoted by $\mu_s(G)$, is either the minimum nonnegative integer n such that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such n. Krisnawati, Ngurah, Hidayat, and Alghofari [1759] investigated the super edge-magic deficiency of forests whose components are subdivided stars or paths. Imran, Afzal, and Baig investigate the super edge-magic deficiency of volvox and dumbbell type graphs in [1316]. Kanwal, Iftikhar, and Azam [1646] found super edge magic total labelings and deficiencies of forests consisting of two components, where one of the components for each forest is a generalized comb and the other component is a star, bistar, comb, or path. They also investigated the super edge magic total strength of generalized combs.

Let G be a graph with p vertices with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and let S_m be the star with m leaves. If in G, every vertex v_i is identified to the center vertex of S_{m_i}, for some $m_i \geqslant 0$, $1 \leqslant i \leqslant n$, where $S_0 = K_1$, then the graph obtained is denoted by $G_{(m_1,m_2,\ldots,m_p)}$. Let $M(G) = \{(m_1, m_2, \ldots, m_p) \mid G_{(m_1,m_2,\ldots,m_p)}$ is a super edge-magic graph$\}$. The *star super edge-magic deficiency* $S\mu^*(G)$ is defined as

$$S\mu^*(G) = \begin{cases} \min_{(m_1,m_2,\ldots,m_p)}(m_1 + m_2 + \cdots + m_p) & \text{if } M(G) \neq \emptyset, \\ +\infty, \text{ if } M(G) = \emptyset. \end{cases}$$

In [1675] Kathiresan and Sabarimalai Madha determine the star super edge-magic deficiency of certain classes of graphs. In [1760] Krisnawati, Ngurah, Hidayat, and Alghofar showed that some subdivisions of double stars have zero (consecutively) super edge-magic deficiency.

Beardon [518] extended the notion of edge-magic total to countable infinite graphs $G(V, E)$ (that is, $V \cup E$ is countable). His main result is that a countably infinite tree that processes an infinite simple path has a bijective edge-magic total labeling using the integers as labels. He asks whether all countably infinite trees have an edge-magic total labeling with the integers as labels and whether the graph with the integers as vertices and an edge joining every two distinct vertices has a bijective edge-magic total labeling using the integers.

Cavenagh, Combe, and Nelson [663] investigate edge-magic total labelings of countably infinite graphs with labels from a countable Abelian group A. Their main result is that if G is a countable graph that has an infinite set of mutually disjoint edges and A is isomorphic to a countable subgroup of the real numbers under addition then for any k in A there is an edge-magic labeling of G with elements from A that has magic constant k.

Balakrishnan and Kumar [413] proved that the join of $\overline{K_n}$ and two disjoint copies of K_2 is edge-magic total if and only if $n = 3$. Yegnanarayanan [3534] has proved the following graphs have edge-magic total labelings: nP_3 where n is odd; $P_n + K_1; P_n \times C_3$ $(n \geqslant 2)$; the crown $C_n \odot K_1$; and $P_m \times C_3$ with n pendent vertices attached to each vertex of the outermost C_3. He conjectures that for all n, $C_n \odot \overline{K_n}$, the n-cycle with n pendent vertices attached at each vertex of the cycle, and nP_3 have edge-magic total labelings. In fact, Figueroa-Centeno, Ichishima, and Muntaner-Batle, [939] have proved the stronger statement that for all $n \geqslant 3$, the corona $C_n \odot \overline{K_m}$ admits an edge-magic labeling where the set of vertex labels is $\{1, 2, \ldots, |V|\}$. (See also [2073].)

Yegnanarayanan [3534] also introduces several variations of edge-magic labelings and provides some results about them. Kotzig [3409] provides some necessary conditions for graphs with an even number of edges in which every vertex has odd degree to have an edge-magic total labeling. Craft and Tesar [755] proved that an r-regular graph with r odd and $p \equiv 4 \pmod 8$ vertices can not be edge-magic total. Wallis [3407] proved that if G is an edge-magic total r-regular graph with p vertices and q edges where $r = 2^t s + 1$ $(t > 0)$ and q is even, then 2^{t+2} divides p.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [933] have proved the following graphs are edge-magic total: $P_4 \cup nK_2$ for n odd; $P_3 \cup nK_2$; $P_5 \cup nK_2$; nP_i for n odd and $i = 3, 4, 5$; $2P_n$; $P_1 \cup P_2 \cup \cdots \cup P_n$; $mK_{1,n}$; $C_m \odot nK_1$; $K_1 \odot nK_2$ for n even; W_{2n}; $K_2 \times \overline{K}_n$, nK_3 for n odd (the case nK_3 for n even and larger than 2 is done in [2114]); binary trees, generalized Petersen graphs (see also [2288]), ladders (see also [3470]), books, fans, and odd cycles with pendent edges attached to one vertex.

In [939] Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima, investigate super edge-magic total labelings of graphs with two components. Among their results are: $C_3 \cup C_n$ is super edge-magic total if and only if $n \geqslant 6$ and n is even; $C_4 \cup C_n$ is super edge-magic total if and only if $n \geqslant 5$ and n is odd; $C_5 \cup C_n$ is super edge-magic total if and only if $n \geqslant 4$ and n is even; if m is even with $m \geqslant 4$ and n is odd with $n \geqslant m/2 + 2$, then $C_m \cup C_n$ is super edge-magic total; for $m = 6, 8$, or 10, $C_m \cup C_n$ is super edge-magic total if and only if $n \geqslant 3$ and n is odd; $2C_n$ is strongly felicitous if and only if $n \geqslant 4$ and n is even (the converse was proved by Lee, Schmeichel, and Shee in [1881]); $C_3 \cup P_n$ is super edge-magic total for $n \geqslant 6$; $C_4 \cup P_n$ is super edge-magic total if and only if $n \neq 3$; $C_5 \cup P_n$ is super edge-magic total for $n \geqslant 4$; if m is even with $m \geqslant 4$ and $n \geqslant m/2+2$ then $C_m \cup P_n$ is super edge-magic total; $P_m \cup P_n$ is super edge-magic total if and only $(m, n) \neq (2, 2)$ or(3,3); and $P_m \cup P_n$ is edge-magic total if and only $(m, n) \neq (2, 2)$. In [2634] Rizvi, Ali, Iqbal, and Gulraze give super edge-magic total labelings of forests whose components are caterpillars and stars, forests whose components are stars and banana trees, and a new families of trees.

Enomoto, Llado, Nakamigawa, and Ringel [891] conjecture that if G is a graph of order $n + m$ that contains K_n, then G is not edge-magic total for $n \gg m$. Wijaya and Baskoro [3470] proved that $P_m \times C_n$ is edge-magic total for odd n at least 3. Ngurah and Baskoro [2288] state that $P_2 \times C_n$ is not edge-magic total. Hegde and Shetty [1225] have shown that every T_p-tree (see §4.4 for the definition) is edge-magic total. Ngurah, Simanjuntak, and Baskoro [2298] show that certain subdivisions of the star $K_{1,3}$ have edge-magic total labelings. Ali, Hussain, Shaker, and Javaid [156] provide super edge-magic total labelings of subdivisions of stars $K_{1,p}$ for $p \geqslant 5$. In [2293] Ngurah, Baskoro, Tomescu gave methods for construction new (super) edge-magic total graphs from old ones by adding some new pendent edges. They also proved that $K_{1,m} \cup {P_n}^m$ is super edge-magic total. Wallis [3407] proves that a cycle with one pendent edge is edge-magic total. In [3407] Wallis poses a large number of research problems about edge-magic total graphs.

For $n \geqslant 3$, López, Muntaner-Batle, and Rius-Font [1993] (see [1994] for (corrigendum) let S_n denote the set of all super edge-magic total 1-regular labeled digraphs of order n where each vertex takes the name of the label that has been assigned to it. For $\pi \in S_n$. they define a generalization of generalized Petersen graphs that they denote by $GGP(n;\pi)$, which consists of an outer n-cycle $x_0, x_1, \ldots, x_{n-1}, x_0$, a set of n-spokes $x_i y_i$, $0 \leqslant i \leqslant n-1$, and n inner edges defined by $y_i y_{\pi(i)}, i = 0, \ldots, n-1$. Notice that, for the permutation π defined by $\pi(i) = i + k \pmod{n}$ we have $GGP(n;\pi) = P(n;k)$. They define a second generalization of generalized Petersen graphs, $GGP(n;\pi_2, \ldots, \pi_m)$, as the graphs with vertex sets $\cup_{j=1}^{m} \{x_i^j : i = 0, \ldots, n-1\}$, an outer n-cycle $x_0^1, x_1^1, \ldots, x_{n-1}^1, x_0^1$, and inner edges $x_i^{j-1} x_i^j$ and $x_i^j x_{\pi_j(i)}^j$, for $j = 2, \ldots, m$, and $i = 0, \ldots, n-1$. Notice that, $GGP(n;\pi_2, \ldots, \pi_m) = P_m \times C_n$, when $\pi_j(i) = i + 1 \pmod{n}$ for every $j = 2, \ldots, m$. Among their results are the Petersen graphs are super edge-magic total; for each m with $1 < l \leqslant m$ and $1 \leqslant k \leqslant 2$, the graph $GGP(5;\pi_2, \ldots, \pi_m)$, where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \sigma_k$, is super edge-magic total; for each $1 \leqslant k \leqslant 2$, the graph $P(5n; k + 5r)$ where r is the smallest integer such that $k + 5r = 1 \pmod{n}$ is super edge-magic total.

A w-graph, $W(n)$, has vertices $\{(c_1, c_2, b, w, d) \cup (x^1, x^2, \ldots, x^n) \cup (y^1, y^2, \ldots, y^n)\}$ and edges $\{(c_1x^1, c_1x^2, \ldots, c_1x^n) \cup (c_2y^1, c_2y^2, \ldots, c_2y^n) \cup (c_1b, c_1w) \cup (c_2w, c_2d)\}$. A w-tree, $WT(n,k)$, is a tree obtained by taking k copies of a w-graph $W(n)$ and a new vertex **a** and joining **a** with in each copy d where $n \geqslant 2$ and $k \geqslant 3$. An *extended w-tree $Ewt(n,k,r)$* is a tree obtained by taking k copies of an extended w-graph $Ew(n,r)$ and a new vertex **a** and joining **a** with the vertex d in each of the k copies for $n \geqslant 2$, $k \geqslant 3$ and $r \geqslant 2$. Super edge-magic total labelings for w-trees, extended w-trees, and disjoint unions of extended w-trees are given in [1363], [1360], and [155]. Javaid, Hussain, Ali, and Shaker [1364] provided super edge-magic total labelings of subdivisions of $K_{1,4}$ and w-trees. Shaker, Rana, Zobair, and Hussain [2856] gave a super edge-magic total labeling for a subdivided star with a center of degree at least 4.

In 1988 Godbod and Slater [1108] made the following conjecture. If n is odd, $n \neq 5$, C_n has an edge magic labeling with valence k, when $(5n+3)/2 \leqslant k \leqslant (7n+3)/2$. If n is even, C_n has an edge-magic labeling with valence k when $5n/2 + 2 \leqslant k \leqslant 7n/2 + 1$. Except for small values of n, very few valences for edge-magic labelings of C_n are known. In [1998] López, Muntaner-Batle, and Rius-Font use the $\otimes_h$-product in order to prove the following two results. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique prime factorization of an odd number n. Then C_n admits at least $1 + \sum_{i=1}^{k} \alpha_i$ edge-magic labelings with at least $1 + \sum_{i=1}^{k} \alpha_i$ mutually different valences. Let $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique prime factorization of an even number n, with $p_1 > p_2 > \cdots > p_k$. Then C_n admits at least $\sum_{i=1}^{k} \alpha_i$ edge-magic labelings with at least $\sum_{i=1}^{k} \alpha_i$ mutually different valences. If $\alpha \geqslant 2$ this lower bound can be improved to $1 + \sum_{i=1}^{k} \alpha_i$. In [1988] López, Muntaner-Batle, and Prabu introduce a new $\otimes_h$ labeling construction that has a wider range of applications and applies it to the magic valences of cycles and crowns.

In [3178] Swita, Rafflesia, Henni Ms, Adji, and Simanihuruk use $B[(C_a, m), (C_b, n), P_t]$ to denote the graph that consists of m cycles C_a and n cycles C_b with a common path P_t. They proved that $B[(C_7, 1), (C_3, n), P_2]$ admits an edge-magic total labeling,

$B[(C_a, 1), (C_3, n), P_2]$ admits a super edge-magic total labeling for all $a \equiv 3 \bmod 4$ $(a > 3)$, and $B[(C_7, 2), (C_3, n), P_2]$ admits a super edge-magic total labeling.

In 1996 Erdős asked for $M(n)$, the maximum number of edges that an edge-magic total graph of order n can have (see [755]). In 1999 Craft and Tesar [755] gave the bound $\lfloor n^2/4 \rfloor \leqslant M(n) \leqslant \lfloor n(n-1)/2 \rfloor$. For large n this was improved by Pikhurko [2397] in 2006 to $2n^2/7 + O(n) \leqslant M(n) \leqslant (0.489 + \cdots + o(1)n^2)$.

Enomoto, Lladó, Nakamigawa, and Muntaner-Batle [891] proved that a super edge-magic total graph $G(V, E)$ with $|V| \geqslant 4$ and with girth at least 4 has at most $2|V| - 5$ edges. They prove this bound is tight for graphs with girth 4 and 5 in [891] and [1308].

In his Ph.D. thesis, Barrientos [438] introduced the following notion. Let $L_1, L_2, \ldots, L_h$ be ordered paths in the grid $P_r \times P_t$ that are maximal straight segments such that the end vertex of L_i is the beginning vertex of L_{i+1} for $i = 1, 2, \ldots, h-1$. Suppose for some i with $1 < i < h$ we have $V(L_i) = \{u_0, v_0\}$ where u_0 is the end vertex of L_{i-1} and the beginning vertex of L_i and v_0 is the end vertex of L_1 and the beginning vertex of L_{i+1}. Let $u \in V(L_{i-1}) - \{u_0\}$ and $v \in V(L_{i+1}) - \{v_0\}$. The replacement of the edge u_0v_0 by a new edge uv is called an *elementary transformation* of the path P_n. A tree is called a *path-like tree* if it can be obtained from P_n by a sequence of elementary transformations on an embedding of P_n in a 2-dimensional grid. In [357] Bača, Lin, and Muntaner-Batle proved that if $T_1, T_2, \ldots, T_m$ are path-like trees each of order $n \geqslant 4$ where m is odd and at least 3, then $T_1 \cup T_2, \cup \cdots \cup T_m$ has a super edge-magic labeling. In [356] Bača, Lin, Muntaner-Batle and Rius-Font proved that the number of such trees grows at least exponentially with m. As an open problem Bača, Lin, Muntaner-Batle and Rius-Font ask if graphs of the form $T_1 \cup T_2 \cup \cdots \cup T_m$ where $T_1, T_2, \ldots, T_m$ are path-like trees each of order $n \geqslant 2$ and m is even have a super edge-magic labeling. In [438] Barrientos proved that all path-like trees admit an α-valuation. Using Barrientos's result, it is very easy to obtain that all path-like trees are a special kind of super edge-magic by using a super edge-magic labeling of the path P_n, and hence they are also super edge-magic. Furthermore, Figueroa-Centeno, Ichishima, and Muntaner-Batle proved that if a tree is super edge-magic, then it is also harmonious. Therefore all path-like trees are also harmonious. In [1990] López, Muntaner-Batle, and Rius-Font also use a variation of the Kronecker product of matrices in order to obtain lower bounds for the number of non isomorphic super edge-magic labeling of some types of path-like trees. As a corollary they obtain lower bounds for the number of harmonious labelings of the same type of trees. López, Muntaner-Batle, and Rius-Font [1999] proved that if $m \geqslant 4$ is an even integer and $n \geqslant 3$ is an odd divisor of m, then $C_m \cup C_n$ is super edge-magic. Lee and Kong conjecture that if n is an odd, then $St(a_1, a_2, \ldots, a_n)$ is super edge-magic, and they proved that the following graphs are super edge-magic: $St(m, n)$ $(n \equiv 0 \bmod (m+1)), St(1, k, n) (k = 1, 2 \text{ or } n)$, $St(2, k, n)$ $(k = 2, 3)$, $St(1, 1, k, n)$ $(k = 2, 3)$, $St(k, 2, 2, n)$ $(k = 1, 2)$. Zhenbin and Chongjin [3595] proved that $St(1, m, n), St(3, m, m+1), St(n, n+1, n+2)$ are super edge-magic, and under some conditions $St(a_1, a_2, \ldots, a_{2n+1})$, $St(a_1, a_2, \ldots, a_{4n+1})$, $St(a_1, a_2, \ldots, a_{4n+3})$ are also super edge-magic. In [932] Figueroa-Centeno, Ichishima, and Muntaner-Batle conjectured that the cycle books $B(4, m)$ that consists of m cycles C_4 with a common path P_2 is super edge-magic total if and only if m is even or $m \equiv 5 \pmod 8$. Simanihuruk, Kusmayadi, Swita, Romala, and Damanik [2957] proved this conjecture for $m \geqslant 36$ and m even.

For a simple graph H we say that $G(V,E)$ admits an H-*covering* if every edge in $E(G)$ belongs to a subgraph of G that is isomorphic to H. In [2001] López, Muntaner-Batle, Rius-Font study a relationship existing among (super) magic coverings and the Kronecker product of matrices. (For a simple graph H, $G(V,E)$ admits an H-*covering* if every edge in $E(G)$ belongs to a subgraph of G that is isomorphic to H.) Their results can be applied to construct S-magic partitions. For m copies of a graph G and a fixed subgraph H of each copy the graph $I(G,H,m)$ is formed by taking of all the G_i's and identifying their subgraph H. Liang [1939] determined which $I(G,H,m)$ and which mG have G supermagic coverings. Farida, Indah, and Sudiby [916] study magic coverings of domino graphs (graphs for which every vertex is contained in at most two maximal cliques) and edge magic labelings of domino graphs. They note that coverings can be applied to secret sharing schemes and edge magic labelings can be applied to ruler models. [916] new

Bača, Lin and Muntaner-Batle in [355] using a generalization of the Kronecker product of matrices prove that the number of non-isomorphic super edge-magic labelings of the disjoint union of m copies of the path P_n, $m \equiv 2 \pmod 4$, $m \geqslant 2$, $n \geqslant 4$, is at least $(m/2)^{(2n-2)}$.

In [1992] López, Muntaner-Batle and Rius-Font proved that every super edge-magic graph with p vertices and q edges where $q \geqslant p-1$ has an even harmonious labeling (See Section 4.6.) In [1997] they stated some open problems concerning relationships among super edge-magic labelings and graceful and harmonious labelings. A *Langford sequence* of order m and defect d is a sequence $(t_1, t_2, \ldots, t_{2m})$ of $2m$ numbers such that (i) for every $k \in [d, d+m-1]$ there exist exactly two subscripts $i, j \in [1, 2m]$ with $ti = tj = k$ and (ii) the subscripts i and j satisfy the condition $|i-j| = k$. López and Muntaner-Batle [1987] provided new lower bounds on the number of distinct Langford sequences with certain properties in terms of the number of 1-regular super edge-magic labeled digraphs of a particular order.

Lee and Lee [1848] prove the following graphs are super edge-magic: $P_{2n} + \overline{K_m}$, $(P_2 \cup nK_1) + \overline{K_2}$, graphs obtained by appending a path to the apex of a fan with at least 4 vertices (*umbrella*), and *jelly fish* graphs $J(m,n)$ obtained from a 4-cycle v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4. In [62] Afzel introduces two new familes of graphs called carrom and jukebox graphs and proves they admit super edge-magic labelings. Carroms are generalizations of $C_n \times P_2$.

Marimuthu and Balakrishnan [2080] define a graph $G(p,q)$ to be *edge magic graceful* if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ such $|f(u)+f(v)-f(uv)|$ is a constant for all edges uv of G. An edge magic graceful graph is said to be *super edge magic graceful* if $V(G) = \{1, 2, \ldots, p\}$. They present some properties of super edge magic graceful graphs, prove some classes of graphs are super edge magic graceful, and prove that every super edge magic graceful graph with either $f(uv) > f(u)+f(v)$ for all edges uv or $f(uv) < f(u)+f(v)$ for all edges uv is sequential, harmonious, super edge magic and not graceful. Marimuthu, Kavitha, and Balakrishnan [2081] proved that the generalized Petersen graphs $P(n,1)$ and $P(n,(n-1)/2)$ are super edge magic graceful when n is odd.

Let $G = (V, E)$ be a (p, q)-linear forest. In [356] Bača, Lin, Muntaner-Batle, and Rius-Font call a labeling f a *strong super edge-magic* labeling of G and G a *strong super edge-magic* graph if $f : V \cup E \to \{1, 2, \dots, p+q\}$ with the extra property that if $uv \in E, u', v' \in V(G)$ and $d_G(u, u') = d_G(v, v') < +\infty$, then we have that $f(u) + f(v) = f(u') + f(v')$. In [110] Ahmad, López, Muntaner-Batle, and Rius-Font define the concept of strong super edge-magic labeling of a graph with respect to a linear forest as follows. Let $G = (V, E)$ be a (p, q)-graph and let F be any linear forest contained in G. A *strong super edge-magic labeling of* G *with respect to* F is a super edge-magic labeling f of G with the extra property with if $uv \in E(F), u', v' \in V(F)$ and $d_F(u, u') = d_F(v, v') < +\infty$ then we have that $f(u) + f(v) = f(u') + f(v')$. If a graph G admits a strong super edge-magic labeling with respect to some linear forest F, they say that G is a *strong super edge-magic graph with respect to* F. They prove that if m is odd and G is an acyclic graph which is strong super edge-magic with respect to a linear forest F, then mG is strong super edge-magic with respect to $F_1 \cup F_2 \cup \cdots \cup F_m$, where $F_i \simeq F$ for $i = 1, 2, \dots, m$ and every regular caterpillar is strong super edge-magic with respect to its spine.

Noting that for a super edge-magic labeling f of a graph G with p vertices and q edges, the magic constant k is given by the formula: $k = (\sum_{u \in V} \deg(u) f(u) + \sum_{i=p+1}^{p+q} i)/q$, López, Muntaner-Batle and Rius-Font [1991] define the set $S_G = \left\{ (\sum_{u \in V} \deg(u) g(u) + \sum_{i=p+1}^{p+q} i)/q : \text{ the function } g : V \to \{i\}_{i=1}^{p} \text{ is bijective} \right\}$. If $\lceil \min S_G \rceil \leqslant \lfloor \max S_G \rfloor$ then the *super edge-magic interval* of G is the set $I_G = [\lceil \min S_G \rceil, \lfloor \max S_G \rfloor] \cap \mathbb{N}$. The *super edge-magic set* of G is $\sigma_G = \{k \in I_G :$ there exists a super edge-magic labeling of G with valence $k\}$. López et al. call a graph G *perfect super edge-magic* if $I_G = \sigma_G$. They show that the family of paths P_n is a family of perfect super edge-magic graphs with $|I_{P_n}| = 1$ if n is even and $|I_{P_n}| = 2$ if n is odd and raise the question of whether there is an infinite family $F_1, F_2, \dots$ of graphs such that each member of the family is perfect super edge-magic and $\lim_{i \to +\infty} |I_{F_i}| = +\infty$. They show that graphs $G \cong C_{p^k} \odot \overline{K_n}$ where $p > 2$ is a prime is such a family. Takahasi, Muntaner-Batle, and Ichishima [3185] investigated the perfect (super) edge-magic deficiency of $K_{1,n}$ [3185] new

In [1992] López et al. define the *irregular crown* $C(n; j_1, j_2, \dots, j_n) = (V, E)$, where $n > 2$ and $j_i \geqslant 0$ for all $i \in \{1, 2, \dots, n\}$ as follows: $V = \{v_i\}_{i=1}^{n} \cup V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_k = \{v_k^1, v_k^2, \dots, v_k^{j_k}\}$, if $j_k \neq 0$ and $V_k = \emptyset$ if $j_k = 0$, for each $k \in \{1, 2, \dots, n\}$ and $E = \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_1 v_n\} \cup (\cup_{k=1, j_k \neq 0}^{n} \{v_k v_k^l\}_{l=1}^{j_k})$. In particular, they denote $C_m^n \cong C(m; j_1, j_2, \dots, j_m)$, where $j_{2i-1} = n$, for each i with $1 \leqslant i \leqslant (m+1)/2$, and $j_{2i} = 0$, for each i, $1 \leqslant i \leqslant (m-1)/2$. They prove that the graphs C_3^n and C_5^n are perfect edge-magic for all $n > 1$.

López et al. [1995] define $\mathfrak{F}^k$-family and $\mathfrak{E}^k$-family of graphs as follows. The infinite family of graphs $(F_1, F_2, \dots)$ is an $\mathfrak{F}^k$*-family* if each element F_n admits exactly k different valences for super edge-magic labelings, and $\lim_{n \to +\infty} |I(F_n)| = +\infty$. The infinite family of graphs $(F_1, F_2, \dots)$ is an $\mathfrak{E}^k$*-family* if each element F_n admits exactly k different valences for edge-magic labelings, and $\lim_{n \to +\infty} |J(F_n)| = +\infty$. An easy observation is that $(K_{1,2}, K_{1,3}, \dots)$ is an $\mathfrak{F}^2$-family and an $\mathfrak{E}^3$-family. They pose the two problems: for which positive integers k is it possible to find $\mathfrak{F}^k$-families and $\mathfrak{E}^k$-families? Their main results in [1995] are that an $\mathfrak{F}^k$-family exits for each $k = 1, 2, 3$; and an $\mathfrak{E}^k$-family exits for each $k = 3, 4$ and 7.

McSorley and Trono [2119] define a relaxed version of edge-magic total labelings of a graph as follows. An *edge-magic injection* μ of a graph G is an injection μ from the set of vertices and edges of G to the natural numbers such that for every edge uv the sum $\mu(u) + \mu(v) + \mu(uv)$ is some constant k_μ. They investigate $\kappa(G)$, the smallest k_μ among all edge-magic injections of a graph G. They determine $\kappa(G)$ in the cases that G is K_2, K_3, K_5, K_6 (recall that these are the only complete graphs that have edge-magic total labelings), a path, a cycle, or certain types of trees. They also show that every graph has an edge-magic injection and give bounds for $\kappa(K_n)$.

Avadayappan, Vasuki, and Jeyanthi [264] define the *edge-magic total strength* of a graph G as the minimum of all constants over all edge-magic total labelings of G. We denote this by $emt(G)$. They use the notation $< K_{1,n} : 2 >$ for the tree obtained from the bistar $B_{n,n}$ (the graph obtained by joining the center vertices of two copies of $K_{1,n}$ with an edge) by subdividing the edge joining the two stars. They prove: $emt(P_{2n}) = 5n+1$; $emt(P_{2n+1}) = 5n+3$; $emt(< K_{1,n} : 2 >) = 4n+9$; $emt(B_{n,n}) = 5n+6$; $emt((2n+1)P_2) = 9n+6$; $emt(C_{2n+1}) = 5n+4$; $emt(C_{2n}) = 5n+2$; $emt(K_{1,n}) = 2n+4$; $emt(P_n^2) = 3n$; and $emt(K_{n,m}) \leqslant (m+2)(n+1)$ where $n \leqslant m$. Using an analogous definition for super edge-magic total strength, Swaninathan and Jeyanthi [3172], [3172], [3173] provide results about the super edge-magic strength of trees, fire crackers, unicyclic graphs, and generalized theta graphs. Basha [289] determined the reverse super edge-magic strength of banana trees.

Ngurah, Simanjuntak, and Baskoro [2298] show that certain subdivisions of the star $K_{1,3}$ have super edge-magic total labelings. In [891] Enomoto, Lladó, Nakamigawa and Ringel conjectured that all trees have a super edge-magic total labeling. Ichishima, Muntaner-Batle, and Rius-Font [1307] have shown that any tree of order p is contained in a tree of order at most $2p - 3$ that has a super edge-magic total labeling.

In [356] Bača, Lin, Muntaner-Batle, and Rius-Font use a generalization of the Kronecker product of matrices introduced by Figueroa-Centeno, Ichishima, Muntaner-Batle, and Rius-Font [941] to obtain an exponential lower bound for the number of non-isomorphic strong super edge-magic labelings of the graph mP_n, for m odd and any n, starting from the strong super edge-magic labeling of P_n. They prove that the number of non-isomorphic strong super edge-magic labelings of the graph mP_n, $n \geqslant 4$, is at least $\frac{5}{2}2^{\lfloor \frac{m}{2} \rfloor} + 1$ where $m \geqslant 3$ is an odd positive integer. This result allows them to generate an exponential number of non-isomorphic super edge-magic labelings of the forest $F \cong \bigcup_{j=1}^{m} T_j$, where each T_j is a path-like tree of order n and m is an odd integer.

López, Muntaner-Batle, and Rius-Font [1989] introduced a generalization of super edge-magic graphs called *super edge-magic models* and prove some results about them.

Yegnanarayanan and Vaidhyanathan [3535] use the term *nice* $(1, 1)$ *edge-magic labeling* for a super edge-magic total labeling. They prove: a super edge-magic total labeling f of a (p, q)-graph G satisfies $2\sum_{v \in V(G)} f(v)deg(v) \equiv 0 \bmod q$; if G is (p, q) r-regular graph $(r > 1)$ with a super edge-magic total labeling then q is odd and the magic constant is

$(4p+q+3)/2$; every super edge-magic total labeling has at least two vertices of degree less than 4; fans $P_n + K_1$ are edge-magic total for all n and super edge-magic total if and only if n is at most 6; books B_n are edge-magic total for all n; a super edge-magic total (p, q)-graph with $q \geqslant p$ is sequential; a super edge-magic total tree is sequential; and a super edge-magic total tree is cordial. These last three results had been proved earlier by Figueroa-Centenoa, Ichishima, and Muntaner-Batle [932].

In [3534] Yegnanarayanan conjectured that the disjoint union of $2t$ copies of P_3 has a $(1, 1)$ edge-magic labeling and posed the problem of determining the values of m and n such that mP_n has a $(1, 1)$ edge-magic labeling. Manickam and Marudai [2073] prove the conjecture and partially settle the open problem.

Hegde and Shetty [1231] (see also [1230]) define the *maximum magic strength* of a graph G as the maximum magic constant over all edge-magic total labelings of G. We use $eMt(G)$ to denote the maximum magic strength of G. Hegde and Shetty call a graph G with p vertices *strong magic* if $eMt(G) = emt(G)$; *ideal magic* if $1 \leqslant eMt(G) - emt(G) \leqslant p$; and *weak magic* if $eMt(G) - emt(G) > p$. They prove that for an edge-magic total graph G with p vertices and q edges, $eMt(G) = 3(p+q+1) - emt(G)$. Using this result they obtain: P_n is ideal magic for $n > 2$; $K_{1,1}$ is strong magic; $K_{1,2}$ and $K_{1,3}$ are ideal magic; and $K_{1,n}$ is weak magic for $n > 3$; $B_{n,n}$ is ideal magic; $(2n+1)P_2$ is strong magic; cycles are ideal magic; and the generalized web $W(t, 3)$ (see §2.2 for the definition) with the central vertex deleted is weak magic.

Santhosh [2713] has shown that for n odd and at least 3, $eMt(C_n \odot P_2) = (27n+3)/2$ and for n odd and at least 3, $(39n+3)/2 \leqslant eMt(C_n \odot P_2) \leqslant (40n+3)/2$. Moreover, he proved that for n odd and at least 3 both $C_n \odot P_2$ and $C_n \odot P_3$ are weak magic. In [711] Chopra and Lee provide an number of families of super edge-magic graphs that are weak magic.

In [2219] Murugan introduces the notions of *almost-magic labeling, relaxed-magic labeling, almost-magic strength*, and *relaxed-magic strength* of a graph. He determines the magic strength of Huffman trees and twigs of odd order and the almost-magic strength of nP_2 (n is even) and twigs of even order. Also, he obtains a bound on the magic strength of the path-union $P_n(m)$ and on the relaxed-magic strength of kS_n and kP_n.

Enomoto, Llado, Nakamigawa, and Ringel [891] call an edge-magic total labeling *super edge-magic* if the set of vertex labels is $\{1, 2, \ldots, |V|\}$ (Wallis [3407] calls these labelings *strongly edge-magic*). They prove the following: C_n is super edge-magic if and only if n is odd; caterpillars are super edge-magic; $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$; and K_n is super edge-magic if and only if $n = 1, 2$, or 3. They also prove that if a graph with p vertices and q edges is super edge-magic then, $q \leqslant 2p - 3$. In [2051] MacDougall and Wallis study super edge-magic (p, q)-graphs where $q = 2p - 3$. Enomoto et al. [891] conjecture that every tree is super edge-magic. Lee and Shan [1889] have verified this conjecture for trees with up to 17 vertices with a computer. Fukuchi, and Oshima, [997] have shown that if T is a tree of order $n \geqslant 2$ such that T has diameter greater than or equal to $n - 5$, then T has a super edge-magic labeling.

Various classes of banana trees that have super edge-magic total labelings have been found by Swaminathan and Jeyanthi [3172] and Hussain, Baskoro, and Slamin [1277]. In

[85] Ahmad, Ali, and Baskoro [85] investigate the existence of super edge-magic labelings of subdivisions of banana trees and disjoint unions of banana trees. They pose three open problems.

Kotzig and Rosa's ([1743] and [1744]) proof that nK_2 is edge-magic total when n is odd actually shows that it is super edge-magic. Kotzig and Rosa also prove that every caterpillar is super-edge magic. Figueroa-Centeno, Ichishima, and Muntaner-Batle prove the following: if G is a bipartite or tripartite (super) edge-magic graph, then nG is (super) edge-magic when n is odd [936]; if m is a multiple of $n+1$, then $K_{1,m} \cup K_{1,n}$ is super edge-magic [936]; $K_{1,2} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 3; $K_{1,m} \cup K_{1,n}$ is edge-magic if and only if mn is even [936]; $K_{1,3} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 4 [936]; $P_m \cup K_{1,n}$ is super edge-magic when $m \geqslant 4$ [936]; $2P_n$ is super edge-magic if and only if n is not 2 or 3; $K_{1,m} \cup 2nK_2$ is super edge-magic for all m and n [936]; $C_3 \cup C_n$ is super edge-magic if and only if $n \geqslant 6$ and n is even [939] (see also [1149]); $C_4 \cup C_n$ is super edge-magic if and only if $n \geqslant 5$ and n is odd [939] (see also [1149]); $C_5 \cup C_n$ is super edge-magic if and only if $n \geqslant 4$ and n is even [939]; if m is even and at least 6 and n is odd and satisfies $n \geqslant m/2+2$, then $C_m \cup C_n$ is super edge-magic [939]; $C_4 \cup P_n$ is super edge-magic if and only if $n \neq 3$ [939]; $C_5 \cup P_n$ is super edge-magic if $n \geqslant 4$ [939]; if m is even and at least 6 and $n \geqslant m/2+2$, then $C_m \cup P_n$ is super edge-magic [939]; and $P_m \cup P_n$ is super edge-magic if and only if $(m,n) \neq (2,2)$ or (3,3) [939]. They [936] conjecture that $K_{1,m} \cup K_{1,n}$ is super edge-magic only when m is a multiple of $n+1$ and they prove that if G is a super edge-magic graph with p vertices and q edges with $p \geqslant 4$ and $q \geqslant 2p-4$, then G contains triangles. In [939] Figueroa-Centeno et al. conjecture that $C_m \cup C_n$ is super edge-magic if and only if $m+n \geqslant 9$ and $m+n$ is odd.

Singgih [2968] gave super edge magic total labelings for unions of books $mB(n)$ for odd m; $m(P_2 \times P_n)$ for m and n odd; $r(P_m \times P_n)$ for odd r and $(m,n) \neq (2,2)$ or (3,3); $r(P_3 \times mP_n)$ for odd r; mP_n for $m \equiv 2 \pmod 4$, $n \neq 2,3$; and mP_{4n} for $m \equiv 2 \pmod 4$, $n > 1$.

In [996] Fukuchi and Oshima describe a construction of super edge-magic labelings of some families of trees with diameter 4. Salman, Ngurah, and Izzati [2688] use S_n^m ($n \geqslant 3$) to denote the graph obtained by inserting m vertices in every edge of the star S_n. They prove that S_n^m is super edge-magic when $m = 1$ or 2.

In [2000] López, Muntaner-Batle, and Ruis-Font introduce a new construction for super edge-magic labelings of 2-regular graphs which allows loops and is related to the knight jump in the game of chess. They also study the super edge-magic properties of cycles with cords.

Muntaner-Batle calls a bipartite graph with partite sets V_1 and V_2 *special super edge-magic* if is has a super edge-magic total labeling f with the property that $f(V_1) = \{1, 2, \ldots, |V_1|\}$. He proves that a tree has a special super edge-magic labeling if and only if it has an α-labeling (see §3.1 for the definition). Figueroa-Centeno, Ichishima, Muntaner-Batle, and Rius-Font [941] use matrices to generate edge-magic total labeling and define the concept of super edge-magic total labelings for digraphs. They prove that if G is a graph with a super edge-magic total labeling then for every natural number d there exists a natural number k such that G has a (k,d)-arithmetic labeling (see §4.2 for the definition). In [1825] Lee and Lee prove that a graph is super edge-magic if and only if it is $(k,1)$-strongly indexable (see §4.3 for the definition of (k,d)-strongly indexable graphs). They also provide a way to construct (k,d)-strongly indexable graphs from two given (k,d)-strongly indexable graphs. This allows them to obtain several existing results about super edge-magic graphs as special cases of their constructions. Acharya and Germina [33] proved that the class of strongly indexable graphs is a proper subclass of super edge-magic graphs.

In [1283] Ichishima, López, Muntaner-Batle and Rius-Font show how one can use the product $\otimes_h$ of super edge-magic 1-regular labeled digraphs and digraphs with harmonious, or sequential labelings to create new undirected graphs that have harmonious, sequential labelings or partitional labelings (see §4.1 for the definition). They define the product $\otimes_h$ as follows. Let $\overrightarrow{D} = (V,E)$ be a digraph with adjacency matrix $A(\overrightarrow{D}) = (a_{ij})$ and let $\Gamma = \{F_i\}_{i=1}^m$ be a family of m digraphs all with the same set of vertices V'. Assume that $h : E \longrightarrow \Gamma$ is any function that assigns elements of Γ to the arcs of D. Then the digraph $\overrightarrow{D} \otimes_h \Gamma$ is defined by $V(D \otimes_h \Gamma) = V \times V'$ and $((a_1,b_1),(a_2,b_2)) \in E(D \otimes_h \Gamma) \Longleftrightarrow [(a_1,a_2) \in E(D) \wedge (b_1,b_2) \in E(h(a_1,a_2))]$. An alternative way of defining the same product is through adjacency matrices, since one can obtain the adjacency matrix of $\overrightarrow{D} \otimes_h \Gamma$ as follows: if $a_{ij} = 0$ then a_{ij} is multiplied by the $p' \times p'$ 0-square matrix, where $p' = |V'|$. If $a_{ij} = 1$ then a_{ij} is multiplied by $A(h(i,j))$ where $A(h(i,j))$ is the adjacency matrix of the digraph $h(i,j)$. They prove the following. Let $\overrightarrow{D} = (V,E)$ be a harmonious (p,q)-digraph with $p \leqslant q$ and let h be any function from E to the set of all super edge-magic 1-regular labeled digraphs of order n, which we denote by S_n. Then the undirected graph $und(\overrightarrow{D} \otimes_h S_n)$ is harmonious. Let $\overrightarrow{D} = (V,E)$ be a sequential digraph and let $h : E \longrightarrow S_n$ be any function. Then $und(\overrightarrow{D} \otimes_h S_n)$ is sequential. Let D be a partitional graph and let $h : E \longrightarrow S_n$ be any function, where $\overrightarrow{D} = (V,E)$ is the digraph obtained by orienting all edges from one stable set to the other one. Then $und(\overrightarrow{D} \otimes_h S_n)$ is partitional.

Marr, Ochel, and Perez [2096] say a digraph D with v vertices and e directed edges has an *in-magic total labeling* if there exists a bijective function λ from $V(D) \cup E(D)$ to $\{1,2,\ldots,v+e\}$ such that for every vertex x we have $\lambda(x)+\sum\lambda(y,x) = k$ for some integer k, where the sum is taken over all directed edges (y,x). They provide such labelings for trees and cycles and discuss some relationships between this labeling and other digraph labelings.

In [1996] López, Muntaner-Batle and Rius-Font introduce the concept of $\{H_i\}_{i\in I}$-*super edge-magic decomposable* as follows: Let $G = (V,E)$ be any graph and let $\{H_i\}_{i\in I}$ be a set of graphs such that $G = \oplus_{i\in I} H_i$ (that is, G decomposes into the graphs in the set $\{H_i\}_{i\in I}$). Then we say that G is $\{H_i\}_{i\in I}$-super edge-magic decomposable if there is a bijection $\beta : V \to [1,|V|]$ such that for each $i \in I$ the subgraph H_i meets the following two requirements: (i) $\beta(V(H_i)) = [1,|V(H_i)|]$ and (ii) $\{\beta(a)+\beta(b) : ab \in E(H_i)\}$ is a set of consecutive integers. Such function β is called an $\{H_i\}_{i\in I}$-*super edge-magic labeling* of G. When $H_i = H$ for every $i \in I$ we just use the notation H-super edge-magic decomposable labeling. Among their results are the following. Let $G = (V,E)$ be a (p,q)-graph which is $\{H_1,H_2\}$-super edge-magic decomposable for a pair of graphs H_1 and H_2. Then G is super edge-bimagic; Let n be an even integer. Then the cycle C_n is $(n/2)K_2$-super edge-magic decomposable if and only if $n \equiv 2 \pmod 4$. Let n be odd. Then for any super edge-magic tree T there exists a bipartite connected graph $G = G(T, n)$ such that G is (nT)-super edge-magic decomposable. Let G be a $\{H_i\}_{i\in I}$-super edge magic decomposable graph, where H_i is an acyclic digraph for each $i \in I$. Assume that $\overrightarrow{G}$ is any orientation of G and $h : E(\overrightarrow{G}) \to S_p$ is any function. Then und$(\overrightarrow{G} \otimes_h S_p)$ is $\{pH_i\}_{i\in I}$-super edge magic decomposable.

As a corollary of the last result they have that if G is a 2-regular, (1-factor)-super edge-magic decomposable graph and $\overrightarrow{G}$ is any orientation of G and $h : E(\overrightarrow{G}) \to S_p$ is any function, then und$(\overrightarrow{G} \otimes_h S_p)$ is a 2-regular, (1-factor)-super edge-magic decomposable graph. Moreover, if we denote the 1-factor of G by F then pF is the 1-factor of und$(\overrightarrow{G} \otimes_h S_p)$.

They pose the following two open questions: Fix $p \in \mathbb{N}$. Find the maximum $r \in \mathbb{N}$ such that there is a r-regular graph of order p which is $(p/2)K_2$-super edge-magic decomposable: and characterize the set of 2-regular graphs of order n, $n \equiv 2 \pmod 4$, such that each component has even order and admits an $(n/2)K_2$-super edge-magic decomposition. In connection to open question 1 they prove: For all $r \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that there exists a k-regular bipartite graph $B(n)$, with $k > r$ and $|V(B(n))| = 2 \cdot 3^n$, such that $B(n)$ is $(3^n K_2)$-super edge-magic decomposable.

Hendy, Sugeng, Salman, and Ayunda [1245] provided a sufficient condition for $\overline{C_n}[\overline{K_m}]$ to have a $P_t[\overline{K_m}]$-magic decompositions, where $n > 3,\ m > 1$, and $t = 3,\ 4,\ n-2$.

An H-magic labeling in an H-decomposable graph G is a bijection $f\, V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that, for every copy H in the decomposition, $\sum_{v\in V(H)} f(v) + \sum_{e\in E(H)} f(e)$ is constant. The function f is said to be an H-V-super magic labeling if $f(V(G)) = \{1, 2, \ldots, p\}$. In [2224] Murugan and Chandra Kumar find the magic constant for H-factorable graphs that are H-V-super magic. Also, they give a necessary and sufficient condition for an H-factorable graph to be H-V-super magic and characterize the even regular graphs with a 2-factor-V-super magic labeling.

A bipartite graph G with partite sets X_1 and X_2 is called *consecutively super edge-magic* if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(X_1) = \{1, 2, \ldots, |X_1|\}$, $f(X_2) = \{|X_1| + 1, |X_1| + 2, \ldots, |V(G)|\}$ and $f(u) + f(v) + f(uv)$ is a constant for each $uv \in E(G)$. In [1294] Ichishima, Muntaner-Batle, and Oshima investigated for which bipartite graphs is it possible to add a finite number of isolated vertices so that the resulting graph is consecutively super edge-magic. If it is possible for a bipartite graph G, then they say that the minimum such number $\mu_c(G)$ of isolated vertices is the *consecutively super edge-magic deficiency* of G; otherwise, it is $+\infty$. Thus, the consecutively super edge-magic deficiency of a graph G is a measure of how close G is to being consecutively super edge-magic. They also include a detailed discussion of other concepts that are closely related to the consecutively super edge-magic deficiency.

In [1297] Ichishima, Muntaner-Batle, and Oshima prove that $\alpha(G) = \mu_c(G) + |V(G)| + 1$. Thus a tree has a consecutively super edge-magic if and only if it has an α-valuation.

They explore the relation between super edge-magic labelings and graceful labelings of trees.

In [1315] Ichishima, Oshima, and Takamashi introduce the notion of strength sum of a non-empty graph as follows. The *strength sum* $\mathrm{strs}_f(G)$ of a numbering $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ is defined by $\mathrm{strs}(G) = \min\{\mathrm{strs}_f(G) \mid f$ is a numbering of $G\}$, where $\mathrm{strs}_f(G) = \sum_{uv \in E(G)}(f(u) + f(v))$. A numbering f of a graph G for which $\mathrm{strs}_f(G) = \mathrm{strs}(G)$ is called a *strength sum labeling of* G. They also discuss relations among invariants on super edge-magic graphs and their its strength sums. Ichishima, Muntaner-Batle, and Oshima [1303] proved that for every $k \in [1, n-1]$, there exists a graph G of order n satisfying $\delta(G) = k$ and $\mathrm{str}(G) = n + k$, where $\delta(G)$ denotes the minimum degree of G.

Avadayappan, Jeyanthi, and Vasuki [263] define the *super magic strength* of a graph G as $sm(G) = \min\{s(L)\}$ where L runs over all super edge-magic labelings of G. They use the notation $< K_{1,n} : 2 >$ for the tree obtained from the bistar $B_{n,n}$ (the graph obtained by joining the center vertices of two copies of $K_{1,n}$ with an edge) by subdividing the edge joining the two stars. They prove: $sm(P_{2n}) = 5n + 1$; $sm(P_{2n+1}) = 5n + 3$; $sm(< K_{1,n} : 2 >) = 4n + 9$; $sm(B_{n,n}) = 5n + 6$; $sm((2n + 1)P_2) = 9n + 6$; $sm(C_{2n+1}) = 5n + 4$; $emt(C_{2n}) = 5n + 2$; $sm(K_{1,n}) = 2n + 4$; and $sm(P_n^2) = 3n$. Note that in each case the super magic strength of the graph is the same as its magic strength.

Santhosh and Singh [2712] proved that $C_n \odot P_2$ and $C_n \odot P_3$ are super edge-magic for all odd $n \geqslant 3$ and prove for odd $n \geqslant 3$, $sm(C_n \odot P_2) = (15n + 3)/2$ and $(20n + 3) \leqslant sm(C_n \odot P_3) \leqslant (21n + 3)/2$.

Gray [1150] proves that $C_3 \cup C_n$ is super edge-magic if and only if $n \geqslant 6$ and $C_4 \cup C_n$ is super edge-magic if and only if $n \geqslant 5$. His computer search shows that $C_5 \cup 2C_3$ does not have a super edge-magic labeling.

In [3407] Wallis posed the problem of investigating the edge-magic properties of C_n with the path of length t attached to one vertex. Kim and Park [1699] call such a graph an (n, t)*-kite*. They prove that an $(n, 1)$-kite is super edge-magic if and only if n is odd and an $(n, 3)$-kite is super edge-magic if and only if n is odd and at least 5. Park, Choi, and Bae [2344] show that $(n, 2)$-kite is super edge-magic if and only if n is even. Wallis [3407] also posed the problem of determining when $K_2 \cup C_n$ is super edge-magic. In [2344] and [1699] Park et al. prove that $K_2 \cup C_n$ is super edge-magic if and only if n is even. Kim and Park [1699] show that the graph obtained by attaching a pendent edge to a vertex of degree one of a star is super-edge magic and that a super edge-magic graph with edge magic constant k and q edges satisfies $q \leqslant 2k/3 - 3$.

Lee and Kong [1845] use $\mathrm{St}(a_1, a_2, \ldots, a_n)$ to denote the disjoint union of the n stars $\mathrm{St}(a_1)$, $\mathrm{St}(a_2)$, ... , $\mathrm{St}(a_n)$. They prove the following graphs are super edge-magic: $\mathrm{St}(m, n)$ where $n \equiv 0 \bmod (m+1)$; $\mathrm{St}(1, 1, n)$; $\mathrm{St}(1, 2, n)$; $\mathrm{St}(1, n, n)$; $\mathrm{St}(2, 2, n)$; $\mathrm{St}(2, 3, n)$; $\mathrm{St}(1, 1, 2, n)$ $(n \geqslant 2)$; $\mathrm{St}(1, 1, 3, n)$; $\mathrm{St}(1, 2, 2, n)$; and $\mathrm{St}(2, 2, 2, n)$. They conjecture that $\mathrm{St}(a_1, a_2, \ldots, a_n)$ is super edge-magic when $n > 1$ is odd. Gao and Fan [1026] proved that $\mathrm{St}(1, m, n)$; $\mathrm{St}(3, m, m + 1)$; and $\mathrm{St}(n, n + 1, n + 2)$ are super edge-magic, and under certain conditions $\mathrm{St}(a_1, a_2, \ldots, a_{2n+1})$, $\mathrm{St}(a_1, a_2, \ldots, a_{4n+1})$, and $\mathrm{St}(a_1, a_2, \ldots, a_{4n+3})$ are also super edge magic.

In [2050] MacDougall and Wallis investigate the existence of super edge-magic labelings of cycles with a chord. They use C_v^t to denote the graph obtained from C_v by joining two vertices that are distance t apart in C_v. They prove: C_{4m+1}^t ($m \geqslant 3$) has a super edge-magic labeling for every t except $4m-4$ and $4m-8$; C_{4m}^t ($m \geqslant 3$) has a super edge-magic labeling when $t \equiv 2 \bmod 4$; and that C_{4m+2}^t ($m > 1$) has a super edge-magic labeling for all odd t other than 5, and for $t = 2$ and 6. They pose the problem of what values of t does C_{2n}^t have a super edge-magic labeling.

Enomoto, Masuda, and Nakamigawa [892] have proved that every graph can be embedded in a connected super edge-magic graph as an induced subgraph. Slamin, Bača, Lin, Miller, Simanjuntak [2996] proved that the friendship graph consisting of n triangles is super edge-magic if and only if n is 3, 4, 5, or 7. Fukuchi proved [994] the generalized Petersen graph $P(n, 2)$ (see §2.7 and at least 5. Baskoro and Ngurah [511] showed that nP_3 is super edge-magic for $n \geqslant 4$ and n even.

Hegde and Shetty [1234] showed that a graph is super edge-magic if and only if it is strongly k-indexable (see §4.1 for the definition). Figueroa-Centeno, Ichishima, and Muntaner-Batle [932] proved that a graph is super edge-magic if and only if it is strongly 1-harmonious and that every super edge-magic graph is cordial. They also proved that P_n^2 and $K_2 \times C_{2n+1}$ are super edge-magic. In [933] Figueroa-Centeno et al. show that the following graphs are super edge-magic: $P_3 \cup kP_2$ for all k; kP_n when k is odd; $k(P_2 \cup P_n)$ when k is odd and $n = 3$ or $n = 4$; and fans F_n if and only if $n \leqslant 6$. They conjecture that kP_2 is not super edge-magic when k is even. This conjecture has been proved by Z. Chen [696] who showed that kP_2 is super edge-magic if and only if k is odd. Figueroa-Centeno et al. proved that the book B_n is not super edge-magic when $n \equiv 1, 3, 7 \pmod 8$ and when $n = 4$. They proved that B_n is super edge-magic for $n = 2$ and 5 and conjectured that for every $n \geqslant 5$, B_n is super edge-magic if and only if n is even or $n \equiv 5 \pmod 8$. Yuansheng, Yue, Xirong, and Xinhong [3566] proved this conjecture for the case that n is even. They prove that every tree with an α-labeling is super edge-magic. Yokomura (see [891]) has shown that $P_{2m+1} \times P_2$ and $C_{2m+1} \times P_m$ are super edge-magic (see also [932]). In [934], Figueroa-Centeno et al. proved that if G is a (super) edge-magic 2-regular graph, then $G \odot \overline{K}_n$ is (super) edge-magic and that $C_m \odot \overline{K}_n$ is super edge-magic. Fukuchi [993] shows how to recursively create super edge-magic trees from certain kinds of existing super edge-magic trees. Ngurah, Baskoro, and Simanjuntak [2292] provide a method for constructing new (super) edge-magic graphs from existing ones. One of their results is that if G has an edge-magic total labeling and G has order p and size p or $p-1$, then $G \odot nK_1$ has an edge-magic total labeling.

Ichishima, Muntaner-Batle, Oshima [1292] enlarged the classes of super edge-magic 2-regular graphs by presenting some constructions that generate large classes of super edge-magic 2-regular graphs from previously known super edge-magic 2-regular graphs or pseudo super edge-magic graphs. By virtue of known relationships among other classes of labelings the 2-regular graphs obtained from their constructions are also harmonious, sequential, felicitous and equitable. Their results add credence to the conjecture of Holden et al. [1256] that all 2-regular graphs of odd order with the exceptions of $C_3 \cup C_4$, $3C_3 \cup C_4$, and $2C_3 \cup C_5$ possess a strong vertex-magic total labeling, which is equivalent to super edge-magic labelings for 2-regular graphs. For a 2-regular graph G with $2m+1$ vertices that has a strong vertex-magic total labeling McQuillan and McQuillan [2115] proved that $G \cup 2mC_3$, $G \cup (2m+2)C_3$, $G \cup mC_8$ and $G \cup (m+1)C_8$ also have a strong vertex-magic total labeling.

Lee and Lee [1847] investigate the existence of total edge-magic labelings and super edge-magic labelings of unicylic graphs. They obtain a variety of positive and negative results and conjecture that all unicyclic are edge-magic total.

Shiu and Lee [2915] investigated edge labelings of multigraphs. Given a multigraph G with q edges they call a bijection from the set of edges of G to $\{1, 2, \ldots, q\}$ with the property that for each vertex v the sum of all edge labels incident to v is a constant independent of v a *supermagic* labeling of G. They use $K_2[n]$ to denote the multigraph consisting of n edges joining 2 vertices and $mK_2[n]$ to denote the disjoint union of m copies of $K_2[n]$. They prove that for m and n at least 2, $mK_2[n]$ is supermagic if and only if n is even or if both m and n are odd.

In 1970 Kotzig and Rosa [1743] defined the *edge-magic deficiency*, $\mu(G)$, of a graph G as the minimum n such that $G \cup nK_1$ is edge-magic total. If no such n exists they define $\mu(G) = \infty$. In 1999 Figueroa-Centeno, Ichishima, and Muntaner-Batle [938] extended this notion to *super edge-magic deficiency*, $\mu_s(G)$, is the analogous way. They prove the following: $\mu_s(nK_2) = \mu(nK_2) = n - 1 \pmod 2$; $\mu_s(C_n) = 0$ if n is odd; $\mu_s(C_n) = 1$ if $n \equiv 0 \pmod 4$; $\mu_s(C_n) = \infty$ if $n \equiv 2 \pmod 4$; $\mu_s(K_n) = \infty$ if and only if $n \geqslant 5$; $\mu_s(K_{m,n}) \leqslant (m-1)(n-1)$; $\mu_s(K_{2,n}) = n - 1$; and $\mu_s(F)$ is finite for all forests F. They also prove that if a graph G has q edges with $q/2$ odd, and every vertex is even, then $\mu_s(G) = \infty$ and conjecture that $\mu_s(K_{m,n}) \leqslant (m-1)(n-1)$. This conjecture was proved for $m = 3, 4,$ and 5 by Hegde, Shetty, and Shankaran [1235] using the notion of strongly k-indexable labelings. Baig, Baskoro, and Semaničová-Feňovčíková [276] investigated the super edge-magic deficiency of a forest consisting of stars. Ngurah investigates the (super) edge-magic deficiency of chain graphs in [2287] and Ngurah and Adiwijaya does the same in [2286].

For an (n, t)-kite graph (a path of length t attached to a vertex of an n-cycle) G Ahmad, Siddiqui, Nadeem, and Imran [117] proved the following: for odd $n \geqslant 5$ and even $t \geqslant 4, \mu_s(G) = 1$; for odd $n \geqslant 5$, $t \geqslant 5$, $t \neq 11$, and $t \equiv 3, 7 \pmod 8$, $\mu_s(G) \leqslant 1$; for $n \geqslant 10$, $n \equiv 2 \pmod 4$ and $t = 4$, $\mu_s(G) \leqslant 1$; and for $t = 5$, $\mu_s(G) = 1$.

In [394] Baig, Ahmad, Baskoro, and Simanjuntak provide an upper bound for the super edge-magic deficiency of a forest formed by paths, stars, combs, banana trees, and subdivisions of $K_{1,3}$. Baig, Baskoro, and Semaničová-Feňovčíková [395] investigate the super edge-magic deficiency of forests consisting of stars. Among their results are: a forest consisting of $k \geqslant 3$ stars has super edge-magic deficiency at most $k - 2$; for every positive integer n a forest consisting of 4 stars with exactly 1, $n, n,$ and $n + 2$ leaves has a super edge-magic total labeling; for every positive integer n a forest consisting of 4 stars with exactly 1, $n + 5, 2n + 6,$ and $n + 1$ leaves has a super edge-magic total labeling; and for every positive integers n and k a forest consisting of k identical stars has super edge-magic deficiency at most 1 when k is even and deficiency 0 when k is odd. In [108] Ahmad, Javaid, Nadeem, and Hasni investigate the super edge-magic deficiency of some families of graphs related to ladder graphs. Kanwal, Javed, and Riasat [1648] give super edge-magic total labelings and the deficiency for forests consisting of extended w-trees, combs, stars and paths. In [112] Ahmad, Nadeem, and Gupta provided bounds for the super edge-magic deficiency of some Toeplitz graphs.

The *generalized Jahangir* graph $J_{n,m}$ for $m \geqslant 3$ is a graph on $nm+1$ vertices, consisting of a cycle C_{nm} with one additional vertex that is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}. In [396] Baig, Imran, Javaid, and Semaničová-Feňovčiková study the super edge-magic deficiencies of the web graph $Wb_{n,m}$, the generalized Jahangir graph $J_{2,n}$, crown products $L_n \odot K_1,\ K_4 \odot nK_1$, and gave the exact value of super edge-magic deficiency for one class of lobsters.

In [937] Figueroa-Centeno, Ichishima, and Muntaner-Batle proved that $\mu_s(P_m \cup K_{1,n}) = 1$ if $m = 2$ and n is odd, or $m = 3$ and n is not congruent to 0 mod 3, whereas in all other cases $\mu_s(P_m \cup K_{1,n}) = 0$. They also proved that $\mu_s(2K_{1,n}) = 1$ when n is odd and $\mu_s(2K_{1,n}) \leqslant 1$ when n is even. They conjecture that $\mu_s(2K_{1,n}) = 1$ in all cases. Other results in [937] are: $\mu_s(P_m \cup P_n) = 1$ when $(m, n) = (2, 2)$ or $(3, 3)$ and $\mu_s(P_m \cup P_n) = 0$ in all other cases; $\mu_s(K_{1,m} \cup K_{1,n}) = 0$ when mn is even and $\mu_s(K_{1,m} \cup K_{1,n}) = 1$ when mn is odd; $\mu(P_m \cup K_{1,n}) = 1$ when $m = 2$ and n is odd and $\mu(P_m \cup K_{1,n}) = 0$ in all other cases; $\mu(P_m \cup P_n) = 1$ when $(m, n) = (2, 2)$ and $\mu(P_m \cup P_n) = 0$ in all other cases; $\mu_s(2C_n) = 1$ when n is even and ∞ when n is odd; $\mu_s(3C_n) = 0$ when n is odd; $\mu_s(3C_n) = 1$ when $n \equiv 0 \pmod 4$; $\mu_s(3C_n) = \infty$ when $n \equiv 2 \pmod 4$; and $\mu_s(4C_n) = 1$ when $n \equiv 0 \pmod 4$. They conjecture the following: $\mu_s(mC_n) = 0$ when mn is odd; $\mu_s(mC_n) = 1$ when $mn \equiv 0 \pmod 4$; $\mu_s(mC_n) = \infty$ when $mn \equiv 2 \pmod 4$; $\mu_s(2K_{1,n}) = 1$; and if F is a forest with two components, then $\mu(F) \leqslant 1$ and $\mu_s(F) \leqslant 1$. Santhosh and Singh [2711] proved: for n odd at least 3, $\mu_s(K_2 \odot C_n) \leqslant (n - 3)/2$; for $n > 1,\ 1 \leqslant \mu_s(P_n[P_2]) = \lceil (n - 1)/2 \rceil$; and for $n \geqslant 1,\ 1 \leqslant \mu_s(P_n \times K_4) \leqslant n$.

Ichishima and Oshima [1313] prove the following: if a graph $G(V, E)$ has an α-labeling and no isolated vertices, then $\mu_s(G) \leqslant |E| - |V| + 1$; if a graph $G(V, E)$ has an α-labeling, is not sequential, and has no isolated vertices, then $\mu_s(G) = |E| - |V| + 1$; and, if m is even, then $\mu_s(mK_{1,n}) \leqslant 1$. As corollaries of the last result they have: $\mu_s(2K_{1,n}) = 1$; when $m \equiv 2 \pmod 4$ and n is odd, $\mu_s(mK_{1,n}) = 1$; $\mu_s(mK_{1,3}) = 0$ when $m \equiv 4 \pmod 8$ or m is odd; $\mu_s(mK_{1,3}) = 1$ when $m \equiv 2 \pmod 4$; $\mu_s(mK_{2,2}) = 1$; for $n \geqslant 4,\ (n-4)2^{n-2}+3 \leqslant \mu_s(Q_n) \leqslant (n-2)2^{n-1} - 4$; and for $s \geqslant 2$ and $t \geqslant 2,\ \mu_s(mK_{s,t}) \leqslant m(st - s - t) + 1$. They conjecture that for $s \geqslant 2$ and $t \geqslant 2,\ \mu_s(mK_{s,t}) = m(st - s - t) + 1$ and pose as a problem determining the exact value of $\mu_s(Q_n)$.

Ichishima and Oshima [1311] determined the super edge-magic deficiency of graphs of the form $C_m \cup C_n$ for m and n even and for arbitrary n when $m = 3, 4, 5,$ and 7. They state a conjecture for the super edge-magic deficiency of $C_m \cup C_n$ in the general case. Afzal and Aslam [64] investigate the super edge-magic deficiency of various disjoint unions of $K_{2,n}$ with stars, paths and disjoint union of paths. The *join product* of two graphs is their graph union with additional edges that connect all vertices of the first graph to each vertex of the second graph. In [2296] Ngurah and Simanjuntak investigate the super edge-magic deficiencies of a wheel minus an edge and join products of a path, a star, and a cycle with isolated vertices. They also show that the join product of a super edge-magic graph with isolated vertices has finite super edge-magic deficiency.

A *block* of a graph is a maximal subgraph with no cut-vertex. The *block-cut-vertex graph* of a graph G is a graph H whose vertices are the blocks and cut-vertices in G; two vertices are adjacent in H if and only if one vertex is a block in G and the other is a cut-vertex in G belonging to the block. A *chain graph* is a graph with blocks $B_1, B_2, B_3, \ldots, B_k$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cut-vertex graph is a path. The chain graph with k blocks where each block is identical and isomorphic to the complete graph K_n is called the *kK_n-path*.

Ngurah, Baskoro, and Simanjuntak [2291] investigate the exact values of $\mu_s(kK_n\text{-path})$ when $n = 2$ or 4 for all values of k and when $n = 3$ for $k \equiv 0, 1, 2 \pmod 4$, and give an upper bound for $k \equiv 3 \pmod 4$. They determine the exact super edge-magic deficiencies for fans, double fans, wheels of small order and provide upper and lower bounds for the general case as well as bounds for some complete partite graphs. They also include some open problems. Lee and Wang [1909] show that various chain graphs with blocks that are complete graphs are super edge-magic. In [107] investigate the super edge-magic deficiency of some kites and $C_n \cup K_2$.

Figueroa-Centeno and Ichishima [930] introduce the notion of the *sequential number* $\sigma(G)$ of a graph G without isolated vertices to be either the smallest positive integer n for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, n\}$ in such a way that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels are $|E(G)|$ consecutive integers or $+\infty$ if there exists no such integer n. They prove that $\sigma(G) = \mu_s(G) + |V(G)| - 1$ for any graph G without isolated vertices, and $\sigma(K_{m,n}) = mn$, which settles the conjecture of Figueroa-Centeno, Ichishima, and Muntaner-Batle [938] that $\mu_s(K_{m,n}) = (m - 1)(n - 1)$.

In [1289] Ichishima and Muntaner-Batle define the *strong sequential number* $\sigma_s(G)$ of G as the smallest positive integer n for which there exists an injective function from the vertices of G to $[0, n]$ such that when each edge uv is labeled $f(u)+f(v)$, the resulting set of edge labels is $[c, c+q-1]$ for some positive integer c and there exists an integer λ so that $\min\{f(u), f(v)\} \leqslant \lambda < \max\{f(u), f(v)\}$ for all edges uv. Note that for G to have finite $\sigma_s(G)$, it must be bipartite. They prove for a graph G of order p, $\sigma(G) = \mu_s(G) + p - 1$. From this it follows that the problems of determining the sequential number and super edge-magic deficiency are equivalent and that for any graph G, $\sigma(G)$ is finite if and only if $\mu_s(G)$ is finite. They also introduced the following parameter as a measure of how close a graph G is to having an α-labeling. The *alpha-number* $\alpha(G)$ of a graph G with q edges is the smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that when each edge uv is labeled $|f(u) - f(v)|$ the resulting set of edge labels is $[c, c + q - 1]$ for some positive integer c, and there exists an integer λ so that $\min\{f(u), f(v)\} \leqslant \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. If no such n exists the alpha-number of G is defined to be $+\infty$. Since a graph that admits an α-labeling is necessarily bipartite, graphs with finite $\alpha(G)$ are bipartite.

Ichishima and Muntaner-Batle [1289] prove: if every vertex of graph G has even degree and $|E(G)| \equiv 2 \pmod 4$, then $\sigma(G) = \sigma_s(G) = +\infty$; for every graph G of order p, $\sigma_s(G) = \mu_c(G)+p-1$; and if G is a super edge-magic graph with at least one edge, then the graph $G + nK_1$ is sequential for every positive integer n. As corollaries they have: for every graph $\sigma_s(G) = \alpha(G)$; a graph G has an α-labeling if and only if $\sigma_s(G) = |E(G)|$; and if a graph G of order p and size $q \geqslant 1$ has a super edge-magic labeling f with $s = \min\{f(u) + f(v) : uv \in E(G)\}$, then $\sigma(G + nK_1) \leqslant s + q + (n-1)p - 2$; if G is a graph of order p and size $q \geqslant 1$ and G has a super edge-magic labeling f with $s = \min\{f(u) + f(v) : uv \in E(G)\}$, then $\mu_s(G + nK_1) \leqslant s + q + (n-2)(p-1) - 3$; and if G is a super edge-magic graph with at least one edge, then the graph $G + nK_1$ is harmonious and felicitous for any positive integer n.

For a graph G order p and size q Ichishima, Muntaner-Batle, and Oshima [1304] prove the following: if $q = p - 1$ and $\beta_s(G) = p - 1$, then $\beta(G \odot nK_1) = \beta_s(G \odot nK_1) = (n+1)p - 1$ for every positive integer n; if $q > p - 1$ and $\beta_s(G) = q$, then there exists a supergraph H of G such that $\beta(H \odot nK_1) = \beta_s(H \odot nK_1) = (n+1)(q+1) - 1$ for every positive integer n; if G has a subgraph H such that $\beta_s(H) = q < p - 1$, then $\beta(H \odot nK_1) = \beta_s(H \odot nK_1) = (n+1)(q+1) - 1$ for every positive integer n; and if G has a subgraph H such that $\beta_s(H) = q < p - k(H')$, where H' is a subgraph of H without isolated vertices, then $\beta(H \odot nK_1) = \beta_s(H \odot nK_1) = (n+1)(q+1) - 1$ for every positive integer n.

As the concept of super magic strength is effectively defined only for super edge-magic graphs, Ichishima, Muntaner-Batle, and Oshima [1299] generalize it for any nonempty graph as follows. A *numbering* f of a graph G of order p is a labeling that assigns distinct elements of the set $[1, p]$ to the vertices of G, where each edge uv of G is labeled $f(u) + f(v)$. The *strength*, $\text{str}_f(G)$, of a numbering $f : V(G) \to [1, p]$ of G is defined by $\text{str}_f(G) = \max\{f(u) + f(v) \,|uv \in E(G)\}$, that is, $\text{str}_f(G)$ is the maximum edge label of G, and the *strength*, $\text{str}(G)$, of a graph G itself is $\text{str}(G) = \min\{\text{str}_f(G) \,|f \text{ is a numbering of } G\}$. A numbering f of a graph G for which $\text{str}_f(G) = \text{str}(G)$ is called a *strength labeling* of G. If G is an empty graph, then $\text{str}(G)$ is undefined. For a graph G of order p they prove the following: if G has order at least 3 and contains a path of order k ($k \in [2, p-1]$) as an induced subgraph, then $\text{str}(G) \leqslant 2p - (k-1)$; if $\Delta(G) + 2 \leqslant \text{str}(G) \leqslant 2p - 1$; and if $p + m + \min\{p, \delta(G) + m\} \leqslant \text{str}(G + mK_1) \leqslant \text{str}(G) + 2m$ for every positive integer m. They determine the exact strength for many basic families of graphs such as paths, cycles complete graphs, ladders, books, and hypercubes. They conclude with six problems and a conjecture.

In [1300] Ichishima, Muntaner-Batle, and Oshima determined the strength of caterpillars and complete n-ary k-level trees. The strength $\text{str}(G)$ is also given for graphs G obtained by taking the corona of certain graphs and arbitrary number of isolated vertices. They further proved if G is a graph of order p with $\delta(G) \geqslant 1$ and $\text{str}(G) = p + \delta(G)$, then $\text{str}(G \odot nK_1) = (n+1)p + 1$. for every positive integer n. In [1302] Ichishima, Muntaner-Batle, Oshima show that for every $k \in [1, n-1]$, there exists a graph G of order n satisfying $\delta(G) = k$ and $\text{str}(G) = n + k$, where $\delta(G)$ denotes the minimum degree of G. In [1306] Ichishima, Muntaner-Batle, and Oshima investigate minimum degree conditions for the strength of graphs. They determine certain degree sequences of graphs that naturally arise and proof that these degree sequences determine a unique graph realization. In addition, they establish a parallel bandwidth result to the one on strength of graphs and enlarge the class of k-stable properties known so far.

The following result Ichishima, Muntaner-Batle, and Oshima in [1294] shows the connection between the alpha-number of a graph and its consecutively super edge-magic deficiency. For every graph G of order p, $\alpha(G) = \mu_c(G) + p - 1$. This result shows that the problems of determining the alpha-number and consecutively super edge-magic deficiency are equivalent.

In [2297] Ngurah and Simanjuntak proved that if G is a cycle-free graph with minimum degree one and $\mu_s(G + K_1) = 0$ then G is either a tree or a forest. They also prove: the join product of some classes of trees and forests with an isolated vertex has zero super edge-magic deficiency; for all but one tree of order at most 6, their join product with an isolated vertex has zero super edge-magic deficiency. For trees T of order at least 7 they proved that if $\mu_s(T + K_1) = 0$, then either $2K_{1,3}$ or $K_3 \cup K_{1,3}$ is a subgraph of $T + K_1$. For the super edge-magic deficiency of the join product of a tree T of order at least 2 with $m \geqslant 2$ isolated vertices, they showed that $\mu_s(T + mK_1) = 0$ if and only if $T = P_2$. For a tree $T \neq P_2$, they proved $\mu_s(T + mK_1) \geqslant \left\lfloor \frac{(m-1)(|V(T)-2|+1)}{2} \right\rfloor$.They also present results for the super edge-magic deficiency of some chain graphs.

Z. Chen [696] has proved: the join of K_1 with any subgraph of a star is super edge-magic; the join of two nontrivial graphs is super edge-magic if and only if at least one of them has exactly two vertices and their union has exactly one edge; and if a k-regular graph is super edge-magic, then $k \leqslant 3$. Chen also obtained the following: there is a connected super edge-magic graph with p vertices and q edges if and only if $p - 1 \leqslant q \leqslant 2p - 3$; there is a connected 3-regular super edge-magic graph with p vertices if and only if $p \equiv 2 \pmod 4$; and if G is a k-regular edge-magic total graph with p vertices and q edges then $(p + q)(1 + p + q) \equiv 0 \pmod{2d}$ where $d = \gcd(k - 1, q)$. As a corollary of the last result, Chen observes that $nK_2 + nK_2$ is not edge-magic total.

Another labeling that has been called "edge-magic" was introduced by Lee, Seah, and Tan in 1992 [1887]. They defined a graph $G = (V, E)$ to be *edge-magic* if there exists a bijection $f\colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $f^+\colon V \to N$ defined by $f^+(u) = \sum_{(u,v)\in E} f(u, v) \pmod{|V|}$ is a constant map. Lee (see [1875]) conjectured that a cubic graph with p vertices is edge-magic if and only if $p \equiv 2 \pmod 4$. Lee, Pigg, and Cox [1875] verified this conjecture for prisms and several other classes of cubic graphs. They also show that $C_n \times K_2$ is edge-magic if and only if n is odd. Shiu and Lee [2915] showed that the conjecture is not true for multigraphs and disconnected graphs. In [2915] Lee's conjecture was modified by restricting it to simple connected cubic graphs. A computer search by Lee, Wang, and Wen [1912] showed that the new conjecture was false for a graph of order 10. Using different methods, Shiu [2893] and Lee, Su, and Wang [1898] gave proofs that it is was false.

Lee, Seah, and Tan [1887] establish that a necessary condition for a multigraph with p vertices and q edges to be edge-magic is that p divides $q(q+1)$ and they exhibit several new classes of cubic edge-magic graphs. They also proved: $K_{n,n}$ ($n \geqslant 3$) is edge-magic and K_n is edge-magic for $n \equiv 1, 2 \pmod 4$ and for $n \equiv 3 \pmod 4$ ($n \geqslant 7$). Lee, Seah, and Tan further proved that following graphs are not edge-magic: all trees except P_2; all unicyclic graphs; and K_n where $n \equiv 0 \pmod 4$. Schaffer and Lee [2725] have proved that $C_m \times C_n$ is always edge-magic. Lee, Tong, and Seah [1904] have conjectured that the total graph of a (p, p)-graph is edge-magic if and only if p is odd. They prove this conjecture for cycles. Lee, Kitagaki, Young, and Kocay [1844] proved that a maximal outerplanar graph with p vertices is edge-magic if and only if $p = 6$. Shiu [2892] used matrices with special properties to prove that the composition of P_n with $\overline{K_n}$ and the composition of P_n with $\overline{K_{kn}}$ where kn is odd and n is at least 3 have edge-magic labelings. Boonklurb, Narissayaporn, and Singhun [598] show that under some conditions the m-node k-uniform hyperpaths and m-node k-uniform hypercycles are super edge-magic.

For a (p, q)-graph a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \dots, p+q\}$ such that for each edge $xy \in E(G)$ the value of $f(x) + f(xy) + f(y)$ is either k_1, k_2 or k_3 is said to be an *edge trimagic total labeling* . Regees and Jayasekaran [2613] prove that $C_m \times P_n$, the generalized web graph, and the generalized web graph without a center are super edge trimagic total graphs. In [2612] proved that the star type graphs $P_3 \odot \overline{K_n}$, $B_{m,n}$, $\langle B_{m,n} : 2\rangle$ and $\langle K_{1,n}\, 3\rangle$ admits edge trimagic total labelings and super edge trimagic total labelings.

A *triangular belt* is obtained from $P_n \times P_2$ with vertices $u_1, u_2, \dots u_n$ and $v_1, v_2, \dots, v_{n_1}$ by adding an edge $\{(u_k v_{k+1}) \mid k = 1, 2, \dots, n-1\}$. The braid graph is obtained from a pair of paths P_n and P_n' with vetrtices $u_1, u_2, \dots u_n$ and $v_1, v_2, \dots u_n$ obtained by joining the ith vertex of P_n and the $(i+1)$th vertex with P_n' and joining the ith vertex of P_n' and the $(i+2)$th vertex with P_n for $1 \leqslant i \leqslant n-2$. A *semi Jahangir* graph is connected graph with a vertex set $\{u, u_k \mid 1 \leqslant k \leqslant n\} \cup \{s_k \mid 1 \leqslant k \leqslant n-1\}$ and edge set $\{u_k s_k \mid 1 \leqslant k \leqslant n-1\} \cup \{s_k u_{k+1} \mid 1 \leqslant k \leqslant n-1\} \cup \{u_k u_i \mid 1 \leqslant k \leqslant n\}$ A graph obtained from a graph G replacing each edge e_i by an H-graph in such a way that the ends of e_i are merged with a pendant vertex in P_2 and pendant vertex in P_2' is called *H super subdivision* of G is denoted by $HSS(G)$, where the H-graph is a tree on 6 vertices in which exactly two vertices have degree 3. Vaghela and Parmar [3243] prove that the H-graph of a path, alternate triangular belt graph, braid graph, semi Jahangir graph, F-trees, (obtained from a path $v_1, v_2, \dots, v_n$ by appending an edge to v_{n-1} and v_n), H-super subdivision of a path are edge magic total graphs, and an H-graph of a path, alternate triangular belt graph, braid graphs, semi Jahangir graph, H-super subdivision of a path, F-trees, the $H \odot K_1$ graph of a path are edge trimagic total graphs.

Amuthavalli and Sugapriya [185] defined a *reverse edge-trimagic* labeling on a graph $G(V, E)$ with p vertices and q edges as a one-to-one map that takes the vertices and edges onto the integers $1, 2, \dots, p+q$ with the property that for every edge e, when the sum of all vertex labels incident to e is subtracted from edge label $f(e)$, the result is one of three constants. A reverse edge-trimagic labeling is said to be a *reverse super edge-trimagic* labeling if $f(V) = \{1, 2, \dots, p\}$ and $f(E) = \{p+1, p+2, \dots, p+q\}$. They investigated the reverse super edge-trimagic labeling of barycentric subdivision of bistars, degree splitting graphs of $K_{1,n} + K_{1,n}$ and $K_{1,n} \cup K_{1,n}$, and the splitting graphs of stars.

Chopra, Dios, and Lee [710] investigated the edge-magicness of joins of graphs. Among their results are: $K_{2,m}$ is edge-magic if and only if $m = 4$ or 10; the only possible edge-magic graphs of the form $K_{3,m}$ are those with $m = 3, 5, 6, 15, 33$, and 69; for any fixed m there are only finitely many n such that $K_{m,n}$ is edge-magic; for any fixed m there are only finitely many trees T such that $T + \overline{K_m}$ is edge-magic; and wheels are not edge-magic.

Lee, Ho, Tan, and Su [1843] define the *edge-magic index* of a graph G to be the smallest positive integer k such that the graph kG is edge-magic. They completely determined the edge-magic indices of graphs which are stars. In [2910] Shiu, Lam, and Lee give the edge-magic index set of the second power of a path.

For any graph G and any positive integer k the graph $G[k]$, called the *k-fold G*, is the hypergraph obtained from G by replacing each edge of G with k parallel edges. Lee, Seah, and Tan [1887] proved that for any graph G with p vertices, $G[2p]$ is edge-magic and, if p is odd, $G[p]$ is edge-magic. Shiu, Lam, and Lee [2909] show that if G is an $(n+1,n)$-multigraph, then G is edge-magic if and only if n is odd and G is isomorphic to the disjoint union of K_2 and $(n-1)/2$ copies of $K_2[2]$. They also prove that if G is a $(2m+1,2m)$-multigraph and $k \geqslant 2$, then $G[k]$ is edge-magic if and only if $2m+1$ divides $k(k-1)$. For a $(2m,2m-1)$-multigraph G and k at least 2, they show that $G[k]$ is edge-magic if $4m$ divides $(2m-1)k((2m-1)k+1)$ or if $4m$ divides $(2m+k-1)k$. In [2907] Shiu, Lam, and Lee characterize the (p,p)-multigraphs that are edge-magic as $mK_2[2]$ or the disjoint union of $mK_2[2]$ and two particular multigraphs or the disjoint union of K_2, $mK_2[2]$, and four particular multigraphs. They also show for every $(2m+1,2m+1)$-multigraph G, $G[k]$ is edge-magic for all k at least 2. Lee, Seah, and Tan [1887] prove that the multigraph $C_n[k]$ is edge-magic for $k \geqslant 2$.

Tables 6 and 7 summarize what is known about edge-magic total labelings and super edge-magic total labelings. We use **SEMT** to indicate the graphs have super edge-magic total labelings and **EMT** to indicate the graphs have edge-magic total labelings. A question mark following SEMT or EMT indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová.

Table 6: **Summary of Edge-magic Total Labelings**

Graph	*Types*	*Notes*
P_n	EMT	[3411]
trees	EMT?	[1744], [2626]
C_n	EMT	for $n \geqslant 3$ [1743], [1108], [2638], [542]
K_n	EMT	iff $n = 1, 2, 3, 4, 5$, or 6 [1744], [755], [891] enumeration of all EMT of K_n [3411]
$K_{m,n}$	EMT	[3411], [1743]
crowns $C_n \odot K_1$	EMT	[3534], [3411]
C_n with a single edge attached to one vertex	EMT	[3411]
wheels W_n	EMT	iff $n \not\equiv 3 \pmod 4$ [891], [995]
fans	EMT	[2996], [932], [933]
(p, q)-graph	not EMT	if q even, $p + q \equiv 2 \pmod 4$ [2626]
nP_2	EMT	iff n odd [1743]
$P_n + K_1$	EMT	[3534]
r-regular graph	not EMT	r odd and $p \equiv 4 \pmod 8$ [755]
$P_3 \cup nK_2$ and $P_5 \cup nK_2$	EMT	[932], [933]
$P_4 \cup nK_2$	EMT	n odd [932], [933]
nP_i	EMT	n odd, $i = 3, 4, 5$ [3534], [932], [933]
nP_3	EMT?	[3534]
$2P_n$	EMT	[932], [933]

Continued on next page

Table 6 – *Continued from previous page*

Graph	*Types*	*Notes*
$P_1 \cup P_2 \cup \cdots \cup P_n$	EMT	[932], [933]
$mK_{1,n}$	EMT	[932], [933]
unicylic graphs	EMT?	[1847]
$K_1 \odot nK_2$	EMT	n even [932], [933]
$K_2 \times \overline{K}_n$	EMT	[932], [933]
nK_3	EMT	iff $n \neq 2$ odd [932], [933], [2114]
binary trees	EMT	[932], [933]
$P(m, n)$ (generalized Petersen graph see §2.7)	EMT	[932], [933], [2288]
ladders	EMT	[932], [933]
books	EMT	[932], [933]
odd cycle with pendent edges attached to one vertex	EMT	[932], [933]
$P_m \times C_n$	EMT	n odd $n \geqslant 3$ [3470]
$P_m \times P_2$	EMT	m odd $m \geqslant 3$ [3470]
$K_{1,m} \cup K_{1,n}$	EMT	iff mn is even [936]
$G \odot \overline{K}_n$	EMT	if G is EMT 2-regular [934]

Table 7: **Summary of Super Edge-magic Labelings**

Graph	*Types*	*Notes*
C_n	SEMT	iff n is odd [891]
caterpillars	SEMT	[891], [1743], [1744]
$K_{m,n}$	SEMT	iff $m = 1$ or $n = 1$ [891]
K_n	SEMT	iff $n = 1, 2$ or 3 [891]
trees	SEMT?	[891]
nK_2	SEMT	iff n odd [696]
nG	SEMT	if G is a bipartite or tripartite SEM graph and n odd [936]
$mB(n)$	SEMT	if m is odd [2968]
$m(P_2 \times P_n)$	SEMT	if $m,\ n$ are odd [2968]
$r(P_m \times P_n)$	SEMT	if r is odd, $(m, n) \neq (2, 2)$ or (3,3) [2968]
$r(P_3 \times mP_n)$	SEMT	if r is odd [2968]
$K_{1,m} \cup K_{1,n}$	SEMT	if m is a multiple of $n + 1$ [936]
$K_{1,m} \cup K_{1,n}$	SEMT?	iff m is a multiple of $n + 1$ [936]
$K_{1,2} \cup K_{1,n}$	SEMT	iff n is a multiple of 3 [936]
$K_{1,3} \cup K_{1,n}$	SEMT	iff n is a multiple of 4 [936]
$P_m \cup K_{1,n}$	SEMT	if $m \geqslant 4$ is even [936]
$2P_n$	SEMT	iff n is not 2 or 3 [936]
$2P_{4n}$	SEMT	for all n [936]
mP_n	SEMT	if $m \equiv 2 \pmod 4$, $n \neq 2, 3$ [2968]

Continued on next page

Table 7 – *Continued from previous page*

Graph	*Types*	*Notes*
mP_{4n}	SEMT	if$m \equiv 2 \pmod 4$, $n > 1$ [2968]
$K_{1,m} \cup 2nK_{1,2}$	SEMT	for all m and n [936]
$C_3 \cup C_n$	SEMT	iff $n \geqslant 6$ even [939], [1149]
$C_4 \cup C_n$	SEMT	iff $n \geqslant 5$ odd [939], [1149]
$C_5 \cup C_n$	SEMT	iff $n \geqslant 4$ even [939]
$C_m \cup C_n$	SEMT	if $m \geqslant 6$ even, n odd $n \geqslant m/2 + 2$ [939]
$C_m \cup C_n$	SEMT?	iff $m + n \geqslant 9$ and $m + n$ odd [939]
$C_4 \cup P_n$	SEMT	iff $n \neq 3$ [939]
$C_5 \cup P_n$	SEMT	if $n \neq 4$ [939]
$C_m \cup P_n$	SEMT	if $m \geqslant 6$ even, $n \geqslant m/2 + 2$ [939]
$P_m \cup P_n$	SEMT	iff $(m, n) \neq (2, 2)$ or $(3, 3)$ [939]
corona $C_n \odot \overline{K}_m$	SEMT	$n \geqslant 3$ [939]
$St(m, n)$	SEMT	$n \equiv 0 \pmod{m + 1}$ [1845]
$St(1, k, n)$	SEMT	$k = 1, 2$ or n [1845]
$St(2, k, n)$	SEMT	$k = 2, 3$ [1845]
$St(1, 1, k, n)$	SEMT	$k = 2, 3$ [1845]
$St(k, 2, 2, n)$	SEMT	$k = 1, 2$ [1845]
$St(a_1, \ldots, a_n)$	SEMT?	for $n > 1$ odd [1845]
C_{4m}^t	SEMT	[2050]

Continued on next page

Table 7 – *Continued from previous page*

Graph	*Types*	*Notes*
C_{4m+1}^t	SEMT	[2050]
friendship graph of n triangles	SEMT	iff $n = 3, 4, 5$, or 7 [2996]
generalized Petersen graph $P(n, 2)$ (see §2.7)	SEMT	if $n \geqslant 3$ odd [993]
nP_3	SEMT	if $n \geqslant 4$ even [511]
P_n^2	SEMT	[932]
$K_2 \times C_{2n+1}$	SEMT	[932]
$P_3 \cup kP_2$	SEMT	for all k [933]
kP_n	SEMT	if k is odd [933]
$k(P_2 \cup P_n)$	SEMT	if k is odd and $n = 3, 4$ [933]
fans F_n	SEMT	iff $n \leqslant 6$ [933]
books B_n	SEMT	if n even [3566]
books B_n	SEMT?	if $n \equiv 5 \pmod 8$[933]
trees with α-labelings	SEMT	[933]
$P_{2m+1} \times P_2$	SEMT	[891], [932]
$C_{2m+1} \times P_m$	SEMT	[932]
$G \odot \overline{K}_n$	SEMT	if G is SEM 2-regular graph [934]
$C_m \odot \overline{K}_n$	SEMT	[934]
join of K_1 with any subgraph of a star	SEMT	[696]
if G is k-regular SEMT		then $k \leqslant 3$ [696]

Continued on next page

Table 7 – *Continued from previous page*

Graph	*Types*	*Notes*
graph		
G is connected (p, q)-graph	SEMT	G exists iff $p - 1 \leqslant q \leqslant 2p - 3$ [696]
G is connected 3-regular graph on p vertices	SEMT	iff $p \equiv 2 \pmod 4$ [696]
$nK_2 + nK_2$	not SEMT	[696]

5.3 Vertex-magic Total Labelings

MacDougall, Miller, Slamin, and Wallis [2047] introduced the notion of a vertex-magic total labeling in 1999. For a graph $G(V, E)$ an injective mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is a *vertex-magic total labeling* if there is a constant k, called the *magic constant*, such that for every vertex v, $f(v) + \sum f(vu) = k$ where the sum is over all vertices u adjacent to v (some authors use the term "vertex-magic" for this concept). They prove that the following graphs have vertex-magic total labelings: C_n; P_n $(n > 2)$; $K_{m,m}$ $(m > 1)$; $K_{m,m} - e$ $(m > 2)$; and K_n for n odd. They also prove that when $n > m + 1$, $K_{m,n}$ does not have a vertex-magic total labeling. They conjectured that $K_{m,m+1}$ has a vertex-magic total labeling for all m and that K_n has vertex-magic total labeling for all $n \geqslant 3$. The latter conjecture was proved by Lin and Miller [1953] for the case that n is divisible by 4 while the remaining cases were done by MacDougall, Miller, Slamin, and Wallis [2047]. McQuillan [2113] provided many vertex-magic total labelings for cycles C_{nk} for $k \geqslant 3$ and odd $n \geqslant 3$ using given vertex-magic labelings for C_k. Gray, MacDougall, and Wallis [1159] then gave a simpler proof that all complete graphs are vertex-magic total. Krishnappa, Kothapalli, and Venkaiah [1734] gave another proof that all complete graphs are vertex-magic total. Senthil Amutha and Murugesan [2755] characterized connected vertex magic total labeling graphs through their ideals in topological spaces. Among other results, Wang and Zhang [3450] settle a 2006 conjecture raised by Slamin et al., which claims the existence of the vertex magic total labeling of disjoint union of multiple copies of $C_n \odot K_1$. Vimal Kumar and Vijayalakshmi [3369] investigated vertex magic total labelings of the middle and total graphs of cycles.

In [2047] MacDougall, Miller, Slamin, and Wallis conjectured that for $n \geqslant 5$, K_n has a vertex-magic total labeling with magic constant h if and only if h is an integer satisfying $n^3 + 3n \leqslant 4h \leqslant n^3 + 2n^2 + n$. In [2116] McQuillan and Smith proved that this conjecture is true when n is odd. Armstrong and McQuillan [223] proved that if $n \equiv 2 \pmod 4$ $(n \geqslant 6)$ then K_n has a vertex-magic total labeling with magic constant h for each integer h satisfying $n^3 + 6n \leqslant 4h \leqslant n^3 + 2n^2 - 2n$. If, in addition, $n \equiv 2 \pmod 8$, then K_n has a vertex-magic total labeling with magic constant h for each integer h satisfying $n^3 + 4n \leqslant 4h \leqslant n^3 + 2n^2$. They further showed that for each odd integer

$n \geqslant 5$, $2K_n$ has a vertex-magic total labeling with magic constant h for each integer h such that $n^3+5n \leqslant 2h \leqslant n^3+2n^2-3n$. If, in addition, $n \equiv 1 (\mathrm{mod}\ 4)$, then $2K_n$ has a vertex-magic total labeling with magic constant h for each integer h such that $n^3+3n \leqslant 2h \leqslant n^3+2n^2-n$.

In [2114] McQuillan and McQuillan investigate the existence of vertex-magic labelings of nC_3. They prove: for every even integer $n \geqslant 4$, nC_3 is vertex-magic (and therefore also edge-magic); for each even integer $n \geqslant 6$, nC_3 has vertex-magic total labelings with at least $2n-2$ different magic constants; if $n \equiv 2$ mod 4, two extra vertex-magic total labelings with the highest possible and lowest possible magic constants exist; if $n = 2\cdot 3^k$, $k > 1$, nC_3 has a vertex-magic total labeling with magic constant k if and only if $(1/2)(15n+4) \leqslant k \leqslant (1/2)(21n+2)$; if n is odd, there are vertex-magic total labelings for nC_3 with $n+1$ different magic constants. In [2112] McQuillan provides a technique for constructing vertex-magic total labelings of 2-regular graphs. In particular, if m is an odd positive integer, $G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_k}$ has a vertex-magic total labeling, and J is any subset of $I = \{1, 2, \ldots, k\}$ then $(\cup_{i\in J}\ mC_{n_i}) \cup (\cup_{i\in I-J}\ mC_{n_i})$ has a vertex-magic total labeling.

In [741] Cichacz, Fronček and Singgih introduced a new method to expand some known vertex magic total labelings of 2-regular graphs. The also proved that for odd values of m, if $(2r+1) \not\equiv 0 \pmod 3$ and $n \not\equiv 0 \pmod{(2r+1)}$, then $2mC_{rn} \cup mC_n$ has a vetex magic total labeling.

Lin and Miller [1953] have shown that $K_{m,m}$ is vertex-magic total for all $m > 1$ and that K_n is vertex-magic total for all $n \equiv 0 \pmod 4$. Phillips, Rees, and Wallis [2394] generalized the Lin and Miller result by proving that $K_{m,n}$ is vertex-magic total if and only if m and n differ by at most 1. Cattell [661] has shown that a necessary condition for a graph of the form $H + \overline{K_n}$ to be vertex-magic total is that the number of vertices of H is at least $n-1$. As a corollary he gets that a necessary condition for $K_{m_1,m_2,\ldots,m_r,n}$ where n is the largest size of any partite set to be vertex-magic total is that $m_1+m_2+\cdots+m_r \geqslant n$. He poses as an open question whether graphs that meet the conditions of the theorem are vertex-magic total. Cattell also proves that $K_{1,n,n}$ has a vertex-magic total labeling when n is odd and $K_{2,n,n}$ has a vertex-magic total labeling when $n \equiv 3 \pmod 4$. In [2548] Rahim and Slamin proved the disjoint union of coronas $C_{t_1} \odot K_1 \cup C_{t_2} \odot K_1 \cup \cdots \cup C_{t_n} \odot K_1$ has a vertex-magic total labeling with magic constant $6\sum_{k=1}^{n} t_k + 1$.

Miller, Bača, and MacDougall [2141] have proved that the generalized Petersen graphs $P(n,k)$ (see §2.7) for the definition) are vertex-magic total when n is even and $k \leqslant n/2-1$. They conjecture that all $P(n,k)$ are vertex-magic total when $k \leqslant (n-1)/2$ and all prisms $C_n \times P_2$ are vertex-magic total. Bača, Miller, and Slamin [372] proved the first of these conjectures (see also [2998] for partial results) while Slamin and Miller prove the second. Slamin, Prihandoko, Setiawan, Rosita and Shaleh [2999] constructed vertex-magic total labelings for the disjoint union of two copies of $P(n,k)$ and Silaban, Parestu, Herawati, Sugeng, and Slamin [2955] extended this to any number of copies of $P(n,k)$. More generally, they proved that for $n_j \geqslant 3$ and $1 \leqslant k_j \leqslant \lfloor (n_j-1)/2 \rfloor$, the union $P(n_1,k_1) \cup P(n_2,k_2) \cup \cdots \cup P(n_t,k_t)$ has a vertex-magic total labeling with vertex magic constant $10(n_1+n_2+\cdots+n_t)+2$. In the same article Silaban et al. define the union of t

special circulant graphs $\cup_{j=1}^{t} C_n(1, m_j)$ as the graph with vertex set $\{v_i^j \mid 0 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant t\}$ and edge set $\{v_i^j v_{i+1}^j \mid 0 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant t\} \cup \{v_i^j v_{i+m_j}^j \mid 0 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant t\}$. They prove that for odd n at least 5 and $m_j \in \{2, 3, \ldots, (n-1)/2\}$, the disjoint union $\cup_{j=1}^{t} C_n(1, m_j)$ has a vertex-magic total labeling with constant $8tn + (n - 10/2 + 3$.

MacDougall et al. ([2047], [2049] and [1157]) have shown: W_n has a vertex-magic total labeling if and only if $n \leqslant 11$; fans F_n have a vertex-magic total labelings if and only if $n \leqslant 10$; friendship graphs have vertex-magic total labelings if and only if the number of triangles is at most 3; $K_{m,n}$ $(m > 1)$ has a vertex-magic total labeling if and only if m and n differ by at most 1. Wallis [3407] proved: if G and H have the same order and $G \cup H$ is vertex-magic total then so is $G + H$; if the disjoint union of stars is vertex-magic total, then the average size of the stars is less than 3; if a tree has n internal vertices and more than $2n$ leaves then it does not have a vertex-magic total labeling. Wallis [3408] has shown that if G is a regular graph of even degree that has a vertex-magic total labeling then the graph consisting of an odd number of copies of G is vertex-magic total. He also proved that if G is a regular graph of odd degree (not K_1) that has a vertex-magic total labeling then the graph consisting of any number of copies of G is vertex-magic total.

Gray, MacDougall, McSorley, and Wallis [1158] investigated vertex-magic total labelings of forests. They provide sufficient conditions for the nonexistence of a vertex-magic total labeling of forests based on the maximum degree and the number of internal vertices, and leaves or the number of components. They also use Skolem sequences to prove a star forest with each component a $K_{1,2}$ has a vertex-magic total labeling.

Recall a helm H_n is obtained from a wheel W_n by attaching a pendent edge at each vertex of the n-cycle of the wheel. A *generalized helm* $H(n,t)$ is a graph obtained from a wheel W_n by attaching a path on t vertices at each vertex of the n-cycle. A *generalized web* $\mathrm{W}(n,t)$ is a graph obtained from a generalized helm $H(n,t)$ by joining the corresponding vertices of each path to form an n-cycle. Thus $\mathrm{W}(n,t)$ has $(t+1)n+1$ vertices and $2(t+1)n$ edges. A *generalized Jahangir graph* $J_{k,s}$ is a graph on $ks+1$ vertices consisting of a cycle C_{ks} and one additional vertex that is adjacent to k vertices of C_{ks} at distance s to each other on C_{ks}. Rahim, Tomescu, and Slamin [2549] prove: H_n has no vertex-magic total labeling for any $n \geqslant 3$; $\mathrm{W}(n,t)$ has a vertex-magic total labeling for $n = 3$ or $n = 4$ and $t = 1$, but it is not vertex-magic total for $n \geqslant 17t + 12$ and $t \geqslant 0$; and $J_{n,t+1}$ is vertex-magic total for $n = 3$ and $t = 1$, but it does not have this property for $n \geqslant 7t + 11$ and $t \geqslant 1$. Recall a flower is the graph obtained from a helm by joining each pendent vertex to the central vertex of the helm. Ahmad and Tomescu [118] proved that flower graph is vertex-magic if and only if the underlying cycle is C_3.

Fronček, Kovář, and Kovářová [974] proved that $C_n \times C_{2m+1}$ and $K_5 \times C_{2n+1}$ are vertex-magic total. Kovář [1746] furthermore proved some general results about products of certain regular vertex-magic total graphs. In particular, if G is a $(2r+1)$-regular vertex-magic total graph that can be factored into an $(r+1)$-regular graph and an r-regular graph, then $G \times K_5$ and $G \times C_n$ for n even are vertex-magic total. He also proved that if G an r-regular vertex-magic total graph and H is a $2s$-regular supermagic graph that can be factored into two s-regular factors, then their Cartesian product $G \times H$ is vertex-magic total if either r is odd, or r is even and $|H|$ is odd.

Ivančo and Polláková [1347] consider supermagic graphs having a saturated vertex (i.e., a vertex that is adjacent to every other vertex). They characterize supermagic graphs $G + K_1$, where G is a regular graph, using a connection to vertex-magic total graphs. They prove that if G is a d-regular graph of order n then the join $G + K_1$ is supermagic if and only if G has a VMT labeling with constant h such that $(n - d - 1)$ is a divisor of the non-negative integer $(n+1)h - n((d+2)/2)(n(d+2)/2) + 1)$. They also prove $K_{1,n,n}$ is supermagic if and only if $n \geqslant 2$; $K_{1,2,2,\dots,2}$ is supermagic except for $K_{1,2}$; and the graph obtained from $K_{n,n}$ ($n \geqslant 5$) by removing all edges in a Hamilton cycle is supermagic. They also consider circulant graphs and prove that the complement of the circulant graph $C_{2n}(1, n)$, $n \geqslant 4$, is supermagic.

In [2534] a novel algorithm is proposed based on the calculation of vertex magic total labelling value for every node in the network. Upon receiving the message from the sender node, the receiver node will quickly detect the faulty node by comparing the vertex magic total labeling pivot value. Experimental results show that the proposed approach leads to high true fault rate detection accuracy compared to the false fault rate detection.

MacDougall, Miller, and Sugeng [2048] define a *super vertex-magic total labeling* of a graph $G(V, E)$ as a vertex-magic total labeling f of G with the additional property that $f(V) = \{1, 2, \dots, |V|\}$ and $f(E) = \{|V| + 1, |V| + 2, \dots, |V| + |E|\}$ (some authors use the term "super vertex-magic" for this concept). They show that a (p, q)-graph that has a super vertex-magic total labeling with magic constant k satisfies the following conditions: $k = (p+q)(p+q+1)/v - (v+1)/2$; $k \geqslant (41p+21)/18$; if G is connected, $k \geqslant (7p-5)/2$; p divides $q(q+1)$ if p is odd, and p divides $2q(q+1)$ if p is even; if G has even order either $p \equiv 0 \pmod 8$ and $q \equiv 0$ or $3 \pmod 4$ or $p \equiv 4 \pmod 8$ and $q \equiv 1$ or $2 \pmod 4$; if G is r-regular and p and r have opposite parity then $p \equiv 0 \pmod 8$ implies $q \equiv 0 \pmod 4$ and $p \equiv 4 \pmod 8$ implies $q \equiv 2 \pmod 4$. They also show: C_n has a super vertex-magic total labeling if and only if n is odd; and no wheel, ladder, fan, friendship graph, complete bipartite graph or graph with a vertex of degree 1 has a super vertex-magic total labeling. They conjecture that no tree has a super vertex-magic total labeling and that K_{4n} has a super vertex-magic total labeling when $n > 1$. The latter conjecture was proved by Gómez in [1121]. In [1122] Gómez proved that if G is a d-regular graph that has a vertex-magic total labeling and k is a positive integer such that $(k-1)(d+1)$ is even, then kG has a super vertex-magic total labeling. As a corollary, we have that if n and k are odd or if $n \equiv 0 \pmod 4$ and $n > 4$, then kK_n has a super vertex-magic total labeling. Gómez also shows how graphs with super vertex-magic total labeling can be constructed from a given graph G with super vertex-magic total labeling by adding edges to G in various ways.

Gray and MacDougall [1156] establish the existence of vertex-magic total labelings for several infinite classes of regular graphs. Their method enables them to begin with any even-regular graph and from it construct a cubic graph possessing a vertex-magic total labeling. A feature of the construction is that it produces strong vertex-magic total labelings many even order regular graphs. The construction also extends to certain families of non-regular graphs. MacDougall has conjectured (see [1747]) that every r-regular ($r > 1$) graph with the exception of $2K_3$ has a vertex-magic total labeling. As a corollary of a general result Kovář [1747] has shown that every $2r$-regular graph with an odd number of vertices and a Hamiltonian cycle has a vertex-magic total labeling.

Gómez and Kovář [1123] proved that a super vertex-magic total labeling of kK_n exists for n odd and any k, for $4 < n \equiv 0 \pmod 4$ and any k, and for $n = 4$ and k even. They also showed kK_{4t+2} does not admit a super vertex-magic total labeling for k odd and provide a large number of super vertex-magic total labelings of kK_{4t+2} for any k based on a super vertex-magic total labeling of kK_{4t+1}.

Beardon [517] has shown that a necessary condition for a graph with c components, p vertices, q edges and a vertex of degree d to be vertex-magic total is $(d+2)^2 \leqslant (7q^2+(6c+5)q+c^2+3c)/p$. When the graph is connected this reduces to $(d+2)^2 \leqslant (7q^2+11q+4)/p$. As a corollary, the following are not vertex-magic total: wheels W_n when $n \geqslant 12$; fans F_n when $n \geqslant 11$; and friendship graphs $C_3^{(n)}$ when $n \geqslant 4$.

Beardon [519] has investigated how vertices of small degree effect vertex-magic total labelings. Let $G(p, q)$ be a graph with a vertex-magic total labeling with magic constant k and let d_0 be the minimum degree of any vertex. He proves $k \leqslant (1 + d_0)(p + q - d_0/2)$ and $q < (1 + d_0)q$. He also shows that if $G(p, q)$ is a vertex-magic graph with a vertex of degree one and t is the number of vertices of degree at least two, then $t > q/3 \geqslant (p-1)/3$. Beardon [519] has shown that the graph obtained by attaching a pendent edge to K_n is vertex-magic total if and only if $n = 2, 3$, or 4.

Meissner and Zwierzyński [2131] used finding vertex-magic total labelings of graphs as a way to compare the efficiency of parallel execution of a program versus sequential processing.

Swaminathan and Jeyanthi [3170] prove the following graphs are super vertex-magic total: P_n if and only if n is odd and $n \geqslant 3$; C_n if and only if n is odd; the star graph if and only if it is P_2; and mC_n if and only if m and n are odd. In [3171] they prove the following: no super vertex-magic total graph has two or more isolated vertices or an isolated edge; a tree with n internal edges and tn leaves is not super vertex-magic total if $t > (n + 1)/n$; if Δ is the largest degree of any vertex in a tree T with p vertices and $\Delta > (-3 + \sqrt{1 + 16p})/2$, then T is not super vertex-magic total; the graph obtained from a comb by appending a pendent edge to each vertex of degree 2 is super vertex-magic total; the graph obtained by attaching a path with t edges to a vertex of an n-cycle is super vertex-magic total if and only if $n + t$ is odd. Ali, Bača, and Bashir [150] proved that mP_3 and mP_4 have no super vertex-magic total labeling

For $n > 1$ and distinct odd integers x, y and z in $[1,n - 1]$ Javaid, Ismail, and Salman [1355] define the *chordal ring* of order n $CR_n(x, y, z)$, as the graph with vertex set Z_n, the additive group of integers modulo n, and edges $(i, i + x), (i, i + y), (i, i + z)$ for all even i. They prove that $CR_n(1, 3, n - 1)$ has a super vertex-magic total labeling when $n \equiv 0 \bmod 4$ and $n \geqslant 8$ and conjecture that for an odd integer Δ, $3 \leqslant \Delta \leqslant n-3, n \equiv 0 \bmod 4$, $CR_n(1, \Delta, n - 1)$ has a super vertex-magic total labeling with magic constant $23n/4 + 2$.

The *Knödel graphs* $W_{\Delta,n}$ with n even and degree Δ, where $1 \leqslant \Delta \leqslant \lfloor \log_2 n \rfloor$ have vertices pairs (i, j) with $i = 1, 2$ and $0 \leqslant j \leqslant n/2 - 1$ where for every $0 \leqslant j \leqslant n/2 - 1$ and there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod n/2)$, for $k = 0, 1, \ldots, \Delta - 1$. Xi, Yang, Mominul, and Wong [3495] have shown that $W_{3,n}$ is super vertex-magic total when $n \equiv 0 \bmod 4$.

Marimuthu and Balakrishnan [2079] called a vertex magic total labeling of $G(V, E)$ *E-super vertex magic* if $f(E(G)) = \{1, 2, 3, \ldots, |E(G)|\}$. The *cocktail party* graph, $H_{m,n}$ $(m, n \geqslant 2)$, is the graph with a vertex set $V = \{v_1, v_2, \ldots, v_{mn}\}$ partitioned into n independent sets $V = \{I_1, I_2, \ldots, I_n\}$ each of size m such that $v_iv_j \in E$ for all $i, j \in \{1, 2, \ldots, mn\}$ where $i \in I_p,\ j \in I_q,\ p \neq q$. (The graph $H_{n,n}$ is the complement of the ladder graph and the dual graph of the n-cube.) Marimuthu and Balakrishnan [2079] gave some basic properties of such labelings and proved that $H_{m,n}$ is E-super vertex magic. Wang and Zhang [3448] show the following: Hamiltonian even regular graphs of odd order are E-super magic; even-regular graphs of odd order that contains a 2-factor consisting of an odd number of odd cycles with the same size are E-super vertex magic; graphs that can be decomposed into the sum of two spanning graphs where one is E-super magic and one is regular of even degree are E-supermagic; even-regular graphs of odd order that contain a 2-factor consisting of an odd number of odd cycles with the same size are E-super vertex magic; and circulant graphs with odd order are E-super vertex magic. Swaminathan and Jeyanthi [3170] proved that mC_n is E-super magic if and only if both m and n are odd.

Samarathunge, Athapattu, and Perera [2690] proved that an E-super vertex labeling of edges adjacent to a leaf vertex must be greater than or equal to $k - (p + q)$ and sum of the parent edge labelings must be greater than or equal to $k - (p + q)$, which k is the magic constant. This implies that an E-super vertex labeling does not been proved that E-super vertex magic labeling does not exist for the perfect binary trees. A broom $B_{n,d}$ is ontained by attaching $n - d$ pendent edges to one of the pendent vertices of P_d. Marimuthu, Suganya, Kalaivani, and Balakrishnan [2085] proved the following: $B_{n,n-2}$ is E-super vertex magic if and only if n is odd; a graph obtained by subdividing all the edges of $K_{1,n}$ is super E-super vertex magic if and only if $n \leqslant 4$, and $K_{1,n}$ is E-super vertex magic if and only if $n = 2$. [2690] new [2085] new

In [2084] Marimuthu and Kumar investigate E-super vertex magic labelings of disconnected graphs. They prove: if a graph with p vertices and q edges and even order has an E-super vertex magic labeling, then either (i) $p \equiv 0 \pmod 8$ and $q \equiv 0$ or $3 \pmod 4$, or (ii) $p \equiv 4 \pmod 8$ and $q \equiv 1$ or $2 \pmod 4$; if an r-regular graph G of order p has an E-super vertex magic labeling, then p and r have opposite parity and (i) if $p \equiv 0 \pmod 8$, then $q \equiv 0 \pmod 4$ (ii) if $p \equiv 4 \pmod 8$, then $q \equiv 2 \pmod 4$; mC_n is E-super vertex magic if and only if $P_n \cup (m-1)C_n$ is E-super vertex magic; $P_m \cup K_{1,m}$ is not E-super vertex magic; $C_m \cup P_n$ is not E-super vertex magic if both m and n have the same parity; the disjoint union of two non-isomorphic suns is not E-super vertex magic; the disjoint union of any number of isomorphic suns is not E-super vertex magic; and mP_3 is not E-super vertex magic for any integer $m > 1$. They conjecture that $K_m \cup P_m$ is E-super vertex magic if $m = 8t + 2$.

In [2229] Mutharasu and Kumar generalized the notion of super vertex-magic total labelings as follows. Let $G(V, E)$ be a graph and k be an integer with $1 \leqslant k \leqslant \text{diam}(G)$. For $e \in E(G)$, let $E_k(e)$ be the set of all vertices that are at a distance at most k from e and let $E_k(v)$ be the set of all edges that are at a distance at most k from v (u and v are at distance 1 from the edge uv). A graph G is said to be *E_k-regular with regularity*

r if, for all edges e, $|E_k(e)| = r$ for some positive integer r. Note that all nontrivial graphs are E_1-regular. Let G be a simple graph with p vertices and q edges. A V-super vertex magic labeling is a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $f(V(G)) = \{1, 2, \ldots, p\}$ and for each vertex $v \in V(G)$, $f(v) + \sum_{u \in N(v)} f(uv) = M$ for some positive integer M. A V_k-*super vertex magic labeling* (V_k-SVML) is a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ with the property that $f(V(G)) = \{1, 2, \ldots, p\}$ and for each $v \in V(G)$, $f(v) + \sum_{e \in E_k(v)} f(e) = M$ for some positive integer M. A graph that admits a V_k-SVML is called V_k-*super vertex magic*. Mutharasu and Kumar gave a necessary and sufficient condition for the existence of V_k-SVML in graphs, determined the magic constant for E_k-regular graphs, and obtained results about V_2-SVML labelings for cycles, complement of cycles, prisms, and a family of circulant graphs.

Balbuena, Barker, Das, Lin, Miller, Ryan, and Slamin [409] call a vertex-magic total labeling of $G(V, E)$ a *strongly vertex-magic total labeling* if the vertex labels are $\{1, 2, \ldots, |V|\}$. They prove: the minimum degree of a strongly vertex-magic total graph is at least 2; for a strongly vertex-magic total graph G with n vertices and e edges, if $2e \geqslant \sqrt{10n^2 - 6n + 1}$ then the minimum degree of G is at least 3; and for a strongly vertex-magic total graph G with n vertices and e edges if $2e < \sqrt{10n^2 - 6n + 1}$ then the minimum degree of G is at most 6. They also provide strongly vertex-magic total labelings for certain families of circulant graphs. In [2112] McQuillan provides a technique for constructing vertex-magic total labelings of 2-regular graphs. In particular, if m is an odd positive integer, $G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_k}$ has a strongly vertex-magic total labeling, and J is any subset of $I = \{1, 2, \ldots, k\}$ then $(\cup_{i \in J}\, mC_{n_i}) \cup (\cup_{i \in I-J}\, mC_{n_i})$ has a strongly vertex-magic total labeling.

Gray [1150] proved that if G is a graph with a spanning subgraph H that possesses a strongly vertex-magic total labeling and $G - E(H)$ is even regular, then G also possesses a strongly vertex-magic total labeling. As a corollary one has that regular Hamiltonian graphs of odd order have a strongly vertex-magic total labelings.

In a series of papers Gray and MacDougall expand on McQuillan's technique to obtain a variety of results. In [1153] Gray and MacDougall show that for any $r \geqslant 4$, every r-regular graph of odd order at most 17 has a strong vertex-magic total labeling. They also showthat several large classes of r-regular graphs of even order, including some Hamiltonian graphs, have vertex-magic total labelings. They conjecture that every 2-regular graph of odd order possesses a strong vertex-magic total labeling if and only if it is not of the form $(2t-1)C_3 \cup C_4$ or $2tC_3 \cup C_5$. They include five open problems.

In [1155] Gray and MacDougall introduce a procedure called a *mutation* that transforms one vertex-magic totaling labeling into another one by swapping sets of edges among vertices that may result in different labeling of the same graph or a labeling of a different graph. Among their results are: a description of all possible mutations of a labeling of the path and the cycle; for all $n \geqslant 2$ and all i from 1 to $n-1$ the graphs obtained by identifying an end points of paths of lengths $i, i+1$, and $2n-2i-1$ have a vertex-magic total labeling; for odd n, the graph obtained by attaching a path of length $n-m$ to an m cycle, (such graphs are called $(m; n-m)$-*kites*) have strong vertex-magic total labelings for $m = 3, \ldots, n-2$; $C_{2n+1} \cup C_{4n+4}$ and $3C_{2n+1}$ have a strong vertex-magic total labeling;

and for $n \geqslant 2$, $C_{4n} \cup C_{6n-1}$ has a strong vertex-magic total labeling. They conclude with three open problems.

Kimberley and MacDougall [1701] studied mutations that involve labelings of regular graphs into labelings of other regular graphs. They present results of extensive computations which confirm how prolific this procedure is. These computations add weight to MacDougall's conjecture that all nontrivial regular graphs are vertex-magic.

Gray and MacDougall [1154] show how to construct vertex-magic total labelings for several families of non-regular graphs, including the disjoint union of two other graphs already possessing vertex-magic total labelings. They prove that if G is a d-regular graph of order v and H a t-regular graph of order u with each having a strong vertex magic total labeling and $vd^2 + 2d + 2v + 2u = 2tvd + 2t + ut^2$ then $G \cup H$ possesses a strong vertex-magic total labeling. They also provide bounds on the minimum degree of a graph with a vertex-magic total labeling.

In [1156] Gray and MacDougall establish the existence of vertex-magic total labelings for several infinite classes of regular graphs. Their method enables them to begin with any even-regular graph and construct a cubic graph possessing a vertex-magic total labeling that produces strong vertex-magic total labelings for many even order regular graphs. The construction also extends to certain families of non-regular graphs.

In [2245] Nagaraj, Ponnappan, and Prabakaran define a vertex-magic total labeling of G to be an *even vertex magic total labeling* if the set of vertex labels is $\{2, 4, 6, \ldots, 2|V(G)|\}$. They prove the following: C_n is even vertex magic total if and only if n is odd; rC_s is even vertex magic total if and only if r and s are odd; $C_n \odot K_1$ is even vertex magic total; wheels are not even vertex magic total; fans (excluding C_3) are not even vertex magic total; kites are not even vertex magic total; and K_{4n} is not even vertex magic total. In [2248] they prove that $C_3 \cup C_{2t}$ $(t > 2)$ and $C_4 \cup C_{2t+1}$ $(t \geqslant 2)$ have even vertex magic total labelings. In [2247] Nagaraj, Ponnappan, and Prabakan prove that the union of any finite numbers of graphs of the form $C_n \odot K_1$ (the sizes may vary) has an even vertex magic total labeling.

Rahim and Slamin [2547] give the bounds for the number of vertices for Jahangir graphs, helms, webs, flower graphs and sunflower graphs when the graphs considered are not vertex-magic total. Thirusangu, Nagar, and Rajeswari [3212] show that certain Cayley digraphs of cyclic groups have vertex-magic total labelings.

Balbuena, Barker, Lin, Miller, and Sugeng [410] call vertex-magic total labeling an *a-vertex consecutive magic* labeling if the vertex labels are $\{a, a+1, \ldots, a+|V|\}$. For an a-vertex consecutive magic labeling of a graph G with p vertices and q edges they prove: if G has one isolated vertex, then $a = q$ and $(p-1)^2 + p^2 = (2q+1)^2$; if $q = p-1$, then p is odd and $a = p-1$; if $p = q$, then p is odd and if G has minimum degree 1, then $a = (p+1)/2$ or $a = p$; if G is 2-regular, then p is odd and $a = 0$ or p; and if G is r-regular, then p and r have opposite parities. They also define an *b-edge consecutive magic* labeling analogously and state some results for these labelings. Marimuthu and Kumar [2087] gave some results related to a-vertex consecutive magic graphs. In [2807] Setiawan, Sugeng, Silaban, and Riama provided sufficient conditions for a graph to admit a super edge magic total labeling and to have a b-edge consecutive edge magic total labeling. They alos gave [2087] new [2807] new the super edge magic total labelings for banana trees, firecrackers, and identified several classes of connected graphs that have both labelings.

Wood [3485] generalizes vertex-magic total and edge-magic total labelings by requiring only that the labels be positive integers rather than consecutive positive integers. He gives upper bounds for the minimum values of the magic constant and the largest label for complete graphs, forests, and arbitrary graphs.

Exoo, Ling, McSorley, Phillips, and Wallis [911] call a function λ a *totally magic labeling* of a graph G if λ is both an edge-magic total and a vertex-magic total labeling of G. A graph with such a labeling is called *totally magic*. Among their results are: P_3 is the only connected totally magic graph that has a vertex of degree 1; the only totally magic graphs with a component K_1 are K_1 and $K_1 \cup P_3$; the only totally magic complete graphs are K_1 and K_3; the only totally magic complete bipartite graph is $K_{1,2}$; nK_3 is totally magic if and only if n is odd; $P_3 \cup nK_3$ is totally magic if and only if n is even. In [3410] Wallis asks: Is the graph $K_{1,m} \cup nK_3$ ever totally magic? That question was answered by Calhoun, Ferland, Lister, and Polhill [652] who proved that if $K_{1,m} \cup nK_3$ is totally magic then $m = 2$ and $K_{1,2} \cup nK_3$ is totally magic if and only if n is even.

McSorley and Wallis [2118] examine the possible totally magic labelings of a union of an odd number of triangles and determine the spectrum of possible values for the sum of the label on a vertex and the labels on its incident edges and the sum of an edge label and the labels of the endpoints of the edge for all known totally magic graphs.

Gray and MacDougall [1151] define an *order n sparse semi-magic square* to be an $n \times n$ array containing the entries $1, 2, \ldots, m$ once (for some $m < n^2$), has its remaining entries equal to 0, and whose rows and columns have a constant sum of k. They prove some basic properties of such squares and provide constructions for several infinite families of squares, including squares of all orders $n \geqslant 3$. Moreover, they show how such arrays can be used to construct vertex-magic total labelings for certain families of graphs.

In Tables 8, 9, and 10, **VMT** means vertex-magic total labeling, **SVMT** means super vertex magic total, and **TM** means totally magic labeling. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2007.

Table 8: **Summary of Vertex-magic Total Labelings**

Graph	*Types*	*Notes*
C_n	VMT	[2047]
P_n	VMT	$n > 2$ [2047]
$K_{m,m} - e$	VMT	$m > 2$ [2047]
$K_{m,n}$	VMT	iff $\lvert m - n\rvert \leqslant 1$ [2394], [2047], [2049]

Continued on next page

Table 8 – *Continued from previous page*

Graph	*Types*	*Notes*
K_n	VMT	for n odd [2047] for $n \equiv 2 \pmod 4), n > 2$ [1953]
nK_3	VMT	iff $n \neq 2$ [932], [933], [2114]
mK_n	VMT	$m \geqslant 1,\ n \geqslant 4$ [2117]
Petersen $P(n,k)$	VMT	[372]
prisms $C_n \times P_2$	VMT	[2998]
W_n	VMT	iff $n \leqslant 11$ [2047], [2049]
F_n	VMT	iff $n \leqslant 10$ [2047], [2049]
friendship graphs	VMT	iff # of triangles $\leqslant 3$ [2047], [2049]
$G + H$	VMT	$\|V(G)\| = \|V(H)\|$ and $G \cup H$ is VMT [3407]
unions of stars	VMT	[3407]
tree with n internal vertices and more than $2n$ leaves	not VMT	[3407]
nG	VMT	n odd, G regular of even degree, VMT [3408] G is regular of odd degree, VMT, but not K_1 [3408]
$C_n \times C_{2m+1}$	VMT	[974]
$K_5 \times C_{2n+1}$	VMT	[974]
$G \times C_{2n}$	VMT	G $2r+1$-regular VMT [1746]
$G \times K_5$	VMT	G $2r+1$-regular VMT [1746]
$G \times H$	VMT	G r-regular VMT, r odd or r even and $\|H\|$ odd, H $2s$-regular supermagic [1746]

Table 9: **Summary of Super Vertex-magic Total Labelings**

Graph	*Types*	*Notes*
P_n	SVMT	iff $n > 1$ is odd [3170]
C_n	SVMT	iff n is odd [3170] and [2048]
$K_{1,n}$	SVMT	iff $n = 1$ [3170]
mC_n	SVMT	iff m and n are odd [3170]
W_n	not SVMT	[2048]
ladders	not SVMT	[2048]
friendship graphs	not SVMT	[2048]
$K_{m,n}$	not SVMT	[2048]
dragons (see §2.2)	SVMT	iff order is even [3171], [3171]
Knödel graphs $W_{3,n}$	SVMT	$n \equiv 0 \pmod 4$ [3495]
graphs with min. deg. 1	not SVMT	[2048]
K_{4n}	SVMT	$n > 1$ [1121]

Table 10: **Summary of Totally Magic Labelings**

Graph	*Types*	*Notes*
P_3	TM	the only connected TM graph with vertex of deg 1 [911]
K_n	TM	iff $n = 1, 3$ [911]
$K_{m,n}$	TM	iff $K_{m,n} = K_{1,2}$ [911]
nK_3	TM	iff n is odd [911]

Continued on next page

Table 10 – *Continued from previous page*

Graph	*Types*	*Notes*
$P_3 \cup nK_3$	TM	iff n is even [911]
$K_{1,m} \cup nK_3$	TM	iff $m = 2$ and n is even [652]

5.4 *H*-Magic Labelings

In 2005 Gutiérrez and Lladó [1167] introduced the notion of an H-magic labeling of a graph, which generalizes the concept of a magic valuation. Let H and $G = (V, E)$ be finite simple graphs with the property that every edge of G belongs to at least one subgraph isomorphic to H. A bijection $f\colon V \cup E \to \{1, \ldots, |V| + |E|\}$ is an *H-magic labeling* of G if there exists a positive integer $m(f)$, called the *magic sum*, such that for any subgraph $H'(V', E')$ of G isomorphic to H, the sum $\sum_{v\in V'} f(v) + \sum_{e\in E'} f(e)$ is equal to the magic sum, $m(f)$. A graph is *H-magic* if it admits an H-magic labeling. If, in addition, the H-magic labeling f has the property that $\{f(v)\}_{v\in V} = \{1, \ldots, |V|\}$, then the graph is *$H$-supermagic*. A K_2-magic labeling is also known as an edge-magic total labeling. Gutiérrez and Lladó investigate the cases where $G = K_n$ or $G = K_{m,n}$ and H is a star or a path. Among their results are: a d-regular graph is not $K_{1,h}$ for any $1 < h < d$; $K_{n,n}$ is $K_{1,n}$-magic for all n; $K_{n,n}$ is not $K_{1,n}$-supermagic for $n > 1$; for any integers $1 < r < s$, $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$; P_n is P_h-supermagic for all $2 \leqslant h \leqslant n$; K_n is not P_h-magic for any $2 < h \leqslant n$; C_n is P_h-magic for any $2 \leqslant h < n$ such that $\gcd(n, h(h-1)) = 1$. They also show that by uniformly gluing copies of H along edges of another graph G, one can construct connected H-magic graphs from a given 2-connected graph H and an H-free supermagic graph G.

Lladó and Moragas [1982] studied cycle-magic graphs. They proved: wheels W_n are C_3-magic for odd n at least 5; for $r \geqslant 3$ and $k \geqslant 2$ the windmill graphs $C_r^{(k)}$ (the one-point union of k copies of C_r) are C_r-supermagic; and if G is C_4-free supermagic graph of odd size, then $G \times K_2$ is C_4-supermagic. As corollaries of the latter result, they have that for n odd, prisms $C_n \times K_2$ and books $K_{1,n} \times K_2$ are C_4-magic. They define a subdivided wheel $W_n(r, k)$ as the graph obtained from a wheel W_n by replacing each radial edge $vv_i, 1 \leqslant i \leqslant n$ by a vv_i-path of size $r \geqslant 1$, and every external edge v_iv_{i+1} by a v_iv_{i+1}-path of size $k \geqslant 1$. They prove that $W_n(r, k)$ is C_{2r+k}-magic for any odd $n \neq 2r/k + 1$ and that $W_n(r, 1)$ is C_{2r+1}-supermagic. They also prove that the graph obtained by joining the end points of any number of internally disjoint paths of length $p \geqslant 2$ is C_{2p}-supermagic. Asif, Ali, Numan, and Semaničová-Feňovčíková [259] proved that if G is C_r-(super)magic, then so is nG and that $P_m \times P_n$ $(m, n \geqslant 4)$ is C_4-supermagic. In [2483] Pradipta and Salman define a *calendula graph*, denoted by $Cl_{m,n}$, as the graph constructed from C_m and m copies of C_n, $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$, and grafting the i-th edge of C_m to an edge of C_{n_i} for each i. They provide some cycle-supermagic labelings of calendula graphs. Chithra, Marimuthu, and Kumar [703] provided some basic results on the magic constant of graphs, on cycle-supermagic labelings of generalized splitting graphs, and proved that mC_n is cycle -supermagic for $m \geqslant 2$ and $n \geqslant 3$. Yang, Rashid, Siddiqui, and Hanif [3516] have provided explicit formulas for various C_n-super magic labelings of pumpkin graphs (graphs with exactly two vertices with multiple edges joining them) and two classes of planar graphs containing 8-sided and 4-sided faces or 6-sided and 4-sided faces.

A decomposition of a graph G into isomorphic copies of a graph H is H-magic if there is a bijection f from $V(G) \cup E(G)$ onto $\{0, 1, \ldots, |V(G)|+|E(G)|-1\}$ such that the sum of labels of edges and vertices of each copy of H in the decomposition is constant. By using the results on the sumset partition problem, Inayah, Lladó, and Moragas [1322] show that K_{2m+1} admits T-magic decompositions by any graceful tree with m edges. They address analogous problems for complete bipartite graphs and for antimagic and (a, d)-antimagic decompositions.

An edge of H-magic graph G is said to be a *good edge* if it belongs to only one subgraph isomorphic to H. For $s \geqslant 1$, B is the collection of good edges obtained by choosing exactly s good edges from each subgraph isomorphic to H in G. A *uniform subdivided* graph $\mathcal{G}$ of the graph G is obtained by subdividing all edges of B with $k \geqslant 1$ vertices. A *nonuniform subdivided* graph is obtained by subdividing the edges of $E(G) \setminus B$. Rizvi, Khalid, Ali, Miller, and Ryan [2635] prove that if a graph G is a C_n-supermagic graph then its uniform subdivided graph $\mathcal{G}$ is C_{n+sk}-(super)magic for positive integers n, s, and k. Using known results on the cycle-supermagicness they immediately obtain that uniform subdivided graphs of fans, antiprisms, triangular ladders, ladders and grids are cycle-(super)magic. They also prove that some special nonuniform subdivisions of fans and triangular ladders are cycle-supermagic.

Jeyanthi and Muthuraja [1482] established that $P_{m,n}$ is C_{2m}-supermagic for all $m, n \geqslant 2$ and the splitting graph of C_n is C_4-supermagic for $n \neq 4$. Nirmalasari Wijaya, Ryan, and Kalinowski [2303] show that for odd n and arbitrary k, the firecracker $F_{k,n}$ is $F_{2,n}$-supermagic, the banana tree $B_{k,n}$ is $B_{1,n}$-supermagic, and flower graphs are C_3-supermagic. Kojima [1725] proved that for two positive integers m and t with $m > t \geqslant 2$, if C_m is P_t-supermagic, then C_{3m} is also P_t-supermagic and for $t = 2, 3, 4,$ or 9 and C_n is P_t-supermagic if and only if n is odd with $n > t$. Nirmalasari Wijaya, Ryan, and Kalinowski [2279] proved that every d-dimensional grid graph ($d > 2$) is Q_d-supermagic where Q_d is the d-cube. Pu, Numan, Butt, Asif, Rafique, and Shao [2528] showed that toroidal fullerenes, Klein-bottle fullerenes, and the disjoint union of toroidal and Klein-bottle fullerenes are C_6-supermagic and the subdivision of toroidal fullerenes, Klein-bottle fullerenes, and any graph homeomorphic to a toroidal fullerene or Klein-bottle fullerene are cyclic-supermagic. Ulfatimah, Roswitha, and Kusmayadi [3237] proved that a star with one or more appended edges at each end-point admits a double star $S_{2,2}$-supermagic labeling and $L_m \odot P_n$ admits supermagic labeling of the one-point union of C_3 and C_4 for $m, n \geqslant 2$. Martini, Roswitha, and Lestari [2092] provided a C_4-supermagic labeling of $P_n \times P_m$ and a C_3 supermagic labeling of $K_1 + \overline{K_2}$. [2092] new

Öner and Erol [2321] showed that the triangular book-snake graph $S(3, n, s)$ admits a C_3-supermagic labeling. Using these labelings and supermagicness of subdivisions given by Taimur, Numan, Mumtaz, and Semaničová-Feňovčíková in [3184] and the fact that

$S(m,n,s)$ can be thought of as a subdivision of $S(3,n,s)$, Öner and Erol showed the C_m-supermagicness of polygonal book-snake graph $S(m,n,s)$.

The *edge corona path graph* $G_m \diamond P_n$ is the graph obtained from one copy of the gear graph G_m and $3m$ copies of P_n, P_n^i, by joining two end vertices of $e_i \in E(G_m)$ to every vertex $v_j inV(P_n)$ in the i-th copy of G_m with $i = 1, 2, \ldots, 3m$ and $j = 1, 2, \ldots, n$. Noviati, Martini, and Indriati [2306] provided a $C_3 \diamond P_n$-supermagic labeling for $f_n \diamond P_n$ and a $P_3 \diamond P_n$-supermagic labeling for $S_n \diamond P_n$ for odd $n \geqslant 3$.

Rizvi, Ali, and Hussian [2633] proved: the disjoint union of two or more copies of G is C_3-supermagic when G is a fan, triangular ladder, wheel, or a generalized antiprism; the disjoint union of two or more copies of G is C_3-supermagic when G is a ladder or a book; $sF_{n+1} \cup kF_n$ is C_3-supermagic; and $sL_{n+1} \cup kL_n$ is C_4-supermagic. Khalid, Rizvi, and Ali [1682] investigated whether the disjoint union of isomorphic copies of a connected cycle-supermagic graph is cycle-supermagic or not. They also study cycle-supermagic labelings for the disjoint union of isomorphic copies of fans, ladders, triangular ladders, wheels, books, and generalized antiprisms as well as disjoint unions of non-isomorphic copies of ladders and fans. Ali, Rizvi, Semaničová-Feňovčíková [157] proved that the disjoint union of an arbitrary number of isomorphic copies of prisms $C_n \times P_m$, $m \geqslant 2$ and $n \geqslant 3$, $n \neq 4$, is C_4-supermagic. They propose an open problem to find a C_4-supermagic labeling of the graph $t(C_4 \times P_m)$ for $m \geqslant 2$ and $t \geqslant 1$.

Liang [1937] proved the following: if there exist an even integer k and $m_i \equiv 0 \pmod{k}$ for every i in [1,n], then there exist $K_{k,k}$- and C_{2k}-supermagic decompositions of $K_{m_1,\ldots,m_n}$; if k and $t_n \geqslant k$ are even integers, then for any positive integers $t_i \equiv 0 \pmod{k}$, i in $[1, n-1]$, there exists a C_{2k}-supermagic decomposition of $K_{t_1,\ldots,t_{n-1},t_n}$; if there exists an even integer k and $K_{m,n}$ is C_{2k}-decomposable, then there exists a C_{2k}-supermagic decomposition of $K_{m,n}$; and if G is a graph with p vertices and p edges, H is a graph with q vertices and q edges, and there is an H-supermagic decomposition of G, then there exists an H-supermagic decomposition of nG. In [3469] Wichianpaisarn and Mato gave necessary and sufficient conditions for the existence of $K_{1,n-1}$-supermagic decomposition of $K_{n,n}$ minus a one-factor.

In [2100] Maryati, Baskoro, and Salman provided P_n-(super) magic labelings of subdivisions of stars, shrubs and banana trees. Ngurah, Salman, and Sudarsana [2294] construct C_n-(super) magic labelings for some fans and ladders. For any connected graph H, Maryati, Salman, Baskoro, and Irawati [2103] proved that the disjoint union of k isomorphic copies of a connected graph H is a H-supermagic graph if and only if $|V(H)| + |E(H)|$ is even or k is odd. In [2101] Maryati, Baskoro, Salman, and Irawati give some necessary conditions for any P_n-magic graph and provide some P_n-supermagic labelings of a cycle with some pendent edges and its subdivisions.

The m-shadow of graph G, $D_m(G)$, is a graph obtained by taking m copies of G, namely, G_1, $G_2, \ldots, G_m$, and then joining every vertex u in G_i, $i \in \{1, 2, \ldots, m-1\}$, to the neighbors of the corresponding vertex v in G_{i+1}. Agustin, Susanto, Dafik, Prihandini, Alfarisi, and Sudarsana [73] studied the H-supermagic labelings of $D_m(G)$ where G are paths and cycles.

Kojima [1725] proved the following. Let G be a C_4-free super edge-magic (p,q)-graph with the minimum degree at least one and $m \geqslant 2$. If q odd and $m = 2$ or $|p-q| \geqslant 2$, then $P_m \times G$ is C_4-supermagic; if p is odd and $m = 2$ or $|p-q| = 1$ and $m \leqslant 5$, then $P_m \times G$ is C_4-supermagic; if $n \geqslant 3$ is odd and m is even, then $P_2 \times (C_n \odot \overline{K_m})$ is C_4-supermagic; if $n \geqslant 3$ is odd and m is odd, then $P_2 \times (C_n \odot \overline{K_m})$ is not C_4-supermagic; if G is a caterpillar, then $P_m \times G$ is C_4-supermagic for $m \geqslant 2$; and $P_m \times C_n$ is C_4-supermagic for $m \geqslant 2$ and $n \geqslant 3$. The latter result solved an open problem in [2295] by Ngurah, Salman, and Susilowati. Kojma also proved that if a C_4-free bipartite $(p, p-1)$-graph G with the minimum degree at least one and partite sets U and V has a super edge-magic labeling f of G such that $f(U) = \{1, 2, \ldots, |U|\}$, then $P_m \times (2G)$ is C_4-supermagic.

Maryati, Salman, Baskoro, Ryan, and Miller [2104] define a *shackle* as a graph obtained from nontrivial connected graphs $G_1, G_2, \ldots, G_k$ $(k \geqslant 2)$ such that G_s and G_t have no common vertex for every s and t in $[1, k]$ with $|s-t| \geqslant 2$, and for every i in $[1, k-1]$, G_i and G_{i+1} share exactly one common vertex that are all distinct. They prove that shackles and amalgamations constructed from copies of a connected graph H is H-supermagic. (Recall for finite collection of graph $G_1, G_2, \ldots, G_k$ with a fixed vertex v_i from each G_i, an *amalgamation*, AmalG_i, v_i), is the graph obtained by identifying the v_i.) Ashari and Salman [252] gave sufficient conditions for (H_1, H_2)-supermagic labelings for shackles involving cycles, flowers, and prisms.

Ngurah, Salman, and Susilowati [2295] proved the following: chain graphs with identical blocks each isomorphic to C_n are C_n-supermagic; fans are C_3-supermagic; ladders and books are C_4-supermagic; $K_{1,n} + K_1$ are C_3-supermagic; grids $P_m \times P_n$ are C_4-supermagic for $m \geqslant 3$ and $n = 3, 4$, and 5. They pose the case that $P_m \times P_n$ are C_4-supermagic for $n > 5$ as an open problem. They also have some results on P_t-(super) magic labelings of cycles. Kathiresan, Marimuthu, and Chithra investagated the existence of C_m-supermagic labelings generalized fans, generalization of a graph obtained by joining of a star K1,n with one isolated vertex, grids, and generalized books. The results given in this article are the generalizations of some results given by Ngurah, Salman, and Susilowati in [2295].

Roswitha, Baskoro, Maryati, Kurdhi, and Susanti [2653] proved: the generalized Jahangir graph $J_{k,s}$ is C_{s+2}-supermagic; $K_{2,n}$ is C_4-supermagic; and W_n for n even and $n \geqslant 4$ is C_3-supermagic. As an open problem they asked if $K_{m,n}$, $2 < m \leqslant n$, admits a C_{2m}-supermagic labeling. Roswitha and Baskoro [2654] proved that double stars, caterpillars, firecrackers, and banana trees admit star-supermagic labelings.

Maryati, Salman, and Baskoro [2102] characterized all graphs G such that the disjoint union of copies of G is G-supermagic. They also showed: the disjoint union of any paths is mP_n-supermagic for certain values of m and n; some subgraph amalgamations of graphs G are G-supermagic; and for any subgraph H of G Amal$(G,\ H,\ k)$ is G-supermagic. Salman and Maryati [2687] proved that Amal(G, P_n, k) is G-supermagic.

Selvagopal and Jeyanthi proved: for any positive integer n, a the k-polygonal snake of length n is C_k-supermagic [2741]; for $m \geqslant 2$, $n = 3$, or $n > 4$, $C_n \times P_m$ is C_4-supermagic [1523]; $P_2 \times P_n$ and $P_3 \times P_n$ are C_4-supermagic for all $n \geqslant 2$ [1523]; the one-point union of any number of copies of a 2-connected H is H-magic [1521]; graphs obtained by taking copies $H_1, H_2, \ldots, H_n$ of a 2-connected graph H and two distinct edges e_i, e_i' from each H_i and identifying e_i' of H_i with e_{i+1} of H_{i+1} where $|V(H)| \geqslant 4, |E(H)| \geqslant 4$ and n is odd or both n and $|V(H)|+|E(H)|$ are even are H-supermagic [1521]. For simple graphs H and G the *H-supermagic strength of G* is the minimum constant value of all H-magic total labelings of G for which the vertex labels are $\{1, 2, \ldots, |V|\}$. Jeyanthi and Selvagopal [1522] found the C_n-supermagic strength of n-polygonal snakes of any length and the H-supermagic strength of a chain of an arbitrary 2-connected simple graph.

Let $H_1, H_2, \ldots, H_n$ be copies of a graph H. Let u_i and v_i be two distinct vertices of H_i for $i = 1, 2, \ldots, n$. The *chain graph* H_n of H of length n is the graph obtained by identifying the vertices u_i and v_{i+1} for $i = 1, 2, \ldots, n-1$. In [1520] Jayanthi and Selvagopal show that a chain graph of any 2-connected simple graph H is H-supermagic and if H is a 2-connected (p, q) simple graph, then H_n is H-supermagic if $p+q$ is even or $p+q+n$ is even.

The *antiprism* on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ (subscripts are taken modulo n). Jeyanthi, Selvagopal, and Sundaram [1525] proved the following graphs are C_3-supermagic: antiprisms, fans, and graphs obtained from the ladders $P_2 \times P_n$ with the two paths $v_{1,1}, \ldots, v_{1,n}$ and $v_{2,1}, \ldots, v_{2,n}$ by adding the edges $v_{1,j}v_{2,j+1}$.

Jeyanthi and Selvagopal [1524] show that for any 2-connected simple graph H the edge amalgamation of a finite number of copies of H is H-supermagic. They also show that the graph obtained by picking one endpoint v_i from each of k copies of $K_{1,k}$ then creating a new graph by joining each v_i to a fixed new vertex v is $K_{1,k}$-supermagic.

An H-magic labeling in an *H-decomposable* of a graph G is a bijection $f : V(G) \cup E(G)$ onto $\{1, 2, \ldots, p+q\}$ such that for every copy of H in the decomposition, the sum of $f(v) + f(e)$ over all v in $V(H)$ and e in $E(H)$ is constant. The labeling f is said to be *$H - V$-super magic* if $f(V(G)) = \{1, 2, \ldots, p\}$. Marimuthu and Kumar [2088] prove that $K_{n,n}$ ($n \geqslant 2$) is H-V-super magic decomposable when H is $K_{1,n}$. Marimuthu and Kumar [2086] provide a necessary and sufficient condition for the existence of V-super vertex-magic labeling and give E-super and V-super vertex-magic total labeling of certain families of generalized Petersen graphs. They also prove that no wheel is E-super vertex-magic, C_3 is the only friendship graph that is V-super vertex-magic, and C_3 is the only friendship graph that is E-super vertex-magic.

An H-magic labeling f is said to be an *H-E-super magic* labeling if $f(E(G)) = \{1, 2, \ldots, q\}$. A graph that admits an H-E-super magic labeling is called an *H-E-super magic decomposable* graph. Subbiah and Pandimadevi [3061] study some elementary properties of H-E-super magic labelings with H an m-factor and provide a necessary and sufficient condition for an even regular graph to be H-E-super magic decomposable where H is a 2-factor.

5.5 Magic Labelings of Type (a, b, c)

A magic-type method for labeling the vertices, edges, and faces of a planar graph was introduced by Lih [1949] in 1983. Lih defines a *magic labeling of type* (1,1,0) of a planar graph $G(V, E)$ as an injective function from $\{1, 2, \ldots, |V|+|E|\}$ to $V \cup E$ with the property that for each interior face the sum of the labels of the vertices and the edges surrounding that face is some fixed value. Similarly, Lih defines a *magic labeling of type* $(1, 1, 1)$ of a planar graph $G(V,E)$ with face set F as an injective function from $\{1,2,\ldots,|V|+|E|+|F|\}$ to $V \cup E \cup F$ with the property that for each interior face the sum of the labels of the face and the vertices and the edges surrounding that face is some fixed value. Lih calls a labeling involving the faces of a plane graph *consecutive* if for every integer s the weights of all s-sided faces constitute a set of consecutive integers. Lih gave consecutive magic labelings of type $(1,1,0)$ for wheels, friendship graphs, prisms, and some members of the Platonic family. In [297] Bača shows that the cylinders $C_n \times P_m$ have magic labelings of type $(1,1,0)$ when $m \geqslant 2, n \geqslant 3, n \neq 4$. In [307] Bača proves that the generalized Petersen graph $P(n,k)$ (see §2.7 for the definition) has a consecutive magic labeling if and only if n is even and at least 4 and $k \leqslant n/2-1$. In [101] Ahmed and Babujee provide face bimagic labelings of type (1,1,0) for certain wheels, cylinders, and disjoint unions of m copies of prism graphs. [101] new

Bača gave magic labelings of type $(1,1,1)$ for fans [291], ladders [291], planar bipyramids (that is, 2-point suspensions of paths) [291], grids [300], hexagonal lattices [299], Möbius ladders [294], and $P_n \times P_3$ [295]. Kathiresan and Ganesan [1669] show that the graph $P_{a,b}$ consisting of $b \geqslant 2$ internally disjoint paths of length $a \geqslant 2$ with common end points has a magic labeling of type $(1,1,1)$ when b is odd, and when $a = 2$ and $b \equiv 0 \pmod 4$. They also show that $P_{a,b}$ has a consecutive labeling of type $(1,1,1)$ when b is even and $a \neq 2$. Ali, Hussain, Ahmad, and Miller [154] study magic labeling of type $(1,1,1)$ for wheels and subdivided wheels. They prove: wheels admits a magic labeling of type and $(1,1,1)$ and $(0,1,1)$, for odd n wheels W_n n admit a magic labeling of type $(0,1,0)$, and subdivided wheels admit a magic labeling of type $(1,1,0)$. As an open problem they ask for a magic labeling of type $(1,1,0)$ for W_n and n even. Ahmad [80] proves that subdivided ladders admit magic labelings of type (1,1,1) and admit consecutive magic labelings of type (1,1,0).

Bača [293], [292], [303], [301], [295], [302] and Bača and Holländer [332] gave magic labelings of type $(1,1,1)$ and type $(1,1,0)$ for certain classes of convex polytopes. Kathiresan and Gokulakrishnan [1671] provided magic labelings of type $(1,1,1)$ for the families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces, and one external infinite face. Bača [298] also provides consecutive and magic labelings of type $(0,1,1)$ (that is, an injective function from $\{1,2,\ldots,|E|+|F|\}$ to $E \cup F$ with the property that for each interior face the sum of the labels of the face and the edges surrounding that face is some fixed value) and a consecutive labeling of type $(1,1,1)$ for a kind of planar graph with hexagonal faces. Tabraiz and Hussain [3181] provide a super magic labeling of type (1, 0, 0) for ladders and a super magic labeling of type (1, 0, 0) for subdivided ladders.

A *magic labeling of type* (1,0,0) of a planar graph G with vertex set V is an injective function from $\{1,2,\ldots,|V|\}$ to V with the property that for each interior face the sum of the labels of the vertices surrounding that face is some fixed value. Kathiresan, Muthuvel, and Nagasubbu [1674] define a *lotus inside a circle* as the graph obtained from the cycle with consecutive vertices $a_1, a_2, \ldots, a_n$ and the star with central vertex b_0 and end vertices $b_1, b_2, \ldots, b_n$ by joining each b_i to a_i and a_{i+1} $(a_{n+1} = a_1)$. They prove that these graphs $(n \geqslant 5)$ and subdivisions of ladders have consecutive labelings of type $(1,0,0)$. Devaraj [809] proves that graphs obtained by subdividing each edge of a ladder exactly the same number of times has a magic labeling of type $(1, 0, 0)$.

In Table 11 we use following abbreviations

M(a, b, c) magic labeling of type (a, b, c)

CM(a, b, c) consecutive magic labeling of type (a, b, c).

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Table 11: **Summary of Magic Labelings of Type (a, b, c)**

Graph	*Labeling*	*Notes*
W_n	CM(1,1,0)	[1949]
friendship graphs	CM(1,1,0)	[1949]
prisms	CM(1,1,0)	[1949]
cylinders $C_n \times P_m$	M(1,1,0)	$m \geqslant 2$, $n \geqslant 3$, $n \neq 4$ [297]
fans F_n	M(1,1,1)	[291]
ladders	M(1,1,1)	[291]
planar bipyramids (see §5.3)	M(1,1,1)	[291]
grids	M(1,1,1)	[300]
hexagonal lattices	M(1,1,1)	[299]
Möbius ladders	M(1,1,1)	[294]
$P_n \times P_3$	M(1,1,1)	[295]
certain classes of convex polytopes	M(1,1,1) M(1,1,0)	[293], [303], [301], [295] [302], [332]
certain classes of planar graphs with hexagonal faces	M(0,1,1) CM(0,1,1) CM(1,1,1)	[298]
lotus inside a circle (see §5.3)	CM(1,0,0)	$n \geqslant 5$ [1674]
subdivisions of ladders	M(1,0,0) CM(1,0,0)	[809] [1674]

5.6 Sigma Labelings/1-vertex magic labelings/Distance Magic

In 1987 Vilfred [3388] (see also [3389]) defined a *sigma-labeling* of a graph G with n vertices as a bijection f from the vertices of G to $\{1, 2, \dots, n\}$ such that there is a constant k with the property that, at any vertex v the sum $\sum f(u)$ taken over all neighbors u of v is k. The concept of sigma labeling was independently studied in 2003 by Miller, Rodger, and Simanjuntak in [2148] under the name 1-*vertex magic*. In a 2009 article Sugeng, Fronček, Miller, Ryan, and Walker [3081] used the term *distance magic labeling.* For convenience, we will use the term distance magic. In [3391] Vilfred and Jinnah give a number of necessary conditions for a graph to have a distance magic labeling. One of them is that if u and v are vertices of a graph with a distance labeling, then the order of the symmetric difference of $N(u)$ and $N(v)$ (neighborhoods of u and v) is not 1 or 2. This condition rules out a large class of graphs as having distance magic labelings. Rao, Singh, and Parameswaran [2594] have shown $C_m \times C_n$ has a distance magic labeling if and only if $m = n \equiv 2 \pmod 4$ and $K_m \times K_n,\ m \geqslant 2, n \geqslant 3$ does not have a distance magic labeling. In [525] Benna gives necessary and sufficient condition for $K_{m,n}$ to be a distance magic graph and proves that if G_1 and G_2 are connected graphs with minimum degree 1 and at least three vertices, then $G_1 \times G_2$ does not have a distance magic labeling. Rao, Sighn, and Parameswaran [49] prove that every graph is an induced subgraph of a regular graph that has a distance magic labeling. As open problems, Rao [2592] asks for a characterize 4-regular graphs that have distance magic labelings and which graphs of the form $C_m \times C_n,\ m = n \equiv 2 \pmod 4$ have distance magic labelings. Kovář, Fronček, and Kovářová [1749] classified all orders n for which a 4-regular distance magic graph exists and also showed that there exists a distance magic graph with $k = 2t$ for every integer $t \geqslant 6$. Acharaya, Rao, Signh, and Parameswaran [48] proved $P_m \times C_n$ does not have a distance magic labeling when m is at least 3 and provide necessary and sufficient conditions for $K_{m,n}$ to have a distance magic labeling. Kovár and Silber [1751] proved that an $(n-3)$-regular distance magic graph with n vertices exists if and only if $n \equiv 3 \pmod 6$ and that its structure is determined uniquely. Moreover, they reduce constructions of Fronček to a single construction and provide another sufficient condition for the existence a distance magic graph with an odd number of vertices. Fronček, Kovář, and Kovářová [975] provide a construction for distance magic graphs arising from arbitrary regular graphs based on an application of magic rectangles. They also solve a problem posed by Shafiq, Ali, and Simanjuntak [2845]. Godinho and Singh [1111] investigate the distance magic labelings for neighborhood expansions of graphs and present a method for embedding regular graphs into distance magic graphs.

Among the results of Miller, Rodger, and Simanjuntak in [2148]: the only trees that have a distance magic labeling are P_1 and P_3; C_n has a distance magic labeling if and only if $n = 4$; K_n has a distance magic labeling if and only if $n = 1$; the wheel $W_n = C_n + P_1$ has a distance magic labeling if and only if $n = 4$; the complete graph $K_{n,n,\dots,n}$ with p partite sets has a distance magic labeling if and only if n is even or both n and p are odd; an r-regular graph where n is odd does not have a distance magic labeling; and $G \times \overline{K_{2n}}$ has a distance magic labeling for any regular graph G. They also give necessary and sufficient conditions for complete tripartite graphs to have a distance magic labeling.

For a given a graph $G = (V, E)$ and a positive integer t, the *generalised Mycielskian graph* $M_t(G)$ is the graph with vertex set $\{V \times \{0, 1, \ldots, t-1\}\} \cup \{u\}$, with edges $(x, 0)(y, 0)$ and $(x, i)(y, i+1)$ where there is an edge $xy \in E$, and an edge $(x, t-1)u$ for all $x \in V$. In [2374] Pawar and Singh gave some results on distance magic labeling of generalised Mycielskian graphs. [2374] new

Generalizing defintions given by Vilfred in [3388] and [3389], Vilfred Kamalappan [3390] defines a labeling $f : V(G) \to \{1, 2, \ldots, n\}$ on a graph G of order $n \geqslant 3$ as a *k-distance magic labeling* (k-DML) if $\sum_{w \in \partial N_k(u)} f(w)$ is a constant and independent of $u \in V(G)$ where $\partial N_k(u) = \{v \in V(G) : d(u, v) = k\}$, $k \in \mathbb{N}$. A graph G is called a *k-distance magic (k-DM)* if it has a k-DML. A *long brush*, denoted by $LP_{n,m}$, is a graph with vertex set $\{u_1, u_2, \ldots, u_n,\ v_1, v_2, \ldots, v_m\}$, a path $P_n = u_1\ u_2 \cdots u_n$ and edge set $E(P_n) \cup \{u_1 v_i : i = 1 \text{ to } m\} \cup E(< v_1, v_2, \ldots, v_m >)$, $m + n \geqslant 3$ and $m, n \in \mathbb{N}$. Using partition techniques, he obtain families of k-DM graphs and prove that (i) For $k, n \geqslant 3$, $m \geqslant 2$ and $k, m, n \in \mathbb{N}$, $LP_{n,m}$ is k-DM if and only if $m(m-1) \leqslant 2n$ and $k = n$; (ii) For a given $m \geqslant 2$ and for every $k \in \mathbb{N}_0$, $LP_{\frac{m(m-1)}{2}+k,m}$ is a $(\frac{m(m-1)}{2} + k)$-DM graph, and (iii) For $m \geqslant 3$, $x \geqslant 2$, $LP_{1,m} = K_1(u_1) + (K_{m_1} \cup K_{m_2} \cup \ldots \cup K_{m_x})$, $1 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_x$, $m_1 + m_2 + \ldots + m_x = m$, $m_1 + m_2 \geqslant 3$ and $m_1, m_2, \ldots, m_x, x \in \mathbb{N}$, $LP_{1,m}$ is 2-DM if and only if u_1 is assigned with a suitable j and $J_{m+1} \setminus \{j\}$ is partitioned into x constant sum partites of orders $m_1, m_2, \ldots, m_x$, $1 \leqslant j \leqslant m + 1$. [3390] new

An orientable Γ-distance magic labeling of a graph, was introduced by Cichacz, Freyberg and Fronček [738] as a generalization of group distance magic labeling for oriented graphs. They showed that an even regular circulant graph of order n is orientable Z_n-distance magic, the direct product $C_n \times C_m$ is orientable Z_{nm}-distance magic. They also considered some products of circulant graphs. Moreover they proved that if G has order $n \equiv 2 \pmod 4$ and all vertices of odd degree, then there does not exist an orientable Γ-distance magic labeling of G for any Abelian group Γ of order n. Dyrlaga and Szopa in [863] gave necessary and sufficient conditions for lexicographic product $K_m \circ \overline{K_n} \cong K_{\underbrace{m, m, \ldots, m}_{n}}$ to be oriantable ζ_{mn}-distance magic. As a consequence, they provide an infinite family of odd regular graphs possessing orientable ζ_n-distance magic labeling. In [958] and [959] Freyberg and Keranen found orientable Z_n-distance magic labelings of the Cartesian product of cycles. In [960] they studied Z_n-distance magic labelings for the strong product of cycles.

In [2194] Mukherjee and Pawa use $L = \{1, 2, \ldots, n_p\}$ to denote the multiset obtained by reducing $1, 2, \ldots, n$ modulo p and replacing 0's, if any, by p. They call a graph G *p-distance magic* if there is a bijective map f from the vertex set V to the multiset L such that the weight of each vertex $w(x)$ is equal to the same element $\mu \pmod p$, where $w(x) = \sum f(y)$ is taken over all $y \in N(x)$. They proved that a graph G is distance magic if and only if it is p-distance magic for all $p \geqslant 1$. They also proved that for $p = n$, p-distance magic labeling becomes a Z_n-distance magic labeling. In some cases, they provide ways to construct different magic constants in for Z_n-distance magic labelings. In some cases, they prove the uniqueness of the magic constant for Z_n-distance magic labeling. [2194] new

Anholcer, Cichacz, Peterin, and Tepeh [205] proved that the direct product of two cycles C_m and C_n is distance magic if and only if $m = 4$ or $n = 4$, or $m, n \equiv 0 \pmod 4$ (the *direct product* of graphs G and H has the vertex set $V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if g is adjacent to g' in G and h is adjacent to h' in H). In [732] Cichacz gave necessary and sufficient conditions for circulant graph $C_n(1, 2, \ldots, p)$ to be distance magic for p odd. In [739] Cichacz and Fronček characterized all distance magic circulant graphs $C_n(1, p)$ for p odd. Cichacz, Fronček, Krop, and Raridan [740] proved that r-partite graph $K_{n,n,\ldots,n} \times C_4$ is distance magic if and only if $r > 1$ and $n > 2$ is even. Anholcer and Cichacz [201] gave necessary and sufficient conditions for lexicographic product of an r-regular graph G and $K_{m,n}$ to be distance magic. Cichacz and Görlich [744] gave necessary and sufficient conditions for the direct product of an r-regular graph G and $K_{m,n}$ to be distance magic. Kamatchi, Ramalakshmi, and Nilavarasi [1570] proved that P_n^2 and the graphs obtained from P_n ($n \neq 5$) by joining each internal vertex v with an end vertex at even distance from v are distance antimagic. In [2943] Shrimali and Parmar discuss the existence of distance magic labelings for the product, direct product, strong product, and carona product of graphs involing C_3^t and C_4. In [734] necessary and sufficient conditions for complete tripartite graphs to be group distance magic was given by Cichacz. In [245] Arumugam, Kamatchi, and Kovář give several results on distance magic graphs and open problems. [1570] new

A graph $\Gamma(V, E)$ of order n is said to be a *closed distance magic labeling* of Γ is a bijection $f : V \to \{1, 2, \ldots, n\}$ for which there exists a positive integer r such that $\sum_{x \in N[u]} f(x) = r$ for all vertices $u \in V$, where $N[u]$ is the closed neighborhood of u. A graph is said to be *closed distance magic* if it admits a closed distance magic labeling. Fernández, Maleki, Miklavič, and Razafimahatratra [925] classified all connected closed distance magic circulants with valency at most 5, that is, Cayley graphs $\mathrm{Cay}(Z_n; S)$, where $|S| \leqslant 5$ and S generates Z_n. [925] new

A finite r-regular graph G has a p-partition (resp. closed p-partition) ($p \geqslant 2$) if there exists a partition of the set $V(G)$ into $V_1, V_2, \ldots, V_p$ such that for every $x \in V(G)$, all $V(x) \cap V_i$ (respectively, $V[x] \cap V_1$) have the same size. In [747] Cichacz and Nikodem proved the following for finite r-regular graphs G. If G is distance magic (resp. closed distance magic) graph with a p-partition and $p(t-1)$ even then tG is also distance (resp. closed distance) magic. If G has order t and H is p-regular such that tH is distance (resp. closed distance) magic, then the lexicographic product of G and H is distance (resp. closed distance) magic. If G has order t and H is such that tH is distance magic, then the lexicographic product of G and H and the direct product of G and H are distance (resp. closed distance) magic. If H is a p-regular distance magic graph with a 2-partition, then the lexicographic product of G and H and the direct product of G and H are distance magic. They further proved that if $G = C_3$ or G is the strong product of C_n and C_m for $n = 3$ and m odd, or $m, n \equiv 3 \pmod 6$, then tG is closed distance magic if and only if t is odd. (The *strong direct product* of G and H has vertex set $V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G.)

In [2761] Seoud, Maqsoud, and Aldiban determined whether or not the following families of graphs have a distance magic vertex labeling: $K_n - \{e\}$; $K_n - \{2e\}$; P_n^k; C_n^2; $K_m \times$

C_n; $C_m + P_n$; $C_m + C_n$; $P_m + P_n$; $K_{1,r,s}$; $K_{1,r,m,n}$; $K_{2,r,m,n}$; $K_{m,n} + P_k$; $K_{m,n} + C_k$; $C_m + \overline{K_n}$; $P_m + \overline{K_n}$; $P_m \times P_n$; $K_{m,n} \times P_k$; $K_m \times P_n$; the splitting graph of $K_{m,n}$; $K_n + G$; $K_m + \overline{K_n}$; $K_m + C_n$; $K_m + P_n$; $K_{m,n} + K_r$; $C_m \times P_n$; $C_m \times K_{1,n}$; $C_m \times K_{n,n}$; $C_m \times K_{n,n+1}$; $K_m \times K_{n,r}$; and $K_m \times K_n$. Typically, distance magic labelings exist only a few low parameter cases.

Miklavic and Sparl [2137] provided a sufficient condition for a Hamming graph to be distance magic and as a they corollary provide an infinite number of pairs (D, q) for which the corresponding Hamming graph with words of length D and over an alphabet of size q is distance magic. They classify distance magic folded hypercubes (a graph obtained from a hypercube by identifying pairs of vertices at maximal distance) by showing that the dimension-D folded hypercube is distance magic if and only if is divisible by 4.

In [965] Frонček defined the notion of a Γ-*distance magic* graph as one that has a bijective labeling of vertices with elements of an Abelian group Γ resulting in constant sums of neighbor labels. A graph that is Γ-distance magic for an Abelian group Γ is called *group distance magic*. Cichacz and Fronček [739] showed that for an r-regular distance magic graph G on n vertices, where r is odd there does not exist an Abelian group Γ of order n having exactly one involution (i.e., an element that is its own inverse) that is Γ-distance magic. Fronček [965] proved that $C_m \times C_n$ is a Z_{mn}-distance magic graph if and only if mn is even. He also showed that $C_{2^n} \times C_{2^n}$ has a $Z_{2^{2n}}$-distance magic labeling. In [728] Cichacz showed some Γ-distance magic labelings for $C_m \times C_n$ where $\Gamma \not\approx Z_{mn}$ and $\Gamma \not\approx Z_{2^{2n}}$. Anholcer, Cichacz, Peterin, and Tepeh [207] proved that if an r_1-regular graph G_1 is Γ_1-distance magic and an r_2-regular graph G_2 is Γ_2-distance magic, then the direct product of graphs G_1 and G_2 is $\Gamma_1 \times \Gamma_2$-distance magic. Moreover they showed that if G is an r-regular graph of order n and $m = 4$ or $m = 8$ and r is even, then $C_m \times G$ is group distance magic. They proved that $C_m \times C_n$ is Z_{mn}-distance magic if and only if $m \in \{4, 8\}$ or $n \in \{4, 8\}$ or $m, n \equiv 0 \pmod 4$. They also showed that if $m, n \not\equiv 0 \pmod 4$ then $C_m \times C_n$ is not Γ-distance magic for any Abelian group Γ of order mn. Cichacz [729] gave necessary and sufficient conditions for complete k-partite graphs of odd order p to be Z_p-distance magic. Moreover she showed that if $p \equiv 2 \pmod 4$ and k is even, then there does not exist a group Γ of order p that has a Γ-distance labeling for a k-partite complete graph of order p. She also proved that $K_{m,n}$ is a group distance magic graph if and only if $n + m \not\equiv 2 \pmod 4$. In [730] Cichacz proved that if G is an Eulerian graph, then the lexicographic product of G and C_4 is group distance magic. In the same paper she also showed that if $m + n$ is odd, then the lexicographic product of $K_{m,n}$ and C_4 is group distance magic. In [731] Cichacz gave necessary and sufficient conditions for direct product of $K_{m,n}$ and C_4 for $m + n$ odd and for $K_{m,n} \times C_8$ to be group distance magic. In [733] Cichacz proved that for n even and $r > 1$ the Cartesian product the complete r-partitie graph $K_{n,n,\ldots,n}$ and C_4 is group distance magic. Godinho and Singh [1110] obtain group distance magic labelings of C_n^r for certain classes of abelian groups and provide necessary conditions for existence of such labelings.

Anholcer, Cichacz, Froncek, Simanjuntak, and Qiu [202] proved that for n odd, there does not exist a Γ-distance magic labeling of Q_n for any abelian group Γ of order $|V(Q_n)|$, whereas for n even there exists a Γ-distance magic labeling of Q_n for every abelian group

Γ of order $|V(Q_n)|$. They also study similar distance antimagic and Γ-distance antimagic labelings where one finds a bijection such that the sums of labels are pairwises distinct for all the vertices. They show that there exists a Γ-distance antimagic labeling of Q_n for any abelian group Γ of order $2n$ where n is odd for these labelings and give some relationships between Γ-closed distance magic and antimagic labelings and Γ-distance antimagic labelings.

Cichacz [735] showed there exists an infinite family of odd regular graphs possessing Γ-distance magic labeling for groups Γ with more than one involution. In [732] Cichacz using a notion of a Γ-magic rectangle set $MRS_\Gamma(a, b; c)$ showed group distance labeling for Cartesian and direct product of complete r-partite graphs. These results supported a conjecture in [739] that says that if G is a distance magic graph, then G is group distance magic.

A *directed* Γ*-distance magic* labeling of an oriented graph $\overrightarrow{G} = (V, A)$ of order n is a bijective mapping f from the vertex set of G to an abelian group Γ of order n with the property that there exists a constant $c \in \Gamma$ such that, for every vertex $v \in V(\overrightarrow{G})$, $w(v) = \sum_{u\in N_G^{\text{in}}(v)} f(u) - \sum_{u\in N_G^{\text{out}}(v)} f(u) = c$, where $N_G^{\text{in}}(v)$ is the open in-neighborhood and $N_G^{\text{out}}(v)$ is the open out-neighborhood of vertex v, that is $N_G^{\text{in}}(v) = \{u : uv \in A\}$ and $N_G^{\text{out}}(v) = \{u : vu \in A\}$. If for a graph G there exists an orientation $\overrightarrow{G}$ such that there is a directed Γ-distance magic labeling f for $\overrightarrow{G}$ the graph G is called *orientable* Γ*-distance magic*. Freyberg and Keranen [959] proved that $C_m \times C_n$ is orientable $\mathbb{Z}_{mn}$-distance magic for all $m, n \geqslant 3$.

In [206] Anholcer, Cichacz, Peterin, and Tepeh introduce the notion of balanced distance magic graphs. They say that a distance magic graph G with an even number of vertices is *balanced* if there exists a bijection f from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ such that for every vertex w the following holds: If $u \in N(w)$ with $f(u) = i$, then there exists $v \in N(w), u \neq v$ with $f(v) = |V(G)| - i + 1$. They prove that a graph G is balanced distance magic if and only if G is regular and $V(G)$ can be partitioned in pairs $(u_i, v_i), i \in \{1, 2, \ldots, |V(G)|/2$, such that $N(u_i) = N(v_i)$ for all i. Using this characterization, the following theorems are proved: if G is a regular graph and H is a graph not isomorphic to $\overline{K_n}$ where n is odd, then $G \odot H$ is a balanced distance magic graph if and only if H is a balanced distance magic graph; $G \times H$ is balanced distance magic if and only if one of G and H is balanced distance magic and the other one is regular; and $C_m \times C_n$ is distance magic if and only if $n = 4$ or $m = 4$ or $m, n \equiv 0 \pmod 4$ and $C_m \times C_n$ is balanced distance magic if and only if $n = 4$ or $m = 4$. In [209] they prove that every balanced distance magic graph is also group distance magic; the Cartesian product of a balanced distance magic graph and a regular graph is group distance magic; the direct product of C_4 or C_8 and a regular graph is group distance magic; and they show that $C_8 \times G$ is also group distance magic for any even-regular graph G. They also prove that $C_{4s} \times C_{4t}$ is $A \times B$-distance magic for any Abelian groups A and B of order $4s$ and $4t$, respectively. Moreover, they conjecture that $C_{4m} \times C_{4n}$ is a group distance magic graph. They prove that $C_m \times C_n$ is Z_{mn}-distance magic if and only if $m \in \{4, 8\}$ or $n \in \{4, 8\}$ or both n and m are divisible by 4, and that $C_m \times C_n$ with orders not divisible by 4 is not

Γ-distance magic for any Abelian group Γ of order mn.

Let $G = (V, E)$ be a graph on n vertices. A bijection f from the vertices of graph G to $\{1, 2, \ldots, |V(G)|\}$ is called a *nearly distance magic* labeling of G if there exists a positive integer k such that $\sum f(x)$ over all $x \in N(v) = k$ or $k+1$ for all v. The constant k is called a *magic constant* of the graph and any graph which admits such a labeling is called a *nearly distance magic* graph. Godinho, Singh, and Arumugam [1113] give several basic results on nearly distance magic graphs and compute the magic constant k in terms of the fractional total domination number of the graph.

In [2098] Marr and Simanjunak defined *D-magic labelings* for oriented graphs where D is a distance set as follows. The vertices of a graph G are distinct integers in $\{1, 2, \ldots, |V(G)|\}$ such that the sum of all the vertex labels that are a distance in D from a given vertex is the same across all vertices. They gave some results related to the magic constant, constructd a few infinite families of D-magic graphs, and examined trees, cycles, and multipartite graphs.

In [1629] Kang, Chen, Li, and Hou investigated D-magic labelings of the halved n-cube ($n \geqslant 2$) (that is, all binary strings of length n with even number of 1's as vertices and edges between any two strings of Hamming distance 2). They show that the folded n-cube has a $\{1\}$-magic labeling (resp., a $\{0, 1\}$-magic labeling) if and only if $n = 0 \pmod 4$ (resp., $n = 3 \pmod 4$).

A survey of results on distance magic (sigma, 1-vertex) labelings through 2009 is given in [241].

5.7 Other Types of Magic Labelings

For a graph $G = (V, E)$ naturally embedded in the torus, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a *C_n-face-magic toroidal* graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for every $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels along C_n is a constant S. Let $x_v = f(v)$ for all $v \in V(G)$. Then $\{x_v \mid v \in V(G)\}$ is called a *C_n-face-magic toroidal labeling* on G. Curran, Low, and Locke [765] show that, for all $m,\ n \geqslant 2,\ C_m \times C_n$ admits a C_4-face-magic toroidal labeling if and only if either $m = 2$ or $n = 2$, or both m and n are even. They say that a C_4-face-magic toroidal labeling $\{x_{i,j} \mid (i, j) \in V(C_{2m} \times C_{2n})\}$ on $C_{2m} \times C_{2n}$ is *antipodal balanced* if $x_{i,j} + x_{i+m,j+n} = S/2$, for all $(i, j) \in V(C_{2m} \times C_{2n})$. They show that there exists an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ if and only if the parity of m and n are the same. Furthermore, when both m and n are even, an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ is both row-sum balanced and column-sum balanced. In addition, when $m = n$ is even, an antipodal balanced C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is diagonal-sum balanced. In [761] Curran and Locke proved that there are 144 distinct C_4-face-magic labelings on the 4×4 projective grid graph $P_{4,4}$ (up to symmetries on the projective plane).

Curran [759] showed that an $m \times n$ projective grid graph admits a C_4-face-magic labeling if and only if both m and n have the same parity. When m and n are even, the C_4-face-magic value of a C_4-face-magic labeling on an $m \times n$ projective grid graph must be $2mn + 2$. Also, when m and n are odd, he proved that the C_4-face-magic value of a

C_4-face-magic labeling on an $m \times n$ projective grid graph is either $2mn+1$, $2mn+2$, or $2mn+3$. In [760] Curran determined a category of the C_4-face-magic labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn+1$ or $2mn+3$. He conjectured that these are the only C_4-face-magic labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn+1$ or $2mn+3$.

In 2004 Baskar Babujee [482] and [483] introduced the notion of *vertex-bimagic* labeling in which there exist two constants k_1 and k_2 such that the sums involved in a specified type of magic labeling is k_1 or k_2. Thus a vertex-bimagic total labeling with bimagic constants k_1 and k_2 is the same as a vertex-magic total labeling except for each vertex v the sum of the label of v and all edges adjacent to v may be k_1 or k_2. Murugesan and Senthil Amutha [2225] proved that the bistar $B_{n,n}$ is vertex-bimagic total labeling for $n > 2$. An edge bimagic total labeling *edge bimagic total* of a graph $G(V, E)$ with p vertices and q edges is a bijection f from the set of vertices and edges to such that for every edge $uv \in E, f(u) + f(uv) + f(v)$ is one of two oconstants k_1 or k_2, independent of the choice of the edge. A bimagic labeling is of interest for graphs that do not have a magic labeling of a particular type. Bimagic labelings for which the number of sums equal to k_1 and the number of sums equal to k_2 differ by at most 1 are called *equitable*. When all sums except one are the same the labeling is called *almost magic*. Although the wheel W_n does not have an edge-magic total labeling when when $n \equiv 3 \pmod 4$, Marr, Phillips and Wallis [2095] showed that these wheels have both equitable bimagic and almost magic labelings. They also show that whereas nK_2 has an edge-magic total labeling if and only if n is odd, nK_2 has an edge-bimagic total labeling when n is even and although even cycles do not have super edge-magic total labelings all cycles have super edge-bimagic total labelings. They conjecture that there is a constant N such that K_n has a edge-bimagic total labeling if and only if n is at most N. They show that such an N must be at least 8. They also prove that if G has an edge-magic total labeling then $2G$ has an edge-bimagic total equitable labeling. Amara Jothi, David, and Baskar Babujee [174] provide edge-bimagic labelings for switching of paths, cycles, stars, crowns and helms. They also examine whether operations on edge magic graphs results in edge bimagic graphs or not.

Baskar Babujee and Babitha [487] call a graph with p vertices *1-vertex bimagic* if there is a bijective labeling f from the vertices to $\{1, 2, \ldots, p\}$ such that for each vertex u the sum of all $f(v)$ where v is adjacent to u is either a constant k_1 or a constant k_2 and $k_1 \neq k_2$. A graph with p vertices is called *odd 1-vertex bimagic* if there is a bijective labeling f from the vertices to $\{1, 3, \ldots, 2p-1\}$ such that for each vertex u the sum of all $f(v)$ where v is adjacent to u is either a constant k_1 or a constant k_2 and $k_1 \neq k_2$. A graph with p vertices is called *even 1-vertex bimagic* if there is a bijective labeling f from the vertices to $\{0, 2, \ldots, 2(p-1)\}$ such that for each vertex u the sum of all $f(v)$ where v is adjacent to u is either a constant k_1 or a constant k_2 and $k_1 \neq k_2$.

Baskar Babujee and Babitha [487] prove that a necessary condition for the existence of a 1-vertex bimagic vertex labeling f of a graph G is $\sum_{x \in V(G)} d(x)f(x) = k_1p_1 + k_2p_2$ where $d(x)$ is the degree of vertex x and p_1 and p_2 are the number of vertices with common count k_1 and k_2, respectively. Among their results are: if G has a 1-vertex bimagic vertex labeling and $G \neq C_4$, then $G+K_1$ admits a 1- vertex bimagic vertex labeling; C_n a 1-vertex bimagic if and only if $n = 4$; $K_{m,n}$ is 1-vertex bimagic; graphs obtained from P_n ($n \geqslant 3$) by adding edges joining every pair of vertices an odd distance apart are 1-vertex bimagic; n-partite graphs of the form $K_{p,p,\ldots,p}$ are 1-vertex bimagic for all $p > 1$ when n is even and 1-vertex bimagic for all even p when n is odd; a regular or biregular graph admits a 1-vertex bimagic labeling if and only if it the admits an odd 1-vertex bimagic labeling and if and only it admits an even 1-vertex bimagic labeling.

In [2753] Semenyuta, Nedilko, and Nedilko introduce the notion of the equivalence of vertex labelings on a given graph. They prove the equivalence of three bimagic labelings for regular graphs and obtain a particular solution for the problem of the existence a 1-vertex bimagic vertex labeling for graphs of isomorphic $K_{n,n,m}$. They prove that the sequence of biregular graphs $K_{n(ij)} = ((K_{n-1} - M) + K_1) - (u_n u_i) - (u_n u_j)$ admits a 1-vertex bimagic vertex labeling, where u_i, u_j is any pair of nonadjacent vertices in the graph $K_{n-1} - M$, u_n is the vertex of K_1, and M is the perfect matching of the complete graph K_{n-1}. They show that if an r-regular graph G of order n is a distance magic graph, then the graph $G + G$ has a 1-vertex bimagic vertex labeling with magic constants $(n+1)(n+r)/2 + n^2$ and $(n+1)(n+r)/2 + nr$. They also define two new types of graphs that do not admit 1-vertex bimagic vertex labeling.

Baskar Babujee and Jagadesh [483], [491], [492], and [490] proved the following graphs have super edge bimagic labelings: cycles of length 3 with a nontrivial path attached; $P_3 \odot K_{1,n}$ n even; $P_n + \overline{K_2}$ (n odd); $P_2 + mK_1$ ($m \geqslant 2$); $2P_n$ ($n \geqslant 2$); the disjoint union of two stars; $3K_{1,n}$ ($n \geqslant 2$); $P_n \cup P_{n+1}$ ($n \geqslant 2$); $C_3 \cup K_{1,n}$; P_n; $K_{1,n}$; $K_{1,n,n}$; the graphs obtained by joining the centers of any two stars with an edge or a path of length 2; the graphs obtained by joining the centers of two copies of $K_{1,n}$ ($n \geqslant 3$) with a path of length 2 then joining the center one of copies of $K_{1,n}$ to the center of a third copy of $K_{1,n}$ with a path of length 2; combs $P_n \odot K_1$; cycles; wheels; fans; gears; K_n if and only if $n \leqslant 5$.

Given positive integers k and λ, Yao, Chen, Yao, and Cheng [3528] say that a total labeling f of a connected graph $G(V, E)$ from $V \cup E$ to $\{1, 2, \ldots, |V| + |E|\}$ such that $f(x) \neq f(y)$ for distinct $x, y \in V \cup E$ and $f(u) + f(v) = k + \lambda f(uv)$ for each edge uv in E is a (k, λ)-*magically total labeling* of G. They provide necessary and sufficient conditions for graphs with (k, λ))-magically total labelings to also have graceful, odd-graceful, felicitous, and (a, d)-edge antimagic total labelings (see §6.2).

In [1993] López, Muntaner-Batle, and Rius-Font give a necessary condition for a complete graph to be edge bimagic in the case that the two constants have the same parity.

In [488] Baskar Babujee, Babitha, and Vishnupriya make the following definitions. For any natural number a, a graph $G(p, q)$ is said to be *a-additive super edge bimagic* if there exists a bijective function f from $V(G) \cup E(G)$ to $\{a+1, a+2, \ldots, a+p+q\}$ such that for every edge uv, $f(u)+f(v)+f(uv) = k_1$ or k_2. For any natural number a, a graph $G(p, q)$ is said to be *a-multiplicative super edge bimagic* if there exists a bijective f from $V(G) \cup E(G)$ to $\{a, 2a, \ldots, (p+q)a\}$ such that for every edge uv, $f(u)+f(v)+f(uv) = k_1$ or k_2. A graph $G(p, q)$ is said to be *super edge-odd bimagic* if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 3, 5, \ldots, 2(p + q) - 1\}$ such that for every edge uv $f(u) + f(v) + f(uv) = k_1$ or k_2. If f is a super edge bimagic labeling, then a function g from $E(G)$ to $\{0, 1\}$ with the property that for every edge uv, $g(uv) = 0$ if $f(u) + f(v) + f(uv) = k_1$ and $g(uv) = 1$ if

$f(u) + f(v) + f(uv) = k_2$ is called a *super edge bimagic cordial labeling* if the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. They prove: super edge bimagic graphs are a-additive super edge bimagic; super edge bimagic graphs are a-multiplicative super edge bimagic; if G is super edge-magic, then $G + K_1$ is super edge bimagic labeling; the union of two super edge magic graphs is super edge bimagic; and P_n, C_{2n} and $K_{1,n}$ are super edge bimagic cordial.

For any nontrivial Abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling f of the edges of G with the nonzero elements of A such that the vertex labeling f^+ defined by $f^+(v) = \Sigma f(vu)$ over all edges vu is a constant. In [3049] and [3050] Stanley noted that Z-magic graphs can be viewed in the more general context of linear homogeneous diophantine equations. Shiu, Lam, and Sun [2911] have shown the following: the union of two edge-disjoint A-magic graphs with the same vertex set is A-magic; the Cartesian product of two A-magic graphs is A-magic; the lexicographic product of two A-magic connected graphs is A-magic; for an Abelian group A of even order a graph is A-magic if and only if the degrees of all of its vertices have the same parity; if G and H are connected and A-magic, G composed with H is A-magic; $K_{m,n}$ is A-magic when $m, n \geqslant 2$ and A has order at least 4; K_n with an edge deleted is A-magic when $n \geqslant 4$ and A has order at least 4; all generalized theta graphs (§4.4 for the definition) are A-magic when A has order at least 4; $C_n + \overline{K_m}$ is A-magic when $n \geqslant 3, m \geqslant 2$ and A has order at least 2; wheels are A-magic when A has order at least 4; flower graphs $C_m@C_n$ are A-magic when $m, n \geqslant 2$ and A has order at least 4 ($C_m@C_n$ is obtained from C_n by joining the end points of a path of length $m - 1$ to each pair of consecutive vertices of C_n).

When the constant sum of an A-magic graph is zero the graph is called *zero-sum A-magic*. The null set $N(G)$ of a graph G is the set of all positive integers h such that G is zero-sum Z_h-magic. Akbari, Ghareghani, Khosrovshahi, and Zare [129] and Akbari, Kano, and Zare [130] proved that the null set $N(G)$ of an r-regular graph G, $r \geqslant 3$, does not contain the numbers 2, 3 and 4. Akbari, Rahmati, and Zare [131] proved the following: if G is an even regular graph then G is zero-sum Z_h-magic for all h; if G is an odd r-regular graph, $r \geqslant 3$ and $r \neq 5$ then $N(G)$ contains all positive integers except 2 and 4; if an odd regular graph is also 2-edge connected then $N(G)$ contains all positive integers except 2; and a 2-edge connected bipartite graph is zero-sum Z_h-magic for $h \geqslant 6$. They also determine the null set of 2-edge connected bipartite graphs, describe the structure of some odd regular graphs, $r \geqslant 3$, that are not zero-sum 4-magic, and describe the structure of some 2-edge connected bipartite graphs that are not zero-sum Z_h-magic for $h = 2, 3$, and 4. They conjecture that every 5-regular graph admits a zero-sum 3-magic labeling.

In [1877] Lee, Saba, Salehi, and Sun investigate graphs that are A-magic where $A = V_4 \approx Z_2 \oplus Z_2$ is the Klein four-group. Many of theorems are special cases of the results of Shiu, Lam, and Sun [2911] given in the previous paragraph. They also prove the following are V_4-magic: a tree if and only if every vertex has odd degree; the star $K_{1,n}$ if and only if n is odd; $K_{m,n}$ for all $m, n \geqslant 2$; $K_n - e$ (edge deleted K_n) when $n > 3$; even cycles with k pendent edges if and only if k is even; odd cycles with k pendent edges if and only if k is odd; wheels; $C_n + \overline{K_2}$; generalized theta graphs; graphs that are copies of C_n that share a common edge; and $G + \overline{K_2}$ whenever G is V_4-magic. In [1947] Libeeshkumar and Anil Kumar discussed induced V_4-magic labelings of some cycle related graphs.

In [709] Choi, Georges, and Mauro explore Z_2^k-magic graphs in terms of even edge-coverings, graph parity, factorability, and nowhere-zero 4-flows. They prove that the minimum k such that bridgeless G is zero-sum Z_2^k-magic is equal to the minimum number of even subgraphs that cover the edges of G, known to be at most 3. They also show that bridgeless G is zero-sum Z_2^k-magic for all $k \geqslant 2$ if and only if G has a nowhere-zero 4-flow, and that G is zero-sum Z_2^k-magic for all $k \geqslant 2$ if G is Hamiltonian, bridgeless planar, or isomorphic to a bridgeless complete multipartite graph, and establish equivalent conditions for graphs of even order with bridges to be Z_2^k-magic for all $k \geqslant 4$. In [1067] Georges, Mauro, and Wang utilized well-known results on edge-colorings in order to construct infinite families that are V_4-magic but not Z_4-magic. In [1948] Libeeshkumar and Kumar discussed some induced V_4-magic labelings of some subdivision graphs.

Low and Roberts [2032] use the combinatorial nullstellensatz to show the existence of Z_p-magic labelings (prime $p \geqslant 3$) for various graphs, without having to construct the Z_p-magic labelings. They give examples illustrating the usefulness and limitations in applying the combinatorial nullstellensatz to the integer-magic labeling problem. They show various classes of graphs have Z_3-magic labelings. Baskar Babujee and Shobana [505] prove that the following graphs have Z_3-magic labelings: C_{2n}; K_n $(n \geqslant 4)$; $K_{m,2m}$ $(m \geqslant 3)$; ladders $P_n \times P_2$ $(n \geqslant 4)$; bistars $B_{3n-1,3n-1}$; and cyclic, dihedral, and symmetric Cayley digraphs for certain generating sets. Siddiqui [2951] proved that generalized prisms, generalized antiprisms, fans and friendship graphs are Z_{3k}-magic for $k \geqslant 1$. In [715] Chou and Lee investigated Z_3-magic graphs.

Chou and Lee [715] showed that every graph is an induced subgraph of an A-magic graph for any nontrivial Abelian group A. Thus it is impossible to find a Kuratowski type characterization of A-magic graphs. Low and Lee [2030] have shown that if a graph is A_1-magic then it is A_2-magic for any subgroup A_2 of A_1 and for any nontrivial Abelian group A every Eulerian graph of even size is A-magic. For a connected graph G, Low and Lee define $T(G)$ to be the graph obtained from G by adding a disjoint uv path of length 2 for every pair of adjacent vertices u and v. They prove that for every finite nontrivial Abelian group A the graphs $T(P_{2k})$ and $T(K_{1,2n+1})$ are A-magic. Shiu and Low [2921] show that $K_{k_1,k_2,\ldots,k_n}(k_i \geqslant 2)$ is A-magic, for all A where $|A| \geqslant 3$. In [2926] Shiu and Low analyze the A-magic property for complete n-partite graphs and composition graphs with deleted edges. Lee, Salehi and Sun [1880] have shown that for $m, n \geqslant 3$ the double star $DS(m, n)$ is Z-magic if and only if $m = n$.

S. M. Lee [1839] calls a graph G *fully magic* if it is A-magic for all nontrivial abelian groups A. Low and Lee [2030] showed that if G is an Eulerian graph of even size, then G is fully magic. In [1839] Lee gives several constructions that produce infinite families of fully magic graphs and proves that every graph is an induced subgraph of a fully magic graph.

In [1783] Kwong and Lee call the set of all k for which a graph is Z_k-magic the *integer-magic spectrum* of the graph. They investigate the integer-magic spectra of the coronas of some specific graphs including paths, cycles, complete graphs, and stars. Low and Sue

[2034] have obtained some results on the integer-magic spectra of tessellation graphs. Shiu and Low [2922] provide the integer-magic spectra of sun graphs. Chopra and Lee [713] determined the integer-magic spectra of all graphs consisting of any number of pairwise disjoint paths with common end vertices (that is, generalized theta graphs). Low and Lee [2030] show that Eulerian graphs of even size are A-magic for every finite nontrivial Abelian group A whereas Wen and Lee [3465] provide two families of Eularian graphs that are not A-magic for every finite nontrivial Abelian group A and eight infinite families of Eulerian graphs of odd sizes that are A-magic for every finite nontrivial Abelian group A. Low and Lee [2030] also prove that if A is an Abelian group and G and H are A-magic, then so are $G \times H$ and the lexicographic product of G and H. Low and Shiu [2033] prove: $K_{1,n} \times K_{1,n}$ has a Z_{n+1}-magic labeling with magic constant 0; if $G \times H$ is Z_2-magic, then so are G and H; if G is Z_m-magic and H is Z_n-magic, then the integer-magic spectra of $G \times H$ contains all common multiples of m and n; if n is even and $k_i \geqslant 3$ then the integer-magic spectra of $P_{k_1} \times P_{k_2} \times \cdots \times P_{k_n} = \{3, 4, 5, \ldots\}$. In [2924] Shiu and Low determine all positive integers k for which fans and wheels have a Z_k-magic labeling with magic constant 0. Shiu and Low [2925] determined for which $k \geqslant 2$ a connected bicyclic graph without a pendent has a Z_k-magic labeling.

Jeyanthi and Jeya Daisy [1442] prove that P_n^2 $(n > 4)$, C_n^2, the total graph of C_n, and the splitting graph of C_{2n} are Z_k-magic graphs. They also prove: the splitting graph of C_n is Z_k-magic when n is even and n is odd and k is even, the middle graph of C_n is Z_k-magic when n and k are odd, the $m\Delta_{2n}$-snake graph is Z_k-magic when $k > m$, the graph obtained by joining the vertices u_i and u_{i+1} of C_n by a path of length m_i for $1 \leqslant i \leqslant n-1$, and u_1 and u_n by a path of length m_n is Z_k-magic if either all $m_1, m_2, \ldots, m_n$ are even or all are odd. In [1443] Jeyanthi and Jeya Daisy prove total graphs of the paths, flower graphs, and $C_m \times P_n$ are Z_k-magic. They also prove closed helms are Z_k-magic when $k > 4$ is even, lotuses inside a circle are Z_{4k}-magic, and graphs consisting of two cycles with a common edge are Z_k-magic when at least one cycle is even. In [1448] Jeyanthi prove the following graphs are Z_k-magic: two odd cycles connected by a path; the graph obtained by identifying a vertex of C_n with a pendent vertex of a star, m-splitting graphs of paths, and m-middle graphs of paths. They prove that if G is Z_m-magic with magic constant a then $G \odot \overline{K_m}$ is Z_m-magic.

Jeyanthi and Jeya Daisy [1441] prove that the subdivision graphs of the following families of graphs are Z_k-magic: ladders, triangular ladders, the shadow graph of paths, the total graph of paths, flowers, generalized prisms $C_m \times P_n$ for m even, $m\Delta_n$-snakes, lotuses inside a circle, the square graph of paths, gears of even cycles, closed helms of even cycles, and antiprisms A_n^m for m even.

Let G be a graph and let $G_1, G_2, \ldots, G_n$ be $n \geqslant 2$ copies of G. The graph obtained by replacing each endpoint vertex of $K_{1,n}$ by the graphs $G_1, G_2, \ldots, G_n$ is called the *open star* of G. Jeyanthi and Jeya Daisy [1445] proved that the open star graphs of shells, flowers, double wheels, cylinders, wheels, generalised Peterson graphs, lotuses inside a circle, and closed helms are Z_k-magic graphs. They also prove that the super subdivision of any graph is Z_k-magic.

Jeyanthi and Jeya Daisy [1446] proved that the path union of $n \geqslant 2$ copies of the following families of graphs are Z_k-magic: odd cycles; generalised Peterson graphs $P(r,m)$ when r is odd and $1 \leqslant m \leqslant \frac{r}{2}$; shell graphs S_r when $r > 3$; wheels W_r when $r > 3$; closed helms CH_r when (i) $r > 3$ is odd and (ii) r is even and k is even; double wheels DW_r when $r > 3$ is odd; flowers Fl_r when $r > 2$; $C_r \times P_2$ when r is odd; total graphs of paths $T(P_r)$ for all $n, r > 4$; lotuses inside a circle LC_r when $r > 3$; and $C_r \odot K_1$ for odd r.

Jeyanthi and Jeya Daisy [1447] proved that the following graphs are k-magic: shell graphs S_n when n is odd or n is even and k is even; generalised Jahangir graphs $J_{n,s}$ when n and s have the same parity or n is even, s is odd, and k is even; $(P_n + P_1) \times P_2$ when n is odd; double wheels $2C_n + K_1$; mongolian tents $M(m,n)$ when m is even; flower snark graphs; slanting ladders (that is, graphs obtained from two paths $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by joining each u_i with v_{i+1}, $1 \leqslant i \leqslant n-1$) when n is even; double step grid graphs; double arrow graphs obtained from $P_m \times P_n$ by joining a new vertex with the m vertices of the first copy of P_m and another new vertex with the m vertices of the last copy of P_m when m is even; semi Jahangir graphs (the connected graph with vertex set $\{p, x_i, y_k : 1 \leqslant i \leqslant n+1, 1 \leqslant k \leqslant n\}$ and the edge set $\{px_i : 1 \leqslant i \leqslant n+1\} \cup \{x_i y_i : 1 \leqslant i \leqslant n\} \cup \{y_i x_{i+1} : 1 \leqslant i \leqslant n\}$); graphs obtained by connecting double wheels DW_{n_1} and DW_{n_2} by a path when n_1 and n_2 are odd; graphs obtained by joining two copies of shell graphs by a path; and the splitting graph of a Z_k magic graph with magic constant 0.

Let G be a graph with n vertices $\{u_1, u_2, \ldots, u_n\}$ and consider n copies of G, $G_1, G_2, \ldots, G_n$, with vertex sets $V(G_i) = \{u_i^j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\}$. The *cycle of a graph* G, denoted by $C(n.G)$, is obtained by identifying the vertex u_1^j of G_j with u_i of G for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n$. Jeyanthi and Jeya Daisy [1448] prove that the following graphs are Z_k-magic: $C(n.C_r)$ except r is even, n is odd, and k is odd; generalised Peterson graphs $C(n.P(r,m))$ except r is even, n is odd, and k is odd; cycles of shell graphs; cycles of wheel graphs; cycles of closed helms; cycles of double wheels $C(n.DW_r)$ except r is even, n is odd, and k is odd; cycles of triangular ladder graphs; cycles of flower graphs; and cycles of lotus inside a circle graphs. Jeyanthi and Jeya Daisy [1448] also prove that if G is Z_k-magic then $C(n.G)$ is Z_k-magic if n or k are even.

Shiu and Low [2923] have introduced the notion of ring-magic as follows. Given a commutative ring R with unity, a graph G is called *R-ring-magic* if there exists a labeling f of the edges of G with the nonzero elements of R such that the vertex labeling f^+ defined by $f^+(v) = \Sigma f(vu)$ over all edges vu and vertex labeling $f^\times$ defined by $f^\times(v) = \Pi f(vu)$ over all edges vu are constant. They give some results about R-ring-magic graphs.

In [647] Cahit says that a graph $G(p,q)$ is *total magic cordial* (TMC) provided there is a mapping f from $V(G) \cup E(G)$ to $\{0,1\}$ such that $(f(a) + f(b) + f(ab)) \bmod 2$ is a constant modulo 2 for all edges $ab \in E(G)$ and $|f(0) - f(1)| \leqslant 1$ where $f(0)$ denotes the sum of the number of vertices labeled with 0 and the number of edges labeled with 0 and $f(1)$ denotes the sum of the number of vertices labeled with 1 and the number of edges labeled with 1. He says a graph G is *total sequential cordial* (TSC) if there is a mapping f from $V(G) \cup E(G)$ to $\{0,1\}$ such that for each edge $e = ab$ with $f(e) = |f(a) - f(b)|$ it is true that $|f(0) - f(1)| \leqslant 1$ where $f(0)$ denotes the sum of the number of vertices labeled with 0 and the number of edges labeled with 0 and $f(1)$ denotes the sum of the number of vertices labeled with 1 and the number of edges labeled with 1. He proves that the following graphs have a TMC labeling: $K_{m,n}$ $(m, n > 1)$, trees, cordial graphs, and K_n if and only if $n = 2, 3, 5$, or 6. He also proves that the following graphs have a TSC labeling: trees; cycles; complete bipartite graphs; friendship graphs; cordial graphs; cubic graphs other than K_4; wheels W_n $(n > 3)$; K_{4k+1} if and only if $k \geqslant 1$ and $\sqrt{k}$ is an integer; K_{4k+2} if and only if $\sqrt{4k+1}$ is an integer; K_{4k} if and only if $\sqrt{4k+1}$ is an integer; and K_{4k+3} if and only if $\sqrt{k+1}$ is an integer. In [1429] Jeyanthi, Angel Benseera, and Cahit prove mP_2 is TMC if $m \not\equiv 2 \pmod 4$, mP_n is TMC for all $m \geqslant 1$ and $n \geqslant 3$, and obtain partial results about TMC labelings of mK_n. Neela and Selvaraj proved that the complete tripartite graphs are TMC and complete multipartite graphs are TMC when the partite sets have even sizes

Parameswari and Jayalakshmi e [2337] proved that *octopus* graphs (the join of a fan F_n and $K_{1,n}$ $(n > 1)$), vanessa graphs $2F_n + K_{1,n}$, lilly graphs $2K_{1,n} + 2P_n$, and lotus graphs admit cordial totally magic labelings. Parameswari [2336] proved that the square graphs and shadow graphs of bistars admit cordial totally magic labelings. In [2339] and [2340] Parameswari and Rajeswari proved that Paley digraphs, fans, gears, and shadow graphs of paths and stars are total magic cordial. Neela and Selvaraj [2267] showed that complete tripartite graphs and complete multipartite graphs are total magic cordial when the partite sets are of even sizes. [2337] new [2336] new [2339] new [2340] new [2267] new

Vaidya and Barasara [3251] proved that every graph can be embedded as an induced subgraph of a totally magic cordial graph thereby ruling out any possibility of obtaining any forbidden subgraph characterization for totally magic cordial graphs. They also proved that every connected graph can be embedded as an induced subgraph of a totally magic cordial connected graph and every planar graph can be embedded as an induced subgraph of a totally magic cordial planner graph. Similar results were also obtained for total sequential cordial labeling.

Jeyanthi and Angel Benseera [1427] investigated the existence of TMC labelings of the one-point unions of copies of cycles, complete graphs and wheels. In [1428] Jeyanthi and Angel Benseera prove that if $G_i(p_i, q_i)$, $i = 1, 2, 3, \ldots, n$ are totally magic cordial graphs with $C = 0$ such that $p_i + q_i$, $i = 1, 2, 3, ..., n$ are even, and $|p_i - 2m_i| \leqslant 1$, where m_i is the number of vertices labeled with 0 in G_i, $i = 1, 2, \ldots, n$, then $G_1 + G_2 + \cdots + G_n$ is TMC; if G is an odd graph with $p + q \equiv 2 \pmod 4$, then G is not TMC; fans F_n are TMC for $n \geqslant 2$; wheels W_n $(n \geqslant 3)$ are TMC if and only if $n \not\equiv 3 \pmod 4$; mW_{4t+3} is TMC if and only if m is even; mW_n is TMC if $n \not\equiv 3 \pmod 4$; $C_n + \overline{K}_{2m+1}$ is TMC if and only if $n \not\equiv 3 \pmod 4$; $C_{2n+1} \odot \overline{K}_m$ is TMC if and only if m is odd; the disjoint union of $K_{1,m}$ and $K_{1,n}$ is TMC if and only if m or n is even.

For a bijection $f : V(G) \cup E(G) \to Z_k$ such that for each edge $uv \in E(G)$, $f(u) + f(v) + f(uv)$ is constant $\pmod k$ $n_f(i)$ denotes the number vertices and edges labeled by i under f. If $|n_f(i) - n_f(j)| \leqslant 1$ for all $0 \leqslant i < j \leqslant k-1$, f is called a *k-totally magic cordial* labeling of G. A graph is said to be *k-totally magic cordial* if it admits a k-totally magic cordial labeling. In [1430] Jeyanthi, Angel Benseera, and Lau provide some ways to construct new families of k-totally magic cordial (k-TMC) graphs from a known k-totally magic cordial graph. Let G (respectively, H) be a (p, q)-graph (respectively, an (n, m)-graph) that admits a k-TMC labeling f (respectively, g) with constant C such that $n_f(i)$

and $v_f(i) = \frac{p}{k}$ (or $n_g(i)$ and $v_g(i) = \frac{n}{k}$) are constants for all $0 \leqslant i \leqslant k-1$, they show that $G+H$ also admits a k-TMC labeling with constant C. They prove the following. If G is an edge magic total graph, then G is k-TMC for $k \geqslant 2$; if G is an odd graph with $p+q \equiv k \pmod{2k}$ and $k \equiv 2 \pmod 4$, then G is not k-TMC; if $n \equiv 7 \pmod 8$, $K_n \odot K_1$ is not $2n$-TMC; if $n \equiv 2 \pmod 4$, $C_n \odot C_3$ is not n-TMC; if $n \equiv 1 \pmod 2$, $C_n \odot K_5$ is not $2n$-TMC; if $n \equiv 2 \pmod 4$, $C_n \times P_2$ is not n-TMC; K_n $(n \geqslant 3)$ is n-TMC; $K_n \odot K_1$ $(n \geqslant 3)$ is n-TMC; S_n is n-TMC for all $n \geqslant 1$; $K_{m,n}$ $(m \geqslant n \geqslant 2)$ is both m-TMC and n-TMC; W_n is n-TMC for all odd $n \geqslant 3$ and is 3-TMC for $n \equiv 0 \pmod 6$; mK_n $(n \geqslant 2)$ is n-TMC if $n \geqslant 3$ is odd; $K_n + K_n$ is n-TMC if $n \geqslant 3$ is odd; $S_n + S_n$ $(n \geqslant 1)$ is $(n+1)$-TMC; and if $m \geqslant 3$ and n is odd, $C_n \times P_m$ $(n \geqslant 3)$ is n-TMC. In [1432] Jeyanthi, Angel Benseera, and Lau call a graph G hypo-k-TMC if $G - \{v\}$ is k-TMC for each vertex v in $V(G)$ and establish that some families of graphs admit and do not admit hypo-k-TMC labeling.

A graph $G(V,E)$ where $V = \{v_i, 1 \leqslant i \leqslant n\}$ and $E = \{v_iv_{i+1}, 1 \leqslant i \leqslant n\}$ is 0-*edge magic* if there exists a bijection $f : V(G) \to \{1,-1\}$ such that the induced edge labeling defined by $f^*(uv) = f(u) + f(v)$ is 0 for all $uv \in E$. Paths, cycles, complete n-ary pseudo trees, $P_m \times C_n$ where $n \equiv 0 \pmod 2$, Q_n, the graph C_m attached to mK_1, $m \equiv 0 \pmod 2$, friendship graphs $C_n^{(m)}$, and the graph $P_m \times P_m \times P_m$ are 0-edge magic graphs [1747], [1369], [2265]. Jayapriya [1367] proved the splitting graphs $spl(P_n)$, $spl(C_n)$, $spl(K_{1,n})$, $spl(B_{m,n})$, and splitting graph of any tree admits 0-edge magic labelings. Laurejas and Pedrano [1819] determine the 0-edge magic labeling of $P_m \times P_n$, $C_m \times C_n$, and the generalized Petersen graph. They also prove that odd cycles are not 0-edge magic.

A *binary magic total* labeling of a graph G is a function $f : V(G) \cup E(G) \to \{0,1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod 2$ for all $ab \in E(G)$. Jeyanthi and Angel Benseera [1431] define the *totally magic cordial deficiency* of G as the minimum number of vertices taken over all binary magic total labeling of G that is necessary to add so that that the resulting graph is totally magic cordial. The totally magic cordial deficiency of G is denoted by $\mu_T(G)$. They provide $\mu_T(K_n)$ for some cases.

Let G be a graph rooted at a vertex u and f_i be a binary magic total labeling of G and $f_i(u) = 0$ for $i = 1, 2, \ldots, k$ and $n_{f_i}(0) = \alpha_i$, $n_{f_i}(1) = \beta_i$ for $i = 1, 2, \ldots, k$. Jeyanthi and Angel Benseera [1431] determine the totally magic cordial deficiency of the one-point union $G^{(n)}$ of n copies of G. They show that for $n \equiv 3 \pmod 4$ the totally magic cordial deficiency of W_n, $W_n^{(4t+1)}$, $W_{4t+1}^{(n)}$ and $C_n + \overline{K}_{2m+1}$ is 1; for m odd, $\mu_T(mW_{4t+3}) = 1$; and for $n \equiv 1 \pmod 4$, $\mu_T(K_4^{(n)}) = 1$.

In 2001, Simanjuntak, Rodgers, and Miller [2148] defined a *1-vertex magic* (also known as *distance magic* labeling vertex labeling of $G(V,E)$ as a bijection from V to $\{1, 2, \ldots, |V|\}$ with the property that there is a constant k such that at any vertex v the sum $\sum f(u)$ taken over all neighbors of v is k. Among their results are: $H \times \overline{K}_{2k}$ has a 1-vertex-magic vertex labeling for any regular graph H; the symmetric complete multipartite graph with p parts, each of which contains n vertices, has a 1-vertex-magic vertex labeling if and only if whenever n is odd, p is also odd; P_n has a 1-vertex-magic vertex labeling if and only if $n = 1$ or 3; C_n has a 1-vertex-magic vertex labeling if and only if $n = 4$; K_n has a 1-vertex-magic vertex labeling if and only if $n = 1$; W_n has a

1-vertex-magic vertex labeling if and only if $n = 4$; a tree has a 1-vertex-magic vertex labeling if and only if it is P_1 or P_3; and r-regular graphs with r odd do not have a 1-vertex-magic vertex labeling.

Miller, Rogers, and Simanjuntak [2148] the complete p-partite $(p > 1)$ graph $K_{n,n,\ldots,n}$ $(n > 1)$ has a 1-vertex-magic vertex labeling if and only if either n is even or np is odd. Shafiq, Ali, Simanjuntak [2845] proved $mK_{n,n,\ldots,n}$ has a 1-vertex-magic vertex labeling if n is even or mnp is odd and $m \geqslant 1, n > 1$, $p > 1$; and $mK_{n,n,\ldots,n}$ does not have a 1-vertex-magic vertex labeling if np is odd, $p \equiv 3 \pmod 4$, and m is even.

Recall if $V(G) = \{v_1, v_2, \ldots, v_p\}$ is the vertex set of a graph G and $H_1, H_2, \ldots, H_p$ are isomorphic copies of a graph H, then $G[H]$ is the graph obtained from G by replacing each vertex v_i of G by H_i and joining every vertex in H_i to every neighbor of v_i. Shafiq, Ali, Simanjuntak [2845] proved if G is an r-regular graph $(r \geqslant 1)$ then $G[C_n]$ has a 1-vertex-magic vertex labeling if and only if $n = 4$. They also prove that for $m \geqslant 1$ and $n > 1$, $mC_p[\overline{K_n}]$ has 1-vertex-magic vertex labeling if and only if either n is even or mnp is odd or n is odd and $p \equiv 3 \pmod 4$.

For a graph G Jeyanthi and Angel Benseera [1426] define a function f from $V(G) \cup E(G)$ to $\{0, 1\}$ to be a *totally vertex-magic cordial labeling* (TVMC) with a constant C if $f(a)+\sum_{b\in N(a)} f(ab) \equiv C \pmod 2$ for all vertices $a \in V(G)$ and $|n_f(0)-n_f(1)| \leqslant 1$, where $N(a)$ is the set of vertices adjacent to the vertex a and $n_f(i)$ is the sum of the number of vertices and edges with label i. They prove the following graphs have totally vertex-magic cordial labelings: vertex-magic total graphs; trees; K_n; $K_{m,n}$ whenever $|m-n| \leqslant 1$; $P_n + P_2$; friendship graphs with C = 0; and flower graphs Fl_n for $n \geqslant 3$ with C= 0. They also proved that if G is TVMC with $C = 1$, then the graph obtained by identifying any vertex of G with any vertex of a tree is TVMC with $C = 1$; if G is a (p, q) graph with $|p - q| \leqslant 1$, then G is TVMC with $C = 1$; and if $G(p, q)$ is a TVMC graph with constant $C = 0$ where p is odd, then $G + \overline{K_{2m}}$ is TVMC with $C = 1$ if m is odd and with $C = 0$ if m is even.

Jeyanthi, Angel Benseera, and Immaculate Mary [1425] showed that the following graphs have totally magic cordial labelings: (p, q) graphs with $|p - q| \leqslant 1$; flower graphs Fl_n for $n \geqslant 3$; ladders; and graphs obtained by identifying a vertex of C_m with each vertex of C_n. They also proved that if $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two disjoint totally magic cordial graphs with $p_1 = q_1$ or $p_2 = q_2$ then $G_1 \cup G_2$ is totally magic cordial. In Theorem 10 in [647] Cahit stated that K_n is totally magic cordial if and only if $n \in \{2, 3, 5, 6\}$. Jeyanthi and Angel Benseera [1431] proved that K_n is totally magic cordial if and only if $\sqrt{4k + 1}$ has an integer value when $n = 4k$; $\sqrt{k + 1}$ or $\sqrt{k}$ have an integer value when $n = 4k + 1$; $\sqrt{4k + 5}$ or $\sqrt{4k + 1}$ have an integer value when $n = 4k + 2$; or $\sqrt{k + 1}$ has an integer value when $n = 4k + 3$.

A graph G is said to have a *totally magic cordial* TMC labeling with constant C if there exists a mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod 2$ for all $ab \in E(G)$ and $|n_f(0) - n_f(1)| \leqslant 1$, where $n_f(i)$ $(i = 0, 1)$ is the sum of the number of vertices and edges with label i. In [1428] Jeyanthi and Angel Benseera prove that if $G_i(p_i, q_i)$, $i = 1, 2, 3, \ldots, n$ are totally magic cordial graphs with $C = 0$ such that $p_i + q_i$, $i = 1, 2, 3, \ldots, n$ are even, and $|p_i - 2m_i| \leqslant 1$, where m_i is the number of vertices labeled with 0 in G_i, $i = 1, 2, \ldots, n$, then $G_1 + G_2 + \cdots + G_n$ is TMC. They also prove the following. If G be an odd graph with $p + q \equiv 2 \pmod 4$, then G is not TMC; fan graph F_n is TMC for $n \geqslant 2$; the wheel graph W_n ($n \geqslant 3$) is TMC if and only if $n \not\equiv 3 \pmod 4$; mW_{4t+3} is TMC if and only if m is even; mW_n is TMC if $n \not\equiv 3 \pmod 4$ and $m \geqslant 1$; $C_n + \overline{K}_{2m+1}$ is TMC if and only if $n \not\equiv 3 \pmod 4$; $C_{2n+1} \odot \overline{K}_m$ is TMC if and only if m is odd; and the disjoint union of $K_{1,m}$ and $K_{1,n}$ is TMC if and only if m or n is even.

Balbuena, Barker, Lin, Miller, and Sugeng [417] call a vertex-magic total labeling of a graph $G(V, E)$ an *a-vertex consecutive magic labeling* if the vertex labels are $\{a+1, a+2, \ldots, a+|V|\}$ where $0 \leqslant a \leqslant |E|$. They prove: if a tree of order n has an a-vertex consecutive magic labeling then n is odd and $a = n-1$; if G has an a-vertex consecutive magic labeling with n vertices and $e = n$ edges, then n is odd and if G has minimum degree 1, then $a = (n+1)/2$ or $a = n$; if G has an a-vertex consecutive magic labeling with n vertices and e edges such that $2a \leqslant e$ and $2e \geqslant \sqrt{6}n - 1$, then the minimum degree of G is at least 2; if a 2-regular graph of order n has an a-vertex consecutive magic labeling, then n is odd and $a = 0$ or n; and if a r-regular graph of order n has an a-vertex consecutive magic labeling, then n and r have opposite parities.

Balbuena et al. also call a vertex-magic total labeling of a graph $G(V, E)$ a *b-edge consecutive magic labeling* if the edge labels are $\{b+1, b+2, \ldots, b+|E|\}$ where $0 \leqslant b \leqslant |V|$. They prove: if G has n vertices and e edges and has a b-edge consecutive magic labeling and one isolated vertex, then $b = 0$ and $(n-1)^2 + n^2 = (2e+1)^2$; if a tree with odd order has a b-edge consecutive magic labeling then $b = 0$; if a tree with even order has a b-edge consecutive magic labeling then it is P_4; a graph with n vertices and e edges such that $e \geqslant 7n/4$ and $b \geqslant n/4$ and a b-edge consecutive magic labeling has minimum degree 2; if a 2-regular graph of order n has a b-edge consecutive magic labeling, then n is odd and $b = 0$ or $b = n$; and if a r-regular graph of order n has an b-edge consecutive magic labeling, then n and r have opposite parities.

Sugeng and Miller [3084] prove: If (V, E) has an a-vertex consecutive edge magic labeling, where $a \neq 0$ and $a \neq |E|$, then G is disconnected; if (V, E) has an a-vertex consecutive edge magic labeling, where $a \neq 0$ and $a \neq |E|$, then G cannot be the union of three trees with more than one vertex each; for each nonnegative a and each positive n, there is an a-vertex consecutive edge magic labeling with n vertices; the union of r stars and a set of $r-1$ isolated vertices has an s-vertex consecutive edge magic labeling, where s is the minimum order of the stars; for every b every caterpillar has a b-edge consecutive edge magic labeling; if a connected graph G with n vertices has a b-edge consecutive edge magic labeling where $1 \leqslant b \leqslant n-1$, then G is a tree; the union of r stars and a set of $r-1$ isolated vertices has an r-edge consecutive edge magic labeling.

Baskar Babujee, Vishnupriya, and Jagadesh [508] introduced a labeling called *a-vertex consecutive edge bimagic total* as a graph $G(V, E)$ for which there are two positive integers k_1 and k_2 and a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V| + |E|\}$ such that $f(u) + f(v) + f(uv) = k_1$ or k_2 for all edges uv and $f(V) = \{a+1, a+2, \ldots, a+|V|\}$, $0 \leqslant a \leqslant |V|$. They proved the following graphs have such labelings: P_n, $K_{1,n}$, combs, bistars $B_{m,n}$, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, trees obtained by joining the centers of two stars with a path of length 2, trees obtained from P_5 by identifying the center of a copy $K_{1,n}$ with the two end vertices and the middle vertex. In [496] Baskar Babujee and Jagadesh proved that cycles, fans, wheels, and gear graphs have a-vertex consecutive edge bimagic total labelings. Baskar Babujee, Jagadesh, Vishnupriya [499] study the properties of a-vertex consecutive edge bimagic total labeling for $P_3 \odot K_{1,2n}$, $P_n + \overline{K_2}$ (n is odd and $n \geqslant 3$), $(P_2 \cup mK_1) + \overline{K_2}$, $(P_2 + mK_1)$ $(m \geqslant 2), C_n$, fans $P_n + K_1$, double fans $P_n + 2K_1$, and graphs obtained by appending a path of length at least 2 to a vertex of C_3. Baskar Babujee and Jagadesh [497] prove the following graphs have a-vertex consecutive edge bimagic total labelings: $2P_n$ $(n \geqslant 2)$, $P_n \cup P_{n+1}$ $(n \geqslant 2)$, $K_{2,n}$, $C_n \odot K_1$, and that $C_3 \cup K_{1,n}$ an a-vertex consecutive edge bimagic labeling for $a = n + 3$.

Vishnupriya, Manimekalai, and Baskar Babujee [3405] define a labeling f of a graph $G(p, q)$ to be a *edge bimagic total labeling* if there exists a bijection f from $V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that for each edge $e = (u, v) \in E(G)$ we have $f(u)+f(e)+f(v) = k_1$ or k_2, where k_1 and k_2 are two constants. They provide edge bimagic total labelings for $B_{m,n}$, $K_{1,n,n}$, and trees obtained from a path by appending an edge to one of the vertices adjacent to an endpoint of the path. An edge bimagic total labeling is $G(V, E)$ is called an *a-vertex consecutive edge bimagic total labeling* if the vertex labels are $\{a+1, a+2, \ldots, a+|V|\}$ where $0 \leqslant a \leqslant |E|$. Baskar Babujee and Jagadesh [494] prove the following graphs a-vertex consecutive edge-bimagic total labelings: the trees obtained from $K_{1,n}$ by adding a new pendent edge to each of the existing n pendent vertices; the trees obtained by adding a pendent path of length 2 to each of the n pendent vertices of $K_{1,n}$; the graphs obtained by joining the centers of two copies of identical stars by a path of length 2; and the trees obtained from a path by adding new pendent edges to one pendent vertex of the path. Baskar Babujee, Vishnupriya, and Jagadesh [508] proved the following graphs have such labelings: P_n, $K_{1,n}$, combs, bistars $B_{m,n}$, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, trees obtained by joining the centers of two stars with a path of length 2, trees obtained from P_5 by identifying the center of a copy $K_{1,n}$ with the two end vertices and the middle vertex. In [496] Baskar Babujee and Jagadesh proved that cycles, fans, wheels, and gear graphs have a-vertex consecutive edge bimagic total labelings. Baskar Babujee, Jagadesh, Vishnupriya [499] study the properties of a-vertex consecutive edge bimagic total labeling for $P_3 \odot K_{1,2n}$, $P_n + \overline{K_2}$ (n is odd and $n \geqslant 3$), $(P_2 \cup mK_1) + \overline{K_2}$, $(P_2 + mK_1)$ $(m \geqslant 2), C_n$, fans $P_n + K_1$, double fans $P_n + 2K_1$, and graphs obtained by appending a path of length at least 2 to a vertex of C_3.

Vishnupriya, Manimekalai, and Baskar Babujee [3405] prove that bistars, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, and trees obtained subdividing each edge of a star have edge bimagic total labelings. Prathap and Baskar Babujee [2517] obtain all possible edge magic total labelings and edge bimagic total labelings for the star $K_{1,n}$. Jayasekaran1 and Flower [1371] proved that the shadow graph and the splitting graph of paths stars and cycles are edge trimagic total and super edge trimagic total.

Let D be a digraph of order p and size q. For an integer $k \geqslant 1$ and $v \in V(D)$, let $w_k(v) = \sum_f(e)$ over the set of all in-arcs that are at distance at most k from v. A *V_k-super vertex in-magic labeling* (V_k-SVIML) of D is an one-to-one onto function f

from $V(D) \cup A(D)$ to $\{1, 2, \ldots, p+q\}$ such that $f(V(D)) = \{1, 2, \ldots, p\}$ and for every $v \in V(D)$, $f(v) + w_k(v) = M$ for some positive integer M. A digraph that admits a V_k-SVIML is called V_k-*super vertex in-magic* (V_k-SVIM). In [2230] Mutharasu, Mary Bernard, Duraisamy kumar study some properties of V_k-SVIML in digraphs. They characterized the digraphs that are V_k-SVIM and find the magic constant for E_k-regular digraphs. They furthermore characterized the unidirectional cycles and union of unidirectional cycles which are V_2-SVIML.

More about magic labelings of directed graphs can be found in [2093] and [580].

6 Antimagic-type Labelings

6.1 Antimagic Labelings

Hartsfield and Ringel [1191] introduced antimagic graphs in 1990. A graph with q edges is called *antimagic* if its edges can be labeled with $1, 2, \dots, q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct.[3] Among the graphs they prove are antimagic are: P_n ($n \geqslant 3$), cycles, wheels, and K_n ($n \geqslant 3$). T. Wang [3431] has shown that the toroidal grids $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$ are antimagic and, more generally, graphs of the form $G \times C_n$ are antimagic if G is an r-regular antimagic graph with $r > 1$. Cheng [701] proved that all Cartesian products or two or more regular graphs of positive degree are antimagic and that if G is j-regular and H has maximum degree at most k, minimum degree at least one (G and H need not be connected), then $G \times H$ is antimagic provided that j is odd and $j^2 - j \geqslant 2k$, or j is even and $j^2 > 2k$. Wang and Hsiao [3432] prove the following graphs are antimagic: $G \times P_n$ ($n > 1$) where G is regular; $G \times K_{1,n}$ where G is regular; compositions $G[H]$ (see §2.3 for the definition) where H is d-regular with $d > 1$; and the Cartesian product of any double star (two stars with an edge joining their centers) and a regular graph. In [700] Cheng proved that $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_t}$ ($t \geqslant 2$) and $C_m \times P_n$ are antimagic. In [3014] Solairaju and Arockiasamy prove that various families of subgraphs of grids $P_m \times P_n$ are antimagic. Liang and Zhu [1932] proved that if G is k-regular ($k \geqslant 2$), then for any graph H with $|E(H)| \geqslant |V(H)| - 1 \geqslant 1$, the Cartesian product $H \times G$ is antimagic. They also showed that if $|E(H)| \geqslant |V(H)| - 1$ and each connected component of H has a vertex of odd degree, or H has at least $2|V(H)| - 2$ edges, then the prism of H is antimagic. Shang [2862] showed that all spiders are antimagic. Lee, Lin, and Tsai [1833] proved that C_n^2 is antimagic and the vertex sums form a set of successive integers when n is odd. Shang, Lin, and Liaw [2865] show that a star forest containing no S_1 and at most one S_2 as components is antimagic. They also prove that if a star forest mS_2 is antimagic then $m = 1$ and $mS_2 \cup S_n$ ($n \geqslant 3$) is antimagic if and only if $m \leqslant \min\{2n+1, 2n-5+\sqrt{8n^2-24n+17}/2\}$. Wang, Miao, and Li [3443] show that certain graphs with even factors are antimagic. Li [1924] gives antimagic labelings for C_n^k for $k = 2, 3$, and 4. In [3449] Wang and Zhang showed that certain classes of regular graphs of odd degree with particular type of perfect matchings are antimagic. As a by-product, they get that generalized Petersen graphs and a subclass of Cayley graphs of Z_n are antimagic. Deng and Li [799] proved that caterpillars with maximum degree 3 are antimagic.

Let $\tau(G)$ to denote the maximum integer such that $G \cup tP_3$ is antimagic for all $t \leqslant \tau(G)$. The existence of such an integer was proved by Chang, Chen, Li, and Pan in [669]. Shang, Lin, Liaw [2865] and Li [1923] gave tight bounds of $\tau(G)$ for star forests and balanced double stars. Their general bound is also tight for many other families of graphs, including cycles C_n ($3 \leqslant 9$) and $2C_3$. Moreover, they discuss properties of $\tau(G)$ and pose open questions, including whether their bound is always tight. Chavez, Le, Der-Fen Liu,

[3]A comprehensive expository treatment of antimagic labelings is given by Bača, Miller, Ryan, and Semaničová-Feňovčíkováin [370].

Shurman [687] generalized the results of Shang et al by providing an upper bound of $\tau(G)$ for all graphs without isolated vertices and P_2 as components.

For a graph G and a vertex v of G, the *vertex switching* graph G_v is the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. Vaidya and Vyas [3327] proved that the graphs obtained by the switching of a pendent vertex of a path, a vertex of a cycle, a rim vertex of a wheel, the center vertex of a helm, or a vertex of degree 2 of a fan are antimagic graphs.

Phanalasy, Miller, Rylands, and Lieby [2392] in 2011 showed that there is a relationship between completely separating systems and labeling of regular graphs. Based on this relationship they proved that some regular graphs are antimagic. Phanalasy, Miller, Iliopoulos, Pissis, and Vaezpour [2390] proved the Cartesian product of regular graphs obtained from [2392] is antimagic. Ryan, Phanalasy, Miller, and Rylands introduced the generalized web and flower graphs in [2661] and proved that these families of graphs are antimagic. Rylands, Phanalasy, Ryan, and Miller extended the concept of generalized web graphs to the single apex multi-generalized web graphs and they proved these graphs to be antimagic in [2663]. Ryan, Phanalasy, Rylands and Miller extended the concept of generalized flower to the single apex multi-(complete) generalized flower graphs and constructed antimagic labeling for this family of graphs in [2662]. For more about antimagicness of generalized web and flower graphs see [2144]. Phanalasy, Ryan, Miller and Arumugam [2391] introduced the concept of generalized pyramid graphs and they constructed antimagic labeling for these graphs. Bača, Miller, Phanalasy, and Feňovčíková proved that some join graphs and incomplete join graphs are antimagic in [367]. Moreover, in [366] they proved that the complete bipartite graph $K_{m,m}$ and complete 3-partite graph $K_{m,m,m}$ are antimagic and if G is a k-regular (connected or disconnected) graph with p vertices and $k \geqslant 2$, then the join of G and $(p-k)K_1$, $G+(p-k)K_1$ is antimagic. Arumugam, Miller, Phanalasy, and Ryan [247] provided antimagic labelings for a family of generalized pyramid graphs. Daykin, Iliopoulas. Miller, and Phanalasy [790] show several families of graphs recursively defined from a sequence of graphs that are generalizations of corona graphs are antimagic. Lozano, Mora, Seara, and Tey [2035] proved that caterpillars are antimagic.

Let G be a k-regular graph with p vertices and q edges. The *generalized sausage graph*, denoted by $S(G;m)$, is the graph obtained from $G \times P_m$ ($G \times P_1 = G$), by joining each end vertex of the P_m to a new vertex (which we call *apexes*) with an edge. In particular, when $m = 1$, each vertex of the graph G joins to two vertices with two edges. The *mixed generalized sausage graph*, denoted by $MS(G;m)$, is the graph obtained from the generalized sausage graph $S(G;m)$, $m \geqslant 3$, by joining each vertex of each copy of the $\lceil m/2 \rceil$ copies of G on the left hand side to the left hand side apex, except the nearest copy to the apex, and similarly for the right hand side apex. The *complete mixed generalized sausage graph*, denoted by $CMS(G;m)$ is the graph obtained from the generalized sausage graph by joining each vertex of each copy of G, except the two nearest copies of G to the apexes, to each apex with an edge, and each corresponding pair of vertices of the two nearest copies of G to the apexes with an edge. The *complete mixed generalized sausage graph* $CMS^{-}(G;m)$ is the graph obtained from $CMS(G;m)$ by deleting the edge connecting each corresponding pair of vertices of the two nearest copies of G to the apexes. In [2389] Phanalasy proved a families of generalized sausage graphs, mixed generalized sausage graphs, and complete mixed generalized sausage graphs are antimagic.

A *split* graph is a graph that has a vertex set that can be partitioned into a clique and an independent set. Tyshkevich (see [476]) defines a canonically decomposable graph as follows. For a split graph S with a given partition of its vertex set into an independent set A and a clique B (denoted by $S(A, B)$), and an arbitrary graph H the composition $S(A, B) \circ H$ is the graph obtained by taking the disjoint union of $S(A, B)$ and H and adding to it all edges having an endpoint in each of B and $V(H)$. If G contains nonempty induced subgraphs H and S and vertex subsets A and B such that $G = S(A, B) \circ H$, then G is *canonically decomposable*; otherwise G is *canonically indecomposable*. Barrus [476] proved that every connected graph on at least 3 vertices that is split or canonically decomposable is antimagic.

Hartsfield and Ringel [1191] conjecture that every tree except P_2 is antimagic and, moreover, every connected graph except P_2 is antimagic. In 2004 Alon, Kaplan, Lev, Roditty, and Yuster [167] use probabilistic methods and analytic number theory to show that this conjecture is true for all graphs with n vertices and minimum degree $\Omega(\log n)$. They also prove that if G is a graph with $n \geqslant 4$ vertices and $\Delta(G) \geqslant n-2$, then G is antimagic and all complete partite graphs except K_2 are antimagic. Slíva [3005] proved the conjecture for graphs with a regular dominating subgraph. In 2016 Eccles [864] improved the result of Alon et al. by proving that there exists an absolute constant d_0 such that if G is a graph with average degree at least d_0 and G contains no isolated edge and at most one isolated vertex, then G is antimagic.

Chawathe and Krishna [689] proved that every complete m-ary tree is antimagic. Yilma [3536] extended results on antimagic graphs that contain vertices of large degree by proving that a connected graph with $\Delta(G) \geqslant |V(G)| - 3,\ |V(G)| \geqslant 9$ is antimagic and that if G is a graph with $\Delta(G) = \deg(u) = |V(G)| - k$, where $k \leqslant |V(G)|/3$ and there exists a vertex v in G such that the union of neighborhoods of the vertices u and v forms the whole vertex set $V(G)$, then G is antimagic. Sethuraman and Shermily [2842] proved that binomial trees and Fibonacci trees are antimagic. Vasuki, Shobana, and Ahmed [3364] proved the existence of face antimagic labelings for double duplication of barycentric subdivisions of cycles and some other graphs. Vasuki ·and Shobana [3365] proved the existence of face antimagic labeling of types (0, 1, 0) and (0, 1, 1) for crown related graphs. [3364] new [3365] new

In [122] Ahmed and Babujee defined a *strong face plane graph* as one that is obtained from a plane graph by adding a new vertex to every face, in such way that the faces of the resulting graph are three-sided. If the faces of original plane graphs are three sided faces, then the number of faces increase. They investigated the existence of (super) d-antimagic labeling of type (1, 1, 1) for some strong face plane graphs. Shawkat and Ahmed [2872] investgated the existence of vertex antimagic edge labelings for strong face ladders, strong face wheels, strong face fans, strong face prisms $C_m \times P_2$, and strong face friendship graphs. Kuppan and Shobana [1777] proved the existence of face antimagic labelings for the double duplication of all vertices by the edges of gear graphs G_n for [122] new [2872] new [1777] new

$n \geqslant 3$, $P_3 \times P_{2n}$ ($n \geqslant 2$), $C_n \times P_2$ ($n \geqslant 5$), and the double duplication of all vertices by the edges of a strong face of the triangular snake graph T_n ($n \geqslant 3$). They noted that an (a, d)-face antimagic labeling for double duplication of special graphs can be used to encrypt and decrypt the messages in real time. Jesintha, Vinodhini, Lakshmi [1414] proved that triangular books and double fans graph admit antimagic labelings. [1414] new

Let $G = (V, E)$ be a graph of order n. Let $f : V \to \{1, 2, \ldots, n\}$ be a bijection. The weight $w(v)$ of a vertex v with respect to the labeling f is defined by $w(v) = \sum_{u \in N(v)} f(u)$, where $N(v)$ is the open neighborhood of v. The labeling f is called a *distance antimagic labeling* if $w(v_1) \neq w(v_2)$ for any two distinct vertices v_1, v_2 in V. Cutinho, Sudha, and Arumugam [767] proved that $K_n \times K_n$ is distance antimagic if and only if $n \neq 2$ and $K_3 \times C_n$ is distance antimagic when $n \geqslant 3$ is odd. They included the case $K_3 \times C_n$ when n is even as a problem.

Fronček [966] defines a *handicap incomplete tournament* of n teams with r rounds, HIT(n, r), as a tournament in which every team plays r other teams and the total strength of the opponents that team i plays is $\vec{S}_{n,r}(i) = t - i$ for every i and some fixed constant t. (This means that the strongest team plays strongest opponents, and the lowest ranked team plays weakest opponents.) In terms of distance magic graphs this restriction corresponds to finding a distance antimagic graph with the additional property that the sequence $w(1), w(2), \ldots, w(n)$ (where team i is again the i-th ranked team) is an increasing arithmetic progression with difference one. These graphs are called *handicap distance antimagic graphs*. A *handicap distance d-antimagic labeling* of a graph $G(V, E)$ with n vertices is a bijection $\vec{f} : V \to \{1, 2, \ldots, n\}$ with the property that $\vec{f}(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an increasing arithmetic progression with difference d. A graph G is a *handicap distance d-antimagic graph* if it admits a handicap distance d-antimagic labeling, and *handicap distance antimagic graph* when $d = 1$. In [966] Fronček establishes a relationship between handicap incomplete tournaments and distance antimagic graphs and construct some new infinite classes of distance antimagic graphs and infinite classes of handicap incomplete round robin tournaments. Fronček and Shepanik [979] construct r-regular handicap distance antimagic graphs of order $n \equiv 0 \pmod 8$ for all feasible values of r. Fronček [970] proved that regular handicap distance antimagic graphs exist for every feasible odd order by proving that there exists a regular handicap graph of an odd order n if and only if $n = 9$ or $n \geqslant 13$. In [969] Fronček constructed a class of regular 2-handicap distance antimagic graphs for every order $n \equiv 0 \pmod{16}$. In [971] he proved that a k-regular 2-handicap distance antimagic graph of order $n \equiv 0 \pmod{16}$ exists if and only if $n \geqslant 16$ and $4 \leqslant k \leqslant n-6$. Fronček and Shepanik [980] constructed k-regular handicap distance antimagic graphs of order $n = 4 \pmod 8$ for all feasible values of k. In [2943] Shrimali and Parmar discuss the existence distance antimagic labelings for the product, direct product, strong product, and carona product of graphs involving ${C_3}^t$ and C_4.

Cranston [756] proved that for $k \geqslant 2$, every k-regular bipartite graph is antimagic. For non-bipartite regular graphs, Liang and Zhu [1933] proved that every cubic graph is antimagic. That result was generalized by Cranston, Liang, and Zhu [757], who proved that odd degree regular graphs are antimagic. Hartsfield and Ringel [1191] proved that every 2-regular graph is antimagic. Bérczi, Bernäth, and Vizer [537] use a slight modification of an argument of Cranston et al. [757] to prove that k-regular graphs are antimagic for $k \geqslant 2$. The same was done by Chang, Liang, Pan, and Zhu [672] proved that every even degree regular graph is antimagic.

Tai, Chia, and Ong [3182] proved that the graphs obtained by starting with two verticies and joining them with at least $r \geqslant 3$ edges then subdividing the edges of this graph arbitrarily so that at most one edge (*multi-bridge graphs*) is not subdivided are antimagic. For a connected, undirected, simple graph $G(V, E)$ a bijection f from V to $\{1, 2, \ldots, |V|\}$ is called a *rainbow antimagic vertex labeling* if, for any two edges uv and $u'v'$ in the same path, $w(uv) \neq w(u'v')$, where $w(uv) = f(u) + f(v)$. The *rainbow antimagic connection number* of G is the smallest number of colors taken over all rainbow colorings induced by a rainbow antimagic labeling of G. In [2805] Septory, Utoyo, Dafik, Sulistiyono, and Agustin determined the exact value of the rainbow antimagic connection numbers of Jahangir graphs, firecrackers, $K_{2,m}$, and double stars. A rainbow antimagic labeling f of a graph G is called a *strong rainbow antimagic labeling* of G if for every two vertices u and v there exists path from u to v in which no two edges of the path have the same weight. The *strong rainbow antimagic connection number* of G is the smallest number of colors taken over all strong rainbow colorings induced by strong rainbow antimagic labelings of G. In [1958] Lestari1, Dafik, Susanto, and Wahab determined the connection number of strong rainbow antimagic coloring for Jahangir graphs, Jahangir semi graphs, friendship graphs, fans, stars, and paths. [2805] new [1958] new

A graph $G(V, E)$ is *k-shifted antimagic* if there exists a bijection f from E to $\{k + 1, \ldots, k + |E|\}$ such that the vertex sums of all verticies are distinct; $G(V, E)$ is *absolutely antimagic* if it is k-shifted antimagic for every integer k. In [671] Chang, Li, Daphne Liu, and Pan proved that certain spider forests (a graph where each component is a spider) are k-shifted antimagic for all $k \geqslant 0$. In addition, they showed that for a spider forest G with m edges, there exists a positive integer $k_0 \leqslant m$ such that G is k-shifted antimagic for all $k \geqslant k_0$ and $k \leqslant -(m + k_0 + 11)$. Li and Wang [1928] proved that every tree of diameter four or five, except for two casess, is k-shifted antimagic for every integer k. Chang, Chen, Li, and Pan [669] proved: forest without a component isomorphic to K_2 are k-shifted-antimagic, graphs consisting of vertices of odd degrees and containing no component isomorphic to K_2 are k-shifted-antimagic for sufficiently large k, and P_n ($n \geqslant 6$) are k-shifted for all k. They also gave necessary and sufficiant conditions for stars, double stars, kP_3, and $2K_{1,3}$ to be k-shifted-antimagic. Dhananjaya and Li [823] proved that P_2, P_3, P_4-free linear forests and $K_{1,2}$-free forests are k-shifted-anntimagic with a few exceptions. This extended the results by Shang et al. in [2862] and [2863]. Moreover, they prove that the odd tree forests are k-shifted-antimagic for all k. [1928] new

Chang, Chen, Li, and Pan [668] established connections among various concepts proposed in the literature of antimagic labelings and extend previous results in three ways: some classes of graphs, including trees and graphs whose vertices are of odd degrees, that have not been verified to be antimagic are shown to be k-shifted-antimagic for sufficiently large k; some graphs are proved k-shifted-antimagic for all k, whereas some are proved not for some particular k; and disconnected graphs are also considered.

Beck and Jackanich [524] showed that every connected bipartite graph except P_2 with $|E|$ edges admits an edge labeling with labels from $\{1, 2, \ldots, |E|\}$, with repetition allowed, such that the sums of the labels of the edges incident to each vertex are distinct. They call such a graph *weak antimagic*.

Wang, Liu, and Li [3441] proved: mP_3 $(m \geqslant 2)$ is not antimagic; $P_n \cup P_n$ $(n \geqslant 4)$ is antimagic; $S_n \cup P_n$ is antimagic; $S_n \cup P_{n+1}$ is antimagic; $C_n \cup S_m$ is antimagic for $m \geqslant 2\sqrt{n} + 2$; mS_n is antimagic; if G and H are graphs of the same order and $G \cup H$ is antimagic, then so is $G + H$; and if G and H are r-regular graphs of even order, then $G+H$ is antimagic. In [3442] Wang, Liu, and Li proved that if G is an n-vertex graph with minimum degree at least r and H is an m-vertex graph with maximum degree at most $2r-1$ $(m \geqslant n)$, then $G+H$ is antimagic. Bača, Kimáková, Semaničová-Feňovčiková, and Umar [344] prove the disjoint union of multiple copies of a $(a, 1)$-(super)-tree-antimagic graph is also a $(b, 1)$-(super)-tree-antimagic for certain a and b.

For any given degree sequence pertaining to a tree, Miller, Phanalasy, Ryan, and Rylands [2146] gave a construction for two vertex antimagic edge trees with the given degree sequence and provided a construction to obtain an antimagic unicyclic graph with a given degree sequence pertaining to a unicyclic graph.

Kaplan, Lev, and Roditty [1652] prove that every non-trivial rooted tree for which every vertex that is a not a leaf has at least two children is antimagic (see [1931]) for a correction of a minor error in the the proof). For a graph G with m vertices and an Abelian group A they define G to be A-*antimagic* if there is a one-to-one mapping from the edges of G to the nonzero elements of A such that the sums of the labels of the edges incident to v, taken over all vertices v of G, are distinct. For any $n \geqslant 2$ they show that a non-trivial rooted tree with n vertices for which every vertex that is a not a leaf has at least two children is Z_n-antimagic if and only if n is odd. They also show that these same trees are A-antimagic for elementary Abelian groups G with prime exponent congruent to 1 (mod 3).

In [665] Chan, Low, and Shiu use $[G, A]$ to denote the class of distinct A-antimagic labelings of G. They prove: for a non-trivial Abelian group A that underlies some commutative ring R with unity, if d is a unit in R and $f \in [G, A]$, then $df \in [G, A]$; if A is an Abelian group that contains a subgroup isomorphic to B and a graph G is B-antimagic, then G is A-antimagic; P_{4m+r} and C_{4m+r} are Z_k-antimagic for $k \geqslant 4m+r$ and $r = 0, 1, 3$; P_{4m+2} is Z_k-antimagic for $k \geqslant 4m + 3$; regular Hamiltonian graphs of order $4m + r$ are Z_k-antimagic for $k \geqslant 4m + r$ and $r = 0, 1, 3$, and Z_k-antimagic for $k \geqslant 4m + 3$ and $r = 2$; for odd n, S_n is Z_k-antimagic for $k \geqslant n > 4$; for even n, S_n is Z_k-antimagic for $k \geqslant n + 2 \geqslant 6$ but not Z_n-antimagic or Z_{n+1}-antimagic; trees of order n with exactly one vertex of even degree are Z_k-antimagic for $k \geqslant n$; trees of order n with exactly two vertices of even degree are Z_k-antimagic for $k \geqslant n + 1$; and double stars of order are Z_k-antimagic for $k \geqslant n+1$ when $n \equiv 2 \pmod 4$ and Z_k-antimagic for $k \geqslant n$ when $n \not\equiv 2 \pmod 4$.

Chang and Liu [675] call a graph G $\mathbb{R}$-*antimagic* if for each subset A of $\mathbb{R}$ with $|A| = |E(G)|$, there is an edge labeling such that the sums of the labels assigned to edges incident to distinct vertices are different. They proved that wheels, cycles, and complete graphs of order at least 3 are $\mathbb{R}$-antimagic, and that Cartesian products with at least two terms are $\mathbb{R}$-antimagic when each term is a complete graph of order at least 2 or a cycle.

The *integer-antimagic spectrum* of a graph G is the set $\{k \mid G$ is Z_k-antimagic $(k \geqslant 2\}$. Shiu, Sun, and Low [2931] determine the integer-antimagic spectra of tadpoles and lollipops. Shiu and Low [2927] determine the integer-antimagic spectra of complete bipartite graphs and complete bipartite graphs with a deleted edge. Shiu [2897] determined the integer-antimagic spectra of disjoint unions of cycles. In [2317] Odabasi, Roberts, and Low determine the integer-antimagic spectra for all Hamiltonian graphs.

Liang, Wong, and Zhu [1931] study trees with many degree 2 vertices with a restriction on the subgraph induced by degree 2 vertices and its complement. Denoting the set of degree 2 vertices of a tree T by $V_2(T)$ Liang, Wong, and Zhu proved that if $V_2(T)$ and $V \setminus V_2(T)$ are both independent sets, or $V_2(T)$ induces a path and every other vertex has an odd degree, then T is antimagic. In [2037] Lozano, Mora, Seara, and Tey extended this result by showing that trees whose vertices of even degree induce a path are antimagic.

In [3331] Vaidya and Vyas proved that the middle graphs, total graphs, and shadow graphs of paths and cycles are antimagic. In [1755] and [1756] Krishnaa provided some results for antimagic labelings for graphs derived from wheels and antimagic labelings of helm related graphs.

Bertault, Miller, Pé-Rosés, Feria-Puron, and Vaezpour [549] approached labeling problems as combinatorial optimization problems. They developed a general algorithm to determine whether a graph has a magic labeling, antimagic labeling, or an (a, d)-antimagic labeling (see Section 6.3). They verified that all trees with fewer than 10 vertices are super edge magic and all graphs of the form $P_2^r \times P_3^s$ with less than 50 vertices are antimagic. Kuppan, Shobana, and Cangul [1776] used an (a, d)-face antimagic labeling of a strong face of the duplication of all vertices by edges of a tree T_n $(n \geqslant 2)$ to encrypt and decrypt thirteen secret numbers that can be extended to the double duplication of graphs to encode and decode the numbers that can be used in applications. [1776] new

In [358] Bača, MacDougall, Miller, Slamin, and Wallis survey results on antimagic, edge-magic total, and vertex-magic total labelings.

A *total labeling* of a graph G is a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$. When $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, we say the total labeling is *super*. For a labeling f the associated edge-weight of an edge uv is defined by $wtf(uv) = f(uv) + f(u) + f(v)$. The associated vertex-weight of a vertex v is defined by $wt_f(v) = \sum_{u \in N(v)} f(uv) + f(v)$, where $N(v)$ is the set of the neighbors of v. A labeling f is called *edge-antimagic total* (*vertex-antimagic total*) if all edge-weights (vertex-weights) are pairwise distinct. A graph that admits an edge-antimagic total (vertex-antimagic total) labeling is called an *edge-antimagic total* (*vertex-antimagic total*) graph. A labeling that is simultaneously edge-antimagic total and vertex-antimagic total is called a *totally antimagic total labeling*. A graph that admits a totally antimagic total labeling is called a *totally antimagic total* graph. A labeling g is said to be *ordered* (*sharp ordered*) if $wt_g(u) \leqslant wt_g(v)$ $(wt_g(u) < wt_g(v))$ holds for every pair of vertices $u, v \in V(G)$ such that $g(u) < g(v)$. A graph that admits a (sharp) ordered labeling is called a (*sharp*) *ordered graph*.

Miller, Phanalasy, and Ryan [2143] proved that all graphs have vertex-antimagic total labelings. Bača, Miller, Phanalasy, Ryan, Semaničová-Feňovčíková, and Abildgaard Sillasen [364] prove that mK_1, mK_2, P_n $(n \geqslant 2)$, and C_n are sharp ordered super totally antimagic total. They prove if G is an ordered super edge-antimagic total graph then $G + K_1$ is a totally antimagic total graph. As a corollary they get that stars, friendship graphs $nK_2 + K_1$, fans, and wheels are totally antimagic total. They also prove that if G is a regular ordered super edge-antimagic total graph then $G \odot nK_1$ is totally antimagic total. As a corollary of this result, they have double-stars $K_2 \odot nK_1$ and crowns $C_m \odot nK_1$ are totally antimagic total. They show that a union of regular totally antimagic total graphs is a totally antimagic total graph.

Ahmed and Baskar Baskar [120] proved that complete bipartite graphs admit a totally antimagic total labeling. The same result was proved by Akwu and Ajayi [140] who also showed that the join of a complete bipartite graph and K_1 is a totally antimagic total graph. Ahmed, Baskar Babujee, Bača, Semaničová-Feňovčíková [119] proved that complete graphs admit totally antimagic total labeling. They also considered the problem of finding total labelings for prisms and for two special classes of graphs related to paths that are simultaneously edge-magic and vertex-antimagic.

Miller, Phanalasy, Ryan, and Rylands [2145] provide a method whereby, given any degree sequence pertaining to a tree, one can construct an antimagic tree based on this sequence. By swapping the roles of edges and vertices with respect to a labeling, they provide a method to construct an edge antimagic vertex labeling for any tree. Ahmad, Semaničová-Feňovčíková, Siddiqui, and Kamran [115] construct α-labelings from graceful labelings of smaller trees and transform this labeling to edge-antimagic vertex labeling of trees. Shang [2863] shows that linear forests without either of the paths P_2 or P_3 as components are antimagic. Shang [2864] proved that P_2, P_3, and P_4-free linear forests are antimagic. In [2930] Shiu and Low analyzed various antimagic properties for Cartesian products, hexagonal nets and theta graphs. [2930] new

A *Fibonacci mean antimagic* labeling of a graph G is an injective function $g : V(G) \to \{f_2, f_3, \ldots, f_{n+1}\}$, where f_n is nth Fibonacci number, and the induced a function $g^* : E(G) \to N$ defined by $g^*(uv) = \lceil (g(u) + g(v))/2 \rceil$ is injective. A graph is called *Fibonacci mean antimagic* if it admits a Fibonacci mean antimagic labeling. Thirugnanasambandam and Chitra [3208] proved the following graphs have Fibonacci mean antimagic labelings: caterpillars, bistars, triangular snakes, quadrilateral snakes, ladders, and the corona product of last three graphs and K_1. [3208] new

In [1206] Hefetz, Mütze, and Schwartz investigate antimagic labelings of directed graphs. An *antimagic* labeling of a directed graph D with n vertices and m arcs is a bijection from the set of arcs of D to the integers $\{1, \ldots, m\}$ such that all n oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of all edges entering that vertex minus the sum of labels of all edges leaving it. Hefetz et al. raise the questions "Is every orientation of any simple connected undirected graph antimagic? and "Given any undirected graph G, does there exist an orientation of G which is antimagic?" They call such an orientation an *antimagic orientation* of G. Regarding the first question, they state that, except for $K_{1,2}$ and K_3, they know of no other counterexamples. They prove that there exists an absolute constant C such that for every undirected graph on n vertices with minimum degree at least $C \log n$ every orientation is antimagic. They also show that every orientation of $S_n,\ n \neq 2$, is antimagic; every orientation of W_n is antimagic; and every orientation of $K_n,\ n \neq 3$, is antimagic. For the second question they prove: for odd r, every undirected r-regular graph has an antimagic orientation; for even r every connected undirected r-regular graph that admits a matching that covers all but at most one vertex has an antimagic orientation; and if G is a graph with $2n$ vertices that admits a perfect matching and has an independent set of size n such that every vertex in the independent set has degree at least 3, then G has an antimagic orientation. They conjecture that every connected undirected graph admits an antimagic orientation and ask if it true that every connected directed graph with at least 4 vertices is antimagic. Shan [2861] supported this conjecture by proving that every bipartite graph with no vertex of degree 0 or 2 admits an antimagic orientation and every graph with minimum degree at least 33 admits an antimagic orientation. In [3515] Yang, Carlson, Owens, Perry, Singgih, Song, Zhang, and Zhang proved that a graph G admits an antimagic orientation if $\Delta(G) \geqslant |G| - 3$ or $\Delta(G) = |G| - t \geqslant 4$ for each $t = 4, 5$.

Motivated by the Hartsfield and Ringel on antimagic labelings of graphs, in 2010 Hefetz, Mütze, and Schwartz [1206] initiated the study of antimagic orientations of graphs, and conjectured that every connected graph admits an antimagic orientation. The conjecture has been verified to be true for regular graphs (see [[1206], [1926], [3514]]), and biregular bipartite graphs with minimum degree at least two by Shan and Yu [2860]. Yang, Carlson, Owens, Perry, Singgih, Song, Zhang, Zhang [3515] proved that every connected graph G on $n \geqslant 9$ vertices with maximum degree at least $n - 5$ admits an antimagic orientation. Li, Song, Wang, Yang, and Zhang [1926] proved that every 2-regular graph has an antimagic orientation and for all integers $d \geqslant 2$, every connected $2d$-regular graph has an antimagic orientation. Gao, Yuping and Shan [1020] proved that every lobster admits an antimagic orientation. Ferraro, Newkirk, and Shan [928] prove that subdivided caterpillars in which all pendant edges are replaced by paths of the same length admit an antimagic orientation.

Sonntag [3033] investigated antimagic labelings of hypergraphs. He shows that certain classes of cacti, cycle, and wheel hypergraphs have antimagic labelings. Javaid and Bhatti [1359] extended some of Sonntag's results to disjoint unions of hypergraphs. In [2254] Nalliah investigated the existence of antimagic labelings of some families of digraphs using hooked Skolem sequences. Marimuthu, Raja Durga, and Durga Devi [2090] investigated the existence of super vertex in-antimagic total labelings of generalized de Bruijn digraphs.

Hefetz [1205] calls a graph with q edges k-*antimagic* if its edges can be labeled with $1, 2, \ldots, q + k$ such that the sums of the labels of the edges incident to each vertex are distinct. In particular, antimagic is the same as 0-antimagic. More generally, given a weight function ω from the vertices to the natural numbers Hefetz calls a graph with q edges (ω, k)-*antimagic* if its edges can be labeled with $1, 2, \ldots, q+k$ such that the sums of the labels of the edges incident to each vertex and the weight assigned to each vertex by ω are distinct. In particular, antimagic is the same as $(\omega, 0)$-antimagic where ω is the zero function. Using Alon's combinatorial nullstellensatz [166] as his main tool, Hefetz has proved the following: a graph with 3^m vertices and a K_3 factor is antimagic; a graph with

q edges and at most one isolated vertex and no isolated edges is $(\omega, 2q-4)$-antimagic; a graph with $p > 2$ vertices that admits a 1-factor is $(p-2)$-antimagic; a graph with p vertices and maximum degree $n-k$, where $k \geqslant 3$ is any function of p is $(3k-7)$-antimagic and, in the case that $p \geqslant 6k^2$, is $(k-1)$-antimagic. Hefetz, Saluz, and Tran [1207] improved the first of Hefetz's results by showing that a graph with p^m vertices, where p is an odd prime and m is positive, and a C_p factor is antimagic.

A graph $G = (V, E)$ is *strongly antimagic* if there is a bijective mapping $f : E \to 1, 2, \ldots, |E|$ such that for any two vertices $u \neq v$, not only $\sum_{e \in E(u)} f(e) \neq \sum_{e \in E(v)} f(e)$ and also $\sum_{e \in E(u)} f(e) < \sum_{e \in E(v)} f(e)$ whenever $\deg(u) < \deg(v)$, where $E(u)$ is the set of edges incident to u. Chang, Chin, Li, and Pan [670] proved double spiders (the trees contains exactly two vertices of degree at least 3) are strongly antimagic. They raise the following two questions. Does there exist a strongly antimagic labellings for every antimagic graph? Is there a k-antimagic graph but not $(k+1)$-antimagic?

A *strong edge antimagic total labeling* of a simple graph $G(V, E)$ is a labeling in which the vertex labels are consecutive integers from 1 to $|V|$ such that the total of the labels of the vertices incident to an edge form an ascending arithmetic sequence. Prasetyo [2516] proved that $mK_{1,n}$ has a strong edge antimagic total labeling.

For a graph $G(V, E)$ of order p and size q having no isolated vertices, a bijection $f : V \to \{1, 2, 3, \ldots, p\}$ is called a *local edge antimagic* labeling if for any two adjacent edges uv and vw of G, we have $w(uv) \neq w(vw)$, where the edge weight $w(uv) = f(u)+f(v)$ and $w(vw) = f(v) + f(w)$. A graph G is called a *local edge antimagic* if G has a local edge antimagic labeling. The local edge antimagic chromatic number number of G is the minimum number of colors taken over all colorings induced by local edge antimagic labelings of G. Agustin, Hasan, Dafik, Alfarisi and Prihandini [72] found the local edge antimagic chromatic numbers of paths, cycles, friendship graphs, ladders, stars, wheels, complete graphs, prisms, $C_m \odot mK_1$, and $G \odot mK_1$, where G is any graph of size at least 3. Moreover, they found a relation between local edge antimagic chromatic number and local antimagic vertex chromatic number. Rajkumar and Nalliah [2566] determined the local edge antimagic chromatic numbers for a friendship graphs, wheels, fans, helms, flower graphs, and closed helms. Ahmad, Bača, Lascsáková and Semaničová-Feňovčíková [93] call a labeling of a plane graph *d-antimagic* if for every positive integer s, the set of s-sided face weights is $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some positive integers as a_s and d, where f_s is the number of the s-sided faces. (They allow different sets W_s for different s). A d-antimagic labeling is called *super* if the smallest possible labels appear on the vertices. In [153] they investigated the existence of super d-antimagic labelings of type $(1, 1, 0)$ for disjoint union of plane graphs for several values of difference d. Bača, Numan, and Semaničová-Feňovčíková [375] invesitgate the existence of super d-antimagic labelings of generalized prisms. Hussainn and Tabraiz [1278] investigated super d-antimagic labeling of type (1,1,1) on the snakes kC_5; subdivided kC_5; and isomorphic copies of kC_5 for strings $(1, 1, \ldots, 1)$ and $(2, 2, \ldots, 2)$.

[72] new

[2566] new

Bača, Baskoro, Jendrol̆, and Miller [318] investigated various k-antimagic labelings for graphs in the shape of hexagonal honeycombs. They use H_n^m to denote the honeycomb graph with m rows, n columns, and mn 6-sided faces. They prove: for n odd H_n^m, has a 0-antimagic vertex labeling and a 2-antimagic edge labeling, and if n is odd and $mn > 1$, H_n^m has a 1-antimagic face labeling. In [2928] Shiu and Low show how to construct k-antimagic graphs from existing graphs G with particular labeling properties by joining G to cycles and dumbell related graphs with an edge.

Huang, Wong, and Zhu [1272] say a graph G is *weighted-k-antimagic* if for any vertex weight function w from the vertices of G to the natural numbers there is an injection f from the edges of G to $\{1, 2, \ldots, |E| + k\}$ such that for any two distinct vertices u and v, $\sum(f(e) + w(v)) \neq \sum(f(e) + w(u))$ over all edges incidence to v. They proved that if G has odd prime power order p^z and has total domination number 2 with the degree of one vertex in the total dominating set not a multiple of p, then G is weighted-1-antimagic, and if G has odd prime power order p^z, $p \neq 3$ and has maximum degree at least $|V(G)| - 3$, then G is weighted-1-antimagic. Wong and Zhu [3430] proved: graphs that have a vertex that is adjacent to all other vertices are weighted-2-antimagic; graphs with a prime number of vertices that have a Hamiltonian path are weighted-1-antimagic; and connected graphs $G \neq K_2$ on n vertices are weighted-$\lfloor 3n/2 \rfloor$-antimagic. Matamala and Zamora [2106] proved that $K_{m,n}$, $3 \leqslant m \leqslant n$, $n \geqslant 3$, is weighted-0-antimagic and described a polynomial time algorithm that computes a $(w, 0)$-antimagic labeling of $K_{m,n}$. They also prove the following. Let H be an arbitrary complete partite graph with $n \geqslant 5$ vertices not isomorphic to $K_{1,n}$. Then, any graph containing H as a spanning subgraph is weighted-0-antimagic and given a weight function w, a $(w, 0)$-antimagic labeling can be computed in polynomial time. They prove that each connected graph G on $n \geqslant 3$ vertices having $K_{1,n}$ as a spanning subgraph is weighted-1-antimagic unless G is isomorphic to $K_{1,n}$ and n is even.

A graph G is *weighted k-list-antimagic* if for any vertex weighting $\omega : V(G) \to \mathbb{R}$ and any list assignment $L : E(G) \to 2^{\mathbb{R}}$ with $|L(e)| \geqslant |E(G)| + k$, there exists an edge labeling f such that $f(e) \in L(e)$ for all $e \in E(G)$, labels of edges are pairwise distinct, and the sum of the labels on edges incident to a vertex plus the weight of that vertex is distinct from the sum at every other vertex. Berikkyzy, Brandt, Jahanbekam, Larsen, and Rorabaugh [541] proved that every graph on n vertices having no K_1 or K_2 component is weighted-$\lfloor \frac{4n}{3} \rfloor$-list-antimagic.

A *distance k-antimagic* labeling of a graph $G(V, E)$ is a bijection ^-f from V to $\{1, 2, \ldots, |V|\}$ with the property that there exists an ordering of the vertices of G such that the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an arithmetic progression with difference k. When $k = 1$, then ^-f is simply called a *distance antimagic* labeling. A *distance k-antimagic* graph is a distance k-antimagic graph that admits a distance k-antimagic labeling, and is called *distance antimagic* when $k = 1$. Cichacz, Froncek, Sugeng and Zhou in [742] gave a necessary condition for a graph with an even number of vertices to be distance antimagic with respect to an Abelian group with a unique involution. They also gave sufficient conditions for a Cayley graph on an Abelian group to be distance antimagic or magic with respect to the same group, and discussed the consequences of these results to Cayley graphs on elementary Abelian groups. In [1175] Handa, Godinho, and Singh investigate the existence of distance antimagic labelings of ladders.

For a positive integer k, define $f_k : V(G) \longrightarrow \{1+k, 2+k, \dots, n+k\}$ by $f_k(x) = f(x)+k$. If $w_{f_k}(u) \neq w_{f_k}(v)$ for every pair of vertices $u, v \in V$, for any $k \geqslant 0$ then f is said to be an *arbitrarily distance antimagic* labeling and the graph which admits such a labeling is said to be an *arbitrarily distance antimagic* graph. Handa, Godinho, and Singh [1176] provide arbitrarily distance antimagic labelings for rP_n, the generalized Petersen graph $P(n,k)$, $n \geqslant 5$, the Harary graph $H_{4,n}$ for $n \neq 6$ and prove that join of these graphs is distance antimagic.

For an arbitrary set of distances $D \subseteq \{0, 1, \dots, diam(G)\}$, a D-weight of a vertex x in a graph G under a vertex labeling $f : V \to \{1, 2, \dots, v\}$ is defined as $w_D(x) = \sum_{y \in N_D(x)} f(y)$, where $N_D(x) = \{y \in V \mid d(x,y) \in D\}$. A graph G is said to be D-*distance magic* if all vertices has the same D-vertex-weight, it is said to be D-*distance antimagic* if all vertices have distinct D-vertex-weights. In [2959] Simanjuntak and Wijaya gave some necessary conditions for the existence of D-distance antimagic graphs and conjectured that those conditions are sufficient. They also gave $\{1\}$-distance antimagic labelings for cycles, suns, prisms, complete graphs, wheels, fans, and friendship graphs.

In [244] Arumugam and Kamatchi introduced the notion of (a,d)-distance antimagic graphs as follows. Let G be a graph with vertex set V and $f : V \to \{1, 2, \dots, |V|\}$ be a bijection. If for all v in G the set of sums $\sum f(u)$ taken over all neighbors u of v is the arithmetic progression $\{a, a+d, a+2d, \dots, a+(|V|-1)d\}$, f is called an (a,d)-*distance antimagic* labeling and G is called a (a,d)-*distance antimagic graph*. Arumugam and Kamatchi [244] proved: C_n is (a,d)-distance antimagic if and only if n is odd and $d = 1$; there is no $(1,d)$-distance antimagic labeling for P_n when $n \geqslant 3$; a graph G is $(1,d)$-distance antimagic graph if and only if every component of G is K_2; $C_n \times K_2$ is $(n+2,1)$-distance antimagic; and the graph obtained from $C_{2n} = (v_1, v_2, \dots, v_{2n})$ by adding the edges v_1v_{n+1} and v_iv_{2n+2-i} for $i = 2, 3, \dots, n$ is $(2n+2,1)$-distance antimagic. In [966] and [968] Froncek proved that disjoint copies of the Cartesian product of two complete graphs and its complement are $(a,2)$-distance antimagic and $(a,1)$-distance antimagic. He also proved that disjoint copies of the hypercube Q_3 is $(a,1)$-distance antimagic. Semeniuta [2751] proved that the crown $P_n \odot P_1$ does not admit an $(a,1)$-distance antimagic labeling for $n \geqslant 2$ and $a \geqslant 2$ and determines the values of a for which P_n can be an $(a,1)$-distance antimagic graph. The circulant graph is also investigated. Semenyuta [2752] proved that $P_n \odot P_1$ is not an (a,d)-distance antimagic graph for all a and d and that Q_n is a $(2^n+n-1, n-2)$-distance antimagic graph. He found two types of graphs that do not allow 1-vertex bimagic vertex labeling and established a relation between the distance magic labeling of a regular graph G with 1-vertex bimagic vertex labeling $G \cup G$. Meganingtyas, Dafik, and Slamin [2129] investigated the existence of super (a,d)-vertex antimagic labeling of directed cycles. [2129] new

Patel and Vasva [2368] use $Circ(n,k)$ $n \geqslant 3$ to denote the graph $Circ(n, \{k\})$ defined as follows: for a subset $S \subset \{1, 2, \dots, n\}$, the circulant graph $Circ(n,S)$ is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and there is an edge between vertices v_i and v_j if and only if $|i-j| \in S \bigcup \{1, n-1\}$). In [2368] they proved the existence or non-existence of (a,d)-distance antimagic labelings for the following graphs: $Circ(2n, \{1,n\})$ is $(2n+2,1)$-distance antimagic for all even n; mK_{2n} is $(n(2mn-2m+1),1)$-distance antimagic for all m and n; $3K_{2n+1}$ is $(6n^2+n-1,1)$-distance antimagic for all n; $2K_{2n+1}$ is not (a,d)-distance antimagic for all n; helms H_n is not (a,d)-distance antimagic for any n; books $B_n = S_n \times P_2$ of order $2n+2$ are not (a,d)-distance antimagic for any n; and $K_n \odot K_1$ is not (a,d)-distance antimagic for $n > 1$.

Kamatchi, Vijayakumar, Ramalakshmi, Nilavarasi, and Arumugam [1572] prove that the hypercube is (a,d)-distance antimagic and the bistar $K_2(n,n)$ is distance antimagic. They also show that if G is a regular distance antimagic graph, then $2G$ is also distance antimagic and several families of disconnected graphs are distance antimagic graphs.

A connected graph $G=(V,E)$ with m edges is called *universal antimagic* if for each set B of m positive integers there is an bijective function $f : E \to B$ such that the function $\tilde{f} : V \to \mathbb{N}$ defined at each vertex v as the sum of all labels of edges incident to v is injective. Matamala and Zamora [2105] proved that paths, cycles, split graphs, and graphs that contains the complete bipartite graph $K_{2,n}$ as a spanning subgraph are universal antimagic. A universal antimagic graph is *weight universal antimagic* if, in addition, for any weight function w on the vertices, all $w(u)+f^u$ are distinct. Generalizing their previous result [2105], in [2107] the authors show constructively that if a graph has a complete bipartite graph $K_{m,n}$ as a spanning subgraph with $m,n \geqslant 3$, then it is weighted universal antimagic (hence universal antimagic). They also show that for all other values of m and n, the graph is universal antimagic.

In 2019 [371] Bača, Miller, Ryan, and Semaničová-Feňovčíková published a monograph that focuses on variations of magic and antimagic type lsbelings and includes new results, techniques, constructions, and open problems and conjectures.

In Table 12 we use the abbreviation **A** to mean antimagic. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2014.

Table 12: **Summary of Antimagic Labelings**

Graph	*Labeling*	*Notes*
P_n	A	for $n \geqslant 3$ [1191]
C_n	A	[1191]
W_n	A	[1191]
K_n	A	for $n \geqslant 3$ [1191]
every tree except K_2	A?	[1191]
caterpillars	A	[2035]

Continued on next page

Table 12 – *Continued from previous page*

Graph	*Labeling*	*Notes*
regular graphs	A	[1933], [1191], [672]
every connected graph except K_2	A?	[1191]
$n \geqslant 4$ vertices $\Delta(G) \geqslant n-2$	A	[167]
all complete partite graphs except K_2	A	[167]
$C_m \times P_n$	A	[700]
$P_{m_1} \times P_{m_2} \times \cdots \times P_{m_k}$	A	[700]
$C_{m_1} \times C_{m_2} \times \cdots \times C_{m_k}$	A	[3431]
C_n^2	A	[1833]
mP_3 $m \geqslant 2$	not A	[3441]

6.2 (a, d)-Antimagic Labelings

The concept of an (a, d)-antimagic labelings was introduced by Bodendiek and Walther [584] in 1993. A connected graph $G = (V, E)$ is said to be (a, d)-*antimagic* if there exist positive integers a, d and a bijection $f \colon E \to \{1, 2, \dots, |E|\}$ such that the induced mapping $g_f \colon V \to N$, defined by $g_f(v) = \sum\{f(uv)|\ uv \in E(G)\}$, is injective and $g_f(V) = \{a, a+d, \dots, a+(|V|-1)d\}$. (In [1954] Lin, Miller, Simanjuntak, and Slamim called these (a, d)-*vertex-antimagic edge labelings*). Bodendick and Walther ([586] and [587]) prove the Herschel graph is not (a, d)-antimagic and obtain both positive and negative results about (a, d)-antimagic labelings for various cases of graphs called *parachutes* $P_{g,p}$. ($P_{g,p}$ is the graph obtained from the wheel W_{g+p} by deleting p consecutive spokes.) In [333] Bača and Holländer prove that necessary conditions for $C_n \times P_2$ to be (a, d)-antimagic are $d = 1,\ a = (7n+4)/2$ or $d = 3,\ a = (3n+6)/2$ when n is even, and $d = 2,\ a = (5n+5)/2$ or $d = 4,\ a = (n+7)/2$ when n is odd. Bodendiek and Walther [585] conjectured that $C_n \times P_2$ $(n \geqslant 3)$ is $((7n+4)/2, 1)$-antimagic when n is even and is $((5n+5)/2, 2)$-antimagic when n is odd. These conjectures were verified by Bača and Holländer [333] who further proved that $C_n \times P_2$ $(n \geqslant 3)$ is $((3n+6)/2, 3)$-antimagic when n is even. Bača and Holländer [333] conjecture that $C_n \times P_2$ is $((n+7)/2, 4)$-antimagic when n is odd and at least 7. Bodendiek and Walther [585] also conjectured that $C_n \times P_2$ ($n \geqslant 7$) is $((n+7)/2, 4)$-antimagic. Miller and Bača [2139] prove that the generalized Petersen graph $P(n, 2)$ is $((3n+6)/2, 3)$-antimagic for $n \equiv 0 \pmod 4$, $n \geqslant 8$ and conjectured that $P(n, k)$ is $((3n + 6)/2, 3)$-antimagic for even n and $2 \leqslant k \leqslant n/2 - 1$ (see §2.7 for the definition of $P(n, k)$). This conjecture was proved for $k = 3$ by Xu, Yang, Xi, and Li [3507]. Jirimutu and Wang proved that $P(n, 2)$ is $((5n + 5)/2, 2)$-antimagic for $n \equiv 3 \pmod 4$ and $n \geqslant 7$. Xu, Xu, Lü, Baosheng, and Nan [3503] proved that $P(n, 2)$ is $((3n + 6)/2, 2)$-antimagic for $n \equiv 2 \pmod 4$ and $n \geqslant 10$. Xu, Yang, Xi, and Li [3507] proved that $P(n, 3)$ is $((3n + 6)/2, 3)$-antimagic for even $n \geqslant 10$ and for $n \equiv 0 \pmod 4$, $n \geqslant 8$. In [1957] Lingqi, Linna, Yuan show that $P(n, 3)$ is $(5n + 5)/2, 2)$-antimagic for odd $n \geqslant 7$. Feng, Hong, Yang, and Jirimutu [922] show that $P(n, 5)$ is $(3n + 6)/2, 3)$-antimagic for even $n \geqslant 12$. Bao, Zhao, Yang, Feng, and Jirimutu [421] proved that $P(n, 7)$ is $(\frac{3n+6}{2}, 3)$-antimagic for even $n \geqslant 16$. Ivančo [1341] investigated $(a, 1)$-antimagic labelings and their connection with supermagic generalized double graphs. Bodendiek and Walther [588] proved that the following graphs are not (a, d)-antimagic: even cycles; paths of even order; stars; $C_3^{(k)}; C_4^{(k)}$; trees of odd order at least 5 that have a vertex that is adjacent to three or more end vertices; n-ary trees with at least two layers when $d = 1$; the Petersen graph; K_4 and $K_{3,3}$. They also prove: P_{2k+1} is $(k, 1)$-antimagic; C_{2k+1} is $(k+2, 1)$-antimagic; if a tree of odd order $2k + 1$ ($k > 1$) is (a, d)-antimagic, then $d = 1$ and $a = k$; if K_{4k} ($k \geqslant 2$) is (a, d)-antimagic, then d is odd and $d \leqslant 2k(4k - 3) + 1$; if K_{4k+2} is (a, d)-antimagic, then d is even and $d \leqslant (2k+1)(4k-1)+1$; and if K_{2k+1} ($k \geqslant 2$) is (a, d)-antimagic, then $d \leqslant (2k + 1)(k - 1)$. Lin, Miller, Simanjuntak, and Slamin [1954] show that no wheel W_n ($n > 3$) has an (a, d)-antimagic labeling. In [3153] Susanto provided super (a, d)-C_n-antimagic total labelings for various cases of mC_n. In [1672], Kathiresan and Laurence posed the problem of characterizing the super $(a, 1) - P_3$-antimagic total labeling of the stars S_n, where $n = 6, 7, 8$, and 9. This problem was completely solved in [2568] by Rajkumar, Nalliah, and Venkataraman.

The Hill cipher is a cryptographic algorithm that uses modulo arithmetic and a matrix as a key to perform encryption and decryption. In [2520] Prihandini and Adawiyah1 discussed how to use a super $(3n + 5, 2)$- edge antimagic total labeling to construct the Hill cipher algorithm. They stated that variations of the edge weight function and the corresponding edge label on the graph will make the constructed lock more complicated to hack. [2520] new

In [1349] Ivančo, and Semaničová show that a 2-regular graph is super edge-magic if and only if it is $(a, 1)$-antimagic. As a corollary we have that each of the following graphs are $(a, 1)$-antimagic: kC_n for n odd and at least 3; $k(C_3 \cup C_n)$ for n even and at least 6; $k(C_4 \cup C_n)$ for n odd and at least 5; $k(C_5 \cup C_n)$ for n even and at least 4; $k(C_m \cup C_n)$ for m even and at least 6, n odd, and $n \geqslant m/2 + 2$. Extending a idea of Kovář they prove if G is $(a_1, 1)$-antimagic and H is obtained from G by adding an arbitrary $2k$-factor then H is $(a_2, 1)$-antimagic for some a_2. As corollaries they observe that the following graphs are $(a, 1)$-antimagic: circulant graphs of odd order; $2r$-regular Hamiltonian graphs of odd order; and $2r$-regular graphs of odd order $n < 4r$. They further show that if G is an $(a, 1)$-antimagic r-regular graph of order n and $n - r - 1$ is a divisor of the non-negative integer $a+n(1+r-(n+1)/2)$, then $G \oplus K_1$ is supermagic. As a corollary of this result they have if G is $(n-3)$-regular for n odd and $n \geqslant 7$ or $(n-7)$-regular for n odd and $n \geqslant 15$, then $G \oplus K_1$ is supermagic.

Bertault, Miller, Feria-Purón, and Vaezpour [549] approached labeling problems as combinatorial optimization problems. They developed a general algorithm to determine whether a graph has a magic labeling, antimagic labeling, or an (a,d)-antimagic labeling. They verified that all trees with fewer than 10 vertices are super edge magic and all graphs of the form $P_2^r \times P_3^s$ with less than 50 vertices are antimagic. Javaid, Hussain, Ali, and Dar [1363] and Javaid, Bhatti, and Hussain [1360] constructed super (a,d)-edge-antimagic total labelings for w-trees and extended w-trees (see 5.2 for the definitions) as well as super (a,d)-edge-antimagic total labelings for disjoint union of isomorphic and non-isomorphic copies of extended w-trees. In [1361] Javaid and Bhatt defined a generalized w-tree and proved that they admit a super (a,d)-edge-antimagic total labeling. In [3439] Wang, Li, and Wang proved that some classes of graphs derived from regular or regular bipartite graphs are antimagic. A subdivided star $T(n_1,n_2,\ldots,n_r)$ is a tree obtained by inserting $n_i \geqslant 1,\ 1 \leqslant i \leqslant r$ with $r \geqslant 3$ vertices. In [2542] Raheem, Javaid, and Baig study a super (a,d)-edge-antimagic total labelings of the subdivided stars $T(n,n+1,n_3,\ldots,n_r)$ when n is even and $T(n,n,n+1,n_4,\ldots,n_r)$ when n is odd for all possible values of d. In [2541] Raheem and Baig proved the super edge antimagicness of subdivided stars for all possible values of d. Bhatti, Tahir, and Javaid [567] give super (a,d)-edge antimagic total labelings of some wheel-like graphs. In [248] investigate the existence of super (a,d)-edge antimagic total labeling for friendship graphs and generalized friendship graphs. Girija and Karthikeyan [1103] proved that 3 copies of the jelly fish graphs are super (a,d)-edge antimagic vertex graphs and super (a,d)-edge antimagic total graphs.

In [65] Afzal, Javaid, Alanazi, and Alshehri investigated the super $(a,0)$ edge-antimagicness of the union of the networks of stars, paths, and copies of paths and the rooted product of C_n with $K_{2,n}$. They also provided super $(a,0)$ edge-antimagic labelings of the rooted product of cycles and planar pancyclic networks, and give super $(a,0)$ edge-antimagic labelings for a pancyclic network containing chains of C_6, and three different symmetrically designed lattices.

For graphs G and F, if every edge of G belongs to a subgraph of G isomorphic to F and there exists a total labeling λ of G such that for every subgraph F' of G that is isomorphic to F, the set $\{\Sigma\lambda(F') :\ F' \cong F,\ F' \subseteq G\}$ forms an arithmetic progression starting with a with common difference d, Lee, Tsai, and Lin [1832] say that G is (a,d)-*F-antimagic.* Furthermore, if $\lambda(V(G)) = \{1,2,\ldots,|V(G)|\}$ then G is said to be *super* (a,d)-*F-antimagic* and λ is said to be a *super* (a,d)-*F-antimagic* labeling of G. Lee, Tsai, and Lin [1832] proved that $P_m \times P_n$ $(m,n \geqslant 2)$ is super $(a,1)$-C_4-antimagic. In [2742] Selvagopal, Jeyanthi, Muthuraja, and Semaničová-Feňovčiková investigated the existence super (a,d)-star-antimagic labelings of a particular class of banana trees and construct a star-antimagic graph. In [1531] Jeyanthi, Selvi, and Ramya proved the existence of super (a,d)-C_n-antimagic labelings of fan graphs and ladders. Inayah [1321] proved the existence of an $(a,b)-P_4$-antimagic decomposition of a generalized Petersen graph $GP_z(n,3)$ for several values of b. Taimur, Ali, Numan, Aslam, and Anoh Yannick [3183] provided super

(a, d)-C_3-antimagic total labelings for the generalized antiprism for $d = 0$ and 1 and a super (a, d)-C_8-antimagic total labeling for the toroidal octagonal map for $d = 1, 2, \ldots, 7$.

The *edge corona path graph* $G_m \diamond P_n$ is the graph obtained from one copy of the gear graph G_m and $3m$ copies of P_n, P_n^i, by joining two end vertices of $e_i \in E(G_m)$ to every vertex $v_j \in V(P_n)$ in the i-th copy of G_m with $i = 1, 2, \ldots, 3m$ and $j = 1, 2, \ldots, n$. The graph $G_m \cdot C_n$ is the graph obtained from G_m and $2m + 1$ copies of C_n namely C_n^i by joined every vertex $v_i \in G_m$ to all vertices $v_i \in C_n$ for $i \in \{1, 2, \ldots, 2m + 1\}$. Nistyawati and Martini [2304] proved that for every odd m, the gear edge corona path graph $G_m \diamond P_n$ is super $C_4 \diamond P_n$-antimagic and for every odd m, the gear corona cycle graph $G_m \cdot C_n$ is super $C_4 \cdot C_n$-antimagic. Roswitha, Martini, and S. A. Nugroho [2655] proved: for $n \geqslant 5$ the double cone $DC_n = C_n + \overline{K_2}$ is $(14 + 7n + (n + 1)2, 1)$-C_3-antimagic and $(a, 1)$-C_3-antimagic; DC_n is (a, d)-W_n-antimagic; DC_{2n} is $(a, 1)$-W_{2n}-antimagic; and DC_{2n+1} is $(a, 2)$-W_{2n+1}-antimagic.

Yegnanarayanan [3534] introduced several variations of antimagic labelings and provides some results about them.

The *antiprism* on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ (subscripts are taken modulo n). For $n \geqslant 3$ and $n \not\equiv 2 \pmod 4$ Bača [305] gives $(6n + 3, 2)$-antimagic labelings and $(4n + 4, 4)$-antimagic labelings for the antiprism on $2n$ vertices. He conjectures that for $n \equiv 2 \pmod 4$, $n \geqslant 6$, the antiprism on $2n$ vertices has a $(6n+3, 2)$-antimagic labeling and a $(4n+4, 4)$-antimagic labeling.

Nicholas, Somasundaram, and Vilfred [2301] prove the following: If $K_{m,n}$ where $m \leqslant n$ is (a, d)-antimagic, then d divides $((m - n)(2a + d(m + n - 1)))/4 + dmn/2$; if $m + n$ is prime, then $K_{m,n}$, where $n > m > 1$, is not (a, d)-antimagic; if $K_{n,n+2}$ is (a, d)-antimagic, then d is even and $n + 1 \leqslant d < (n + 1)^2/2$; if $K_{n,n+2}$ is (a, d)-antimagic and n is odd, then a is even and d divides a; if $K_{n,n+2}$ is (a, d)-antimagic and n is even, then d divides $2a$; if $K_{n,n}$ is (a, d)-antimagic, then n and d are even and $0 < d < n^2/2$; if G has order n and is unicylic and (a, d)-antimagic, then $(a, d) = (2, 2)$ when n is even and $(a, d) = (2, 2)$ or $(a, d) = ((n + 3)/2, 1)$ when n is odd; a cycle with m pendent edges attached at each vertex is (a, d)-antimagic if and only if $m = 1$; the graph obtained by joining an endpoint of P_m with one vertex of the cycle C_n is $(2, 2)$-antimagic if $m = n$ or $m = n - 1$; if $m + n$ is even the graph obtained by joining an endpoint of P_m with one vertex of the cycle C_n is (a, d)-antimagic if and only if $m = n$ or $m = n - 1$. They conjecture that for n odd and at least 3, $K_{n,n+2}$ is $((n + 1)(n^2 - 1)/2, n + 1)$-antimagic and they have obtained several results about (a, d)-antimagic labelings of caterpillars.

In [2036] Lozano, Mora, and Seara prove that any caterpillar of order n is $(\lfloor(n - 1)/2, \rfloor - 2)$-antimagic. Furthermore, if C is a caterpillar with a spine of order s, they prove that when C has at least $\lfloor(3s + 1)/2\rfloor$ leaves or $\lfloor(s - 1)/2\rfloor$ consecutive vertices of degree at most 2 at one end of a longest path, then C is antimagic. As a consequence of a result of Wong and Zhu [3484], they also prove that if p is a prime number, any caterpillar with a spine of order p, $p - 1$ or $p - 2$ is 1-antimagic.

In [3392] Vilfred and Florida proved the following: the one-sided infinite path is $(1, 2)$-antimagic; P_{2n} is not (a, d)-antimagic for any a and d; P_{2n+1} is (a, d)-antimagic if and only if $(a,d) = (n,1)$; C_{2n+1} has an $(n+2,1)$-antimagic labeling; and that a 2-regular graph G is (a,d)-antimagic if and only if $|V(G)| = 2n+1$ and $(a,d) = (n+2,1)$. They also prove that for a graph with an (a,d)-antimagic labeling, q edges, minimum degree δ and maximum degree Δ, the vertex labels lie between $\delta(\delta+1)/2$ and $\Delta(2q-\Delta+1)/2$.

Chelvam, Rilwan, and Kalaimurugan [690] proved that Cayley digraph of any finite group admits a super vertex (a,d)-antimagic labeling depending on d and the size of the generating set. They provide algorithms for constructing the labelings. Thirusangu and Bhrathiraja [3210] proved the existence of super vertex (a,d)- antimagic labeling and vertex magic total labeling for Cayley digraphs arising from the groups $Z_{p^n} \oplus Z_{p^n}$ and $Z_m \oplus Z_n$.

Irfan and Semaničová-Feňovčiková [1335] provide some classes of graphs that admit a labeling that is simultaneously a super edge-magic total and a super vertex-antimagic total and give some results for fans, sun graphs, caterpillars, and prisms.

For $n > 1$ and distinct odd integers x, y and z in $[1,n-1]$ Javaid, Ismail, and Salman [1355] define the *chordal ring* of order n, $CR_n(x,y,z)$, as the graph with vertex set Z_n, the additive group of integers modulo n, and edges $(i,i+x), (i,i+y), (i,i+z)$ for all even i. They prove that $CR_n(1,3,7)$ and $CR_n(1,5,n-1)$ have (a,d)-antimagic labelings when $n \equiv 0 \bmod 4$ and conjecture that for an odd integer Δ, $3 \leqslant \Delta \leqslant n-3, n \equiv 0 \bmod 4$, $CR_n((1,\Delta,n-1)$ has an $((7n+8)/4,1)$-antimagic labeling.

For an arbitrary set of distances $D \subseteq \{0,1,\ldots,diam(G)\}$, a D-weight of a vertex x in a graph G under a vertex labeling $f: V \rightarrow \{1,2,\ldots,v\}$ is defined as $w_D(x) = \sum_{y \in N_D(x)} f(y)$, where $N_D(x) = \{y \in V | d(x,y) \in D\}$. A graph G is said to be D-distance magic if all vertices have the same D-vertex-weight, it is said to be D-distance antimagic index$D$-distance antimagic if all vertices have distinct D-vertex-weights, and it is called $(a,d)-D$-distance antimagic if the D-vertex-weights constitute an arithmetic progression with difference d and starting value a. In [2959] Simanjuntak and Wijaya gave some necessary conditions for the existence of D-distance antimagic graphs and conjectured that those conditions are sufficient. They also gave $\{1\}$-distance antimagic labelings for cycles, suns, prisms, complete graphs, wheels, fans, and friendship graphs. Arumugam and Kamatchi [244] characterized (a,d)-distance antimagic cycles and (a,d)-distance antimagic labelings for paths and prisms. In [966] and [968] Fronček proved that disjoint copies of the Cartesian product of two complete graphs and its complement are $(a,2)$-distance antimagic and $(a,1)$-distance antimagic. He also proved that disjoint copies of the hypercube Q_3 is $(a,1)$-distance antimagic. In [1175] Handa, Godinho and Singh investigate the existence of distance antimagic labeling of ladders.

In [3393] Vilfred and Florida call a graph $G = (V,E)$ *odd antimagic* if there exist a bijection $f: E \rightarrow \{1,3,5,\ldots,2|E|-1\}$ such that the induced mapping $g_f: V \rightarrow N$, defined by $g_f(v) = \sum\{f(uv) |\ uv \in E(G)\}$, is injective and *odd* (a,d)*-antimagic* if there exist positive integers a, d and a bijection $f: E \rightarrow \{1,3,5,\ldots,2|E|-1\}$ such that the induced mapping $g_f: V \rightarrow N$, defined by $g_f(v) = \sum\{f(uv) |\ uv \in E(G)\}$, is injective and $g_f(V) = \{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. Although every (a,d)-antimagic graph is antimagic, C_4 has an antimagic labeling but does not have an (a,d)-antimagic labeling. They prove: P_{2n+1} is not odd (a,d)-antimagic for any a and d; C_{2n+1} has an odd

$(2n+2,2)$-antimagic labeling; if a 2-regular graph G has an odd (a,d)-antimagic labeling, then $|V(G)|=2n+1$ and $(a,d)=(2n+2,2)$; C_{2n} is odd magic; and an odd magic graph with at least three vertices, minimum degree δ, maximum degree Δ, and $q \geqslant 2$ edges has all its vertex labels between δ^2 and $\Delta(2q-\Delta)$.

Combining the notions of 1-vertex-magic vertex labelings and antimagic labelings Swaminathan and Jeyanthi [3175] introduced a new labeling as follows. For a graph with p vertices a 1-1 mapping from the vertices to $\{1,2,\dots,p\}$ is called an (a,d)-1-*vertex-antimagic vertex labeling* if the sums of the labels of the vertices adjacent to each vertex taken over all vertices form the set $\{a,a+d,a+2d,\dots,a+(p-1)d\}$. They give some basic properties of such labelings and provide some results for some classes of regular graphs.

For a graph $G=(V,E)$, a bijection g from $V(G)\cup E(G)$ into $\{1,2,\dots,|V(G)|+|E(G)|\}$ is called a (a,d)-*edge-antimagic graceful* labeling of G if the edge-weights $w(xy)=|g(x)+g(y)-g(xy)|$, $xy \in E(G)$, form an arithmetic progression starting from a and having a common difference d. An (a,d)-edge-antimagic graceful labeling is called *super* (a,d)-*edge-antimagic graceful* if $g(V(G))=\{1,2,\dots,|V(G)|\}$. Marimuthu and Krishnaveni [2082] proved mC_n has a super $(0,1)$-edge-antimagic graceful labeling for every $m \geqslant 2$ and $n \geqslant 3$; and mK_n and MP_n have a super $(a,1)$-edge-antimagic graceful labeling for every $m \geqslant 2$ and $n \geqslant 2$.

In [123] Ahmed, Semaničová-Feňovčíková, Bača, Baskar Babujee, and Shobana introduced the notion of graceful antimagic graphs as follows. A graceful labeling that is simultaneously antimagic (that is, the sums of labels of all edges incident to a given vertex are pairwise distinct for different vertices) is said to be *graceful antimagic* . They obtain results about paths, stars, double stars, cycles, complete graphs, complete graphs with deleted edges, and complete bipartite graphs. They include many conjectures and open probelems. [123] new

In [251] Arumugam, Premalatha, Bacă, and Semaničová-Feňovčíková introduced a new graph coloring parameter as follows. Let f be a local antimagic labeling of a connected graph G. The assignment of $w(u)$ to u for each vertex u of G induces naturally a proper vertex coloring of G, called a of G. The , denoted $\chi_{la}(G)$, is the minimum number of colors taken over all local antimagic colorings of G. Arumugam, Lee, Premalatha, Wang [246] proved that for $m>1$, $\chi_{la}(C_3 \odot O_m)=3m+3$, where O_m is the null graph on m verticies, and for $n>1$, $\chi_{la}(K_n \odot K_1)=2n-1$. In [1807] Lau and Nalliah correct a mistake in [1248] for the lower bounds of the local antimagic chromatic number of the corona product of friendship and fan graphs with null graph and obtain a sharp lower bound that gives the exact local antimagic chromatic number of the corona product of friendship and null graph. In [1808] Lau, Ng, and Shiu give counterexamples to the lower bound of $\chi_{la}(G \vee O_2)$ that was obtained in [251]. A sharp lower bound of $\chi_{la}(G \vee O_n)$ and sufficient conditions for the given lower bound to be attained are obtained. Moreover, they improve a theorem and solve a problem stated in [251] and determine the local antimagic chromatic number of complete bipartite graphs. In [1814] Lau, Shiu, and Ng obtain a sharp lower bound of the local antimagic chromatic number of a graph with cut-vertices given by pendants is obtained. They also determine the exact value of the local antimagic chromatic number of many families of graphs with cut-vertices (possibly given by pendant edges). In [1813] Lau, Shiu, and Ng provide sharp upper and lower bounds of $\chi_{la}(G)$ for G with pendant vertices, and give sufficient conditions for equality. They show that there are infinitely many graphs with $k \geqslant \chi(G) - 1$ pendant vertices and $\chi_{la}(G) = k + 1$. They conjecture that every tree T_k, other than certain caterpillars, spiders and lobsters, with k pendant vertices has $\chi_{la}(T_k) = k + 1$. Lau, Shiu, and Soo [1816] proved that a d-leg spider graph has $d + 1 \leqslant \chi_{la} \leqslant d + 2$ and a 3-leg spider has $\chi_{la} = 4$, if not all legs are of odd length. They also obtain many sufficient conditions such that both the values are attainable. They conjecture that almost all d-leg spiders of size q that satisfy $d(d + 1) \leqslant 2(2q - 1)$ with each leg length at least 2 has $\chi_{la} = d + 1$. Bača, Semaničová-Feňovčíková, Lai, and Wang [381] verified a conjecture by Arumugam, Lee, Premalatha, and Wang [246] that for every tree T the local antimagic chromatic number satisfies $l + 1 \leqslant \chi_{la}(T) \leqslant l + 2$, where l is the number of leaves of T, by determining the exact value for the local antimagic chromatic number of all complete full t-ary trees is $l + 1$ for odd t. In [56] Adawiyah, Makhfudloh, and Prihandini obtained results about the local (a, d)-antimagic chromatic number of sunflower graphs and umbrella graphs. [56] new

For a connected graph G with q edges a bijection $f : E \to \{1, 2, \ldots, q\}$ is called a *local antimagic labeling* if for any two adjacent vertices u and v, $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$, and $E(u)$ is the set of edges incident to u. In [251] Arumugam, Premalatha, Bača, and Semaničová-Feňovčíková proved several basic results on this new parameter and conjectured that any connected graph other than K_2 admits a local antimagic labeling. This conjecture was proved by Haslegrave [1193] using the probabilistic method, proves that the local antimagic conjecture is true. Lau [1801] proved that every graph admits a local antimagic total labeling. For any graph G, the graph $H = G \vee O_n$, $n \geqslant 1$, is defined by $V(H) = V(G) \cup \{v_i : 1 \leqslant i \leqslant n\}$ and $E(H) = E(G) \cup \{uv_i : u \in V(G)\}$. In [251, Theorem 2.16], it was claimed that for any G with order $m \geqslant 4$,

$$\chi_{la}(G) + 1 \leqslant \chi_{la}(G \vee O_2) \leqslant \begin{cases} \chi_{la}(G) + 1 & \text{if } m \text{ is even,} \\ \chi_{la}(G) + 2 & \text{if } m \text{ is odd.} \end{cases}$$

Lau and Shiu [1809] determined the local antimagic total chromatic number of graphs that are the amalgamation of complete graphs.

The number of distinct induced vertex labels under a local antimagic labeling f is denoted by $c(f)$, and is called the *color number* of f. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is $\min\{c(f) : f \text{ is a local antimagic labeling of } G\}$. In [1193], Haslegrave proved that the local antimagic chromatic number is well-defined for every connected graph other than K_2. Thus, for every connected graph $G \neq K_2$, $\chi_{la}(G) \geqslant \chi(G)$, the chromatic number of G. In [1811] Lau, Shiu, and Ng provided several sufficient conditions for $\chi_{la}(H) \leqslant \chi_{la}(G)$, where H is obtained from G with a certain edge deleted or added. The further determined the exact value of the local antimagic chromatic number of many cycle-related join graphs.

Let $G(V, E)$ be a simple graph and f be a bijection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ where $f(V) = \{1, 2, \ldots, |V|\}$. For a vertex $x \in V$, define the weight of x, $w(x)$, as the sum of labels of all edges incident with x and the vertex label itself. Such an f is called a *super vertex local antimagic total* labeling (SLAT) if for every two adjacent vertices their weights are different. The *super vertex local antimagic total chromatic number* $\chi_{slat}(G)$ is the minimum number of colors taken over all colorings induced by super vertex local antimagic total labelings of G. Hadiputra, Sugeng, Silaban, Maryati, and Fronček classify all trees T that have $\chi_{slat}(T) = 2$, present a class of trees that have $\chi_{slat}(T) = 3$, and show that for any positive integer $n \geqslant 2$ there is a tree T with $\chi_{slat}(T) = n$.

In Table 13 we use the abbreviation (a, d)-**A** to mean that the graph has an (a, d)-antimagic labeling. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 13: **Summary of (a, d)-Antimagic Labelings**

Graph	*Labeling*	*Notes*
P_{2n}	not (a, d)-A	[588]
P_{2n+1}	iff $(n, 1)$-A	[588]
C_{2n}	not (a, d)-A	[588]
C_{2n+1}	$(n+2, 1)$-A	[588]
stars	not (a, d)-A	[588]
$C_3^{(k)}, C_4^{(k)}$	not (a, d)-A	[588]
$K_{3,3}$	not (a, d)-A	[588]
K_4	not (a, d)-A	[588]
Petersen graph	not (a, d)-A	[588]
W_n	not (a, d)-A	$n > 3$ [1954]
antiprism on $2n$	$(6n+3, 2)$-A	$n \geqslant 3$, $n \not\equiv 2 \pmod 4$ [305]
vertices (see §6.1)	$(4n+4, 4)$-A	$n \geqslant 3$, $n \not\equiv 2 \pmod 4$ [305]
	$(2n+5, 6)$-A?	$n \geqslant 4$ [305]
	$(6n+3, 2)$-A?	$n \geqslant 6$, $n \not\equiv 2 \pmod 4$ [305]
	$(4n+4, 4)$-A?	$n \geqslant 6$, $n \not\equiv 2 \pmod 4$ [305]
Hershel graph (see [684])	not (a, d)-A	[584], [586]
parachutes $P_{g,p}$ (see §6.2)	(a, d)-A	for certain classes [584], [586]
prisms $C_n \times P_2$	$((7n+4)/2, 1)$-A	$n \geqslant 3$, n even [585], [333]
	$((5n+5)/2, 2)$-A	$n \geqslant 3$, n odd [585], [333]
	$((3n+6)/2, 3)$-A	$n \geqslant 3$, n even [333]
	$((n+7)/2, 4)$-A?	$n \geqslant 7$, [586], [333]
generalized Petersen graph $P(n, 2)$	$((3n+6)/2, 3)$-A	$n \geqslant 8$, $n \equiv 0 \pmod 4$ [334]

6.3 (a, d)-Antimagic Total Labelings

Bača, Bertault, MacDougall, Miller, Simanjuntak, and Slamin [323] introduced the notion of a (a, d)-vertex-antimagic total labeling in 2000. For a graph $G(V, E)$, an injective mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is a (a, d)-*vertex-antimagic total labeling* if the set $\{f(v) + \sum f(vu)\}$ where the sum is over all vertices u adjacent to v for all v in G is $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. In the case where the vertex labels are 1,2, $\ldots$, $|V|$, (a, d)-vertex-antimagic total labeling is called a *super* (a, d)-*vertex-antimagic total labeling*. Among their results are: every super-magic graph has an $(a, 1)$-vertex-antimagic total labeling; every (a, d)-antimagic graph $G(V, E)$ is $(a+|E|+1, d+1)$-vertex-antimagic total; and, for $d > 1$, every (a, d)-antimagic graph $G(V, E)$ is $(a+|V|+|E|, d-1)$-vertex-antimagic total. They also show that paths and cycles have (a, d)-vertex-antimagic total labelings for a wide variety of a and d. In [324] Bača et al. use their results in [323] to obtain numerous (a, d)-vertex-antimagic total labelings for prisms, and generalized Petersen graphs (see §2.7 for the definition). (See also [340] and [3086] for more results on generalized Petersen graphs.)

Sugeng, Miller, Lin, and Bača [3086] prove: C_n has a super (a, d)-vertex-antimagic total labeling if and only if $d = 0$ or 2 and n is odd, or $d = 1$; P_n has a super (a, d)-vertex-antimagic total labeling if and only if $d = 2$ and $n \geqslant 3$ is odd, or $d = 3$ and $n \geqslant 3$; no even order tree has a super $(a, 1)$-vertex antimagic total labeling; no cycle with at least one tail and an even number of vertices has a super $(a, 1)$-vertex-antimagic labeling; and the star S_n, $n \geqslant 3$, has no super (a, d)-super antimagic labeling. As open problems they ask whether $K_{n,n}$ has a super (a, d)-vertex-antimagic total labeling and the generalized Petersen graph has a super (a, d)-vertex-antimagic total labeling for specific values a, d, and n. In [2540] Raheem proved that various subclasses of stars admit super (a, d)-edge antimagic total labelings for $d = 1, 2$, and 3. Lin, Miller, Simanjuntak, and Slamin [1954] have shown that for $n > 20$, W_n has no (a, d)-vertex-antimagic total labeling. Tezer and Cahit [3203] proved that neither P_n nor C_n has (a, d)-vertex-antimagic total labelings for $a \geqslant 3$ and $d \geqslant 6$. Kovář [1747] has shown that every $2r$-regular graph with n vertices has an $(s, 1)$-vertex antimagic total labeling for $s \in \{(rn + 1)(r + 1) + tn \mid t = 0, 1, \ldots, r\}$. Dafik, Slamin, Romdhani, and Arianti [777] studied the super (a, d)-antimagicness of generalized flower and disk brake graphs. Dafik, Slamin, Eka, and Sya'diyah [776] proved that triangular books and diamond ladder graphs admit a super (a, d)-edge-antimagic total labeling of graph for $d \in \{0, 1, 2\}$. [776] new

Several papers have been written about vertex-antimagic total labeling of graphs that are the disjoint union of suns. The sun graph S_n is $C_n \odot K_1$. Rahim and Sugeng [2546] proved that $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is $(a, 0)$-vertex-antimagic total (or vertex magic total). Parestu, Silaban, and Sugeng [2342] and [2343] proved $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is (a, d)-vertex-antimagic total for $d = 1, 2, 3, 4$, and 6 and particular values of a. In [2544] Rahim, Ali, Kashif, and Javaid provide (a, d)-vertex antimagic total labelings of disjoint unions of cycles, sun graphs, and disjoint unions of sun graphs. In [891] Enomoto et al. proposed the conjecture that every tree is a super $(a, 0)$-edge-antimagic total graph. Javaid [1357] gave (a, d)-edge-antimagic total labelings for certain subclasses of subdivided stars. Javaid [1358] gave a super (a, d)-edge-antimagic total labeling for the subdivided star $T(n, n, n+4, n+4, n_5, n_6, \ldots, n_r)$ for $d = 0, 1, 2$, where $n_p = 2^{p-4}(n+3)+1,\ 5 \leqslant p \leqslant r$ and $n \geqslant 3$ is odd.

In [2289] Ngurah, Baskova, and Simanjuntak provide (a, d)-vertex-antimagic total labelings for the generalized Petersen graphs $P(n, m)$ for the cases: $n \geqslant 3,\ 1 \leqslant m \leqslant \lfloor (n-1)/2 \rfloor,\ (a, d) = (8n+3, 2)$; odd $n \geqslant 5,\ m = 2,\ (a, d) = ((15n+5)/2, 1)$; odd $n \geqslant 5,\ m = 2,\ (a, d) = ((21n+5)/2, 1)$; odd $n \geqslant 7,\ m = 3,\ (a, d) = ((15n+5)/2, 1)$; odd $n \geqslant 7,\ m = 3,\ (a, d) = ((21n+5)/2, 1)$; odd $n \geqslant 9,\ m = 4,\ (a, d) = ((15n+5)/2, 1)$; and $(a, d) = ((21n+5)/2, 1)$. They conjecture that for n odd and $1 \leqslant m \leqslant \lfloor (m-1)/2 \rfloor$, $P(n, m)$ has an $((21n+5)/2, 1)$-vertex-antimagic labeling. In [3091] Sugeng and Silaban show: the disjoint union of any number of odd cycles of orders $n_1, n_2, \ldots, n_t$, each at least 5, has a super $(3(n_1 + n_2 + \cdots + n_t) + 2, 1)$-vertex-antimagic total labeling; for any odd positive integer t, the disjoint union of t copies of the generalized Petersen graph $P(n, 1)$ has a super $(10t+2)n - \lfloor n/2 \rfloor + 2, 1)$-vertex-antimagic total labeling; and for any odd positive integers t and n ($n \geqslant 3$), the disjoint union of t copies of the generalized Petersen graph $P(n, 2)$ has a super $(21tn+5)/2, 1)$-vertex-antimagic total labeling. Ahmad, Ali, Bača, Kovar, and Semaničová-Feňovčíková, investigated the vertex-antimagicness of regular graphs and the existence of (super) (a, d)-vertex antimagic total labelings for regular graphs in general.

Ali, Bača, Lin, and Semaničová-Feňovčiková [153] investigated super-(a, d)-vertex antimagic total labelings of disjoint unions of regular graphs. Among their results are: if m and $(m-1)(r+1)/2$ are positive integers and G is an r-regular graph that admits a super-vertex magic total labeling, then mG has a super-$(a, 2)$-vertex antimagic total labeling; if G has a 2-regular super-$(a, 1)$-vertex antimagic total labeling, then mG has a super-$(m(a-2)+2, 1), 1)$-vertex antimagic total labeling; mC_n has a super-(a, d)-vertex antimagic total labeling if and only if either d is 0 or 2 and m and n are odd and at least 3 or $d = 1$ and $n \geqslant 3$; and if G is an even regular Hamilton graph, then mG has a super-$(a, 1)$-vertex antimagic total labeling for all positive integers m.

In [385] Bača, A. Semaničová-Feňovčíková, Wang, and Zhang investigate the existence of $(a, 1)$-vertex-antimagic edge labelings for disconnected 3-regular graphs. As an extension of (a, d)-vertex-antimagic edge labeling they also introduce the concept of (a, d)-vertex-antimagic edge deficiency for measuring how close a graph is away from being an (a, d)-antimagic graph. In [250] Arumugam and Nalliah investigate the existence of a super (a, d)-edge-antimagic total labelings of disconnected graphs.

Ahmad, Ali, Bača, Kovář and Semaničová-Feňovčíková [83] provided a technique that allows one to construct several (a, r)-vertex-antimagic edge labelings for any $2r$-regular graph G of odd order provided the graph is Hamiltonian or has a 2-regular factor that has $(b, 1)$-vertex-antimagic edge labeling. A similar technique allows them to construct a super (a, d)-vertex-antimagic total labeling for any $2r$-regular Hamiltonian graph of odd order with differences $d = 1, 2, \ldots, r$ and $d = 2r + 2$.

For $n \geqslant 2$ Dafik, Setiawani, and Azizah [839] define a *shackle* as a graph constructed from connected graphs $G_1, G_2, \ldots, G_n$, all isomorphic to G, such that G_s and G_t are disjoint when $|s-t| \geqslant 2$ and for every $i = 1, 2, \ldots, n-1$, G_i and G_{i+1} share exactly one common vertex v. In a *generalized shackle* a common subgraph is shared by each G_i and

G_{i+1}. Dafik, Setiawani, and Azizah prove that the generalized shackle of a fan of order four and five admits a super (a, d)-edge antimagic total labeling for $d = 0, 1, 2$.

Sugeng and Bong [3080] show how to construct super (a, d)-vertex antimagic total labelings for the circulant graphs $C_n(1, 2, 3)$, for $d = 0, 1, 2, 3, 4, 8$. Thirusangu, Nagar, and Rajeswari [3212] show that certain Cayley digraphs of dihedral groups have (a, d)-vertex-magic total labelings.

For a simple graph H we say that $G(V, E)$ admits an H-*covering* if every edge in $E(G)$ belongs to a subgraph of G that is isomorphic to H. Inayah, Salman, and Simanjuntak [1323] define an (a, d)-H-*antimagic total labeling* of G as a bijective function ξ from $V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that for all subgraphs H' isomorphic to H, the H-weights $w(H') = \sum_{v\in V(H')} \xi(v) + \sum_{e\in E(H')} \xi(e)$ constitute an arithmetic progression $a, a + d, a + 2d, \ldots, a + (t - 1)d$ where a and d are positive integers and t is the number of subgraphs of G isomorphic to H. Such a labeling ξ is called a *super* (a, d)-H-*antimagic total labeling*, if $\xi(V) = \{1, 2, \ldots, |V|\}$. Inayah et al. study some basic properties of such labeling and give (a, d)-cycle-antimagic labelings of fans. Taimur, Numan, Mumtaz, and Semaničová-Feňovčíková [3184] proved that if a graph G is super cycle-antimagic then the subdivided graph of G also admits a super cycle-antimagic labeling and they showed that the subdivided wheel is super (a, d)-cycle-antimagic for wide range of values. Laurence and Kathiresan [1817] investigated super (a, d)-P_n-antimagic total labeling of stars. In [2521] Prihandini, Dafik, Agustin, Alfarisi, Adawiyah, and Santoso investigated the existence of super (a, d)-$P_2 \triangleright H$-antimagic total labelings of the disjoint union graph $C_n \triangle H$. [2521] new Bača, Jeyanthi, Selvagopal, Muthu Raja, and Semaničová-Feňovčíková [269] proved the existence of super (a, d)-H-antimagic labelings of fan graphs and ladders for H isomorphic to a cycle.

In [2750] Semaničová-Feňovčíková, Bača, Lascsáková, Miller, and Ryan investigated the super (a, d)-C_n-antimagic total labelings of wheels and super (a, d)-P_n-antimagic total labelings of cycles and paths. Ovais, Umar, Bača, and Semaničová-Feňovčíková [2323] proved that fans admits a super (a, d)-C_k-antimagic labeling for $d = 1, 3, 2k - 5, 2k - 1, 3k - 1, k - 7, k + 1, 3k - 9$. They also prove that fans admits a super (a, d)-C_3-antimagic labeling for $d = 0, 1, 2, 3, 4, 5, 6, 8$, and a super (a, d)-C_4-antimagic labeling for $d = 0, 1, 2, 3, 4, 5, 6, 7, 11$. They propose an open problem to find a super (a, d)-C_k-antimagic labeling of fans for $d \neq 1, 3, k - 7, k + 1, 2k - 5, 2k - 1, 3k - 1, 3k - 9$. Bača, Miller, Ryan, and Semaničová-Feňovčíková [369] study super (a, d)-H-antimagic labelings of a disjoint union of graphs for $d = |E(H)| - |V(H)|$.

For a vertex u of a graph G, $G_u[S_n]$ is the graph obtained by identifying u with the center of S_n. Then for any vertex w of S_n $G + e$, $e = uw$ is a subgraph of $G_u[S_n]$. Kathiresan and Laurence [1672] prove that the graph $G_u[S_n]$ admits a super-(a, d)-$(G+e)$-antimagic total labeling if and only if $d \in \{0, 1, 2, \ldots, |V(G)| + |E(G)| + 2\}$. Moreover, they show that a caterpillar $S_{n_1,n_2,\ldots,n_k}$ has a super-$(a, 4n^2)$-$S_{n,n}$-antimagic total labeling for $n_1 = n_2 = \cdots = n_k = n$. In [2567] Rajkumar, Nalliah, Uma Maheswari provided a partial solution to the problem of characterizing the super $(a, d) - G + e$-antimagic total labeling of the graph $G_u[S_n]$, where $n \geqslant 3$ and $4 \leqslant d \leqslant p + q + 2$ posed in 2015 by Kathiresan and Laurence in [1672].

Jeyanthi, Muthuraja, Semaničová-Feňovčíková, and Dharshikha proved [1483] proved that fans, triangular ladders, and middle graphs of cycles are super (a,d)-C_3-antimagic for some values of a and d. They also proved that ladder are super (a,d)-C_4-antimagic for $1 \leqslant d \leqslant 8$. Inayah, Simanjuntak, and Salman [1324] proved that there exists a super $(a,d) - H$-antimagic total labelings for shackles of a connected graph H. Nadzima and Martini [2243] determined (a,d)-H-antimagic total labeling for certains cases of $W_n \odot P_n$ with H as $C_3 \odot P_n$ and $W_n \odot C_n$ with H as $C_3 \odot C_n$.

A graph G is said to have an $(H_1, H_2, \ldots, H_k)$-*covering* if every edge in G belongs to at least one of the H_i's. Susilowati, Sania, and Estuningsih [3164] investigated such antimagic labelings for the ladders $P_n \times P_2$ with C_t-coverings for $t = 4,\ 6$, and 8 for some value of d.

Simanjuntak, Bertault, and Miller [2958] define an (a,d)-*edge-antimagic vertex labeling* for a graph $G(V,E)$ as an injective mapping f from V onto the set $\{1,2,\ldots,|V|\}$ such that the set $\{f(u)+f(v)|uv \in E\}$ is $\{a, a+d, a+2d, \ldots, a+(|E|-1)d\}$. (The equivalent notion of (a,d)-*indexable* labeling was defined by Hegde in 1989 in his Ph. D. thesis–see [1210].) Similarly, Simanjuntak et al. define an (a,d)-*edge-antimagic total labeling* for a graph $G(V,E)$ as an injective mapping f from $V \cup E$ onto the set $\{1,2,\ldots,|V|+|E|\}$ such that the set $\{f(v)+f(vu)+f(v)|uv \in E\}$ where v ranges over all of V is $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. Among their results are: C_{2n} has no (a,d)-edge-antimagic vertex labeling; C_{2n+1} has a $(n+2,1)$-edge-antimagic vertex labeling and a $(n+3,1)$-edge-antimagic vertex labeling; P_{2n} has a $(n+2,1)$-edge-antimagic vertex labeling; P_n has a $(3,2)$-edge-antimagic vertex labeling; C_n has $(2n+2,1)$- and $(3n+2,1)$-edge-antimagic total labelings; C_{2n} has $(4n+2,2)$- and $(4n+3,2)$-edge-antimagic total labelings; C_{2n+1} has $(3n+4,3)$- and $(3n+5,3)$-edge-antimagic total labelings; P_{2n+1} has $(3n+4,2)$-, $(3n+4,3)$-, $(2n+4,4)$-, $(5n+4,2)$-, $(3n+5,2)$-, and $(2n+6,4)$-edge-antimagic total labelings; P_{2n} has $(6n,1)$- and $(6n+2,2)$-edge-antimagic total labelings; and several parity conditions for (a,d)-edge-antimagic total labelings. They conjecture: C_{2n} has a $(2n+3,4)$- or a $(2n+4,4)$-edge-antimagic total labeling; C_{2n+1} has a $(n+4,5)$- or a $(n+5,5)$-edge-antimagic total labeling; paths have no (a,d)-edge-antimagic vertex labelings with $d > 2$; and cycles have no (a,d)-antimagic total labelings with $d > 5$. The first and last of these conjectures were proved by Zhenbin in [3594] and the last two were verified by Bača, Lin, Miller, and Simanjuntak [352] who proved that a graph with v vertices and e edges that has an (a,d)-edge-antimagic vertex labeling must satisfy $d(e-1) \leqslant 2v-1-a \leqslant 2v-4$. As a consequence, they obtain: for every path there is no (a,d)-edge-antimagic vertex labeling with $d > 2$; for every cycle there is no (a,d)-edge-antimagic vertex labeling with $d > 1$; for K_n $(n > 1)$ there is no (a,d)-edge-antimagic vertex labeling (the cases for $n = 2$ and $n = 3$ are handled individually); $K_{n,n}$ $(n > 3)$ has no (a,d)-edge-antimagic vertex labeling; for every wheel there is no (a,d)-edge-antimagic vertex labeling; for every generalized Petersen graph there is no (a,d)-edge-antimagic vertex labeling with $d > 1$. They also study the relationship between graphs with (a,d)-edge-antimagic labelings and magic and antimagic labelings. They conjecture that every tree has an $(a,1)$-edge-antimagic total labeling.

Bača and Barrientos [309] prove that if a tree T has an α-labeling and $\{A,B\}$ is the bipartition of the vertices of T, then T also admits an $(a,1)$-edge-antimagic vertex labeling and it admits a $(3,2)$-edge-antimagic vertex labeling if and only if $||A|-|B|| \leqslant 1$.

In [352] Bača, Lin, Miller, and Simanjuntak prove: if P_n has an (a,d)-edge-antimagic total labeling, then $d \leqslant 6$; P_n has $(2n+2,1)$-, $(3n,1)$-, $(n+4,3)$-, and $(2n+2,3)$-edge-antimagic total labelings; P_{2n+1} has $(3n+4,2)$-,$(5n+4,3)$-, $(2n+4,4)$-, and $(2n+6,4)$-edge-antimagic total labelings; and P_{2n} has $(3n+3,2)$- and $(5n+1,2)$-edge-antimagic total labelings. Ngurah [2285] proved P_{2n+1} has $(4n+4,1)$-, $(6n+5,3)$-,$(4n+4,2)$-,$(4n+5,2)$-edge-antimagic total labelings and C_{2n+1} has $(4n+4,2)$- and $(4n+5,2)$-edge-antimagic total labelings. Silaban and Sugeng [2956] prove: P_n has $(n+4,4)$- and $(6,6)$-edge-antimagic total labelings; if $C_m \odot \overline{K_n}$ has an (a,d)-edge-antimagic total labeling, then $d \leqslant 5$; $C_m \odot \overline{K_n}$ has (a,d)-edge-antimagic total labelings for $m > 3, n > 1$ and $d = 2$ or 4; and $C_m \odot \overline{K_n}$ has no (a,d)-edge-antimagic total labelings for m and d and $n \equiv 1 \bmod 4$. They conjecture that P_n $(n \geqslant 3)$ has $(a,5)$-edge-antimagic total labelings. In [3093] Sugeng and Xie use adjacency methods to construct super edge magic graphs from (a,d)-edge-antimagic vertex graphs. Pushpam and Saibulla [2533] determined super (a,d)-edge antimagic total labelings for graphs derived from copies of generalized ladders, fans, generalized prisms and web graphs. Liu, Aslam, Javaid, and Raheen [1970] compute bounds of the minimum and maximum edge-weights for super (a,d)-edge-antimagic labelings on a generalized class of subdivided caterpillars. They also investigate the existence of super (a,d)-edge-antimagic total labeling for the validation of the obtained bounds.

In [389] Bača and Youssef used parity arguments to find a large number of conditions on p,q and d for which a graph with p vertices and q edges cannot have an (a,d)-edge-antimagic total labeling or vertex-antimagic total labeling. Bača and Youssef [389] made the following connection between (a,d)-edge-antimagic vertex labelings and sequential labelings: if G is a connected graph other than a tree that has an (a,d)-edge-antimagic vertex labeling, then $G+K_1$ has a sequential labeling.

In [3066] Sudarsana, Ismaimuza, Baskoro, and Assiyatun prove: for every $n \geqslant 2$, $P_n \cup P_{n+1}$ has a $(6n+1,1)$- and a $(4n+3,3)$-edge-antimagic total labeling, for every odd $n \geqslant 3$, $P_n \cup P_{n+1}$ has a $(6n,1)$- and a $(5n+1,2)$-edge-antimagic total labeling, for every $n \geqslant 2$, $nP_2 \cup P_n$ has a $(7n,1)$- and a $(6n+1,2)$-edge-antimagic total labeling. In [3063] the same authors show that $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ $(n \geqslant 2)$, and $nP_2 \cup P_{n+2}$ are super edge-magic total. They also show that under certain conditions one can construct new super edge-magic total graphs from existing ones by joining a particular vertex of the existing super edge-magic total graph to every vertex in a path or every vertex of a star and by joining one extra vertex to some vertices of the existing graph. Baskoro, Sudarsana, and Cholily [512] also provide algorithms for constructing new super edge-magic total graphs from existing ones by adding pendent vertices to the existing graph. A corollary to one of their results is that the graph obtained by attaching a fixed number of pendent edges to each vertex of a path of even length is super edge-magic. Baskoro and Cholily [510] show that the graphs obtained by attaching any numbers of pendent edges to a single vertex or a fix number of pendent edges to every vertex of the following graphs are super edge-magic total graphs: odd cycles, the generalized Petersen graphs $P(n,2)$ (n odd and at least 5), and $C_n \times P_m$ (n odd, $m \geqslant 2$).

Arumugam and Nalliah [249] proved: the friendship graph $C_3^{(n)}$ with $n \equiv 0, 8 \pmod{12}$ has no super $(a, 2)$-edge-antimagic total labeling; $C_n^{(n)}$ with $n \equiv 2 \pmod 4$ has no super $(a, 2)$-edge-antimagic total labeling; and the generalized friendship graph $F_{2,p}$ consisting of 2 cycles of various lengths, having a common vertex, and having order p where $p \geqslant 5$, has a super $(2p + 2, 1)$-edge-antimagic total labeling if and only if p is odd.

An (a, d)-edge-antimagic total labeling of $G(V, E)$ is called a *super* (a, d)*-edge-antimagic total* if the vertex labels are $\{1, 2, \ldots, |V(G)|\}$ and the edge labels are $\{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$. Bača, Baskoro, Simanjuntak, and Sugeng [322] prove the following: C_n has a super (a, d)-edge-antimagic total labeling if and only if either d is 0 or 2 and n is odd, or $d = 1$; for odd $n \geqslant 3$ and $m = 1$ or 2, the generalized Petersen graph $P(n, m)$ has a super $(11n + 3)/2, 0)$-edge-antimagic total labeling and a super $((5n + 5)/2, 2)$-edge-antimagic total labeling; for odd $n \geqslant 3$, $P(n, (n-1)/2)$ has a super $((11n + 3)/2, 0)$-edge-antimagic total labeling and a super $((5n + 5)/2, 2)$-edge-antimagic total labeling. They also prove: if $P(n, m), n \geqslant 3,\ 1 \leqslant m \leqslant \lfloor (n-1)/2 \rfloor$ is super (a, d)-edge-antimagic total, then $(a, d) = (4n + 2, 1)$ if n is even, and either $(a, d) = ((11n + 3)/2, 0)$, or $(a, d) = (4n + 2, 1)$, or $(a, d) = ((5n + 5)/2, 2)$, if n is odd; and for odd $n \geqslant 3$ and $m = 1, 2$, or $(n - 1)/2$, $P(n, m)$ has an $(a, 0)$-edge-antimagic total labeling and an $(a, 2)$-edge-antimagic total labeling. (In a personal communication MacDougall argues that "edge-magic" is a better term than "$(a, 0)$-edge-antimagic" for while the latter is technically correct, "antimagic" suggests different weights whereas "magic" emphasizes equal weights and that the edge-magic case is much more important, interesting, and fundamental rather than being just one subcase of equal value to all the others.) They conjecture that for odd $n \geqslant 9$ and $3 \leqslant m \leqslant (n-3)/2$, $P(n, m)$ has a $(a, 0)$-edge-antimagic total labeling and an $(a, 2)$-edge-antimagic total labeling. Ngurah and Baskoro [2288] have shown that for odd $n \geqslant 3$, $P(n, 1)$ and $P(n, 2)$ have $((5n + 5)/2, 2)$-edge-antimagic total labelings and when $n \geqslant 3$ and $1 \leqslant m < n/2$, $P(n, m)$ has a super $(4n + 2, 1)$-edge-antimagic total labeling. In [2289] Ngurah, Baskova, and Simanjuntak provide (a, d)-edge-antimagic total labelings for the generalized Petersen graphs $P(n, m)$ for the cases $m = 1$ or 2, odd $n \geqslant 3$, and $(a, d) = ((9n + 5)/2, 2)$.

In [3064] Sudarsana, Baskoro, Uttunggadewa, and Ismaimuza show how to construct new larger super (a, d)-edge-antimagic-total graphs from existing smaller ones.

In [2290] Ngurah, Baskoro, and Simanjuntak prove that mC_n $(n \geqslant 3)$ has an (a, d)-edge-antimagic total in the following cases: $(a, d) = (5mn/2 + 2, 1)$ where m is even; $(a, d) = (2mn+2, 2)$; $(a, d) = ((3mn+5)/2, 3)$ for m and n odd; and $(a, d) = ((mn+3), 4)$ for m and n odd; and mC_n has a super $(2mn + 2, 1)$-edge-antimagic total labeling.

Bača and Barrientos [310] have shown that mK_n has a super (a, d)-edge-antimagic total labeling if and only if (i) $d \in \{0, 2\}$, $n \in \{2, 3\}$ and $m \geqslant 3$ is odd, or (ii) $d = 1$, $n \geqslant 2$ and $m \geqslant 2$, or (iii) $d \in \{3, 5\}$, $n = 2$ and $m \geqslant 2$, or (iv) $d = 4$, $n = 2$, and $m \geqslant 3$ is odd. In [309] Bača and Barrientos proved the following: if a graph with q edges and $q+1$ vertices has an α-labeling, than it has an $(a, 1)$-edge-antimagic vertex labeling; a tree has a $(3, 2)$-edge-antimagic vertex labeling if and only if it has an α-labeling and the number of vertices in its two partite sets differ by at most 1; if a tree with at least two vertices has a super (a, d)-edge-antimagic total labeling, then d is at most 3; if a graph has an

$(a, 1)$-edge-antimagic vertex labeling, then it also has a super $(a_1, 0)$-edge-antimagic total labeling and a super $(a_2, 2)$-edge-antimagic total labeling.

Bača and Youssef [389] proved the following: if G is a connected (a, d)-edge-antimagic vertex graph that is not a tree, then $G+K_1$ is sequential; mC_n has an (a, d)-edge-antimagic vertex labeling if and only if m and n are odd and $d = 1$; an odd degree (p, q)-graph G cannot have a (a, d)-edge-antimagic total labeling if $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$, or $p \equiv 0 \pmod 4$, $q \equiv 2 \pmod 4$, and d is even; a (p, q)-graph G cannot have a super (a, d)-edge-antimagic total labeling if G has odd degree, $p \equiv 2 \pmod 4$, q is even, and d is odd, or G has even degree, $q \equiv 2 \pmod 4$, and d is even; C_n has a $(2n + 2, 3)$- and an $(n+4, 3)$-edge-antimagic total labeling; a (p, q)-graph is not super (a, d)-vertex-antimagic total if: $p \equiv 2 \pmod 4$ and d is even; $p \equiv 0 \pmod 4$, $q \equiv 2 \pmod 4$, and d is odd; $p \equiv 0 \pmod 8$ and $q \equiv 2 \pmod 4$.

In [3066] Sudarsana, Ismaimuza, Baskoro, and Assiyatun prove: for every $n \geqslant 2$, $P_n \cup P_{n+1}$ has super $(n + 4, 1)$- and $(2n + 6, 3)$-edge antimagic total labelings; for every odd $n \geqslant 3$, $P_n \cup P_{n+1}$ has super $(4n + 5, 1)$-,$(3n + 6, 2)$-, $(4n + 3, 1)$- and $(3n + 4, 2)$-edge antimagic total labelings; for every $n \geqslant 2$, $nP_2 \cup P_n$ has super $(6n+2, 1)$- and $(5n+3, 2)$-edge antimagic total labelings; and for every $n \geqslant 1$, $nP_2 \cup P_{n+2}$ has super $(6n+6, 1)$- and $(5n + 6, 2)$-edge antimagic total labelings. They pose a number of open problems about constructing (a, d)-edge antimagic labelings and super (a, d)-edge antimagic labelings for the graphs $P_n \cup P_{n+1}$, $nP_2 \cup P_n$, and $nP_2 \cup P_{n+2}$ for specific values of d.

Dafik, Miller, Ryan, and Bača [773] investigated the super edge-antimagicness of the disconnected graph mC_n and mP_n. For the first case they prove that mC_n, $m \geqslant 2$, has a super (a, d)-edge-antimagic total labeling if and only if either d is 0 or 2 and m and n are odd and at least 3, or $d = 1$, $m \geqslant 2$, and $n \geqslant 3$. For the case of the disjoint union of paths they determine all feasible values for m, n and d for mP_n to have a super (a, d)-edge-antimagic total labeling except when m is even and at least 2, $n \geqslant 2$, and d is 0 or 2. In [775] Dafik, Miller, Ryan, and Bača obtain a number of results about super edge-antimagicness of the disjoint union of two stars and state three open problems. Nalliah and Arumugam [2256] proved that $K_{1,6} \cup K_{1,5}$ does not have such a labeling and prove that some special cases of $K_{1,n+1} \cup K_{1,n}$ do have them.

Sudarsana, Hendra, Adiwijaya, and Setyawan [3065] show that the t-joint copies of wheel W_n have a super edge antimagic $((2n+2)t+2, 1)$-total labeling for $n \geqslant 4$ and $t \geqslant 2$.

In [347] Bača, Lascsáková, and Semaničová investigated the connection between graphs with α-labelings and graphs with super (a, d)-edge-antimagic total labelings. Among their results are: If G is a graph with n vertices and $n - 1$ edges $(n \geqslant 3)$ and G has an α-labeling, then mG is super (a, d)-edge-antimagic total if either d is 0 or 2 and m is odd, or $d = 1$ and n is even; if G has an α-labeling and has n vertices and $n - 1$ edges with vertex bipartition sets V_1 and V_2 where $|V_1|$ and $|V_2|$ differ by at most 1, then mG is super (a, d)–edge-antimagic total for $d = 1$ and $d = 3$. In the same paper Bača et al. prove: caterpillars with odd order at least 3 have super $(a, 1)$-edge-antimagic total labelings; if G is a caterpillar of odd order at least 3 and G has a super $(a, 1)$-edge-antimagic total labeling, then mG has a super $(b, 1)$-edge-antimagic total labeling for some b that is a function of a and m.

In [772] Dafik, Miller, Ryan, and Bača investigated the existence of antimagic labelings of disjoint unions of s-partite graphs. They proved: if $s \equiv 0$ or 1 (mod 4), $s \geqslant 4, m \geqslant 2, n \geqslant 1$ or mn is even , $m \geqslant 2,\ n \geqslant 1,\ s \geqslant 4$, then the complete s-partite graph $mK_{n,n,\dots,n}$ has no super $(a,0)$-edge-antimagic total labeling; if $m \geqslant 2$ and $n \geqslant 1$, then $mK_{n,n,n,n}$ has no super $(a,2)$-antimagic total labeling; and for $m \geqslant 2$ and $n \geqslant 1$, $mK_{n,n,n,n}$ has an $(8mn+2,1)$-edge-antimagic total labeling. They conjecture that for $m \geqslant 2,\ n \geqslant 1$ and $s \geqslant 5$, the complete s-partite graph $mK_{n,n,\dots,n}$ has a super $(a,1)$-antimagic total labeling.

In [373] Bača, Muntaner-Batle, Semaničová-Feňovčiková, and Shafiq investigate super (a,d)-edge-antimagic total labelings of disconnected graphs. Among their results are: If G is a (super) $(a,2)$-edge-antimagic total labeling and m is odd, then mG has a (super) $(a',2)$-edge-antimagic-total labeling where $a' = m(a-3)+(m+1)/2+2$; and if d a positive even integer and k a positive odd integer, G is a graph with all of its vertices having odd degree, and the order and size of G have opposite parity, then $2kG$ has no (a,d)-edge-antimagic total labeling. Bača and Brankovic [325] have obtained a number of results about the existence of super (a,d)-edge-antimagic totaling of disjoint unions of the form $mK_{n,n}$. In [329] Bača, Dafik, Miller, and Ryan provide (a,d)-edge-antimagic vertex labelings and super (a,d)-edge-antimagic total labelings for a variety of disjoint unions of caterpillars. Bača and Youssef [389] proved that mC_n has an (a,d)-edge-antimagic vertex labeling if and only if m and n are odd and $d=1$. Bača, Dafik, Miller, and Ryan [330] constructed super (a,d)-edge-antimagic total labeling for graphs of the form $m(C_n \odot \overline{K}_s)$ and $mP_n \cup kC_n$ while Dafik, Miller, Ryan, and Bača [774] do the same for graphs of the form $mK_{n,n,n}$ and $K_{1,m} \cup 2sK_{1,n}$. Both papers provide a number of open problems. In [357] Bača, Lin, and Muntaner-Batle provide super (a,d)-edge-antimagic total labeling of forests in which every component is a specific kind of tree. In [345] Bača, Kovǎr, Semaničová-Feňovčiková, and Shafiq prove that every even regular graph and every odd regular graph with a 1-factor are super $(a,1)$-edge-antimagic total and provide some constructions of non-regular super $(a,1)$-edge-antimagic total graphs. Bača, Lin, and Semaničová-Feňovčiková [359] show: the disjoint union of m graphs with super $(a,1)$-edge antimagic total labelings have super $(m(a-2)+2,1)$-edge antimagic total labelings; the disjoint union of m graphs with super $(a,3)$-edge antimagic total labelings have super $(m(a-3)+3,3)$-edge antimagic total labelings; if G has a $(a,1)$-edge antimagic total labelings then mG has an $(b,1)$-edge antimagic total labeling for some b; and if G has a $(a,3)$-edge antimagic total labelings then mG has an $(b,3)$-edge antimagic total labeling for some b.

Bača, Miller, Ryan, and Semaničová-Feňovčíková [369] prove that if G admits a (super) (a,d)-H-antimagic labeling, where $d = |E(H)| - |V(H)|$, then mG admits a (super) (b,d)-H-antimagic labelling. By considering special H-coverings of a given H-antimagic graph G they derive many corollaries. In [2749] Semaničová-Feňovčíková, Bača, and Lascsáková provide two constructions of (super) H-antimagic graphs obtained from smaller (super) H'-antimagic graphs. Dafik, Slamin, Tana, Semaničová-Feňovčíková, and Bača [778] show a connection between a constructions of H-antimagic labelings of graph and edge-antimagic total labelings and describe how to obtain the H-antimagic graph using smaller edge-antimagic graph. Bača, Semaničová-Feňovčikovǎ, Umar, and Welyyanti [384] gave sufficient conditions for $G_1 \times G_2$ to admit an H-supermagic or a super (a,d)-H-antimagic labeling but provide no examples of graphs that satisfy the given conditions.

For $t \geqslant 2$ and $n \geqslant 4$ the *Harary graph*, C_p^t, is the graph obtained by joining every two vertices of C_p that are at distance t in C_p. In [2544] Rahim, Ali, Kashif, and Javaid provide super (a,d)-edge antimagic total labelings for disjoint unions of Harary graphs and disjoint unions of cycles. In [1275] Hussain, Ali, Rahim, and Baskoro construct various (a,d)-vertex-antimagic labelings for Harary graphs and disjoint unions of identical Harary graphs. For p odd and at least 5, Balbuena, Barker, Das, Lin, Miller, Ryan, Slamin, Sugeng, and Tkac [409] give a super $((17p+5)/2)$-vertex-antimagic total labeling of C_p^t. MacDougall and Wallis [2050] have proved the following: C_{4m+3}^t, $m \geqslant 1$, has a super $(a,0)$-edge-antimagic total labeling for all possible values of t with $a = 10m+9$ or $10m+10$; C_{4m+1}^t, $m \geqslant 3$, has a super $(a,0)$-edge-antimagic total labeling for all possible values except $t = 5, 9, 4m-4$, and $4m-8$ with $a = 10m+4$ and $10m+5$; C_{4m+1}^t, $m \geqslant 1$, has a super $(10m+4,0)$-edge-antimagic total labeling for all $t \equiv 1 \pmod 4$ except $4m-3$; C_{4m}^t, $m > 1$, has a super $(10m+2,0)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$; C_{4m+2}^t, $m > 1$, has a super $(10m+7,0)$-edge-antimagic total labeling for all odd t other than 5 and for $t = 2$ or 6. In [1276] Hussain, Baskoro, and Ali prove the following: for any $p \geqslant 4$ and for any $t \geqslant 2$, C_p^t admits a super $(2p+2,1)$-edge-antimagic total labeling; for $n \geqslant 4$, $k \geqslant 2$ and $t \geqslant 2$, kC_n^t admits a super $(2nk+2,1)$-edge-antimagic total labeling; and for $p \geqslant 5$ and $t \geqslant 2$, C_p^t admits a super $(8p+3,1)$-vertex-antimagic total labeling, provided if $p \neq 2t$.

Bača and Murugan [379] have proved: if C_n^t, $n \geqslant 4, 2 \leqslant t \leqslant n-2$, is super (a,d)-edge-antimagic total, then $d = 0, 1$, or 2; for $n = 2k+1 \geqslant 5$, C_n^t has a super $(a,0)$-edge-antimagic total labeling for all possible values of t with $a = 5k+4$ or $5k+5$; for $n = 2k+1 \geqslant 5$, C_n^t has a super $(a,2)$-edge-antimagic total labeling for all possible values of t with $a = 3k+3$ or $3k+4$; for $n \equiv 0 \pmod 4$, C_n^t has a super $(5n/2+2,0)$-edge-antimagic total labeling and a super $(3n/2+2,0)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$; for $n = 10$ and $n \equiv 2 \pmod 4$, $n \geqslant 18$, C_n^t has a super $(5n/2+2,0)$-edge-antimagic total labeling and a super $(3n/2+2,0)$-edge-antimagic total labeling for all $t \equiv 3 \pmod 4$ and for $t = 2$ and 6; for odd $n \geqslant 5$, C_n^t has a super $(2n+2,1)$-edge-antimagic total labeling for all possible values of t; for even $n \geqslant 6$, C_n^t has a super $(2n+2,1)$-edge-antimagic total labeling for all odd $t \geqslant 3$; and for even $n \equiv 0 \pmod 4$, $n \geqslant 4$, C_n^t has a super $(2n+2,1)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$. They conjecture that there is a super $(2n+2,1)$-edge-antimagic total labeling of C_n^t for $n \equiv 0 \pmod 4$ and for $t \equiv 0 \pmod 4$ and for $n \equiv 2 \pmod 4$ and for t even.

In [353] Bača, Lin, Miller, and Youssef prove: if the friendship $C_3^{(n)}$ is super (a,d)-antimagic total, then $d < 3$; $C_3^{(n)}$ has an $(a,1)$-edge antimagic vertex labeling if and only if $n = 1, 3, 4, 5$, and 7; $C_3^{(n)}$ has a super (a,d)-edge-antimagic total labelings for $d = 0$ and 2; $C_3^{(n)}$ has a super $(a,1)$-edge-antimagic total labeling; if a fan F_n $(n \geqslant 2)$ has a super (a,d)-edge-antimagic total labeling, then $d < 3$; F_n has a super (a,d)-edge-antimagic total labeling if $2 \leqslant n \leqslant 6$ and $d = 0, 1$ or 2; the wheel W_n has a super (a,d)-edge-antimagic total labeling if and only if $d = 1$ and $n \not\equiv 1 \pmod 4$; K_n, $n \geqslant 3$, has a super (a,d)-edge-antimagic total labeling if and only if either $d = 0$ and $n = 3$, or $d = 1$ and $n \geqslant 3$, or

$d = 2$ and $n = 3$; and $K_{n,n}$ has a super (a,d)-edge antimagic total labeling if and only if $d = 1$ and $n \geqslant 2$.

Bača, Lin, and Muntaner-Batle [354] have shown that if a tree with at least two vertices has a super (a,d)-edge-antimagic total labeling, then d is at most three and P_n, $n \geqslant 2$, has a super (a,d)-edge-antimagic total labeling if and only if $d = 0, 1, 2$, or 3. They also characterize certain path-like graphs in a grid that have super(a,d)-edge-antimagic total labelings.

In [3085] Sugeng, Miller, and Bača prove that the ladder, $P_n \times P_2$, is super (a,d)-edge-antimagic total if n is odd and $d = 0, 1$, or 2 and $P_n \times P_2$ is super $(a,1)$-antimagic total if n is even. They conjecture that $P_n \times P_2$ is super $(a,0)$- and $(a,2)$-edge-antimagic when n is even. Sugeng, Miller, and Bača [3085] prove that $C_m \times P_2$ has a super (a,d)-edge-antimagic total labeling if and only if either $d = 0, 1$ or 2 and m is odd and at least 3, or $d = 1$ and m is even and at least 4. They conjecture that if m is even, $m \geqslant 4$, $n \geqslant 3$, and $d = 0$ or 2, then $C_m \times P_n$ has a super (a,d)-edge-antimagic total labeling. In [1831] M.-J. Lee studied super $(a,1)$-edge-antimagic properties of $m(P_4 \times P_n)$ for $m, n \geqslant 1$ and $m(C_n \odot \overline{K_t})$ for n even and $m, t \geqslant 1$. He also proved that for $n \geqslant 2$ the graph $P_4 \times P_n$ has a super $(8n + 2, 1)$-edge antimagic total labeling.

In [145] and [146] Azizu, Yulianti, Sy, Narwen proved that $(C_m \times P_2) \odot \overline{K_n}$ (branched-prism) for odd $m \geqslant 3$ and $n \geqslant 1$ admits an edge magic and a super (a,d)-edge antimagic total labelings for odd $m \geqslant 3$ and $n \geqslant 1$. [145] new [146] new

Sugeng, Miller, and Bača [3085] define a variation of a ladder, $\mathbb{L}_n$, as the graph obtained from $P_n \times P_2$ by joining each vertex u_i of one path to the vertex v_{i+1} of the other path for $i = 1, 2, \ldots, n-1$. They prove $\mathbb{L}_n$, $n \geqslant 2$, has a super (a,d)-edge-antimagic total labeling if and only if $d = 0, 1$, or 2.

In [771] Dafik, Miller, and Ryan investigate the existence of super (a,d)-edge-antimagic total labelings of $mK_{n,n,n}$ and $K_{1,m} \cup 2sK_{1,n}$. Among their results are: for $d = 0$ or 2, $mK_{n,n,n}$ has a super (a,d)-edge-antimagic total labeling if and only if $n = 1$ and m is odd and at least 3; $K_{1,m} \cup 2sK_{1,n}$ has a super (a,d)-edge-antimagic labeling for $(a,d) = (4n+5)s + 2m + 4, 0), ((2n+5)s + m + 5, 2), ((3n+5)s + (3m+9)/2, 1)$ and $(5s + 7, 4)$.

In [313] Bača, Bashir, and Semaničová showed that for $n \geqslant 4$ and $d = 0, 1, 2, 3, 4, 5$, and 6 the antiprism A_n has a super d-antimagic labeling of type $(1,1,1)$. The *generalized antiprism* A_m^n is obtained from $C_m \times P_n$ by inserting the edges $\{v_{i,j+1}, v_{i+1,j}\}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n-1$ where the subscripts are taken modulo m. Sugeng et al. prove that A_m^n, $m \geqslant 3$, $n \geqslant 2$, is super (a,d)-edge-antimagic total if and only if $d = 1$.

A *toroidal polyhex* (toroidal fullerene) is a cubic bipartite graph embedded on the torus such that each face is a hexagon. Note that the torus is a closed surface that can carry a toroidal polyhex such that all its vertices have degree 3 and all faces of the embedding are hexagons. Bača and Shabbir [386] proved the toroidal polyhex $\mathbb{H}_m^n$ with mn hexagons, $m, n \geqslant 2$, admits a super (a,d)-edge-antimagic total labeling if and only if $d = 1$ and $a = 4mn + 2$.

Bača, Miller, Phanalasy, and A. Semaničová-Feňovčíková [365] investigated the existence of (super) 1-antimagic labelings of type $(1,1,1)$ for disjoint union of plane graphs.

They prove that if a plane graph $G(V,E,F)$ has a (super) 1-antimagic labeling h of type $(1,1,1)$ such that $h(z_{ext}) = |V(G)| + |E(G)| + |F(G)|$ where z_{ext} denotes the unique external face then, for every positive integer m, the graph mG also admits a (super) 1-antimagic labeling of type $(1,1,1)$; and if a plane graph $G(V,E,F)$ has 4-sided inner faces and h is a (super) d-antimagic labeling of type $(1,1,1)$ of G such that $h(z_{ext}) = |V(G)| + |E(G)| + |F(G)|$ where $d = 1,3,5,7,9$ then, for every positive integer m, the graph mG also admits a (super) d-antimagic labeling of type $(1,1,1)$. They also give a similar result about plane graphs with inner faces that are 3-sided.

Sugeng, Miller, Slamin, and Bača [3088] proved: the star S_n has a super (a,d)-antimagic total labeling if and only if either $d = 0,1$ or 2, or $d = 3$ and $n = 1$ or 2; if a nontrivial caterpillar has a super (a,d)-edge-antimagic total labeling, then $d \leqslant 3$; all caterpillars have super $(a,0)$-, $(a,1)$- and $(a,2)$-edge-antimagic total labelings; all caterpillars have a super $(a,1)$-edge-antimagic total labeling; if m and n differ by at least 2 the double star $S_{m,n}$ (that is, the graph obtained by joining the centers of $K_{1,m}$ and $K_{1,n}$ with an edge) has no $(a,3)$-edge-antimagic total labeling.

Sugeng and Miller [3083] show how to manipulate adjacency matrices of graphs with (a,d)-edge-antimagic vertex labelings and super (a,d)-edge-antimagic total labelings to obtain new (a,d)-edge-antimagic vertex labelings and super (a,d)-edge-antimagic total labelings. Among their results are: every graph can be embedded in a connected (a,d)-edge-antimagic vertex graph; every (a,d)-edge-antimagic vertex graph has a proper (a,d)-edge-antimagic vertex subgraph; if a graph has a $(a,1)$-edge-antimagic vertex labeling and an odd number of edges, then it has a super $(a,1)$-edge-antimagic total labeling; every super edge magic total graph has an $(a,1)$-edge-antimagic vertex labeling; and every graph can be embedded in a connected super (a,d)-edge-antimagic total graph.

Rahmawati, Sugeng, Silaban, Miller, and Bača [2550] construct new larger (a,d)-edge-antimagic vertex graphs from an existing (a,d)-edge-antimagic vertex graph using adjacency matrix for difference $d = 1,2$. The results are extended for super (a,d)-edge-antimagic total graphs with differences $d = 0,1,2,3$.

Ajitha, Arumugan, and Germina [161] show that $(p,p-1)$ graphs with α-labelings (see §3.1) and partite sets with sizes that differ by at most 1 have super (a,d)-edge antimagic total labelings for $d = 0,1,2$ and 3. They also show how to generate large classes of trees with super (a,d)-edge-antimagic total labelings from smaller graceful trees.

Bača, Lin, Miller, and Ryan [351] define a *Möbius grid*, M_n^m, as the graph with vertex set $\{x_{i,j}|\ i = 1,2,\dots,m+1, j = 1,2,\dots,n\}$ and edge set $\{x_{i,j}x_{i,j+1}|\ i = 1,2,\dots,m+1, j = 1,2,\dots,n-1\} \cup \{x_{i,j}x_{i+1,j}|\ i = 1,2,\dots,m,\ j = 1,2,\dots,n\} \cup \{x_{i,n}x_{m+2-i,1}|\ i = 1,2,\dots,m+1\}$. They prove that for $n \geqslant 2$ and $m \geqslant 4$, M_n^m has no d-antimagic vertex labeling with $d \geqslant 5$ and no d-antimagic-edge labeling with $d \geqslant 9$.

Ali, Bača, and Bashir, [150] investigated super (a,d)-vertex-antimagic total labelings of the disjoint unions of paths. They prove: mP_2 has a super (a,d)-vertex-antimagic total labeling if and only if m is odd and $d = 1$; mP_3, $m > 1$, has no super $(a,3)$-vertex-antimagic total labeling; mP_3 has a super $(a,2)$-vertex-antimagic total labeling for $m \equiv 1 \pmod 6$; and mP_4 has a super $(a,2)$-vertex-antimagic total labeling for $m \equiv 3 \pmod 4$.

Lee, Tsai, and Lin [1834] denote the subdivision of a star S_n obtained by inserting m

vertices into every edge of the star S_n by S_m^n. They proved that for $n \geqslant 3$, the graph kS_m^n is super (a,d)-edge antimagic total for certain values. In [1283] Ichishima, López, Muntaner-Batle and Rius-Font proved that if G is tripartite and has a (super) (a,d)-edge antimagic total labeling, then nG ($n \geqslant 3$) has a (super) (a,d)-edge antimagic total labeling for $d = 1$ and for $d = 0, 2$ when n is odd.

Let $p, t_1, t_2, \ldots, t_k$ be integers such that $1 \leqslant t_1 < t_2 < \cdots < t_k < p$. A *Toeplitz* graph, denoted by $T_p\langle t_1 \ldots, t_k\rangle$, is a graph with vertex set $\{v_1, v_2, \ldots, v_p\}$ and edge set $\{v_iv_j : |i-j| \in \{t_1, t_2, \ldots, t_k\}\}$. Bača, Bashir, Nadeem, and Shabbir [312] give an upper bound on the difference d when a Toeplitz graph $T_p\langle t_1, t_2, \ldots t_k\rangle$ is super (a,d)-edge-antimagic total. They also construct a super $(a,1)$-edge-antimagic total labeling for an arbitrary Toeplitz graph without isolated vertices and prove that the Toeplitz graph $T_p\langle t_1\rangle$ admits a super $(a,3)$-edge-antimagic total labeling. Moreover, when p and t_1 satisfy certain conditions $T_p\langle t_1\rangle$ also admits a super (a,d)-edge-antimagic total labeling for $d = 0$ and $d = 2$. When $k = 2$ they show the existence of a super $(a,2)$-edge-antimagic total labeling for the Toeplitz graph $T_p\langle t_1, t_1+1\rangle$.

In [178] Amudha, Jayapriya, and Gowri provide an algorithmic encryption method that employs antimagic labelings of graphs.

Pandimadevi and Subbiah [2330] show the existence and nonexistence of (a,d)-vertex antimagic total labeling for several class of digraphs and show how to construct labelings for generalized de Bruijn digraphs.

In [1077] Getzimah and Palani define vertex antimagic total labeling, edge antimagic total labeling on Z_{p+q}, and discuss these labelings for cycles, stars, complete bipartite graphs, the subdivision graphs of ladders, and combs. They also investigate totally (a,d)-edge antimagic graphs, totally super vertex graphs, edge antimagic graphs, and determine the bounds for the vertices and the edges under total labelings.

The book [363] by Bača and Miller has a wealth of material and open problems on super edge-antimagic labelings. In [321] Bača, Baskoro, Miller, Ryan, Simanjuntak, and Sugeng provide detailed survey of results on edge antimagic labelings and include many conjectures and open problems. In 2015 Nalliah [2255] published a list of open problems on super (a,d)-edge antimagic total labelings of graphs. In 2017 Brankovic, Jendrol', Lin, Phanalasy, Ryan, Semaničová-Feňovčíková, Slamin, and Sugeng [314] provided a survey of recent results on face-antimagic labelings. It was dedicated to the memory of Mirka Miller, who introducted the concept of face-antimagic labeling of plane graphs in 2003.

In Tables 14, 15, 16 and 17 we use the abbreviations

(a,d)-VAT (a,d)-vertex-antimagic total labeling

(a,d)-SVAT super (a,d)-vertex-antimagic total labeling

(a,d)-EAT (a,d)-edge-antimagic total labeling

(a,d)-SEAT super (a,d)-edge-antimagic total labeling

(a,d)-EAV (a,d)-edge-antimagic vertex labeling

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 14: **Summary of (a,d)-Vertex-Antimagic Total and Super (a,d)-Vertex-Antimagic Total Labelings**

Graph	*Labeling*	*Notes*
P_n	(a,d)-VAT	wide variety of a and d [323]
P_n	(a,d)-SVAT	iff $d=3,\ d=2, n \geqslant 3$ odd or $d=3,\ n \geqslant 3$ [3086]
C_n	(a,d)-VAT	wide variety of a and d [322]
C_n	(a,d)-SVAT	iff $d=0,2$ and n odd or $d=1$ [3086]
generalized Petersen graph $P(n,k)$	(a,d)-VAT $(a,1)$-VAT	[324] $n \geqslant 3,\ 1 \leqslant k \leqslant n/2$ [3087]
prisms $C_n \times P_2$	(a,d)-VAT	[324]
antiprisms	(a,d)-VAT	[324]
$S_{n_1} \cup \ldots \cup S_{n_t}$	(a,d)-VAT	$d=1,2,3,4,6$ [2343], [2546]
W_n	not (a,d)-VAT	for $n > 20$ [1954]
$K_{1,n}$	not (a,d)-SVAT	$n \geqslant 3$ [3086]

Table 15: **Summary of (a, d)-Edge-Antimagic Total Labelings**

Graph	*Labeling*	*Notes*
trees	$(a, 1)$-EAT?	[352]
P_n	not (a, d)-EAT	$d > 2$ [352]
P_{2n}	$(6n, 1)$-EAT	[2958]
	$(6n + 2, 2)$-EAT	[2958]
P_{2n+1}	$(3n + 4, 2)$-EAT	[2958]
	$(3n + 4, 3)$-EAT	[2958]
	$(2n + 4, 4)$-EAT	[2958]
	$(5n + 4, 2)$-EAT	[2958]
	$(3n + 5, 2)$-EAT	[2958]
	$(2n + 6, 4)$-EAT	[2958]
C_n	$(2n + 2, 1)$-EAT	[2958]
	$(3n + 2, 1)$-EAT	[2958]
	not (a, d)-EAT	$d > 5$ [352]
C_{2n}	$(4n + 2, 2)$-EAT	[2958]
	$(4n + 3, 2)$-EAT	[2958]
	$(2n + 3, 4)$-EAT?	[2958]
	$(2n + 4, 4)$-EAT?	[2958]
C_{2n+1}	$(3n + 4, 3)$-EAT	[2958]
	$(3n + 5, 3)$-EAT	[2958]
	$(n + 4, 5)$-EAT?	[2958]
	$(n + 5, 5)$-EAT?	[2958]
K_n	not (a, d)-EAT	$d > 5$ [352]
$K_{n,n}$	(a, d)-EAT	iff $d = 1, n \geqslant 2$ [353]
caterpillars	(a, d)-EAT	$d \leqslant 3$ [3088]
W_n	not (a, d)-EAT	$d > 4$ [352]
generalized Petersen	not (a, d)-EAT	$d > 4$ [352]
graph $P(n, k)$	$((5n + 5)/2, 2)$-EAT	$n \geqslant 3$ odd, $k = 1, 2$ [2288]
	super $(4n + 2, 1)$-EAT	$n \geqslant 3$, $1 \leqslant k \leqslant n/2$ [2288]

Table 16: **Summary of (a,d)-Edge-Antimagic Vertex Labelings**

Graph	*Labeling*	*Notes*
P_n	$(3,2)$-EAV	[2958]
	not (a,d)-EAV	$d > 2$ [2958]
P_{2n}	$(n+2,1)$-EAV	[2958]
C_n	not (a,d)-EAV	$d > 1$ [352]
C_{2n}	not (a,d)-EAV	[2958]
C_{2n+1}	$(n+2,1)$-EAV	[2958]
	$(n+3,1)$-EAV	[2958]
K_n	not (a,d)-EAV	for $n > 1$ [352]
$K_{n,n}$	not (a,d)-EAV	for $n > 3$ [352]
W_n	not (a,d)-EAV	[352]
$C_3^{(n)}$ (friendship graph)	$(a,1)$-EAV	iff $n = 1,3,4,5,7$ [353]
generalized Petersen graph $P(n,k)$	not (a,d)-EAV	$d > 1$ [352]

Table 17: **Summary of (a, d)-Super-Edge-Antimagic Total Labelings**

Graph	*Labeling*	*Notes*
$C_n \odot K_1$	(a,d)-SEAT	variety of cases [302], [379]
$P_n \times P_2$ (ladders)	(a,d)-SEAT	n odd, $d \leqslant 2$ [3085]
		n even, $d = 1$ [3085]
	(a,d)-SEAT?	$d = 0, 2,\ n$ even [3085]
$C_n \times P_2$	(a,d)-SEAT	iff $d \leqslant 3$ n odd [3085]
		or $d = 1,\ n \geqslant 4$ even [3085]
$C_m \times P_n$	(a,d)-SEAT?	$m \geqslant 4$ even,
		$n \geqslant 3,\ d = 0, 2$ [3085]
caterpillars	$(a,1)$-SEAT	[3088]
$C_3^{(n)}$ (friendship graphs)	(a,d)-SEAT	$d = 0, 1, 2$ [353]
F_n $(n \geqslant 2)$ (fans)	(a,d) SEAT	only if $d < 3$ [353]
	(a,d)-SEAT	$2 \leqslant n \leqslant 6, d = 0, 1, 2$ [353]
W_n	(a,d)-SEAT	iff $d = 1, n \not\equiv 1 \pmod 4$ [353]
K_n $(n \geqslant 3)$	(a,d) SEAT	iff $d = 0, n = 3$ [353]
		$d = 1, n \geqslant 3$ [353]
		$d = 2, n = 3$ [353]
trees	(a,d)-SEAT	only if $d \leqslant 3$ [354]
P_n $(n > 1)$	(a,d)-SEAT	iff $d \leqslant 3$ [354]
mK_n	(a,d)-SEAT	iff $d \in \{0, 2\}, n \in \{2, 3\}$,
		$m \geqslant 3$ odd [310]
		$d = 1, m, n \geqslant 2$ [310]
		$d = 3$ or 5,$n = 2, m \geqslant 2$ [310]
		$d = 4, n = 2,\ m \geqslant 3$ odd [310]
C_n	(a,d)-SEAT	iff $d = 0$ or 2, n odd [354]
		$d = 1$ [322]
$P(m,n)$	(a,d)-SEAT	many cases [322]

6.4 Face Antimagic Labelings and d-antimagic Labeling of Type (1,1,1)

Bača [304] defines a connected plane graph G with edge set E and face set F to be (a,d)-*face antimagic* if there exist positive integers a and d and a bijection $g\colon E \to \{1,2,\ldots,|E|\}$ such that the induced mapping $\psi_g\colon F \to \{a, a+d, \ldots, a+(|F(G)|-1)d\}$, where for a face f, $\psi_g(f)$ is the sum of all $g(e)$ for all edges e surrounding f is also a bijection. In [306] Bača proves that for n even and at least 4, the prism $C_n \times P_2$ is $(6n+3,2)$-face antimagic and $(4n+4,4)$-face antimagic. He also conjectures that $C_n \times P_2$ is $(2n+5,6)$-face antimagic. In [349] Bača, Lin, and Miller investigate (a,d)-face antimagic labelings of the convex polytopes $P_{m+1} \times C_n$. They show that if these graphs are (a,d)-face antimagic then either $d=2$ and $a=3n(m+1)+3$, or $d=4$ and $a=2n(m+1)+4$, or $d=6$ and $a=n(m+1)+5$. They also prove that if n is even, $n \geqslant 4$ and $m \equiv 1 \pmod 4$, $m \geqslant 3$, then $P_{m+1} \times C_n$ has a $(3n(m+1)+3,2)$-face antimagic labeling and if n is at least 4 and even and m is at least 3 and odd, or if $n \equiv 2 \pmod 4$, $n \geqslant 6$ and m is even, $m \geqslant 4$, then $P_{m+1} \times C_n$ has a $(3n(m+1)+3,2)$-face antimagic labeling and a $(2n(m+1)+4,4)$-face antimagic labeling. They conjecture that $P_{m+1} \times C_n$ has $(3n(m+1)+3,2)$- and $(2n(m+1)+4,4)$-face antimagic labelings when $m \equiv 0 \pmod 4$, $n \geqslant 4$, and for m even and $m \geqslant 4$, that $P_{m+1} \times C_n$ has a $(n(m+1)+5,6)$-face antimagic labeling when n is even and at least 4. Bača, Baskoro, Jendroľ, and Miller [318] proved that graphs in the shape of hexagonal honeycombs with m rows, n columns, and mn 6-sided faces have d-antimagic labelings of type $(1,1,1)$ for $d = 1,2,3$, and 4 when n odd and $mn > 1$.

In [361] Bača and Miller define the class Q_n^m of convex polytopes with vertex set $\{y_{j,i} : i = 1,2,\ldots,n; j = 1,2,\ldots,m+1\}$ and edge set $\{y_{j,i}y_{j,i+1} : i = 1,2,\ldots,n; j = 1,2,\ldots,m+1\} \cup \{y_{j,i}y_{j+1,i} : i = 1,2,\ldots,n; j = 1,2,\ldots,m\} \cup \{y_{j,i+1}y_{j+1,i} : 1 + 1,2,\ldots,n; j = 1,2,\ldots,m, j \text{ odd}\} \cup \{y_{j,i}y_{j+1,i+1} : i = 1,2,\ldots,n; j = 1,2,\ldots,m, j \text{ even}\}$ where $y_{j,n+1} = y_{j,1}$. They prove that for m odd, $m \geqslant 3, n \geqslant 3$, Q_n^m is $(7n(m+1)/2+2,1)$-face antimagic and when m and n are even, $m \geqslant 4, n \geqslant 4$, Q_n^m is $(7n(m+1)/2+2,1)$-face antimagic. They conjecture that when n is odd, $n \geqslant 3$, and m is even, then Q_n^m is $((5n(m+1)+5)/2,2)-$face antimagic and $((n(m+1)+7)/2,4)$-face antimagic. They further conjecture that when n is even, $n > 4, m > 1$ or n is odd, $n > 3$ and m is odd, $m > 1$, then Q_n^m is $(3n(m+1)/2+3,3)$-face antimagic. In [308] Bača proves that for the case $m = 1$ and $n \geqslant 3$ the only possibilities for (a,d)-antimagic labelings for Q_n^m are $(7n+2,1)$ and $(3n+3,3)$. He provides the labelings for the first case and conjectures that they exist for the second case. Bača [304] and Bača and Miller [360] describe (a,d)-face antimagic labelings for a certain classes of convex polytopes.

In [317] Bača et al. provide a detailed survey of results on face antimagic labelings and include many conjectures and open problems.

For a plane graph G, Bača and Miller [362] call a bijection h from $V(G) \cup E(G) \cup F(G)$ to $\{1,2,\ldots,|V(G)|+|E(G)| \cup |F(G)|\}$ a *d-antimagic labeling of type* $(1,1,1)$ if for every number s the set of s-sided face weights is $W_s = \{a_s, a_s+d, a_s+2d, \ldots, a_s+(f_s-1)d\}$ for some integers a_s and d, where f_s is the number of s-sided faces (W_s varies with s). They show that the prisms $C_n \times P_2$ $(n \geqslant 3)$ have a 1-antimagic labeling of type $(1,1,1)$ and that for $n \equiv 3 \pmod 4$, $C_n \times P_2$ have a d-antimagic labeling of type $(1,1,1)$ for $d = 2,3,4$, and 6. They conjecture that for all $n \geqslant 3$, $C_n \times P_2$ has a d-antimagic labeling of type

$(1,1,1)$ for $d = 2,3,4,5$, and 6. This conjecture has been proved for the case $d = 3$ and $n \neq 4$ by Bača, Miller, and Ryan [368] (the case $d = 3$ and $n = 4$ is open). The cases for $d = 2,4,5$, and 6 were done by Lin, Slamin, Bača, and Miller [1955]. Bača, Lin, and Miller [350] prove: for $m,n > 8$, $P_m \times P_n$ has no d-antimagic edge labeling of type $(1,1,1)$ with $d \geqslant 9$; for $m \geqslant 2, n \geqslant 2$, and $(m,n) \neq (2,2)$, $P_m \times P_n$ has d-antimagic labelings of type $(1,1,1)$ for $d = 1,2,3,4$, and 6. They conjecture the same is true for $d = 5$. Butt, Numan, Shah, and Ali [634] prove that the generalized prims $C_n \times P_m$ have d-antimagic face labelings of type (1,1,1) for $n \geqslant 5$ and $m \geqslant 2$.

Bača, Miller, and Ryan [368] also prove that for $n \geqslant 4$ the antiprism (see §6.1 for the definition) on $2n$ vertices has a d-antimagic labeling of type $(1,1,1)$ for $d = 1,2$, and 4. They conjecture the result holds for $d = 3,5$, and 6 as well. Lin, Ahmad, Miller, Sugeng, and Bača [1952] did the cases that $d = 7$ for $n \geqslant 3$ and $d = 12$ for $n \geqslant 11$. Sugeng, Miller, Lin, and Bača [3087] did the cases: $d = 7,8,9,10$ for $n \geqslant 5$; $d = 15$ for $n \geqslant 6$; $d = 18$ for $n \geqslant 7$; $d = 12,14,17,20,21,24,27,30,36$ for n odd and $n \geqslant 7$; and $d = 16,26$ for n odd and $n \geqslant 9$.

Baca, Numan, and Semaničová-Feňovčíková [376] investigated the problem of labeling the vertices, edges, and faces of a disjoint union of r copies $C_n \times P_m$ by the consecutive integers starting from 1 in such a way that the sum of the labels of a face and the labels of vertices and edges surrounding that face for all s-sided faces form an arithmetic progression with common difference d.

Ali, Bača, Bashir, and Semaničová-Feňovčíková [152] investigated antimagic labelings for disjoint unions of prisms and cycles. They prove: for $m \geqslant 2$ and $n \geqslant 3$, $m(C_n \times P_2)$ has no super d-antimagic labeling of type $(1,1,1)$ with $d \geqslant 30$; for $m \geqslant 2$ and $n \geqslant 3$, $n \neq 4$, $m(C_n \times P_2)$ has super d-antimagic labeling of type $(1,1,1)$ for $d = 0,1,2,3,4$, and 5; and for $m \geqslant 2$ and $n \geqslant 3$, mC_n has $(m(n+1)+3,3)$- and $(2mn+2,2)$-vertex-antimagic total labeling. Bača and Bashir [311] proved that for $m \geqslant 2$ and $n \geqslant 3$, $n \neq 4$, $m(C_n \times P_2)$ has super 7-antimagic labeling of type $(1,1,1)$ and for $n \geqslant 3$, $n \neq 4$ and $2 \leqslant m \leqslant 2n$ $m(C_n \times P_2)$ has super 6-antimagic labeling of type $(1,1,1)$.

Bača, Numan and Siddiqui [378] investigated the existence of the super d-antimagic labeling of type $(1,1,1)$ for the disjoint union of m copies of antiprism mA_n. They proved that for $m \geqslant 2$, $n \geqslant 4$, mA_n has super d-antimagic labelings of type $(1,1,1)$ for $d = 1,2,3,5,6$. Ahmad, Bača, Lascsáková, and Semaničová-Feňovčíková [93] investigated super d-antimagicness of type $(1,1,0)$ for mG in a more general sense. They prove: if there exists a super 0-antimagic labeling of type $(1,1,0)$ of a plane graph G then, for every positive integer m, the graph mG also admits a super 0-antimagic labeling of type $(1,1,0)$; if a plane graph G with 3-sided inner faces admits a super d-antimagic labeling of type $(1,1,0)$ for $d = 0,6$ then, for every positive integer m, the graph mG also admits a super d-antimagic labeling of type $(1,1,0)$; if a plane graph G with 3-sided inner faces is a tripartite graph with a super d-antimagic labeling of type $(1,1,0)$ for $d = 2,4$ then, for every positive integer m, the graph mG also admits a super d-antimagic labeling of type $(1,1,0)$; if a plane graph G with 4-sided inner faces admits a super d-antimagic labeling of type $(1,1,0)$ for $d = 0,4,8$ then the disjoint union of arbitrary number of copies of G also admits a super d-antimagic labeling of type $(1,1,0)$; if a plane graph G with k-sided inner faces, $k \geqslant 3$, admits a super d-antimagic labeling of type $(1,1,0)$ for $d = 0, 2k$ then, for every positive integer m, the graph mG also admits a super d-antimagic labeling of type $(1,1,0)$; if a plane graph G with k-sided inner faces admits a super k-antimagic labeling of type $(1,1,0)$ for k even then, for every positive integer m, the graph mG also admits a super k-antimagic labeling of type $(1,1,0)$.

Bača, Jendraľ, Miller, and Ryan [340] prove: for n even, $n \geqslant 6$, the generalized Petersen graph $P(n,2)$ has a 1-antimagic labeling of type $(1,1,1)$; for n even, $n \geqslant 6$, $n \neq 10$, and $d = 2$ or 3, $P(n,2)$ has a d-antimagic labeling of type $(1,1,1)$; and for $n \equiv 0 \pmod 4$, $n \geqslant 8$ and $d = 6$ or 9, $P(n,2)$ has a d-antimagic labeling of type $(1,1,1)$. They conjecture that there is an d-antimagic labeling of type (1,1,1) for $P(n,2)$ when $n \equiv 2 \pmod 4$, $n \geqslant 6$, and $d = 6$ or 9.

In [327] Bača, Brankovic, and A. Semaničová-Feňovčikovă provide super d-antimagic labelings of type (1,1,1) for friendship graphs F_n ($n \geqslant 2$) and several other families of planar graphs.

Bača, Brankovic, Lascsáková, Phanalasy, and Semaničová-Feňovčíková [326] provided super d-antimagic labeling of type $(1,1,0)$ for friendship graphs F_n, $n \geqslant 2$, for $d \in \{1,3,5,7,9,11,13\}$. Moreover, they show that for $n \equiv 1 \pmod 2$ the graph F_n also admits a super d-antimagic labeling of type $(1,1,0)$ for $d \in \{0,2,4,6,8,10\}$.

Bača, Baskoro, and Miller [319] have proved that hexagonal planar honeycomb graphs with an even number of columns have 2-antimagic and 4-antimagic labelings of type $(1,1,1)$. They conjecture that these honeycombs also have d-antimagic labelings of type $(1,1,1)$ for $d = 3$ and 5. They pose the odd number of columns case for $1 \leqslant d \leqslant 5$ as an open problem. Bača, Baskoro, and Miller [320] give d-antimagic labelings of a special class of plane graphs with 3-sided internal faces for $d = 0$, 2, and 4. Bača, Lin, Miller, and Ryan [351] prove for odd $n \geqslant 3$, $m \geqslant 1$ and $d = 0,1,2$ or 4, the Möbius grid M_n^m has an d-antimagic labeling of type $(1,1,1)$. Siddiqui, Numan, and Umar [2954] examined the existence of super d-antimagic labelings of type (1,1,1) for Jahangir graphs for certain differences d.

Bača, Numan, and Shabbir [377] studied the existence of super d-antimagic labelings of type $(1,1,1)$ for the toroidal polyhex $\mathbb{H}_m^n$. They labeled the edges of a 1-factor by consecutive integers and then in successive steps they labeled the edges of $2m$-cycles (respectively $2n$-cycles) in a 2-factor by consecutive integers. This technique allowed them to construct super d-antimagic labelings of type $(1,1,1)$ for $\mathbb{H}_m^n$ with $d = 1,3,5$. They suppose that such labelings exist also for $d = 0,2,4$.

Kathiresan and Ganesan [1670] define a class of plane graphs denoted by P_a^b ($a \geqslant 3, b \geqslant 2$) as the graph obtained by starting with vertices $v_1, v_2, \ldots, v_a$ and for each $i = 1,2\ldots,a-1$ joining v_i and v_{i+1} with b internally disjoint paths of length $i+1$. They prove that P_a^b has d-antimagic labelings of type $(1,1,1)$ for $d = 0,1,2,3,4$, and 6. Lin and Sugen [1956] prove that P_a^b has a d-antimagic labeling of type $(1,1,1)$ for $d = 5, 7a-2, a+1, a-3, a-7, a+5, a-4, a+2, 2a-3, 2a-1, a-1, 3a-3, a+3, 2a+1, 2a+3, 3a+1, 4a-1, 4a-3, 5a-3, 3a-1, 6a-5, 6a-7, 7a-7$, and $5a-5$. Similarly, Bača, Baskoro, and Cholily [316] define a class of plane graphs denoted by C_a^b as the graph obtained by starting with vertices $v_1, v_2, \ldots, v_a$ and for each $i = 1,2\ldots,a$ joining v_i and v_{i+1} with b

internally disjoint paths of length $i+1$ (subscripts are taken modulo a). In [316] and [315] they prove that for $a \geqslant 3$ and $b \geqslant 2$, C_a^b has a d-antimagic labeling of type $(1,1,1)$ for $d = 0, 1, 2, 3, a+1, a-1, a+2$, and $a-2$.

In [328] Bača, Brankovic, and Semaničová-Feňovčiková investigated the existence of super d-antimagic labelings of type (1,1,1) for plane graphs containing a special kind of Hamilton path. They proved: if there exists a Hamilton path in a plane graph G such that for every face except the external face, the Hamilton path contains all but one of the edges surrounding that face, then G is super d-antimagic of type (1,1,1) for $d = 0, 1, 2, 3, 5$; if there exists a Hamilton path in a plane graph G such that for every face except the external face, the Hamilton path contains all but one of the edges surrounding that face and if $2(|F(G)|-1) \leqslant |V(G)|$, then G is super d-antimagic of type $(1,1,1)$ for $d = 0, 1, 2, 3, 4, 5, 6$; if G is a plane graph with $M = \lfloor \frac{|V(G)|}{|F(G)|-1} \rfloor$ and a Hamilton path such that for every face, except the external face, the Hamilton path contains all but one of the edges surrounding that face, then for $M = 1$, G admits a super d-antimagic labeling of type (1,1,1) for $d = 0, 1, 2, 3, 5$; and for $M \geqslant 2$, G admits a super d-antimagic labeling of type (1,1,1) for $d = 0, 1, 2, 3, \ldots, M+4$. They also proved that $P_n \times P_2$ $(n \geqslant 3)$ admits a super d-antimagic labeling of type (1,1,1) for $d \in \{0, 1, 2, \ldots, 15\}$ and the graph obtained from $P_n \times P_m$ $(n \geqslant 2)$ by adding a new edge in every 4-sided face such that the added edges are "parallel" admits a super d-antimagic labeling of type (1,1,1) for $d \in \{0, 1, 2, \ldots, 9\}$.

In [1320] Imran, Siddiqui, and Numan examine the existence of super d-antimagic labelings of type (1,1,1) for uniform subdivision of wheel for certain differences d.

In the following tables we use the abbreviations

(a, d)-FA (a, d)-face antimagic labeling

d-AT(1,1,1) d-antimagic labeling of type $(1, 1, 1)$.

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 18: **Summary of Face Antimagic Labelings**

Graph	*Labeling*	*Notes*
Q_n^m (see §6.4)	$(7n(m+1)/2+2,1)$-FA	$m \geqslant 3$, $n \geqslant 3$, m odd [361]
	$(7n(m+1)/2+2,1)$-FA	$m \geqslant 4$, $n \geqslant 4$, m, n even [361]
	$((5n(m+1)+5)/2,2)$-FA?	$m \geqslant 2$, $n \geqslant 3$, m even, n odd [361]
	$((n(m+1)+7)/2,4)$-FA?	$m \geqslant 2$, $n \geqslant 3$, m even, n odd [361]
	$(3n(m+1)/2+3,3)$-FA?	$m > 1$, $n > 4$, n even [361]
	$(3n(m+1)/2+3,3)$-FA?	$m > 1$, $n > 3$, m odd, n odd [361]
$C_n \times P_2$	$(6n+3,2)$-FA	$n \geqslant 4$, n even [306]
	$(4n+4,4)$-FA	$n \geqslant 4$, n even [306]
	$(2n+5,6)$-FA?	[306]
$P_{m+1} \times C_n$	$(3n(m+1)+3,2)$-FA	$n \geqslant 4$, n even and [349] $m \geqslant 3$, $m \equiv 1 \pmod 4$,
	$(3n(m+1)+3,2)$-FA and $(2n(m+1)+4,4)$-FA	$n \geqslant 4$, n even and [349] $m \geqslant 3$, m odd [349], or $n \geqslant 6$, $n \equiv 2 \pmod 4$ and $m \geqslant 4$, m even
	$(3n(m+1)+3,2)$-FA?	$m \geqslant 4$, $n \geqslant 4$, $m \equiv 0 \pmod 4$ [349]
	$(2n(m+1)+4,4)$-FA?	$m \geqslant 4$, $n \geqslant 4$, $m \equiv 0 \pmod 4$ [349]
	$(n(m+1)+5,6)$-FA?	$n \geqslant 4$, n even [349]

Table 19: **Summary of d-antimagic Labelings of Type (1,1,1)**

Graph	*Labeling*	*Notes*
$P_m \times P_n$	not d-AT(1,1,1)	$m, n, d \geqslant 9$, [350]
$P_m \times P_n$	d-AT(1,1,1)	$d = 1, 2, 3, 4, 6$; $m, n \geqslant 2$, $(m, n) \neq (2, 2)$ [350]
$P_m \times P_n$	5-AT(1,1,1)	$m, n \geqslant 2$, $(m, n) \neq (2, 2)$ [350]
$C_n \times P_2$	1-AT(1,1,1)	[362]
	d-AT(1,1,1)	$d = 2, 3, 4$ and 6 [362] for $n \equiv 3 \pmod 4$
	d-AT(1,1,1)	$d = 2, 4, 5, 6$ for $n \geqslant 3$ [1955]
	d-AT(1,1,1)	$d = 3$ for $n \geqslant 5$ [368]

Continued on next page

Table 19 – *Continued from previous page*

Graph	*Labeling*	*Notes*
$P_m \times P_n$	5-AT(1,1,1)? not d-AT	[1955] $m, n > 8,\ d \geqslant 9$ [1955]
antiprism on $2n$ vertices	d-AT(1,1,1) d-AT(1,1,1)?	$d = 1, 2$ and 4 for $n \geqslant 4$ [368] $d = 3, 5$ and 6 for $n \geqslant 4$ [368]
M_n^m (Möbius grids)	d-AT(1,1,1)	$n \geqslant 3$ odd, $d = 0, 1, 2, 4$ [351] $d = 7,\ n \geqslant 3$ [1952] $d = 12,\ n \geqslant 11$ [1952] $d = 7, 8, 9, 10,\ n \geqslant 5$ [3087] $d = 15,\ n \geqslant 6$ [3087] $d = 18\ n \geqslant 7$ [3087]
$P(n, 2)$	d-AT(1,1,1)	$d = 1;\ d = 2, 3,\ n \geqslant 6,$ $n \neq 10$ [340]
$P(4n, 2)$	d-AT(1,1,1)	$d = 6, 9,\ n \geqslant 2,\ n \neq 10$ [340]
$P(4n + 2, 2)$	d-AT(1,1,1)?	$d = 6, 9,\ n \geqslant 1,\ n \neq 10$ [340]
honeycomb graphs with even number of columns	d-AT(1,1,1) d-AT(1,1,1)?	$d = 2, 4$ [319] $d = 3, 5$ [319]
$C_n \times P_2$	d-AT(1,1,1)	$d = 1, 2, 4, 5, 6$ [1955], [362]
$C_n \times P_2$	3-AT(1,1,1)	$n \neq 4$ [368]

6.5 Product Antimagic Labelings

Figueroa-Centeno, Ichishima, and Muntaner-Batle [931] have introduced multiplicative analogs of magic and antimagic labelings. They define a graph G of size q to be *product magic* if there is a labeling from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that, at each vertex v, the product of the labels on the edges incident with v is the same. They call a graph G of size q *product antimagic* if there is a labeling f from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that the products of the labels on the edges incident at each vertex v are distinct. They prove: a graph of size q is product magic if and only if $q \leqslant 1$ (that is, if and only if it is $K_2, \overline{K_n}$ or $K_2 \cup \overline{K_n}$); P_n $(n \geqslant 4)$ is product antimagic; every 2-regular graph is product antimagic;

and, if G is product antimagic, then so are $G + K_1$ and $G \odot \overline{K}_n$. They conjecture that a connected graph of size q is product antimagic if and only if $q \geqslant 3$. Kaplan, Lev, and Roditty [1651] proved the following graphs are product antimagic: the disjoint union of cycles and paths where each path has least three edges; connected graphs with n vertices and m edges where $m \geqslant 4n\ln n$; graphs $G = (V, E)$ where each component has at least two edges and the minimum degree of G is at least $8\sqrt{\ln |E| \ln (\ln |E|)}$; all complete k-partite graphs except K_2 and $K_{1,2}$; and $G \odot H$ where G has no isolated vertices and H is regular. Wang and Gao [3429] show that caterpillars with at least three edges are product antimagic by an $O(m \log m)$ algorithm.

In [2399] Pikhurko characterizes all large graphs that are product antimagic graphs. More precisely, it is shown that there is an n_0 such that a graph with $n \geqslant n_0$ vertices is product antimagic if and only if it does not belong to any of the following four classes: graphs that have at least one isolated edge; graphs that have at least two isolated vertices; unions of vertex-disjoint of copies of $K_{1,2}$; graphs consisting of one isolated vertex; and graphs obtained by subdividing some edges of the star $K_{1,k+l}$.

In [931] Figueroa-Centeno, Ichishima, and Muntaner-Batle also define a graph G with p vertices and q edges to be *product edge-magic* if there is a labeling f from $V(G) \cup E(G)$ onto $\{1, 2, \dots, p+q\}$ such that $f(u) \cdot f(v) \cdot f(uv)$ is a constant for all edges uv and *product edge-antimagic* if there is a labeling f from $V(G) \cup E(G)$ onto $\{1, 2, \dots, p+q\}$ such that for all edges uv the products $f(u) \cdot f(v) \cdot f(uv)$ are distinct. They prove $K_2 \cup \overline{K}_n$ is product edge-magic, a graph of size q without isolated vertices is product edge-magic if and only if $q \leqslant 1$ and every graph other than K_2 and $K_2 \cup \overline{K}_n$ is product edge-antimagic.

7 Miscellaneous Labelings

7.1 Sum Graphs

In 1990, Harary [1183] introduced the notion of a sum graph. A graph $G(V, E)$ is called a *sum graph* if there is an bijection f from V to a set of positive integers S such that $xy \in E$ if and only if $f(x) + f(y) \in S$. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. In 1991 Harary, Hentzel, and Jacobs [1185] defined a *real sum graph* in an analogous way by allowing S to be any finite set of positive real numbers. However, they proved that every real sum graph is a sum graph. Bergstrand, Hodges, Jennings, Kuklinski, Wiener, and Harary [540] defined a *product graph* analogous to a sum graph except that 1 is not permitted to belong to S. They proved that every product graph is a sum graph and vice versa.

For a connected graph G, let $\sigma(G)$, the *sum number of* G, denote the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph (some authors use $s(G)$ for the sum number of G). A labeling that makes G together with $\sigma(G)$ isolated points a sum graph is called an *optimal sum graph* labeling. Ellingham [869] proved the conjecture of Harary [1183] that $\sigma(T) = 1$ for every tree $T \neq K_1$. Smyth [3009] proved that there is no graph G with e edges and $\sigma(G) = 1$ when $n^2/4 < e \leqslant n(n-1)/2$. Smyth [3010] conjectures that the disjoint union of graphs with sum number 1 has sum number 1. More generally, Kratochvil, Miller, and Nguyen [1753] conjecture that $\sigma(G \cup H) \leqslant \sigma(G) + \sigma(H) - 1$. Hao [1177] has shown that if $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ is the degree sequence of a graph G, then $\sigma(G) > \max(d_i - i)$ where the maximum is taken over all i. Bergstand et al. [539] proved that $\sigma(K_n) = 2n - 3$. Hartsfield and Smyth [1192] claimed to have proved that $\sigma(K_{m,n}) = \lceil 3m + n - 3\rceil/2$ when $n \geqslant m$ but Yan and Liu [3512] found counterexamples to this assertion when $m \neq n$. Pyatkin [2535], Liaw, Kuo, and Chang [1946], Wang, and Liu [3455], and He, Shen, Wang, Chang, Kang, and Yu [1201] have shown that for $2 \leqslant m \leqslant n$, $\sigma(K_{m,n}) = \lceil \frac{n}{p} + \frac{(p+1)(m-1)}{2}\rceil$ where $p = \lceil\sqrt{\frac{2n}{m-1} + \frac{1}{4}} - \frac{1}{2}\rceil$ is the unique integer such that $\frac{(p-1)p(m-1)}{2} < n \leqslant \frac{(p+1)p(m-1)}{2}$.

Miller, Ryan, Slamin, and Smyth [2150] proved that $\sigma(W_n) = \frac{n}{2} + 2$ for n even and $\sigma(W_n) = n$ for $n \geqslant 5$ and n odd (see also [3168]). Miller, Ryan, and Smyth [2152] prove that the complete n-partite graph on n sets of 2 nonadjacent vertices has sum number $4n - 5$ and obtain upper and lower bounds on the complete n-partite graph on n sets of m nonadjacent vertices. Fernau, Ryan, and Sugeng [927] proved that the generalized friendship graphs $C_n^{(t)}$ (see §2.2) has sum number 2 except for C_4. Gould and Rödl [1141] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [1129] proved that $\sigma(K_n - \text{edge}) \leqslant 2n - 4$. The same group of six students also investigated the difference between the largest and smallest labels in a sum graph, which they called the *spum*. They proved spum of K_n is $4n - 6$ and the spum of C_n is at most $4n - 10$. Kratochvil, Miller, and Nguyen [1753] have proved that every sum graph on n vertices has a sum labeling such that every label is at most 4^n. Konečný, Kučera, Novotná, Pekárek, Šimsa, and Töpfer [1727] showed that if one allows for non-injective labelings or graphs with loops then there are sum graphs without a minimal sum labeling, which partially answers the question posed by Miller, Ryan and Smyth in [2152]. At a conference in 2000 Miller [2138] posed the following two problems: Given any graph G, does there exist an optimal sum graph labeling that uses the label 1; Find a class of graphs G that have sum number of the order $|V(G)|^s$ for $s > 1$. (Such graphs were shown to exist for $s = 2$ by Gould and Rödl in [1141]). Fernau and Gajjar [924] provided a complete characterization of the sum number of graphs of maximum degree two. As an immediate corollary they have that all graphs that are the disjoint unions of paths and cycles are sum labeled graphs. [924] new

Elizeche and Tripathi [3228] characterized sum and integral sum labelings of complete graphs, symmetric complete bipartite graphs, star graphs, and deduced the spum, integral spum, and integral radius for these classes of graphs.

In [2995] Slamet, Sugeng, and Miller show how one can use sum graph labelings to distribute secret information to set of people so that only authorized subsets can reconstruct the secret.

Chang [673] generalized the notion of sum graph by permitting $x = y$ in the definition of sum graph. He calls graphs that have this kind of labeling *strong sum graphs* and uses $i^*(G)$ to denote the minimum positive integer m such that $G \cup mK_1$ is a strong sum graph. Chang proves that $i^*(K_n) = \sigma(K_n)$ for $n = 2, 3$, and 4 and $i^*(K_n) > \sigma(K_n)$ for $n \geqslant 5$. He further shows that for $n \geqslant 5$, $3n^{\log_2 3} > i^*(K_n) \geqslant 12\lfloor n/5 \rfloor - 3$.

In [924] Fernau and Gajjar initiate the study of sum graphs from the viewpoint of computational complexity. They show that every n-vertex, m-edge, d-degenerate graph (a graph is d-degenerate if all its subgraphs have a vertex of degree at most d) can be made a sum graph by adding at most m isolated vertices to it, such that the largest numbers used as vertex labels grows as $\Theta(n^2 d)$. As a consequence, we have that such a graph can be stored using $\Theta(m \log n)$ bits of memory. The previously best known upper bound on the numbers needed for labeling general graphs with the minimum number of isolated vertices was $\Theta(4n)$, due to Kratochvíl, Miller and Nguyen [1753] in 2001. Moreover, Fernau and Gajjar's labeling can be constructed in polynomial time. Their results show the ever-expanding database that is gradually built up can be efficiently implemented and then accessed using their sum-labeling scheme. [924] new

In 1994 Harary [1184] generalized sum graphs by permitting S to be any set of integers. He calls these graphs *integral sum graphs*. Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [2867] has shown that C_n and W_n are integral sum graphs for all $n \neq 4$. Chen [695] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum graphs. In [695] and [697] Chen gives methods for constructing new connected integral sum graphs from given integral sum graphs by identifying vertices. Chen [697] has shown that every graph is an induced subgraph of a connected integral sum graph. Chen [697] calls a vertex of a graph *saturated* if it is adjacent to every other vertex of the graph. He proves that every integral sum graph except K_3 has at most two saturated vertices and gives the exact structure of all integral sum graphs that have exactly two saturated vertices. Chen [697] also proves that a connected integral sum graph with $p > 1$ vertices and q edges and no saturated vertices satisfies $q \leqslant p(3p-2)/8-2$. Wu, Mao, and Le [3486] proved that mP_n are integral sum graphs. They also show that the conjecture of Harary [1184] that the sum number of C_n equals the integral sum number of C_n if and only if $n \neq 3$ or 5 is false and that for $n \neq 4$ or 6 the integral sum number of C_n is at most 1. Vilfred and Nicholas [3385] prove that graphs G of order n with $\Delta(G) = n-1$ and $|V_\Delta(G)| > 2$ are not integral sum graphs, except K_3, and that integral sum graphs G of order n with $\Delta(G) = n-1$ and $|V_\Delta(G)| = 2$ exist and are unique up to isomorphism. Chen [699] proved that if $G(V, E)$ is an integral sum other than K_3 that has vertex of degree $|V|-1$, then the edge-chromatic number of G is $|V|-1$.

In 2021 Singla, Tiwari, and Tripathi [2965] were the first to investigated the spum and integral spum (denoted ispum(G). They provided results on complete graphs, symmetric complete bipartite graphs, star graphs, cycles, and paths and gave sharp lower bounds for the spum and the integral spum of connected graphs. Shortly after that paper appeared, Li [1925] used some of the techniques in [2965], as well as his own, to correct multiple errors and improve many results in [2965]. The paper by Singla, Tiwari, and Tripathi and the one by Li open up new areas for research.

Following are some of the corrected and improved results. For $3 \leqslant n \leqslant 6$, $\text{spum}(P_n) = 2n-3$, and for $n \geqslant 7$, $2n-2 \leqslant \text{spum}(P_n) \leqslant 2n+1$ if n is odd, and $2n-2 \leqslant \text{spum}(P_n) \leqslant 2n-1$ if n is even. Li conjectures that for $n \geqslant 8$, we have $\text{spum}(P_n) = 2n+1$ if n is odd and $\text{spum}(P_n) = 2n-1$ if n is even. He verified the conjecture for n up to 15. In [2965] Singla, Tiwari, and Tripathi gave a flawed proof that $\text{spum}(C_n) = 2n-1$ for all $n \geqslant 13$. Li [1925] proved $\text{spum}(C_3) = 6$ and $\text{spum}(C_n) = 2n-1$ for $n \geqslant 4$.

Li [1925] improved the previous best upper bounds given for $\text{ispum}(C_n)$ in [2132] as follows. For $n \geqslant 12$, we have $\text{ispum}(C_n) \leqslant 8(n-9)$ if n is odd and $\frac{3}{2}(3n-14)$ if n is even.

In 1990 Harary [1183] showed that $\zeta(nK_2) = 0$ for all positive integers n, but Li is the first to investigate the spum and ispum of nK_2. He proved that for all positive integers n $\sigma(nK_2) = 1$, $\text{spum}(nK_2) = 4n-2$, and $\text{ispum}(2K_2) = 4$ and $\text{ispum}(nK_2) = 4n-3$ for $n \neq 2$.

Motivated by the fact that there is no obvious relationship between ispum(G) and spum(G), due to $\zeta(G)$ possibly being strictly smaller than $\sigma(G)$, i.e., when the inequality $\zeta(G) \leqslant \sigma(G)$ is strict, Li introduced the following modification of spum. The *sum-diameter* of a graph G, denoted sd(G), is the minimum possible value of range(L) for a set L of positive integer labels, such that the induced sum graph of L consists of the disjoint union of G with any number of isolated vertices. The difference between the definition of sum-diameter and spum is that in spum, one is required to use the minimum number of additional isolated vertices possible. Similarly, Li defines the *integral sum-diameter* of a graph G, denoted isd(G), by allowing the labels to be arbitrary distinct integers. He notes that $\text{sd}(G) \leqslant \text{spum}(G)$, as any set of labels L from the definition of spum(G), i.e., on $n + \sigma(G)$ vertices, is a valid labeling for the definition of sd(G). Similarly, $\text{isd}(G) \leqslant \text{sd}(G)$. Noting that any labeling L from the definition of sd(G) is a valid labeling for the definition of isd(G), yields $\text{isd}(G) \leqslant \text{sd}(G)$ for all graphs G.

Li observes that the following stronger result, which was initially stated for spum(G)

in [2965] rather than $\mathrm{sd}(G)$, follows from the same proof. For graphs G of order n without any isolated vertices, with maximum and minimum vertex degree Δ and δ, respectively, $\mathrm{sd}(G) \geqslant 2n - (\Delta - \delta) - 2$.

Similarly, he noticed that the proof of Theorem 2.2 in [2965] does not use the assumption that exactly $\zeta(G)$ isolated vertices are present when bounding $\mathrm{ispum}(G)$, and thus the following strengthening is true. For graphs G of order n without any isolated vertices, with maximum degree Δ, we have $\mathrm{isd}(G) \geqslant 2n - \Delta - 3$.

Li uses the notion of Sidon sets to prove that if G is a graph with n vertices, then $\mathrm{sd}(G) \leqslant 64n^2 - 64n + 9$. Because $\mathrm{isd}(G) \leqslant \mathrm{sd}(G)$, we also have that $\mathrm{isd}(G) \leqslant 64n^2 - 64n + 9$. To show this upper bound is asymptotically tight, for a graph G with n vertices, he lets $f(n)$ be the maximum value of $\mathrm{sd}(G)$ over all graphs G with n vertices. Then Li proves $f(n) \geqslant \frac{n^2}{4} - O(n \log n) = \Omega(n^2)$.

For K_n, Li proves $\mathrm{sd}(K_n) = \mathrm{spum}(K_n) = 4n-6$ for $n \geqslant 2$ and $\mathrm{isd}(K_n) = \mathrm{spum}(K_n) = 4n-6$ for all $n \geqslant 2$ and $\mathrm{isd}(K_n) = \mathrm{ispum}(K_n) = n-1$ for $n \leqslant 3$ and $\mathrm{ispum}(K_n) = 4n-6$ when $n \geqslant 4$.

For cycles Li proves: for all $n \geqslant 4$, $2n-2 \leqslant \mathrm{sd}(C_n) \leqslant 2n-1$ and $2n-5 \leqslant \mathrm{isd}(C_n) \leqslant 2n-1$.

The current best bounds for $\mathrm{ispum}(P_n)$ are due to Singla, Tiwari, and Tripathi [2965]: $2n-5 \leqslant \mathrm{ispum}(P_n) \leqslant \frac{5}{2}(n-3)$ if n is odd and $2n-5 \leqslant \mathrm{ispum}(P_n) \leqslant 2n-3$ if n is even. For the sum-diameter, Li obtains a much tighter result that for $n \geqslant 3$, we have $2n-3 \leqslant \mathrm{sd}(P_n) \leqslant 2n-2$. An exhaustive computer search yields that $\mathrm{sd}(P_n) = 2n-3$ for $3 \leqslant n \leqslant 6$ and $\mathrm{sd}(P_n) = 2n-2$ for $7 \leqslant n \leqslant 13$. This led Li to conjecture that $\mathrm{sd}(P_n) = 2n-3$ for $3 \leqslant n \leqslant 6$ and $\mathrm{sd}(P_n) = 2n-2$ for $n \geqslant 7$. For $\mathrm{isd}(P_n)$ Li obtains: $2n-5 \leqslant \mathrm{isd}(P_n) \leqslant 2n-2$ if n is odd and $2n-5 \leqslant \mathrm{isd}(P_n) \leqslant 2n-3$ if n is even.

Li provides upper bounds on the sum-diameter for the disjoint union of two graphs and the join of two graphs as follows. For disjoint graphs G_1 and G_2 with no isolated vertices $\mathrm{sd}(G_1 \cup G_2) \leqslant 10\max\{\mathrm{sd}(G_1), \mathrm{sd}(G_2)\} + \mathrm{sd}(G_1) + \mathrm{sd}(G_2) + 2$. If one wishes to add isolated vertices to a graph G, Li proves that if G is a graph with no isolated vertices, and N_k is the empty graph on k vertices, then $\mathrm{sd}(G \cup N_k) \leqslant \max\{k, 4\mathrm{sd}(G)\} + k - 5$. If one wishes to add a vertex with arbitrary edges incident to a graph G with no isolated vertices to form a new graph G' obtained by adding a vertex to G along with any desired edges incident to this new vertex, then $\mathrm{sd}(G') \leqslant 4\mathrm{sd}(G) - 1$. For two graphs G_1 and G_2 with no isolated vertices, where without loss of generality $\mathrm{sd}(G_1) \leqslant \mathrm{sd}(G_2)$, Li proves $\mathrm{sd}(G_1 + G_2) \leqslant 8\mathrm{sd}(G_2) + 11\mathrm{sd}(G_1) - 5$.

For a subset U of vertices from a graph $G(V, E)$ with no isolated vertices and $G[U]$ denoting the subgraph of G consisting of U and all of the edges connecting pairs of vertices in U, Li proves that $\mathrm{sd}(G[U]) \leqslant 2\mathrm{sd}(G) - 2$. As an open question he asks is: For any induced subgraph $G[U]$ of G, is $\mathrm{sd}(G[U]) \leqslant \mathrm{sd}(G)$? For graphs obtained by deleting an edge e from a graph G with no isolated vertices (denoted by $G \setminus e$) or by contracting an edge e from G (denoted by G/e), Li proves that $\mathrm{sd}(G \setminus e) \leqslant 4\mathrm{sd}(G) - 1$ and $\mathrm{sd}(G/e) \leqslant 4\mathrm{sd}(G) - 1$. For a graph G with no isolated vertices and an edge $e = (u, v)$ between two vertices u and v of G that is not present in G, Li proves that $\mathrm{sd}(G + e) \leqslant 4\mathrm{sd}(G) - 1$.

Li concludes his paper by introducing the generalization of sum-diameter to hypergraphs, and generalizes his previous general upper and lower bounds to provide preliminary bounds on both sides for arbitrary k-uniform hypergraphs and with some remarks on areas for further research and some open questions.

He, Wang, Mi, Shen, and Yu [1199] say that a graph has a *tail* if the graph contains a path for which each interior vertex has degree 2 and an end vertex of degree at least 3. They prove that every tree with a tail of length at least 3 is an integral sum graph.

B. Xu [3499] has shown that the following are integral sum graphs: the union of any three stars; $T \cup K_{1,n}$ for all trees T; mK_3 for all m; and the union of any number of integral sum trees. Xu also proved that if $2G$ and $3G$ are integral sum graphs, then so is mG for all $m > 1$. Xu poses the question as to whether all disconnected forests are integral sum graphs. Nicholas and Somasundaram [2299] prove that all banana trees (see Section 2.1 for the definition) and the union of any number of stars are integral sum graphs.

Liaw, Kuo, and Chang [1946] proved that all caterpillars are integral sum graphs (see also [3486] and [3499] for some special cases of caterpillars). This shows that the assertion by Harary in [1184] that $K(1,3)$ and $S(2,2)$ are not integral sum graphs is incorrect. They also prove that all cycles except C_4 are integral sum graphs and they conjecture that every tree is an integral sum graph. Singh and Santhosh show that the crowns $C_n \odot K_1$ are integral sum graphs for $n \geqslant 4$ [2978] and that the subdivision graphs of $C_n \odot K_1$ are integral sum graphs for $n \geqslant 3$ [2714]. Wang, Li, and Wei [3420] proved that there exists a connected integral sum graph with any minimum degree and give an upper bound for the relation between the vertex number and the edge number of a connected integral sum graph with no saturated vertex.

For graphs with n vertices, Tiwari and Tripathi [3218] show that there exist sum graphs with m edges if and only if $m \leqslant \lfloor (n-1^2)/4 \rfloor$ and that there exists integral sum graphs with m edges if and only if $m \leqslant \lceil 3(n-1)^2/8 \rceil + \lfloor (n-1)/2 \rfloor$, except for $m = \lceil 3(n-1)^2/8 \rceil + \lfloor (n-1)/2 \rfloor - 1$ when n is of the form $4k+1$. They also characterize sets of positive integers (respectively, integers) that are in bijection with sum graphs (respectively, integral sum graphs) of maximum size for a given order.

The *integral sum number*, $\zeta(G)$, of G is the minimum number of isolated vertices that must be added to G so that the resulting graph is an integral sum graph. Thus, by definition, G is a integral sum graph if and only if $\zeta(G) = 0$. Harary [1184] conjectured that $\zeta(K_n) = 2n-3$ for $n \geqslant 4$. This conjecture was verified by Chen [694], by Sharary [2867], and by B. Xu [3499]. Yan and Liu proved: $\zeta(K_n - E(K_r)) = n-1$ when $n \geqslant 6$, $n \equiv 0 \pmod 3$ and $r = 2n/3 - 1$ [3513]; $\zeta(K_{m.m}) = 2m-1$ for $m \geqslant 2$ [3513]; $\zeta(K_n \backslash -\text{edge}) = 2n-4$ for $n \geqslant 4$ [3513], [3499]; if $n \geqslant 5$ and $n-3 \geqslant r$, then $\zeta(K_n \backslash E(K_r)) \geqslant n-1$ [3513]; if $\lceil 2n/3 \rceil - 1 > r \geqslant 2$, then $\zeta(K_n \backslash E(K_r)) \geqslant 2n-r-2$ [3513]; and if $2 \leqslant m < n$, and $n = (i+1)(im-i+2)/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = (m-1)(i+1)+1$ while if $(i+1)(im-i+2)/2 < n < (i+2)[(i+1)m-i+1]/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = \lceil ((m-1)(i+1)(i+2)+2n)/(2i+2) \rceil$ [3513]. Wang [3415] proved that $\sigma(K_{n+1} \backslash E(K_{1,r})) = \zeta(K_{n+1} \backslash E(K_{1,r})) = 2n-2$ when $r+1$, $2n-3$ when $2 \leqslant r \leqslant n-1$, and $2n-4$ when $r = n$.

Nagamochi, Miller, and Slamin [2237] have determined upper and lower bounds on the sum number a graph. For most graphs $G(V,E)$ they show that $\sigma(G) = \Omega(|E|)$. He,

Yu, Mi, Sheng, and Wang [1200] investigated $\zeta(K_n\backslash E(K_r))$ where $n \geqslant 5$ and $r \geqslant 2$. They proved that $\zeta(K_n\backslash E(K_r)) = 0$ when $r = n$ or $n-1$;
$\zeta(K_n\backslash E(K_r)) = n-2$ when $r = n-2; \zeta(K_n\backslash E(K_r)) = n-1$ when $n-3 \geqslant r \geqslant \lceil 2n/3\rceil - 1; \zeta(K_n\backslash E(K_r)) = 3n-2r-4$ when $\lceil 2n/3\rceil - 1 > r \geqslant n/2; \zeta(K_n\backslash E(K_r)) = 2n-4$ when $\lceil 2n/3\rceil - 1 \geqslant n/2 > r \geqslant 2$. Moreover, they prove that if $n \geqslant 5, r \geqslant 2$, and $r \neq n-1$, then $\sigma(K_n\backslash E(K_r)) = \zeta(K_n\backslash E(K_r))$.

Dou and Gao [851] prove that for $n \geqslant 3$, the fan $F_n = P_n + K_1$ is an integral sum graph, $\rho(F_4) = 1, \rho(F_n) = 2$ for $n \neq 4$, and $\sigma(F_4) = 2, \sigma(F_n) = 3$ for $n = 3$ or $n \geqslant 6$ and n even, and $\sigma(F_n) = 4$ for $n \geqslant 6$ and n odd.

Wang and Gao [3416] and [3417] determined the sum numbers and the integral sum numbers of the complements of paths, cycles, wheels, and fans as follows: $0 = \zeta(\overline{P_4}) < \sigma(\overline{P_4}) = 1$; $1 = \zeta(\overline{P_5}) < \sigma(\overline{P_5}) = 2$; $3 = \zeta(\overline{P_6}) < \sigma(\overline{P_6}) = 4$; $\zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 0,\ n = 1,2,3$; $\zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 2n-7,\ n \geqslant 7$. $\zeta(\overline{C_n}) = \sigma(\overline{C_n}) = 2n-7,\ n \geqslant 7$. $\zeta(\overline{W_n}) = \sigma(\overline{W_n}) = 2n-8,\ n \geqslant 7$. $0 = \zeta(\overline{F_5}) < \sigma(\overline{F_5}) = 1$; $2 = \zeta(\overline{F_6}) < \sigma(\overline{F_6}) = 3$; $\zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 0, n = 3,4$; $\zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 2n-8,\ n \geqslant 7$.

Wang, Yang and Li [3421] proved: $\zeta(K_n\backslash E(C_{n-1}) = 0$ for $n = 4,5,6,7$; $\zeta(K_n\backslash E(C_{n-1}) = 2n-7$ for $n \geqslant 8$; $\sigma(K_4\backslash E(C_{n-1}) = 1$; $\sigma(K_5\backslash E(C_{n-1}) = 2$; $\sigma(K_6\backslash E(C_{n-1}) = 5$; $\sigma(K_7\backslash E(C_{n-1}) = 7$; $\sigma(K_n\backslash E(C_{n-1}) = 2n-7$ for $n \geqslant 8$.

Wang and Li [3419] proved: a graph with $n \geqslant 6$ vertices and degree greater than $(n+1)/2$ is not an integral sum graph; for $n \geqslant 8$, $\zeta(K_n \setminus E(2P_3)) = \sigma(K_n \setminus E(2P_3)) = \epsilon(K_n \setminus E(2P_3)) = \epsilon(K_n \setminus E(2P_3)) = 2n-7$; for $n \geqslant 7$, $\zeta(K_n \setminus E(K_2)) = \sigma(K_n \setminus E(K_2)) = 2n-4$; and for $n \geqslant 7$ and $1 \leqslant r \leqslant \lceil \frac{n}{2}\rceil$, $\zeta(K_n \setminus E(rK_2)) = \sigma(K_n \setminus E(rK_2)) = 2n-5$.

Chen [694] has given some properties of integral sum labelings of graphs G with $\Delta(G) < |V(G)|-1$ whereas Nicholas, Somasundaram, and Vilfred [2301] provided some general properties of connected integral sum graphs G with $\Delta(G) = |V(G)|-1$. They have shown that connected integral sum graphs G other than K_3 with the property that G has exactly two vertices of maximum degree are unique and that a connected integral sum graph G other than K_3 can have at most two vertices with degree $|V(G)|-1$ (see also [3398]).

Vilfred and Florida [3395] have examined one-point unions of pairs of small complete graphs. They show that the one-point union of K_3 and K_2 and the one-point union of K_3 and K_3 are integral sum graphs whereas the one-point union of K_4 and K_2 and the one-point union of K_4 and K_3 are not integral sum graphs. In [3396] Vilfred and Florida defined and investigated properties of maximal integral sum graphs.

Vilfred and Nicholas [3399] have shown that the following graphs are integral sum graphs: banana trees, the union of any number of stars, fans $P_n + K_1$ $(n \geqslant 2)$, Dutch windmills $K_3^{(m)}$, and the graph obtained by starting with any finite number of integral sum graphs $G_1, G_2, \ldots, G_n$ and any collections of n vertices with $v_i \in G_i$ and creating a graph by identifying $v_1, v_2, \ldots, v_n$. The same authors [3400] also proved that $G+v$ where G is a union of stars is an integral sum graph.

Melnikov and Pyatkin [2132] have shown that every 2-regular graph except C_4 is an integral sum graph and that for every positive integer r there exists an r-regular integral sum graph. They also show that the cube is not an integral sum graph. For any integral sum graph G, Melnikov and Pyatkin define the *integral radius of* G as the smallest natural number $r(G)$ that has all its vertex labels in the interval $[-r(G), r(G)]$. For the family of all integral sum graphs of order n they use $r(n)$ to denote maximum integral radius among all members of the family. Two questions they raise are: Is there a constant C such that $r(n) \leqslant C_n$ and for $n > 2$, is $r(n)$ equal to the $(n-2)$th prime?

The concepts of sum number and integral sum number have been extended to hypergraphs. Sonntag and Teichert [3035] prove that every hypertree (i.e., every connected, non-trivial, cycle-free hypergraph) has sum number 1 provided that a certain cardinality condition for the number of edges is fulfilled. In [3036] the same authors prove that for $d \geqslant 3$ every d-uniform hypertree is an integral sum graph and that for $n \geqslant d+2$ the sum number of the complete d-uniform hypergraph on n vertices is $d(n-d)+1$. They also prove that the integral sum number for the complete d-uniform hypergraph on n vertices is 0 when $d = n$ or $n-1$ and is between $(d-1)(n-d-1)$ and $d(n-d)+1$ for $d \leqslant n-2$. They conjecture that for $d \leqslant n-2$ the sum number and the integral sum number of the complete d-uniform hypergraph are equal. Teichert [3197] proves that hypercycles have sum number 1 when each edge has cardinality at least 3 and that hyperwheels have sum number 1 under certain restrictions for the edge cardinalities. (A *hypercycle* $\mathcal{C}_n = (\mathcal{V}_n, \mathcal{E}_n)$ has $\mathcal{V}_n = \cup_{i=1}^{n}\{v_1^i, v_2^i, \ldots, v_{d_i-1}^i\}, \mathcal{E}_n = \{e_1, e_2, \ldots, e_n\}$ with $e_i = \{v_1^i, \ldots, v_{d_i}^i = v_1^{i+1}\}$ where $i+1$ is taken modulo n. A *hyperwheel* $\mathcal{W}_n = (\mathcal{V}'_n, \mathcal{E}'_n)$ has $\mathcal{V}'_n = \mathcal{V}_n \cup \{c\} \cup_{i=1}^{n} \{{v_2}^{n+i}, \ldots, {v_{d_{n+i}-1}}^{n+i}\}, \mathcal{E}'_n = \mathcal{E}_n \cup \{e_{n+1}, \ldots, e_{2n}\}$ with $e_{n+i} = \{{v_1}^{n+i} = c, {v_2}^{n+i}, \ldots, {v_{d_{n+i}-1}}^{n+i}, {v_{d_{n+i}}}^{n+i} = {v_1}^{i}\}$.)

Teichert [3196] determined an upper bound for the sum number of the d-partite complete hypergraph $K^d_{n_1,\ldots,n_d}$. In [3198] Teichert defines the *strong hypercycle* $\mathcal{C}^d_n$ to be the d-uniform hypergraph with the same vertices as C_n where any d consecutive vertices of C_n form an edge of $\mathcal{C}^d_n$. He proves that for $n \geqslant 2d+1 \geqslant 5$, $\sigma(\mathcal{C}^d_n) = d$ and for $d \geqslant 2$, $\sigma(\mathcal{C}^d_{d+1}) = d$. He also shows that $\sigma(\mathcal{C}^3_5) = 3$; $\sigma(\mathcal{C}^3_6) = 2$, and he conjectures that $\sigma(\mathcal{C}^d_n) < d$ for $d \geqslant 4$ and $d+2 \leqslant n \leqslant 2d$.

In [2302] Nicholas and Vilfred define the *edge reduced sum number* of a graph as the minimum number of edges whose removal from the graph results in a sum graph. They show that for K_n, $n \geqslant 3$, this number is $(n(n-1)/2 + \lfloor n/2 \rfloor)/2$. They ask for a characterization of graphs for which the edge reduced sum number is the same as its sum number. They conjecture that an integral sum graph of order p and size q exists if and only if $q \leqslant 3(p^2-1)/8 - \lfloor (p-1)/4 \rfloor$ when p is odd and $q \leqslant 3(3p-2)/8$ when p is even. They also define the *edge reduced integral sum number* in an analogous way and conjecture that for K_n this number is $(n-1)(n-3)/8 + \lfloor (n-1)/4 \rfloor$ when n is odd and $n(n-2)/8$ when n is even.

For certain graphs G Vilfred and Florida [3394] investigated the relationships among $\sigma(G), \zeta(G), \chi(G)$, and $\chi'(G)$ where $\chi(G)$ is the chromatic number of G and $\chi'(G)$ is the edge chromatic number of G. They prove: $\sigma(C_4) = \zeta(C_4) > \chi(C_4) = \chi'(C_4)$; for $n \geqslant 3$, $\zeta(C_{2n}) < \sigma(C_{2n}) = \chi(C_{2n}) = \chi'(C_{2n})$; $\zeta(C_{2n+1}) < \sigma(C_{2n+1}) < \chi(C_{2n+1}) = \chi'(C_{2n+1})$; for $n \geqslant 4$, $\chi'(K_n) \leqslant \chi(K_n) < \zeta(K_n) = \sigma(K_n)$; and for $n \geqslant 2$, $\chi(P_n \times P_2) < \chi'(P_n \times P_2) = \zeta(P_n \times P_2) = \sigma(P_n \times P_2)$.

Alon and Scheinermann [168] generalized sum graphs by replacing the condition

$f(x) + f(y) \in S$ with $g(f(x), f(y)) \in S$ where g is an arbitrary symmetric polynomial. They called a graph with this property a *g-graph* and proved that for a given symmetric polynomial g not all graphs are g-graphs. On the other hand, for every symmetric polynomial g and every graph G there is some vertex labeling such that G together with at most $|E(G)|$ isolated vertices is a g-graph.

Boland, Laskar, Turner, and Domke [595] investigated a modular version of sum graphs. They call a graph $G(V, E)$ a *mod sum graph* (MSG) if there exists a positive integer n and an injective labeling from V to $\{1, 2, \ldots, n-1\}$ such that $xy \in E$ if and only if $(f(x) + f(y)) \pmod{n} = f(z)$ for some vertex z. Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [595] have shown the following graphs are MSG: all trees on 3 or more vertices; all cycles on 4 or more vertices; and $K_{2,n}$. They further proved that K_p ($p \geqslant 2$) is not MSG (see also [1100]) and that W_4 is MSG. They conjecture that W_p is MSG for $p \geqslant 4$. This conjecture was refuted by Sutton, Miller, Ryan, and Slamin [3169] who proved that for $n \neq 4$, W_n is not MSG (the case where n is prime had been proved in 1994 by Ghoshal, Laskar, Pillone, and Fricke [1100]. In the same paper Sutton et al. also showed that for $n \geqslant 3$, $K_{n,n}$ is not MSG. Ghoshal, Laskar, Pillone, and Fricke [1100] proved that every connected graph is an induced subgraph of a connected MSG graph and any graph with n vertices and at least two vertices of degree $n-1$ is not MSG.

In [550] Beste, de Wiljes, and Kreh investigated the sum and mod sum graphs using arithmetic progressions and prime numbers and characterized the induced sum graphs and mod sum graphs. [550] new

Sutton, Miller, Ryan, and Slamin [3169] define the *mod sum number*, $\rho(G)$, of a connected graph G to be the least integer r such that $G \cup \overline{K_r}$ is MSG. Recall the cocktail party graph $H_{m,n}$, $m, n \geqslant 2$, as the graph with a vertex set $V = \{v_1, v_2, \ldots, v_{mn}\}$ partitioned into n independent sets $V = \{I_1, I_2, \ldots, I_n\}$ each of size m such that $v_i v_j \in E$ for all $i, j \in \{1, 2, \ldots, mn\}$ where $i \in I_p$, $j \in I_q$, $p \neq q$. The graphs $H_{m,n}$ can be used to model relational database management systems (see [3165]). Sutton and Miller [3167] prove that $H_{m,n}$ is not MSG for $n > m \geqslant 3$ and $\rho(K_n) = n$ for $n \geqslant 4$. In [3166] Sutton, Draganova, and Miller prove that for n odd and $n \geqslant 5$, $\rho(W_n) = n$ and when n is even, $\rho(W_n) = 2$. Wang, Zhang, Yu, and Shi [3453] proved that fan $F_n (n \geqslant 2)$ are not mod sum graphs and $\rho(F_n) = 2$ for even n at least 6. They also prove that $\rho(K_{n,n}) = n$ for $n \geqslant 3$.

Dou and Gao [852] obtained exact values for $\rho(K_{m,n})$ and $\rho(K_m - E(K_n))$ for some cases of m and n and bounds in the remaining cases. They call a graph $G(V, E)$ a *mod integral sum graph* if there exists a positive integer n and an injective labeling from V to $\{0, 1, 2, \ldots, n-1\}$ (note that 0 is included) such that $xy \in E$ if and only if $(f(x) + f(y)) \pmod{n} = f(z)$ for some vertex z. They define the *mod integral sum number*, $\psi(G)$, of a connected graph G to be the least integer r such that $G \cup \overline{K_r}$ is a mod integral sum graph. They prove that for $m + n \geqslant 3$, $\psi(K_{m,n}) = \rho(K_{m,n})$ and obtained exact values for $\psi(K_m - E(K_n))$ for some cases of m and n and bounds in the remaining cases.

Wallace [3406] has proved that $K_{m,n}$ is MSG when n is even and $n \geqslant 2m$ or when n is odd and $n \geqslant 3m - 3$ and that $\rho(K_{m,n}) = m$ when $3 \leqslant m \leqslant n < 2m$. He also proves that the complete m-partite $K_{n_1,n_2,\ldots,n_m}$ is not MSG when there exist n_i and n_j such that $n_i < n_j < 2n_i$. He poses the following conjectures: $\rho(K_{m,n}) = n$ when $3m-3 > n \geqslant m \geqslant 3$; if $K_{n_1,n_2,\ldots,n_m}$ where $n_1 > n_2 > \cdots > n_m$, is not MSG, then $(m-1)n_m \leqslant \rho(K_{n_1,n_2,\ldots,n_m}) \leqslant (m-1)n_1$; if G has n vertices, then $\rho(G) \leqslant n$; and determining the mod sum number of a graph is NP-complete (Sutton has observed that Wallace probably meant to say 'NP-hard'). Miller [2138] has asked if it is possible for the mod sum number of a graph G be of the order $|V(G)|^2$.

In a sum graph G, a vertex w is called a *working vertex* if there is an edge uv in G such that $w = u+v$. If $G = H \cup \overline{H_r}$ has a sum labeling such that H has no working vertex the labeling is called an *exclusive sum labeling of H with respect G*. The *exclusive sum number*, $\epsilon(H)$, of a graph H is the smallest integer r such that $G \cup \overline{K_r}$ has an exclusive sum labeling. The exclusive sum number is known in the following cases (see [2142] and [2151]): for $n \geqslant 3$, $\epsilon(P_n) = 2$; for $n \geqslant 3$, $\epsilon(C_n) = 3$; for $n \geqslant 3$, $\epsilon(K_n) = 2n-3$; for $n \geqslant 4$, $\epsilon(F_n) = n$ (fan of order $n+1$); for $n \geqslant 4$, $\epsilon(W_n) = n$; $\epsilon(C_3^{(n)}) = 2n$ (friendship graph–see §2.2); $m \geqslant 2$, $n \geqslant 2$, $\epsilon(K_{m,n}) = m+n-1$; for $n \geqslant 2$, $S_n = n$ (star of order $n+1$); $\epsilon(S_{m,n}) = \max\{m,n\}$ (double star); $H_{2,n} = 4n-5$ (cocktail party graph); and ϵ(caterpillar G) $= \Delta(G)$. Dou [850] showed that $H_{m,n}$ is not a mod sum graph for $m \geqslant 3$ and $n \geqslant 3$; $\rho(H_{m,3}) = m$ for $m \geqslant 3$; $H_{m,n} \cup \rho(H_{m,n})K_1$ is exclusive for $m \geqslant 3$ and $\geqslant 4$; and $m(n-1) \leqslant \rho(H_{m,n}) \leqslant mn(n-1)/2$ for $m \geqslant 3$ and $n \geqslant 4$. Vilfred and Florida [3397] proved that $\epsilon(P_3 \times P_3) = 4$ and $\epsilon(P_n \times P_2) = 3$. In [1239] Hegde and Vasudeva provide an $O(n^2)$ algorithm that produces an exclusive sum labeling of a graph with n vertices given its adjacency matrix.

In 2001 Kratochvil, Miller, and Nguyen proved that $\sigma(G \cup H) \leqslant \sigma(G) + \sigma(H) - 1$. In 2003 Miller, Ryan, Slamin, Sugeng, and Tuga [2147] posed the problem of finding the exclusive sum number of the disjoint union of graphs. In 2010 Wang and Li [3418] proved the following. Let G_1 and G_2 be graphs without isolated vertices, L_i be an exclusive sum labeling of $G_i \cup \epsilon(G_i)K_1$, and C_i be the isolated set of L_i for $i = 1$ and 2. If maxC_1 and minC_2 are relatively prime, then $\epsilon(G_1 \cup G_2) \leqslant \epsilon(G_1) + \epsilon(G_2) - 1$. Wang and Li also proved the following: $\epsilon(K_{r,s}) = s+r-1$; $\epsilon(K_{r,s} - E(K_2)) = s-1$; for $s \geqslant r \geqslant 2$, $\epsilon(K_{r,s} - E(rK_2)) = s+r-3$. For $n \geqslant 5$ they prove: $\epsilon(K_n - E(K_n)) = 0$; $\epsilon(K_n - E(K_{n-1})) = n-1$; for $2 \leqslant r < n/2$, $\epsilon(K_n - E(K_r)) = 2n-4$; for $n/2 \leqslant r \leqslant n-2$, $\epsilon(K_n - E(K_r)) = 3n-2r-4$, and $\epsilon(C_n \odot K_1)$ is 3 or 4. They show that $\epsilon(C_3 \odot K_1) = 3$ and guess that for $n \geqslant 4$ the class E_k of hypergraphs with a k-exclusive sum labeling is hereditary, but nontrivial to characterise even for $k = 1$. In [2532] Purcell, Ryan, Rjyáá̌ček, and Skyvová provided a complete description of the minimal forbidden induced subhypergraphs of E_1 that are 3-uniform with maximum vertex degree 2. They also show that every hypertree has a 1-exclusive sum labeling and every combinatorial design does not.

A survey of exclusive sum labelings of graphs is given by Ryan in [2660].

If $\epsilon(G) = \Delta(G)$, then G is said to be an Δ-*optimum summable graph.* An exclusive sum labeling of a graph G using $\Delta(G)$ isolates is called a Δ-*optimum exclusive sum labeling* of G. Tuga, Miller, Ryan, and Ryjáček [3232] show that some families of trees that are Δ-optimum summable and some that are not. They prove that if G is a tree that has at least one vertex that has two or more neighbors that are not leaves then $\epsilon(G) = \Delta(G)$.

Koh, Miller, Smyth, and Wang [1709] show the following: the graphs obtained by identifying one end of a q-path with a vertex of a p-cycle are 1-optimum summable, and that two of these graphs can be joined via a new edge to create a 2-optimum summable graph; generalized θ-graphs are 2-optimum summable; $\theta(p, q, r)$ which consists of a pair of vertices joined by 3 independent paths of lengths p, q and r (with a few small exceptions) are 2-optimum summable; there exists a 3-optimum summable graph of order $4l + 3$ for all $l \geqslant 1$; how to construct for all $k \geqslant 4$ a k-optimum summable graph; and if G is a k-optimum summable graph of order n, then $n \geqslant 2k$.

In [1356] Javaid, Khalid, Ahmad, and Imran introduce a weaker version of sum labeling of graphs as follows. Let $H = (V, E)$ be a simple, finite, undirected graph with $|V| = p$. H is a *weak sum* graph if there exists a labeling L (called a w-sum) of the vertices of V by distinct positive integers such that $(u, v) \in E$ if there exists a vertex $w \in V$ such that $L(w) = L(u) + L(v)$. (A sum graph also requires the "only if" condition). If H is a w-sum graph with the additional constraint that the labels L all fall in the range $1, \ldots, p$, then H is called a *super weak sumgraph* (sw-sumgraph). Because sumgraphs must have isolated vertices we may write $H = G + K_\delta$, where G is connected and K_δ denotes δ isolated vertices If δ is a minimum with respect to G, we say that the sumgraph (respectively, w-sumgraph, sw-sumgraph) H is δ-optimal and that G is δ-optimal summable (respectively, w-summable, sw-summable). Javaid et al. prove: paths are 1-optimal sw-summable; cycles are 2-optimal sw-summable; wheels are 3-optimal sw-summable; K_n is $(n-1)$-optimal sw-summable; and $G = K_{n_1,n_2,\ldots,n_q}$ are t-optimal sw-summable, where t is the minimum degree of any vertex in G. They also prove that for $n \geqslant 5$, the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \pm 1, \pm 2)$ is 4-optimal w-summable. They conjecture that all connected graphs are δ-optimal w-summable for some δ. See also [1709] and [2147].

Grimaldi [1160] has investigated labeling the vertices of a graph $G(V, E)$ with n vertices with distinct elements of the ring Z_n so that $xy \in E$ whenever $(x + y)^{-1}$ exists in Z_n.

In his 2001 Ph. D. thesis Sutton [3165] introduced two methods of graph labelings with applications to storage and manipulation of relational database links specifically in mind. He calls a graph $G = (V_p \cup V_i, E)$ a *sum* graph* of $G_p = (V_p, E_p)$ if there is an injective labeling λ of the vertices of G with non-negative integers with the property that $uv \in E_p$ if and only if $\lambda(u) + \lambda(v) = \lambda(z)$ for some vertex $z \in G$. The *sum* number*, $\sigma^*(G_p)$, is the minimum cardinality of a set of new vertices V_i such that there exists a sum* graph of G_p on the set of vertices $V_p \cup V_i$. A *mod sum* graph* of G_p is defined in the identical fashion except the sum $\lambda(u) + \lambda(v)$ is taken modulo n where the vertex labels of G are restricted to $\{0, 1, 2, \ldots, n-1\}$. The *mod sum* number*, $\rho^*(G_p)$, of a graph G_p is defined in the analogous way. Sum* graphs are a generalization of sum graphs and mod sum* graphs are a generalization of mod sum graphs. Sutton shows that every graph is an induced subgraph of a connected sum* graph. Sutton [3165] poses the following conjectures: $\rho(H_{m,n}) \leqslant mn$ for $m, n \geqslant 2$; $\sigma^*(G_p) \leqslant |V_p|$; and $\rho^*(G_p) \leqslant |V_p|$.

The following table summarizes what is known about sum graphs, mod sum graphs, sum* graphs, and mod sum* graphs is reproduced from Sutton's Ph. D. thesis [3165]. It was updated by J. Gallian in 2006. A question mark indicates the value is unknown. The results on sum* and mod sum* graphs are found in [3165].

Table 20: **Summary of Sum Graph Labelings**

Graph	$\sigma(G)$	$\rho(G)$	$\sigma^*(G)$	$\rho^*(G)$
$K_2 = S_1$	1	1	0	0
stars, $S_n,\ n \geqslant 2$	1	0	0	0
trees $T_n,\ n \geqslant 3$ when $T_n \neq S_n$	1	0	1	0
C_3	2	1	1	0
C_4	3	0	2	0
$C_n,\ n > 4$	2	0	2	0
W_4	4	0	2	0
$W_n,\ n \geqslant 5,\ n$ odd	n	n	2	0
$W_n,\ n \geqslant 6,\ n$ even	$\frac{n}{2}+2$	2	2	0
fan, F_4,	2	1	1	0
fans, $F_n,\ n \geqslant 5,\ n$ odd	?	2	1	0
fans, $F_n,\ n \geqslant 6,\ n$ even	3	2	1	0
$K_n,\ n \geqslant 4$	$2n-3$	n	$n-2$	0
cocktail party graphs, $H_{2,n}$	$4n-5$	0	?	0
$C_n^{(t)}$ $(n,t) \neq (4,1)$ (see §2.2)	2	?	?	?
$K_{n,n}$	$\lceil \frac{4n-3}{2} \rceil$	$n (n \geqslant 3)$	?	?
$K_{m,n},\ 2nm \geqslant n \geqslant 3$	?	n	?	?
$K_{m,n}$ $m \geqslant 3n-3,\ n \geqslant 3,\ m$ odd	?	0	?	0
$K_{m,n},\ m \geqslant 2n,\ n \geqslant 3,\ m$ even	?	0	?	0
$K_{m,n},\ m < n$ $k = \lceil \sqrt{1+(8m+n-1)(n-1)/2}\ \rceil$	$\lceil \frac{kn-k}{2} + \frac{m}{k-1} \rceil$	?	?	?
$K_{n,n} - E(nK_2),\ n \geqslant 6$	$2n-3$	$n-2$	?	?

7.2 Prime and Vertex Prime Labelings[4]

The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla [3221]. A graph with vertex set V is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, \dots, |V|$ such that for each edge xy the labels assigned to x and y are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. Little progress was made on this conjecture until 2011 when Haxell, Pikhurko, Taraz [1197] proved that all large trees are prime. Also, their method allowed them to determine the smallest size of a non-prime connected order-n graph for all large n, proving a conjecture of Rao [2595] in this range. Among the classes of trees known to have prime labelings are: paths, stars, complete binary trees, spiders (i.e., trees with one vertex of degree at least 3 and with all other vertices with degree at most 2), olive trees (i.e., a rooted tree consisting of k branches such that the ith branch is a path of length i), all trees of order up to 50, palm trees (i.e., trees obtained by appending identical stars to each vertex of a path), banana trees, and binomial trees (the binomial tree B_0 of order 0 consists of a single vertex; the binomial tree B_n of order n has a root vertex whose children are the roots of the binomial trees of order $0, 1, 2, \dots, n-1$ (see [2689], [2398], [3221], [990], and [2636]). Tout, Dabboucy, and Howalla [3221] showed t-toe caterpillars (the internal vertices on the spine are regular in degree) are prime and that all caterpillars with maximum degree at most 5 are prime.

Seoud, Sonbaty, and Mahran [2794] provide necessary and sufficient conditions for a graph to be prime. They also give a procedure to determine whether or not a graph is prime. Other graphs with prime labelings include all cycles and the disjoint union of C_{2k} and C_n [804]. The complete graph K_n does not have a prime labeling for $n \geqslant 4$ and W_n is prime if and only if n is even (see [1921]). Lee, Wui, and Yeh [1921] proved that friendship graphs have prime labelings. Diefenderfer et al. [838] and [837] proved that the graph obtained by identifying a vertex of C_n with an endpoint of the star S_m where $1 \leqslant m \leqslant 9$, chains of C_n where $n = 4, 6$, or 8, $C_n \times P_2$ where $n-1$ is prime and $n \geqslant 4$, generalized books $S_n \times P_m$ where $3 \leqslant m \leqslant 7$, and other families of uncylic graphs have prime vertex labelings.

Seoud, Diab, and Elsakhawi [2769] have shown the following graphs are prime: fans; helms; flowers (see §2.2); stars; $K_{2,n}$; and $K_{3,n}$ unless $n = 3$ or 7. They also shown that $P_n + \overline{K_m}$ ($m \geqslant 3$) is not prime. Berliner, Dean, Hook, Marr, Mbirka, and McBee give consecutive cyclic prime labelings of certain classes of ladders. Although $K_{n,n}$ does not have a prime labeling when $n > 2$, Berliner et al. give minimal prime labelings for all n-values $1 \leqslant n \leqslant 23$ and give conditions on m and n for which $K_{m,n}$ are prime. They provide specific values of n for m up to 13. Dissanayake, Abeysekara, Dhananjaya, Perera, and Ranasinghe [841] provide necessary and sufficient conditions for $K_{1,m,n}$ to have a prime labeling. In [570] Bigham, Donovan, Pack, Turley, and Wiglesworth [570] investigated the existence of prime labelings of certain snake graphs. [570] new [570] new

Tout, Dabboucy, and Howalla [3221] proved that $C_m \odot \overline{K_n}$ is prime for all m and n.

[4] I am grateful to John Asplund and N. Bradley Fox for their helpful comments on the results in this section.

Vaidya and Prajapati [3297] proved that the graphs obtained by duplication of a vertex by a vertex in P_n and $K_{1,n}$ are prime graphs and the graphs obtained by duplication of a vertex by an edge, duplication of an edge by a vertex, duplication of an edge by an edge in $P_n,\ K_{1,n}$, and C_n are prime graphs. They also proved that graph obtained by duplication of every vertex by an edge in $P_n,\ K_{1,n}$, and C_n are not prime graphs. Ghorbani and Kamali [1094] proved that ladders have prime labelings.

For m and n at least 3, Seoud and Youssef [2797] define $S_n^{(m)}$, the (m,n)*-gon star*, as the graph obtained from the cycle C_n by joining the two end vertices of the path P_{m-2} to every pair of consecutive vertices of the cycle such that each of the end vertices of the path is connected to exactly one vertex of the cycle. Seoud and Youssef [2797] have proved the following graphs have prime labelings: books; $S_n^{(m)}$; $P_n + \overline{K_2}$ if and only if $n = 2$ or n is odd; $C_n \odot K_1$ with a complete binary tree of order $2^k - 1$ $(k \geqslant 2)$ attached at each pendent vertex, and that C_m-snakes are prime (see §2.2) for the definition). They also prove that every spanning subgraph of a prime graph is prime and every graph is a subgraph of a prime graph. They conjecture that all unicycle graphs have prime labelings. Diefenderfer, Hastings, Heath, Prawzinsky, Preston, White, and Whittemore [837] proved that certain families of graphs that are special cases of Seoud and Youssef's conjecture [2797] have prime labelings. Seoud and Youssef [2797] proved the following graphs are not prime: $C_m + C_n$; C_n^2 for $n \geqslant 4$; P_n^2 for $n = 6$ and for $n \geqslant 8$; and Möbius ladders M_n for n even (see §2.3 for the definition). They also give an exact formula for the maximum number of edges in a prime graph of order n and an upper bound for the chromatic number of a prime graph.

Youssef and Elsakhawi [3559] have shown: the union of stars $S_m \cup S_n$, are prime; the union of cycles and stars $C_m \cup S_n$ are prime; $K_m \cup P_n$ is prime if and only if m is at most 3 or if $m = 4$ and n is odd; $K_n \odot K_1$ is prime if and only if $n \leqslant 7$; $K_n \odot \overline{K_2}$ is prime if and only if $n \leqslant 16$; $6K_m \cup S_n$ is prime if and only if the number of primes less than or equal to $m + n + 1$ is at least m; and that the complement of every prime graph with order at least 20 is not prime. Michael and Youssef [2136] determined all self-complementary graphs that have prime labelings.

Salmasian [2689] has shown that every tree with n vertices $(n \geqslant 50)$ can be labeled with n integers between 1 and $4n$ such that every two adjacent vertices have relatively prime labels. Pikhurko [2398] has improved this by showing that for any $c > 0$ there is an N such that any tree of order $n > N$ can be labeled with n integers between 1 and $(1 + c)n$ such that labels of adjacent vertices are relatively prime.

Baskar Babujee and Vishnupriya [506] proved the following graphs have prime labelings: $nP_2,\ P_n \cup P_n \cup \cdots \cup P_n$, bistars (that is, the graphs obtained by joining the centers of two identical stars with an edge), and the graph obtained by subdividing the edge joining edge of a bistar. Baskar Babujee [485] obtained prime labelings for the graphs: $(P_m \cup nK_1) + \overline{K_2},\ (C_m \cup nK_1) + \overline{K_2},\ (P_m \cup C_n \cup \overline{K_r}) + \overline{K_2},\ C_n \cup C_{n+1},\ (2n-2)C_{2n}$ $(n > 1)$, $C_n \cup mP_k$ and the graph obtained by subdividing each edge of a star once. In [495] Baskar Babujee and Jagadesh prove the following graphs have prime labelings: bistars $B_m, n; P_3 \odot K_{1,n}$; the union of $K_{1,n}$ and the graph obtained from $K_{1,n}$ by appending a pendent edge to every pendent edge of $K_{1,n}$; and the graph obtained by identifying the center of $K_{1,n}$ with the two endpoints and the middle vertex of P_5.

In [3293] Vaidya and Prajapati prove the following graphs have prime labelings: a t-ply graph of prime order; graphs obtained by joining center vertices of wheels W_m and W_n to a new vertex w where m and n are even positive integers such that $m+n+3=p$ and p and $p-2$ are twin primes; the disjoint union of the wheel W_{2n} and a path; the graph obtained by identifying any vertex of a wheel W_{2n} with an end vertex of a path; the graph obtained from a prime graph of order n by identifying an end vertex of a path with the vertex labeled with 1 or n; the graph obtained by identifying the center vertices of any number of fans (that is, a "multiple shell"); the graph obtained by identifying the center vertices of m wheels $W_{n_1}, W_{n_2}, \ldots, W_{n_m}$ where each $n_i \geqslant 4$ is an even integer and each n_i is relatively prime to $2 + \sum_{k=1}^{i-1} n_k$ for each $i \in \{2, 3, \ldots, m\}$. Prajapati and Suther [2510] provided results about the existence of prime labelings of graphs obtained from $K_{2,n}$ by the duplication of vertices and edges. In [2484] Prajapati and Gajjar provided conditions under which the disjoint union of two graphs admit a prime labeling. They showed that $C_{2n+1} \times P_2$ is not prime, $\overline{W_n}$ is prime if and only if $3 \leqslant n \leqslant 6$, and, for a prime $p \geqslant 3$, $C_{p-1} \times P_2$ is prime and a wheel graph of odd order is switching invariant. In [2496] they proved that generalized Petersen graph $P(n,k)$ is prime then n must be even and k must be odd and found some classes of generalized Petersen graphs that admit prime labelings.

In [3479] Wilson and Jini define the torch O_n as the graph that has $n+4$ vertices and $2n+3$ edges with $V(O_n) = \{v - i \mid 1 \leqslant i \leqslant n+4\}$ and $E(O_n) = \{v_i v_{n+1} \mid 2 \leqslant i \leqslant n-2\} \cup \{v_i v_{n+3} \mid 2 \leqslant i \leqslant n-2\} \cup \{v_1 v_i \mid n \leqslant i \leqslant n+4\} \cup \{v_{n-1} v_n, v_n v_{n+2}, v_n v_{n+4}, v_{n+1} v_{n+3}\}$. They proved that the torch graph O_n is a prime graph.

The *Knödel graphs* $W_{\Delta,n}$ with n even and degree Δ, where $1 \leqslant \Delta \leqslant \lfloor \log_2 n \rfloor$ have vertices pairs (i,j) with $i = 1, 2$ and $0 \leqslant j \leqslant n/2 - 1$ where for every $0 \leqslant j \leqslant n/2 - 1$ and there is an edge between vertex $(1,j)$ and every vertex $(2, (j + 2^k - 1) \bmod n/2)$, for $k = 0, 1, \ldots, \Delta - 1$. Haque, Lin, Yang, and Zhao [1182] have shown that $W_{3,n}$ is prime when $n \leqslant 130$.

In [2727] Schuchter and Wilson gave evidence for their conjecture that a generalized Petersen graph $P(n,k)$ is prime if and only if it is bipartite, which occurs for n even and k. They show that it is true for all even n and odd k such that $n \leqslant 9000$ and $1 \leqslant k \leqslant \frac{n}{2}$. They conjectured that all cubic bipartite graphs with at least 8 vertices are prime and verified it for all such graphs G, connected or not, satisfying $8 \leqslant V(G) \leqslant 22$.

In the 24th edition of this survey, the following was stated."Schroeder [2729] proved that every bipartite graph is prime except $K_{3,3}$. This result establishes that the generalized Petersen graph $P(n,k)$ is prime precisely when it is bipartite, the Knődel graph $W_{3,n}$ is prime for all even $n \geqslant 4$, and the union of any number of even cycles is prime." Schroeder's result about bipartite graphs was later modified [2730] to state that for $4 \leqslant n \leqslant 32$, cubic bipartite graphs on $2n$ vertices are prime. Consequently, the statements about the generalized Petersen graph, the Knődel graph, and the union of even cycles are open. In [2193] Mostafa and Ghorbani characterized Hamiltonian k-coprime graphs, which implied a conjecture of Schroeder in [2728] that all 2-regular graphs are prime if every union of even cycles is prime. They then cited Schroeder's original result to claim the 1982 conjecture by Tout, Dabboucy, Howalla [3221] that all 2-regular graphs are prime. So, that conjecture is still open as well.

Prajapati and Shah [2509] investigated the existence of prime labelings for graphs obtained by the duplication of paths, cycles, stars, and wheels. Sundaram, Ponraj, and Somasundaram [3138] investigated the prime labeling behavior of all graphs of order at most 6 and established that only one graph of order 4, one graph of order 5, and 42 graphs of order 6 are not prime.

Kansagara and Patel [1644] define the *web graph without a center* as the graph obtained by joining the n pendent vertices of $C_n \odot K_1$ to form an n-cycle and then adding a single pendent edge to each vertex of this outer cycle. They investigated prime labelings for web graphs without a center and graphs resulting from the subdivision of some specific edges in it. They also provide results on prime labeling of the union of a web graph without center with a wheel, generalized Jahangir graphs, and drum graphs.

Given a collection of graphs $G_1, \ldots, G_n$ and some fixed vertex v_i from each G_i, Lee, Wui, and Yeh [1921] define Amal$\{(G_i, v_i)\}$, the amalgamation of $\{(G_i, v_i)|\ i = 1, \ldots, n\}$, as the graph obtained by taking the union of the G_i and identifying $v_1, v_2, \ldots, v_n$. They proved $Amal\{(G_i, v_i)\}$ has a prime labeling when G_i are paths and when G_i are cycles. They also showed that the amalgamation of any number of copies of W_n, n odd, with a common vertex is not prime. They conjecture that for any tree T and any vertex v from T, the amalgamation of two or more copies of T with v in common is prime. They further conjecture that the amalgamation of two or more copies of W_n that share a common point is prime when n is even ($n \neq 4$). Vilfred, Somasundaram, and Nicholas [3403] have proved this conjecture for the case that $n \equiv 2 \pmod 4$ where the central vertices are identified.

Dean [791], and independently Ghorbani and Kamali [1094], showed that the ladder $P_n \times P_2$ has a prime labeling. In [766] Curran and Ollis pointed out a flaw in Dean's proof [766] new
and showed that a stronger condition is needed for it to hold. They conjecture that this stronger condition is true. They also offer an alternative construction inspired by Dean's approach that shows that if the Even Goldbach Conjecture and a particular strengthening of Lemoine's Conjecture are true, then the Prime Ladder Conjecture follows.

Vilfred, Somasundaram, and Nicholas [3403] proved the following graphs are prime: helms; $P_m \times P_n$ where n is prime, $m \leqslant 3$; double fans $P_n + \overline{K_2}$ if and only if n is odd; and cycles with a P_k-chord. They conjecture that $P_m \times P_n$ where $m < n$ and n is prime is prime and ladders $P_n \times P_2$ are prime. The conjecture about grids was proved by Sundaram, Ponraj, and Somasundaram [3136]. In the same article they also showed that $P_n \times P_n$ is prime when n is prime. Kanetkar [1627] proved: $P_6 \times P_6$ is prime; $P_{n+1} \times P_{n+1}$ is prime when n is a prime with $n \equiv 3$ or $9 \pmod{10}$ and $(n+1)^2+1$ is also prime; and $P_n \times P_{n+2}$ is prime when n is an odd prime with $n \not\equiv 2 \pmod 7$. Curran [758] showed that $P_p \times P_n$ has a prime labeling for any odd prime p and any integer n such that $p < n \leqslant p^2$. Combining the results of Sundaram, Ponraj, and Somasundaram [3136] and Curran, we have that $P_p \times P_n$ has a prime labeling for any odd prime p and any integer n such that $1 \leqslant n \leqslant p^2$.

Seoud, El Sonbaty, and Abd El Rehim [2770] proved that for $m = p_{n+t-1} - (t+n)$ where p_i is the i^{th} prime number in the natural order $K_n \cup K_{t,m}$ is prime and graphs obtained from $K_{2,n}$, $(n \geqslant 2)$ by adding p and q edges out from the two vertices of degree

n of $K_{2,n}$ are prime. They also proved that if G is not prime, then $G \cup K_{1,n}$ is prime if $\pi(n+m+1) \geqslant m$ where m is the order of G and $\pi(x)$ is the number of primes less than or equal to x.

Recall that $C_n^{(k)}$ is the graph obtained from the $k \geqslant 2$ copies of the cycle C_n by identifying exactly one vertex of each of these k copies of C_n. Patel and Vasava [2369] proved the following: $C_n^{(j)} \cup C_m^{(k)}$ is a prime graph if and only if either n is even or m is even; $C_{2n}^{(2)} \cup C_{2m}^{(2)} \cup C_k^{(2)}$ is a prime graph for all n, m and k; $C_{2n} \cup C_{2n} \cup C_{2n} \cup C_{2n} \cup C_{2m} \cup C_k$ is a prime graph for all n, m and k; and $G = \left(\bigcup_{k=1}^{N} C_{n_k}^{(2)}\right) \cup \left(\bigcup_{j=1}^{M} C_{m_j}^{(2)}\right)$ is not a prime graph if $M \leqslant N-2$ They also provided conditions for which $G = C_{2n}^{(2)} \cup C_{2m+1}^{(2)} \cup C_{2k+1}^{(2)}$ is a prime graph.

For any finite collection $\{G_i, u_iv_i\}$ of graphs G_i, each with a fixed edge u_iv_i, Carlson [658] defines the edge amalgamation $Edgeamal\{(G_i, u_iv_i)\}$ as the graph obtained by taking the union of all the G_i and identifying their fixed edges. The case where all the graphs are cycles she calls *generalized books*. She proves that all generalized books are prime graphs. Moreover, she shows that graphs obtained by taking the union of cycles and identifying in each cycle the path P_n are also prime.

In [484] Baskar Babujee proves that the maximum number of edges in a simple graph with n vertices that has a prime labeling is $\sum_{k=2}^{n} \phi(k)$. He also shows that the planar graphs having n vertices and $3(n-2)$ edges (i.e., the maximum number of edges for a planar graph with n vertices) obtained from K_n $(n \geqslant 5)$ with vertices $v_1, v_2, \ldots, v_n$ by deleting the edges joining v_s and v_t for all s and t satisfying $3 \leqslant s \leqslant n-2$ and $s+2 \leqslant t \leqslant n$ has a prime labeling if and only if n is odd.

By showing that for every even $n \leqslant 2.468 \times 10^9$ there exists $1 \leqslant s \leqslant n-1$ such that both $n+s$ and $2n+s$ are prime, Schluchter, Schroeder, Cokus, Ellingson, Harris, Rarity, and Wilson [2726] prove the generalized Peterson graph $P(n,1)$ (which is isomorphic to $C_n \times P_2$) is prime for all even $4 \leqslant n \leqslant 2.468 \times 10^9$. For a fixed n they also describe a method for labeling $P(n,k)$ that is a prime labeling for multiple values of k. Using this method, they prove $P(n,k)$ is prime for all even $n \leqslant 50$ and odd $k < n/2$.

Yao, Cheng, Zhongfu, and Yao [3530] have shown: a tree of order p with maximum degree at least $p/2$ is prime; a tree of order p with maximum degree at least $p/2$ has a vertex subdivision that is prime; if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$ and $d_{T_2}(u_2) \geqslant |T_2|/2$, then T is prime when $|T_1| + |T_2|$ is prime; if a tree T has two edges u_1u_2 and u_2u_3 such that the three components T_1, T_2, and T_3 of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \geqslant |T_1|/2$, $d_{T_2}(u_2) \geqslant |T_2|/2$, and $d_{T_3}(u_3) \geqslant |T_3|/2$, then T is prime when $|T_1| + |T_2| + |T_3|$ is prime.

Vaidya and Prajapati [3294] define a *vertex switching* G_v of a graph G as the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex that is not adjacent to v in G. They say a prime graph G is *switching invariant* if for every vertex v of G, the graph G_v obtained by switching the vertex v in G is also a prime graph. They prove: P_n and $K_{1,n}$ are switching invariant; the graph obtained by switching the center of a wheel is a prime graph; and the graph obtained by switching a rim vertex of W_n is a prime graph if $n+1$ is a prime. They also prove that the graph obtained by switching a rim vertex in W_n is not a prime graph if $n+1$ is an even integer greater than 9.

Prajapati and Gajjar [2484] prove the following graphs are prime: graphs obtained from P_{m+1} and m copies of C_n by identifying each edge of P_{m+1} with an edge of a corresponding copy of C_n; graphs obtained from C_m and m copies of C_n by identifying each edge of C_m with an edge of corresponding copy of C_n; for a prime $p \geqslant 3$ and $p-2$ copies of C_{p+1}, the graph obtained by identifying one vertex of each copy of C_{p+1} with corresponding pendent vertex of $K_{1,p-2}$; for a prime $p \geqslant 3$, $C_{p-1} \times P_2$; and for a prime $p \geqslant 3$, the graphs obtained by joining every rim vertex of a wheel graph W_{p-1} with the corresponding vertex of C_{p-1}. They also prove that the complement of W_n is prime if and only if $3 \leqslant n \leqslant 6$; for odd $n \geqslant 3$ $C_n \times P_2$ is not prime; and W_{2n} is switching invariant.

Selvaraju and Moha [2743] proved that the one-point union of any number of cycles and the one-point union of any number of wheels at the center are prime graphs. Haque, Xiaohui, Yuansheng, and Pingzhong proved that the generalized Petersen graph $P(n,k)$ is prime for all even $n \leqslant 2500$ when $k=1$ [1179] and for all even $n \leqslant 100$ when $k=3$ [1181]. They show $P(n,3)$ is not prime for odd n and conjecture that $P(n,3)$ are prime for all even n.

In [2775] Seoud, El-Sonbaty, and Mahran discuss the primality of some corona graphs $G \odot H$ and conjecture that $K_n \odot \overline{K_m}$ is prime if and only if $n \leqslant \pi(nm+n)+1$, where $\pi(x)$ is the number of primes less than or equal to x. For $m \leqslant 20$ they give the exact values of n for which $K_n \odot \overline{K_m}$ is prime. They also show that $K_{m,n}$ is prime if and only if $\min\{m,n\} \leqslant \pi(m+n) - \pi((m+n)/2)+1$.

In [2369] Patel and Vasava provided the following results about prime graphs: $W_n \cup C_m$ is a prime graph if and only if n and m both are even; $(P_n + \overline{K_2}) \cup C_m$ is a prime graph if and only if either $n=2$, or n is odd and m is even; $C_n^{(2)} \cup C_m$ is a prime graph if and only if at least one of n and m is even; $S_n^{(m)} \cup S_k^{(j)}$ is not a prime graph if m and j are even and n and k are odd ($S_n^{(m)}$ denotes the (m,n)-gon star obtained from C_n and n copies of P_{m-2} by joining the two end vertices of P_{m-2} to each pair of consecutive vertices of the cycle such that each of the end vertices of the path is adjacent to exactly one vertex of the cycle); $S_{2n}^{(2m)} \cup S_k^{(2m)}$ is a prime graph for all n, m, and k; $H_n \bigcup B_n$ is a prime graph for all n and m; $Circ(n,k)$ is not a prime graph when n and k both are even; and $Circ(n,k)$ is not a prime graph when n is odd.

In [2360] Patel and Kansagara prove that the following graphs are prime: $(C_n \odot K_1) \cup (C_n \odot K_1)$ for all n; $G_n \cup G_n$ for all n (G_n is the gear graph); $H_n \cup H_n$ for all n (H_n is the helm graph); $B_n \cup (K_{1,n} \times P_2)$ for all n; $H_n \cup G_n$ for all n; $(C_n \odot K_1) \cup G_n$ for all n; $(C_n \odot K_1) \cup H_n$ for all n; and $C_n(C_n) \cup C_n(C_n)$ if $n \not\equiv 1 \pmod 3$ ($C_n(C_n)$ is the graph obtained by taking barycentric subdivision of C_n and joining each newly inserted vertices of incident edges by an edge).

In [2362] Patel, Kansagava, and Vasva provided prime labeling of graphs obtained by taking barycentric subdivision of W_n, flower graphs, and $(C_n \odot K_1) \cup (C_n \odot K_1)$; and the graphs by obtained by taking subdivision of certain edges W_n, H_n, $(C_n \odot K_1)$ and $(C_n \odot K_1) \cup (C_n \odot K_1)$.

Hamm, Hamm, and Way [1173] proved that if a complete k-partite k-uniform hypergraph has enough vertices and every pod of vertices is large enough, then it does not have a prime labeling. They also proved that if a pod of vertices in a complete k-partite k-uniform is small enough, the graph it does have a prime labeling. [1173] new

Klee, Lehmann, and Park [1705] we extended the notion of prime labeling to the Gaussian integers. They showed that paths, stars, spiders, graphs obtained by joining the centers of two stars with a path, and some firecrackers admit Gaussian prime labelings. In [2946] Shrimali and Singh proved the following graphs have Gaussian vertex prime labelings: books; kayaks; the one point union of k copies of W_n; the one point union of k copies of a gear graph; the one point union of k copies a graph obtained from a wheel by replacing each cycle edge by P_n; and the one point union of k copies a graph obtained from a wheel by replacing each spoke by P_n. Kavitha and Jayalalitha [1677] proved the following graphs admit Gaussian prime distance labeling: grids, paths, bipartite graphs, non-trivial Dutch windmills, non-trivial stars, double stars, and spider trees. [1677] new

The Prime Ladder Conjecture states that every ladder $P_n \times P_2$ is prime. This was proved by Dean [791] in 2017. He conjectures that every integer $n \geqslant 50$ has a canonical partition with at most three terms and he states that this conjecture was verified by computer up to 5,000,000.

Lau, Chu, Suhadak, Foo, and Ng [1803] introduced SD-prime cordial labelings as follows. Given a finite, simple graph G with n vertices and a bijection $f : V(G) \to \{1, 2, \ldots, n\}$, for each edge uv let $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$. For each edge uv define f' induced by f by assigning $f'(uv) = 1$ if $\gcd(S, D) = 1$ and $f'(uv) = 0$ otherwise. Then f' is said to be SD-*prime cordial* if $f'(uv) = 1$ for all edges uv. They provide results about paths, complete bipartite graphs, stars, double stars, wheels, fans, double fans, ladders, and grids. They conjecture that $P_m \times P_n$ is SD-prime for all $m \geqslant 2$ and $n \geqslant 2$. Lau, Shiu, Ng, and Jeyanthi [1812] give sufficient conditions for a theta graph to have an SD-prime cordial labeling, provide a way to construct new SD-prime cordial graphs from existing ones, and investigate SD-prime cordialness of some general graphs. Lourdusamy and Patrick [2011] provide a way to construct SD-prime cordial graphs from an existing graph G with an SD-prime cordial labeling by identifying a vertex of G having a particular label with a vertex of maximum degree of a star or fan or with an endpoint of a path. In [2012] Lourdusamy and Patrick investigated SD-prime cordial labelings of subdivision graphs, splitting graphs, shadow graphs of stars and bistars, $T(P_n), T(C_n)$, the graph obtained by duplication of each vertex of a path and a cycle by an edge, Q_n, $A(T_n)$, triangular ladders, $P_n \odot K_1$, $C_n \odot K_1$, and jewel graphs. Shiu and Lau [2914] provided SD-prime labelings for some one-point unions of gear graphs. [2914] new

The following definitions appear in [2444], [2431], [2432], and [2433]. A *double triangular snake* DT_n consists of two triangular snakes that have a common path; a *double quadrilateral snake* DQ_n consists of two quadrilateral snakes that have a common path; an *alternate triangular snake* $A(T_n)$ is the graph obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertex v_i (that is, every alternate edge of a path is replaced by C_3); a *double alternate triangular snake* $DA(T_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i and w_i;

an *alternate quadrilateral snake* $A(Q_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i and w_i respectively and then joining v_i and w_i (that is, every alternate edge of a path is replaced by a cycle C_4); a *double alternate quadrilateral snake* $DA(Q_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, x_i and w_i and y_i respectively and then joining v_i and w_i and x_i and y_i.

Prajapati and Vantiya [2511] proved that the following snake graphs have SD-prime cordial labelings: triangular (except for $n = 3$), alternate triangular, quadrilateral, alternate quadrilateral, double triangular, double alternate triangular, double quadrilateral, and double alternate quadrilateral. Lourdusamy, Wency, and Patrick [2022] proved that the union of stars and paths, subdivision of combs, subdivision of ladders, and the graph obtained by attaching a star at one end of a path are SD-prime graphs. They proved that the union of two SD-prime cordial graphs need not be SD-prime cordial graphs. Also, they proved that given a positive integer n, there is SD-prime cordial graph with n vertices. In [2512] Prajapati and Vantiya investigated SD-prime cordial labelings of various kinds of snakes involving K_4. In [2513] they investigate the SD-prime cordial labeling of alternate k-polygonal snake graphs of type-1, type-2 and type-3.

Vaidya and Prajapati [3293] have introduced the notion of k-prime labeling. A *k-prime labeling* of a graph G is an injective function $f : V(G) \to \{k+1, k+2, k+3, \ldots, k+|V(G)|-1\}$ for some positive integer k that induces a function f^+ on the edges of G defined by $f^+(uv) = \gcd(f(u), f(v))$ such that $\gcd(f(u), f(v)) = 1$ for all edges uv. A graph that admits a k-prime labeling is called a *k-prime* graph. They prove the following are prime graphs: a tadpole (that is, a graph obtained by identifying a vertex of a cycle to an end vertex of a path); the union of a prime graph of order n and a $(n+1)$-prime graph; the graph obtained by identifying the vertex labeled with n in an n-prime graph with either of the vertices labeled with 1 or n in a prime graph of order n. Arockiamary and Vijayalakshmi [239] proved that shell graphs, uniform shellbutterfly graphs, and shell-flower graphs admit k-prime total labelings. [239] new

For connected graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$, $G_1\ \hat{o}\ G_2$ is the graph obtained by superimposing any selected vertex of G_2 on any selected vertex of G_1. Then for any graph k-prime graph G, Vijayalashmi and Arockiamary [3384] proved the following graphs have k-prime labelings: $C_{2n} \cup C_{2n}$, $G \cup tP_m (m > 1)$, $G \cup K_{1,n}, G\ \hat{o}\ P_n, G \hat{o}\ K_{1,n}$ and $G\ \hat{o}\ (P_n + K_1)$.

A dual of prime labelings has been introduced by Deretsky, Lee, and Mitchem [804]. They say a graph with edge set E has a *vertex prime labeling* if its edges can be labeled with distinct integers $1, \ldots, |E|$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky, Lee, and Mitchem show the following graphs have vertex prime labelings: forests; all connected graphs; $C_{2k} \cup C_n$; $C_{2m} \cup C_{2n} \cup C_{2k+1}$; $C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$; and $5C_{2m}$. They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles. Let $G = \bigcup_{i=1}^{t} C_{2n_i}$ and $N = \sum_{i=1}^{t} n_i$. In [600] Borosh, Hensley and Hobbs proved that there is a positive constant n_0 such that the conjecture of Deretsky et al. is true for the following cases: G is the disjoint union of at most seven cycles; G is a union of cycles all of the same even length $2n$ where $n \leqslant 150\,000$ or where $n \geqslant n_0$; $n_i \geqslant (\log N)^{4 \log\log\log n}$ for all $i = 1, \ldots, t$; and when each C_{2n_i} is repeated at most n_i times. They end their paper with a discussion of graphs whose components are all even cycles, and of graphs with some components that are not cycles and some components that are odd cycles. In [1820] Lavanya and Ganesa provided vertex prime labelings for graphs related to $C_n \odot K_1$.

In [283] Bapat proved the following graphs have vertex prime labelings: kayak paddles $KP(k, m, l)$; books; irregular books not necessarily with pages of the same size; triangular snakes; m-fold triangular snakes of length n obtained from a path $v_1, v_2, \ldots, v_n, v_{n+1}$ by joining v_i and v_{i+1} to new m vertices $w_1^i, w_2^i, \ldots, w_m^i$, for $i = 1, 2, \ldots, n$ giving edges $v_i w_j^i$ and $w_j^i v_{i+1)}$, for $j = 1, \ldots, m,\ i = 1, 2, \ldots, n$; m-fold petal sunflowers obtained from a cycle $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to new m vertices $w_1^i, w_2^i, \ldots, w_m^i$, for $i = 1, 2, \ldots, n$ giving edges $v_i w_j^i$ and $w_j^i v_{i+1)}$ for $j = 1, \ldots, m,\ i = 1, 2, \ldots, n(v_{n+1} = v_1)$; and one-point unions of cycles not necessarily of the same length.

A bijection f from $V(G$ to $\{1, 2, \ldots, |V| + |E|\}$ is said to be a *total prime* if for each edge uv the labels assigned to u and v are relatively prime and for each vertex of degree at least 2, the labels on the incident edges are relatively prime. A graph that admits a total prime labeling is called a *total prime graph*. In [2493] Prajapati and Gajjar defined a *braided star* graph as follows. Let a_0 be the apex vertex and $a_1, a_2, \ldots, a_{n-1}, a_n$ be consecutive n rim vertices of W_n, $(n \geqslant 3)$. Let $b_1, b_2, b_3, \ldots, b_{2n-1}, b_{2n}$ be consecutive $2n$ vertices of C_{2n} $(n > 1)$; and let $c_1, c_2, c_3, \ldots, c_{2n-1}, c_{2n}$ be consecutive $2n$ vertices of a second copy of C_{2n}. Join each a_i to b_{2i-1} by an edge and b_{2i} to c_{2i} by an edge. For each i, join a new vertcies d_i to each c_{2i-1} and c_{2i+1} by an edge taking the subscripts modulo n. They proved that braided stars are prime, total prime, and vertex prime.

In [224] Arockiamary, Baskar Babujee, and Vijayalakshmi say a graph G has a k-*prime total labeling* if k is a positive integer and there is a bijection $f : V(G) \cup E(G) \to \{k, k+1, k+2, \ldots, k+V(G) \cup E(G)-1\}$ such that for every edge uv the labels $f(u), f(v)$, and $f(uv)$ are pairwise relatively prime. A graph that admits a k-prime total labeling is called a k-*prime total graph*. They proved that P_n is k-prime total for odd k, $K_{1,n}$ is k-prime total for odd k and $n \geqslant 2$, the bistar $B(m, n)$ is a $k-$prime total for odd k and $m,\ n \geqslant 2$, and various other tree-related graphs are k-prime total. [224] new

Jothi [1559] calls a graph G *highly vertex prime* if its edges can be labeled with distinct integers $\{1, 2, \ldots, |E|\}$ such that the labels assigned to any two adjacent edges are relatively prime. Such labeling is called a *highly vertex prime* labeling. He proves: if G is highly vertex prime then the line graph of G is prime; cycles are highly vertex prime; paths are highly vertex prime; K_n is highly vertex prime if and only if $n \leqslant 3$; $K_{1,n}$ is highly vertex prime if and only if $n \leqslant 2$; even cycles with a chord are highly vertex prime; $C_p \cup C_q$ is not highly vertex prime when both p and q are odd; and crowns $C_n \odot K_1$ are highly vertex prime.

For a finite simple graph $G(V, E)$ with n vertices and $v \in V$ let $N(v)$ denote the open neighborhood of v. Patel and Shrimali [2364] say a bijective function $f :\to \{1, 2, 3, \ldots, n\}$ is a *neighborhood-prime labeling* of G, if for every vertex $v \in V$ with $deg(v) > 1$,

$gcd\{f(u) : u \in N(v)\} = 1$. A graph that admits a neighborhood-prime labeling is called a *neighborhood-prime* graph. In [2364], [2365], and [2366] they prove the following graphs have a prime-neighborhood labeling: graphs with a vertex of degree $|V| - 1$; paths; C_n if and only if $n \not\equiv 2 \pmod 4$; helms; closed helms; flowers; graphs obtained by the duplication of an arbitrary vertex of cycle or path; $G_1 + G_2$ where each of G_1 and G_2 have at least 2 verticies; $C_n \cup C_m$ is a neighborhood-prime graph if and only if $n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 4$, or $n \equiv 0 \pmod 4$ and $m \equiv 1 \pmod 2$; $W_m \cup W_n$; the union of a finite number of paths; $P_m \times P_n$; and the tensor product of two paths of the same order. They also prove that if G is neighborhood-prime graph and v is a vertex in G that is not adjacent to any pendent vertices, then the graph obtained by duplicating the vertex v is neighborhood-prime [2364].

Patel [2357] showed that the generalized Petersen graph $P(n, k)$ is neighborhood-prime when the greatest common divisor of n and k is $1, 2$, or 4 and that $P(n, 8)$ is neighborhood-prime for all n. Patel and Kansagara [2361] proved that the snake graphs of the type C_k^m are neighborhood-prime if and only if either $k \not\equiv 2 \pmod 4$ or $m \not\equiv 1 \pmod 4$ and that $C_{k,2}^m$ and $C_{k,3}^m$ are neighborhood-prime for all $m \geqslant 2$. Rozario Raj and Sheriff [2659] gave neighborhood-prime labelings for books with triangular and rectangle pages. Shrimali, Rathod, and Vihol [2945] proved the following graphs are neighborhood-prime: graphs obtained from the helm H_n by identifying each vertex of degree 1 with a vertex of W_n, graphs obtained from the helm H_n by identifying each vertex of degree 1 with a vertex of the fan F_n graphs obtained by identifying each pendent vertex of H_n with a vertex of outer cycle of closed helm of H_n, and graphs obtained by identifying each pendent vertex of H_n by vertex of outer cycle of the Petersen graph.

Let $G(V, E)$ be a graph with p vertices and Q edges. Rajesh Kumar and Mathew Varkey [2565] call a bijection from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ an *total neighborhood prime* labeling if for for each vertex of degree at least two, the gcd of labeling on its neighborhood vertices is 1 and for each vertex of degree at least two, the gcd of labeling on the induced edges is 1. They proved that paths, combs, and C_{4n+2} are total neighborhood-prime graphs. Shrimali and Pandya [2940] proved that the following graphs have total neighborhood-prime labelings: combs $P_n \odot K_1$, $P_m \cup P_n$, $(P_m \odot K_1) \cup (P_n \odot K_1)$, $W_m \cup W_n$, graphs obtained from a copy of P_n and n copies $K_{1,n}$ by joining the ith vertex of P_n with an edge to the center vertex of the ith copy of $K_{1,m}$, $C_n \odot mK_1$, and subdivisions of bistars $B_{m,n}$. Shrimali, Rathod, and Vihol [2945] proved that following graphs are neighborhood-prime graphs: the graph obtained by identifying each pendent vertex of a helm H_n with a rim vertex of the wheel W_n; the graph obtained by identifying each pendent vertex of a helm H_n with a vertex of maximum degree of the fan $P_n + K_1$; and the graph obtained by identifying each pendent vertex of H_n with a vertex of outer cycle of closed helm graph $\overline{H_n}$. Rajesh Kumar [1778] proved that double stars, spiders, caterpillars, and firecrackers admit total neighborhood-prime labelings.

In [2564] Rajesh Kumar and Mathew Varkey extend the neighborhood-prime labeling concept to Gaussian integers. Using the spiral order on the Gaussian integers, they showed the following graphs have Gaussian neighborhood-prime labelings: graphs obtained by connecting the centers of two stars with a path, combs $P_n \odot K_1$, spiders, C_n where $n \neq 2$

mod 4, C_4 ($n \geqslant 4$) with a cord, graphs obtained by switching of any vertex C_n, and graphs obtained by duplicating arbitrary vertex of C_n. Rajesh Kumar, Jerome, and Santhosh Kumar [2562] extended the Gaussian neighborhood-prime labeling concept to Gaussian total neighborhood-prime labeling. Among the graphs they prove to have a Gaussian neighborhood-prime labeling are: paths, stars, trees with one vertex of degree at least 3 and all other vertices having degree 1 or 2, and caterpillars. In [2947] Shrimali and Trivedi provided results depending upon the Hamiltonicity of a graph that guarantee that the graph is Gaussian neighborhood-prime graph under the spiral order and proved that generalized Petersen graphs are Gaussian neighborhood-prime graphs under the spiral order. [2947] new

In [2331] Pandya and Shrimali introduce a vertex-edge neighborhood-prime labeling of graphs as follows. For a graph G an injective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G|\}$ is said to be a *vertex-edge neighborhood-prime labeling* if it has the following properties: If u has degree 1, then $\gcd\{f(w), f(uw)\} = 1$ taken over all vertices w adjacent to u; if u has degree greater than 1, then $\gcd\{f(w)\} = 1$ taken over all vertices w adjacent to u and $\gcd\{f(wu)\} = 1$ taken over all vertices w adjacent to u. A graph that admits vertex-edge neighborhood-prime labeling is called a *vertex-edge neighborhood-prime* graph. They give vertex-edge neighborhood-prime labelings for paths, helms, $C_n \oplus K_1$, bistars, the central edge subdivision of bistars, and subdivisions of edges of bistars. They observe that every vertex-edge neighborhood-prime graph is a total neighborhood-prime graph and that a total neighborhood-prime graph that does not have a vertex of degree 1 is vertex-edge neighborhood-prime. Shrimali and Rathod [2944] proved that coconut trees, double coconut trees, spiders with legs of equal length, olive trees, combs, and $F(n, 2)$-firecrackers have vertex-edge neighborhood-prime labelings.

Patel and Ghodasara [2358] proved the following graphs are neighborhood-prime: one point union $C_n^{(k)}$ ($k \geqslant 2, n \geqslant 3$) of k copies of cycle C_n, the barycentric subdivision of wheels and gears, the middle and total graph of crowns $C_n \odot K_1$ ($n \geqslant 3$), the square of crowns, tadpoles $T(n, l)$ ($n \geqslant 3, l \geqslant 1$), cycles, and umbrellas. In [796] Delman, Koilraj, and Raj gave neighborhood-prime labelings of arbitrary super subdivision of helms, tadpoles, and triangular snakes.

In [2507] Prajapati and Shah introduce an odd prime labeling as follows. Let $G(V, E)$ be a graph. A bijection f from V to $\{1, 3, \ldots, 2|V| - 1\}$ is called an *odd prime labeling* if for each edge uv, $\gcd(f(u), f(v)) = 1$. A graph that admits odd prime labeling is called an *odd prime graph.* They prove paths, ladders, complete bipartite graphs, wheels, gears, flowers, helms, closed helms, and generalized Petersen graphs $P(n, 2)$ are odd prime and conjecture that generalized Petersen graphs $P(n, k)$ and every prime graph is an odd prime graph. In [2508] they proved the following graphs are odd prime graphs: graphs obtained by duplication of a vertex of paths, stars, and wheels, and graphs obtained by dupilcation of an edge of cycles, stars, and wheels. Carter and Fox [660] prove that the disjoint union of cycles; cycle chains; snakes; books; triangular, pentagonal, and hexagonal stacked prisms; spiders; perfect binary trees; special cases of caterpillars; and firecrackers are odd prime. They characterize when odd prime labelings exist for powers of paths and cycles and make progress towards proving the conjecture that all prime graphs are also odd prime. Meena and Gajalakshmi [2125] proved that the corona product of $C_n \times P_2$ and $K_1, \overline{K_2}$, and $\overline{K_3}$ are odd prime graphs. [2125] new

For a graph $G(V,E)$ with p vertices and q edges Shiu, Lau, and Lee [2913] call a bijection f from E to $\{1,2,\ldots,q\}$ an *edge-prime labeling* if for each edge uv in E, we have $\gcd(f^+(u), f^+(v)) = 1$, where $f^+(u) = \Sigma f(uw)$ over all $uw \in E$. A graph that admits an edge-prime labeling is called an *edge-prime graph.* A bijection f from E to $\{1,2,\ldots,q\}$ is an *semi-edge-prime labeling* if for each edge uv in E, we have $\gcd(f^+(u), f^+(v)) = 1$ or $f^+(u) = f^+(v)$. They obtained a necessary and sufficient condition for the disjoint union of paths to be edge-prime, proved that all 2-regular graphs are edge-prime, proved that many bipartite and tripartite graphs are edge-prime (or not edge-prime), and showed that certain bipartite and tripartite graphs are semi-edge-prime graphs. In [1806] Lau, Lee, and Shiu proved that if G is a cubic graph and every component is of order 4, 6 or 8, then G is edge-prime if and only if $G \not\approx K_4$ or $nK_{3,3}$ and $n = 2$ or 3 (mod 4). They conjectured that a connected cubic graph G is not edge-prime if and only if $G \approx K_4$.

In [1352] Jagadesh and Baskar Babujee introduced an *edge vertex prime* labeling of a graph G as an injection f from $V(G) \cup E(G)$ to $\{1,2,\ldots,|V(G)|+|E(G)|\}$ such that for every edge uv, the labels $f(u), f(v)$, and $f(uv)$ are pairwise relatively prime. A graph that admits such a labeling is called an *edge vertex prime* graph. They proved that paths, cycles, and stars are are edge vertex prime. In [2346] and [2347] Parmar proved that wheels, fans, friendship graphs, and $K_{2,n}$ are edge vertex prime. Simaringa and Muthukumaran [2960] proved that following graphs have edge vertex prime labelings: triangular and rectangular books, butterfly graphs, $\overline{K_n} \cup K_{1,m}$, $K_{1,m} + K_1$, $\overline{K_m} \cup \overline{K}_n$, Jahangir graphs $J_{n,3}$ and $J_{n,4}$. In [2961] Simaringa and Muthukumaran investigated the existence of edge vertex prime labelings for crowns, unions of cycles, and wheel relate graphs. Shrimali and Parmar [2941] proved that the following graphs have edge vertex prime labelings: bistars $B(m,n)$, n-centipede trees, coconut trees obtained from the path P_n by appending m new pendent edges at an end vertex of P_n), double coconut trees (graphs obtained by attaching $n > 1$ pendent vertices to one end of the path P_r and $m > 1$ pendent vertices to the other end of path P_r), and special classes of banana trees and firecrackers. In [2942] Shrimali and Parmar showed that helms and graphs obtained by joining a fan ($n \geqslant 2$) and C_{2n} by sharing a common vertex are edge vertex prime graphs and that $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k}$ and $K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$ are not an edge vertex prime graphs. They also provided a necessary condition for being edge vertex prime graph. Rilwan and Radha [2624] investigated edge vertex prime labelings and super edge vertex prime labelings of Cayley graphs and Cayley digraphs of finite groups. Baskar Babujee and Jagadesh [498] proved generalized stars $K_{1,n_1,n_2,n_3,\ldots,n_m}$ and generalized cycle stars $C_n * (P_{n_1} \cup P_{n_2} m \cup \ldots \cup P_{n_m)})$ that are obtained by joining one of the pendant vertices of each of the paths with an edge to any vertex of C_n, admit edge vertex prime labelings. Simaringa and Muthukumaran [2962] proved that the following graphs are edge vertex prime graphs: $C_m \cup K_{1,n}, C_m \cup P_n, K_{2,m} \cup C_n, C_n \cup C_n$ when $n = 0$ or 2 mod n, and the one point union of wheel and cycle-related graphs. They also proved that for graphs G with $|V(G)| + |E(G)|$ even $G \cup K_{1,n}$ and $G \cup P_n$ are edge vertex prime graphs. [498] new [2962] new

In [3558] Youssef and Almoreed gave a new variation of the prime labeling as follows:

A graph $G(V,E)$ has an *odd prime labeling* if its vertices can be labeled with distinct odd integers from 1 to $2|V(G)|-1$ such that for every edge xy in E the labels assigned to the vertices of x and y are relatively prime. A graph that admits an odd prime labeling is called an *odd prime graph*. They provided some families of odd prime graphs and give some necessary conditions for a graph to be odd prime. They conjectured that every prime graph is odd prime graph.

A *coprime* labeling of vertices of a graph G with distinct labels from the set $\{1,\ldots,m\}$ for some integer $m \geqslant n$ such that adjacent labels are relatively prime. The minimum value m for which G has a coprime labeling is defined as the *minimum coprime number* , denoted by $\mathrm{pr}(G)$, and a coprime labeling of G with largest label being $\mathrm{pr}(G)$ is called a *minimum coprime* labeling of G. Obviously, if G is a prime graph of order n, then $\mathrm{pr}(G)=n$. In [261] Asplund and Fox focus on the problem of determining the minimum coprime number for graphs that have been shown to not be prime. Amomg them are K_n ($n \geqslant 4$), W_{2n+1}, and the union of odd cycles. C. Lee [1828] determined the minimum coprime number of coronas of complete graphs with empty graphs, the joins of two paths, and prisms. She also proved that gears are prime, double wheels DW_n are prime if and only if n is even, and the graph that obtained by attaching P_2 to each vertex of C_n followed by attaching the star S_m at its center to each pendent vertex is prime.

In [2384] Periasamy, Venugopal, and Rozario Raj introduced the kth Fibonacci prime labeling of graphs as follows. A kth *Fibonacci prime labeling of* a graph $G(V,E)$ with $|V(G)|=n$ is an injective function g from $V(G)$ to $\{f_k, f_{k+1},\ldots,f_{k+n-1}\}$ where f_k is the kth Fibonacci number, that induces a function g^* from $E(G)$ to the non-negative integers defined by $g(uv)=\gcd(g(u),g(v))=1$ for all $uv \in E(G)$. A graph that admits a kth Fibonacci prime labeling and is called a kth *Fibonacci prime* graph. They proved that paths, cycles, $P_n \odot K_1$, triangular snakes, quadrilateral snakes, and tadpoles are kth Fibonacci prime graphs.

The tables following summarize the state of knowledge about prime labelings and vertex prime labelings. In the table, **P** means prime labeling exists, and **VP** means vertex prime labeling exists. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property.

Table 21: **Summary of Prime Labelings**

Graph	*Types*	*Notes*
P_n	P	[990]
stars	P	[990]
complete binary trees	P	[990]
spiders	P	[990]
trees	P?	[1921]
C_n	P	[804]
$C_n \cup C_{2m}$	P	[804]
K_n	P	iff $n \leqslant 3$ [1921]
W_n	P	iff n is even [3221]
helms	P	[2769]
fans	P	[2769]
flowers	P	[2769]
$K_{2,n}$	P	[2769]
$K_{3,n}$	P	$n \neq 3, 7$ [2769]
$P_n + \overline{K_m}$	not P	$n \geqslant 3$ [2769]
$P_n + \overline{K_2}$	P	iff $n = 2$ or n is odd [2769]
books	P	[2797]
$C_m + C_n$	not P	[2797]
C_n^2	not P	$n \geqslant 4$ [2797]
P_n^2	not P	$n \geqslant 6,\ n \neq 7$ [2797]

Continued on next page

Table 21 – *Continued from previous page*

Graph	*Types*	*Notes*
M_n (Möbius ladders)	not P	n even [2797]
$S_m \cup S_n$	P	[3559]
$C_m \cup S_n$	P	[3559]
$K_m \cup S_n$	P	iff no. of primes $\leqslant m+n+1$ is at least m [3559]
$K_n \odot K_1$	P	iff $n \leqslant 7$ [3559]
$P_m \times P_n$ (grids)	P	$m \leqslant 3,\ n$ prime [3403] m prime $> 2,\ p < n \leqslant p^2$ [758]
$C_n \odot \overline{K_i}$ (crowns)	P	[3221]
$P_n \odot \overline{K_2}$	P	iff $n \neq 2$ [3403]
C_m-snakes (see §2.2)	P	[658]
unicyclic	P?	[2769]

Table 22: **Summary of Vertex Prime Labelings**

Graph	*Types*	*Notes*
$C_m + C_n$	not P	[2797]
C_n^2	not P	$n \geqslant 4$ [2797]
P_n	not P	$n = 6,\ n \geqslant 8$ [2797]
M_{2n} (Möbius ladders)	not P	[2797]
connected graphs	VP	[804]
forests	VP	[804]

Continued on next page

Table 22 – *Continued from previous page*

Graph	*Types*	*Notes*
$C_{2m} \cup C_n$	VP	[804]
$C_{2m} \cup C_{2n} \cup C_{2k+1}$	VP	[804]
$C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$	VP	[804]
$5C_{2m}$	VP	[804]
$G \cup H$	VP	if G, H are connected and one is not an odd cycle [804]
2-regular graph G	not VP	G has at least 2 odd cycles
[804]		
	VP?	iff G has at most 1 odd cycle [804]

7.3 Edge-graceful Labelings

In 1985, Lo [1984] introduced the notion of edge-graceful graphs. A graph $G(V, E)$ is said to be *edge-graceful* if there exists a bijection f from E to $\{1, 2, \ldots, |E|\}$ such that the induced mapping f^+ from V to $\{0, 1, \ldots, |V|-1\}$ given by $f^+(x) = (\sum f(xy)) \pmod{|V|}$ taken over all edges xy is a bijection. Note that an edge-graceful graph is antimagic (see §6.1). A necessary condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv p(p+1)/2 \pmod{p}$. Lee [1838] notes that this necessary condition extends to any multigraph with p vertices and q edges. It was conjectured by Lee [1838] that any connected simple (p, q)-graph with $q(q+1) \equiv p(p-1)/2 \pmod{p}$ vertices is edge-graceful. Lee, Kitagaki, Young, and Kocay [1844] prove that the conjecture is true for maximal outerplanar graphs. Lee and Murthy [1830] proved that K_n is edge-graceful if and only if $n \not\equiv 2 \pmod{4}$. (An edge-graceful labeling given in [1984] for K_n for $n \not\equiv 2 \pmod{4}$ is incorrect.) Lee [1838] notes that a multigraph with $p \equiv 2 \pmod{4}$ vertices is not edge-graceful and conjectures that this condition is sufficient for the edge-gracefulness of connected graphs. Lee [1837] has conjectured that all trees of odd order are edge-graceful. Small [3006] has proved that spiders for which every vertex has odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [1687] proved that all spiders of odd order with exactly three end vertices are edge-graceful. Cabaniss, Low, and Mitchem [636] have shown that regular spiders of odd order are edge-graceful. For a (p, q) connected edge-graceful graph G with $q = kp + r$, where $k$is aninteger and $0 \leqslant r < p$.

Kayathri and Amutha [1679] proved that every edge-graceful labeling f of G induces $((k+1)!)^r(k!)^{p-r}$ edge-graceful labelings of G.

Lee and Seah [1883] have shown that $K_{n,n,\ldots,n}$ is edge-graceful if and only if n is odd and the number of partite sets is either odd or a multiple of 4. Lee and Seah [1882] have also proved that C_n^k (the kth power of C_n) is edge-graceful for $k < \lfloor n/2 \rfloor$ if and only if n is odd and C_n^k is edge-graceful for $k \geqslant \lfloor n/2 \rfloor$ if and only if $n \not\equiv 2 \pmod 4$ (see also [636]). Lee, Seah, and Wang [1888] gave a complete characterization of edge-graceful P_n^k graphs. Shiu, Lam, and Cheng [2906] proved that the composition of the path P_3 and any null graph of odd order is edge-graceful. Uma and Mazuda Shanofer [3234] prove that $C_{2n+1} \odot \overline{K_2}$ is edge graceful and the graphs obtained by starting with C_n and, for each edge of C_n, adjoining a copy of C_n that shares an edge with the starting copy (the *flower graph* FL_n) is not edge-graceful.

Lo [1984] proved that all odd cycles are edge-graceful and Wilson and Riskin [3480] proved the Cartesian product of any number of odd cycles is edge-graceful. Lee, Ma, Valdes, and Tong [1858] investigated the edge-gracefulness of grids $P_m \times P_n$. The necessity condition of Lo [1984] that a (p,q) graph must satisfy $q(q+1) \equiv 0$ or $p/2 \pmod p$ severely limits the possibilities. Lee et al. prove the following: $P_2 \times P_n$ is not edge-graceful for all $n > 1$; $P_3 \times P_n$ is edge-graceful if and only if $n = 1$ or $n = 4$; $P_4 \times P_n$ is edge-graceful if and only if $n = 3$ or $n = 4$; $P_5 \times P_n$ is edge-graceful if and only if $n = 1$; $P_{2m} \times P_{2n}$ is edge-graceful if and only if $m = n = 2$. They conjecture that for all $m, n \geqslant 10$ of the form $m = (2k+1)(4k+1)$, $n = (2k+1)(4k+3)$, the grids $P_m \times P_n$ are edge-graceful. Riskin and Weidman [2631] proved: if G is an edge-graceful $2r$-regular graph with p vertices and q edges and $(r, kp) = 1$, then kG is edge-graceful when k is odd; when n and k are odd, kC_n^r is edge-graceful; and if G is the cartesian product of an odd number of odd cycles and k is odd, then kG is edge-graceful. They conjecture that the disjoint union of an odd number of copies of a $2r$-regular edge-graceful graph is edge-graceful.

Shiu, Lee, and Schaffer [2917] investigated the edge-gracefulness of multigraphs derived from paths, combs, and spiders obtained by replacing each edge by k parallel edges. Lee, Ng, Ho, and Saba [1868] construct edge-graceful multigraphs starting with paths and spiders by adding certain edges to the original graphs. Lee and Seah [1884] have also investigated edge-gracefulness of various multigraphs.

Lee and Seah (see [1838]) define a *sunflower* graph $SF(n)$ as the graph obtained by starting with an n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ and creating new vertices $w_1, w_2, \ldots, w_n$ with w_i connected to v_i and v_{i+1} (v_{n+1} is v_1). In [1885] they prove that $SF(n)$ is edge-graceful if and only if n is even. In the same paper they prove that C_3 is the only triangular snake that is edge-graceful. Lee and Seah [1882] prove that for $k \leqslant n/2$, C_n^k is edge-graceful if and only if n is odd, and for $k \geqslant n/2$, C_n^k is edge-graceful if and only if $n \not\equiv 2 \pmod 4$. Lee, Seah, and Lo (see [1838]) have proved that for n odd, $C_{2n} \cup C_{2n+1}, C_n \cup C_{2n+2}$, and $C_n \cup C_{4n}$ are edge-graceful. They also show that for odd k and odd n, kC_n is edge-graceful. Lee and Seah (see [1838]) prove that the generalized Petersen graph $P(n,k)$ (see Section 2.7 for the definition) is edge-graceful if and only if n is even and $k < n/2$. In particular, $P(n,1) = C_n \times P_2$ is edge-graceful if and only if n is even.

Schaffer and Lee [2725] proved that $C_m \times C_n$ $(m > 2, n > 2)$ is edge-graceful if and only if m and n are odd. They also showed that if G and H are edge-graceful regular graphs of odd order then $G \times H$ is edge-graceful and that if G and H are edge-graceful graphs where G is c-regular of odd order m and H is d-regular of odd order n, then $G \times H$ is edge-magic if $\gcd(c, n) = \gcd(d, m) = 1$. They further show that if H has odd order, is $2d$-regular and edge-graceful with $\gcd(d, m) = 1$, then $C_{2m} \times H$ is edge-magic, and if G is odd-regular, edge-graceful of even order m that is not divisible by 3, and G can be partitioned into 1-factors, then $G \times C_m$ is edge-graceful.

In 1987 Lee (see [1886]) conjectured that $C_{2m} \cup C_{2n+1}$ is edge-graceful for all m and n except for $C_4 \cup C_3$. Lee, Seah, and Lo [1886] have proved this for the case that $m = n$ and m is odd. They also prove: the disjoint union of an odd number copies of C_m is edge-graceful when m is odd; $C_n \cup C_{2n+2}$ is edge-graceful; and $C_n \cup C_{4n}$ is edge-graceful for n odd. Bu [617] gave necessary and sufficient conditions for graphs of the form $mC_n \cup P_{n-1}$ to be edge-graceful.

Kendrick and Lee (see [1838]) proved that there are only finitely many n for which $K_{m,n}$ is edge-graceful and they completely solve the problem for $m = 2$ and $m = 3$. Ho, Lee, and Seah [1252] use $S(n; a_1, a_2, \ldots, a_k)$ where n is odd and $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k < n/2$ to denote the (n, nk)-multigraph with vertices $v_0, v_1, \ldots, v_{n-1}$ and edge set $\{v_i v_j |\ i \neq j, i - j \equiv a_t \pmod{n}$ for $t = 1, 2, \ldots, k\}$. They prove that all such multigraphs are edge-graceful. Lee and Pritikin (see [1838]) prove that the Möbius ladders (see §2.2 for definition) of order $4n$ are edge-graceful. Lee, Tong, and Seah [1904] have conjectured that the total graph of a (p, p)-graph is edge-graceful if and only if p is even. They have proved this conjecture for cycles. In [1695] Khodkar and Vinhage proved that there exists a super edge-graceful labeling of the total graph of $K_{1,n}$ and the total graph of C_n. Wang and Zang [3451] proved that a regular graph of odd degree is edge-graceful if it contains either a quasi-prism factor or a claw factor.

Kuang, Lee, Mitchem, and Wang [1768] have conjectured that unicyclic graphs of odd order are edge-graceful. They have verified this conjecture in the following cases: graphs obtained by identifying an endpoint of a path P_m with a vertex of C_n when $m + n$ is even; crowns with one pendent edge deleted; graphs obtained from crowns by identifying an endpoint of P_m, m odd, with a vertex of degree 1; amalgamations of a cycle and a star obtained by identifying the center of the star with a cycle vertex where the resulting graph has odd order; graphs obtained from C_n by joining a pendent edge to $n - 1$ of the cycle vertices and two pendent edges to the remaining cycle vertex.

In [3452] Wang and Zhang introduced the notion called edge-graceful deficiency, which is a parameter to measure how close a graph is away from being an edge-graceful graph. The *edge-graceful deficiency* of a graph G is the minimum value of k such that the edge labeling $f E \to \{1, 2, \ldots, q+k\}$ is edge-graceful. They proved that an odd regular graph is edge-graceful if it contains a quasi-prism factor or a claw factor and completely determine the edge-graceful deficiency of Hamiltonian regular graphs of even degree.

Gayathri and Subbiah [1062] say a graph $G(V, E)$ has a *strong edge-graceful* labeling if there is an injection f from the E to $\{1, 2, 3, \ldots, \langle 3|E|/2\rangle\}$ such that the induced mapping f^+ from V defined by $f^+(u) = (\Sigma f(uv)) \pmod{2|V|}$ taken all edges uv is an injection. They proved the following graphs have strong edge graceful labelings: $P_n (n \geqslant 3), C_n, K_{1,n} (n \geqslant 2)$, crowns $C_n \odot K_1$, and fans $P_n + K_1 (n \geqslant 2)$. In his Ph.D. thesis [3060] Subbiah provided edge-graceful and strong edge-graceful labelings for a large variety of graphs. Among them are bistars, twigs, y-trees, spiders, flags, kites, friendship graphs, mirror of paths, flowers, sunflowers, graphs obtained by identifying a vertex of a cycle with an endpoint of a star, and $K_2 \odot C_n$, and various disjoint unions of path, cycles, and stars.

Hefetz [1205] has shown that a graph $G(V, E)$ of the form $G = H \cup f_1 \cup f_2 \cup \cdots \cup f_r$ where $H = (V, E')$ is edge-graceful and the f_i's are 2-factors is also edge-graceful and that a regular graph of even degree that has a 2-factor consisting of k cycles each of length t where k and t are odd is edge-graceful.

Bača and Holländer [334] investigated a generalization of edge-graceful labeling called (a, b)-consecutive labelings. A connected graph $G(V, E)$ is said to have an (a, b)-*consecutive labeling* where a is a nonnegative integer and b is a positive proper divisor of $|V|$, if there is a bijection from E to $\{1, 2, \ldots, |E|\}$ such that if each vertex v is assigned the sum of all edges incident to v the vertex labels are distinct and they can be partitioned into $|V|/b$ intervals
$W_j = [w_{\text{min}} = (j-1)b + (j-1)a, w_{\text{min}} + jb + (j-1)a - 1]$, where $1 \leqslant j \leqslant p/b$ and w_{min} is the minimum value of the vertices. They present necessary conditions for (a, b)-consecutive labelings and describe (a, b)-consecutive labelings of the generalized Petersen graphs for some values of a and b.

A graph with p vertices and q edges is said to be k-*edge-graceful* if its edges can be labeled with $k, k+1, \ldots, k+q-1$ such that the sums of the edges incident to each vertex are distinct modulo p. In [1907] Lee and Wang show that for each $k \neq 1$ there are only finitely many trees that are k-edge graceful (there are infinitely many 1-edge graceful trees). They describe completely the k-edge-graceful trees for $k = 0, 2, 3, 4$, and 5. Gayathri and Sarada Devi [1037] obtained some necessary conditions and characterizations for k-edge-gracefulness of trees. They also proved that specific families of trees are edge-graceful and k-edge-graceful and conjecture that all odd trees are k-edge-graceful.

Gayathri and Sarada Devi [822] defined a k-*even edge-graceful* labeling of a (p, q) graph $G(V, E)$ as an injection f from E to $\{2k-1, 2k, 2k+1, \ldots, 2k+2q-2\}$ such that the induced mapping f^+ of V defined by $f^+(x) = \sum f(xy) \pmod{2s}$ taken over all edges xy, are distinct and even, where $s = \max\{p, q\}$ and k is a positive integer. A graph G that admits a k-even-edge-graceful labeling is called a k-*even-edge-graceful* graph. In [822], [1038], [1039], and [1040] Gayathri and Sarada Devi investigate the k-even edge-gracefulness of a wide variety of graphs. Among them are: paths; stars; bistars; cycles with a pendent edge; cycles with a cord; crowns $C_n \odot K_1$; graphs obtained from P_n by replacing each edge by a fixed number of parallel edges; and sparklers (paths with a star appended at an endpoint of the path).

In 1991 Lee [1838] defined the *edge-graceful spectrum* of a graph G as the set of all nonnegative integers k such that G has a k-edge graceful labeling. In [1910] Lee, Wang, Ng, and Wang determine the edge-graceful spectrum of the following graphs: $G \odot K_1$ where G is an even cycle with one chord; two even cycles of the same order joined by an edge; and two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex (*butterfly graph*). Lee, Chen, and Wang [1841] have determined the edge-graceful spectra for various cases of cycles with a chord and for certain cases of graphs obtained by joining two disjoint cycles with an edge (i.e., *dumbbell graphs*). More generally, Shiu, Ling, and Low [2919] call a connected with p vertices and $p+1$ edges *bicyclic*. In particular, the family of bicyclic graphs includes the one-point union of two cycles, two cycles joined by a path and cycles with one cord. In [2920] they determine the edge-graceful spectra of bicyclic graphs that do not have pendent edges. Kang, Lee, and Wang [1632] determined the edge-graceful spectra of wheels and Wang, Hsiao, and Lee [3434] determined the edge-graceful spectra of the square of P_n for odd n. Results about the edge-graceful spectra of three types of $(p, p+1)$-graphs are given by Chen, Lee, and Wang [691]. In [3435] Wang and Lee determine the edge-graceful spectra of the one-point union of two cycles, the corona product of the one-point union of two cycles with K_1, and the cycles with one chord.

Lee, Levesque, Lo, and Schaffer [1852] investigate the edge-graceful spectra of cylinders. They prove: for odd $n \geqslant 3$ and $m \equiv 2 \pmod 4$, the spectra of $C_n \times P_m$ is $\emptyset$; for $m = 3$ and $m \equiv 0, 1$ or $3 \pmod 4$, the spectra of $C_4 \times P_m$ is $\emptyset$; for even $n \geqslant 4$, the spectra of $C_n \times P_2$ is all natural numbers; the spectra of $C_n \times P_4$ is all odd positive integers if and only if $n \equiv 3 \pmod 4$; and $C_n \times P_4$ is all even positive integers if and only if $n \equiv 1 \pmod 4$. They conjecture that $C_4 \times P_m$ is k-edge-graceful for some k if and only if $m \equiv 2 \pmod 4$. Shiu, Ling, and Low [2920] determine the edge-graceful spectra of all connected bicyclic graphs without pendent edges.

A graph $G(V, E)$ is called *super edge-graceful* if there is a bijection f from E to $\{0, \pm1, \pm2, \ldots, \pm(|E|-1)/2\}$ when $|E|$ is odd and from E to $\{\pm1, \pm2, \ldots, \pm|E|/2\}$ when $|E|$ is even such that the induced vertex labeling f^* defined by $f^*(u) = \Sigma f(uv)$ over all edges uv is a bijection from V to $\{0, \pm1, \pm2, \ldots, \pm(p-1)/2\}$ when p is odd and from V to $\{\pm1, \pm2, \ldots, \pm p/2\}$ when p is even. Lee, Wang, Nowak, and Wei [1911] proved the following: $K_{1,n}$ is super-edge-magic if and only if n is even; the double star $DS(m, n)$ (that is, the graph obtained by joining the centers of $K_{1,m}$ and $K_{1,n}$ with an edge) is super edge-graceful if and only if m and n are both odd. They conjecture that all trees of odd order are super edge-graceful. In [1899] Lee, Su, and Wei exhibit a family of trees of odd orders which are super edge-graceful. Chung, Lee, Gao and Schaffer [726] posed the problems of characterizing the paths and tress of diameter 4 that are super edge-graceful.

In [725] Chung, Lee, and Gao prove various classes of caterpillars, combs, and amalgamations of combs and stars of even order are super edge-graceful. Lee, Sun, Wei, Wen, and Yiu [1900] proved that trees obtained by starting with the paths the P_{2n+2} or P_{2n+3} and identifying each internal vertex with an endpoint of a path of length 2 are super edge-graceful.

Shiu [2891] has shown that $C_n \times P_2$ is super-edge-graceful for all $n \geqslant 2$. More generally, he defines a family of graphs that includes $C_n \times P_2$ and generalized Petersen graphs are follows. For any permutation θ on n symbols without a fixed point the *θ-Petersen graph* $P(n; \theta)$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i u_{i+1}, u_i w_i, w_i w_{\theta(i)} \mid 1 \leqslant i \leqslant n\}$ where addition of subscripts is done modulo n. (The graph $P(n;\theta)$ need not be simple.) Shiu proves that $P(n;\theta)$ is super-edge-graceful for all $n \geqslant 2$. He also shows that certain other families of connected cubic multigraphs are super-edge-graceful and conjectures that every connected cubic of multigraph except K_4 and the graph with 2 vertices and 3 edges is super-edge-graceful.

In [2904] Shiu and Lam investigated the super-edge-gracefulness of fans and wheel-like graphs. They showed that fans F_{2n} and wheels W_{2n} are super-edge-graceful. Although F_3 and W_3 are not super-edge-graceful the general cases F_{2n+1} and W_{2n+1} are open. For a positive integer n_1 and even positive integers $n_2, n_3, \ldots, n_m$ they define an *m-level wheel* as follows. A wheel is a 1-level wheel and the cycle of the wheel is the 1-level cycle. An i-level wheel is obtained from an $(i-1)$-level wheel by appending $n_i/2$ pairs of edges from any number of vertices of the $i-1$-level cycle to n_i new vertices that form the vertices in the i-level cycle. They prove that all m-level wheels are super-edge-graceful. They also prove that for n odd $C_m \odot \overline{K_n}$ is super-edge-graceful, for odd $m \geqslant 3$ and even $n \geqslant 2$ $C_m \odot \overline{K_n}$ is edge-graceful, and for $m \geqslant 3$ and $n \geqslant 1$ $C_m \odot \overline{K_n}$ is super-edge-graceful. For a cycle C_m with consecutive vertices $v_1, v_2, \ldots, v_m$ and nonnegative integers $n_1, n_2, \ldots, n_m$ they define the graph $A(m; n_1, n_2, \ldots, n_m)$ as the graph obtained from C_m by attaching n_i edges to the vertex v_i for $1 \leqslant i \leqslant m$. They prove $A(m; n_1, n_2, \ldots, n_m)$ is super-edge-graceful if m is odd and $A(m; n_1, n_2, \ldots, n_m)$ is super-edge-graceful if m is even and all the n_i are positive and have the same parity. Chung, Lee, Gao, and Schaffer [726] provide super edge-graceful labelings for various even order paths, spiders and disjoint unions of two stars. In [723] Chung and Lee characterize spiders of even orders that are not super-edge-graceful and exhibit some spiders of even order of diameter at most four that are super-edge-graceful. They raised the question of which paths are super edge-graceful. This was answered by Cichacz, Fronček, and Xu [743] who showed that the only paths that are not super edge-graceful are P_2 and P_4. Cichacz et al. also proved that the only cycles that are not super edge-graceful are C_4 and C_6. Gao and Zhang [1030] proved that some cases of caterpillars are super edge-graceful.

In [726] Chung, Lee, Gao, and Schaffer asked for a characterize trees of diameter 4 that are super edge-graceful. Krop, Mutiso, and Raridan [1765] provide a super edge-graceful labelings for all caterpillars and even size lobsters of diameter 4 that permit such labelings. They also provide super edge-graceful labelings for several families of odd size lobsters of diameter 4. They were unable to find general methods that describe super edge-graceful labelings for a few families of odd size lobsters of diameter 4, although they are able to show that certain lobsters in these families are super-edge graceful. They conclude with three conjectures about rooted trees of height 2 and diameter 4.

Although it is not the case that a super edge-graceful graph is edge-graceful, Lee, Chen, Yera, and Wang [1840] proved that if G is a super edge-graceful with p vertices and q edges and $q \equiv -1 \pmod{p}$ when q is even, or $q \equiv 0 \pmod{p}$ when q is odd, then G is also edge-graceful. They also prove: the graph obtained from a connected super edge-graceful unicyclic graph of even order by joining any two nonadjacent vertices by an edge is super edge-graceful; the graph obtained from a super edge-graceful graph with p vertices and $p+1$ edges by appending two edges to any vertex is super edge-graceful; and the one-point union of two identical cycles is super edge-graceful. Collins, Magnant, and Wang [752] present a stronger concept of "tight" super-edge-graceful labeling. Such a super-edge graceful labeling has an additional constraint on the edge and vertices with the largest and smallest labels. They use this concept to recursively construct tight super-edge graceful trees of any order.

Gayathri, Duraisamy, and Tamilselvi [1042] calls a (p, q)-graph with $q \geqslant p$ *even edge-graceful* if there is an injection f from the set of edges to $\{1, 2, 3, \ldots, 2q\}$ such that the values of the induced mapping f^+ from the vertex set to $\{0, 1, 2, \ldots, 2q-1\}$ given by $f^+(x) = (\Sigma f(xy))(\text{mod } 2q)$ over all edges xy are distinct and even. In [1042] and [1041] Gayathri et al. prove the following: cycles are even edge-graceful if and only if the cycles are odd; even cycles with one pendent edge are even edge-graceful; wheels are even edge-graceful; gears (see §2.2 for the definition) are not even edge-graceful; fans $P_n + K_1$ are even edge-graceful; $C_4 \cup P_m$ for all m are even edge-graceful; $C_{2n+1} \cup P_{2n+1}$ are even edge-graceful; crowns $C_n \odot K_1$ are even edge-graceful; $C_n^{(m)}$ (see §2.2 for the definition) are even edge-graceful; sunflowers (see §3.7 for the definition) are even edge-graceful; triangular snakes (see §2.2 for the definition) are even edge-graceful; closed helms (see §2.2 for the definition) with the center vertex removed are even edge-graceful; graphs decomposable into two odd Hamiltonian cycles are even edge-graceful; and odd order graphs that are decomposable into three Hamiltonian cycles are even edge-graceful.

In [1041] Gayathri and Duraisamy generalized the definition of even edge-graceful to include (p, q)-graphs with $q < p$ by changing the modulus from $2q$ the maximum of $2q$ and $2p$. With this version of the definition, they have shown that trees of even order are not even edge-graceful whereas, for odd order graphs, the following are even edge-graceful: banana trees (see §2.1 for the definition); graphs obtained joining the centers of two stars by a path; $P_n \odot K_{1,m}$; graphs obtained by identifying an endpoint from each of any number of copies of P_3 and P_2; bistars (that is, graphs obtained by joining the centers of two stars with an edge); and graphs obtained by appending the endpoint of a path to the center of a star. They define odd edge-graceful graphs in the analogous way and provide a few results about such graphs. Kathiresan, Muthumari, and Ramalakshmi [1673] proved that $K_n^c \vee 2K_2$ and the flower graph FL_n $(n \geqslant 3)$ admit edge odd graceful labelings. They further prove that $P_{a,b}$ (the graph obtained by identifying the end points of b internally disjoint paths each of length a) admit odd edge graceful labelings when a and b are odd, whereas $P_{a,2}$ is not edge odd graceful when $a \geqslant 2$.

Paley graphs are dense undirected graphs raised from the vertices as elements of an appropriate finite field by joining pairs of vertices that differ by a quadratic residue. In [1571], Kamaraj and Thangakani study the construction of edge even (odd) graceful labeling for Paley graphs and prove that Paley graphs of prime order are edge even (odd) graceful. Lee, Pan, and Tsai [1874] call a graph G with p vertices and q edges *vertex-graceful* if there exists a labeling $f\, V(G) \to \{1, 2, \ldots, p\}$ such that the induced labeling f^+ from $E(G)$ to Z_q defined by $f^+(uv) = f(u) + f(v) \pmod q$ is a bijection. Vertex-graceful graphs can be viewed the dual of edge-graceful graphs. They call a vertex-graceful graph *strong vertex-graceful* if the values of $f^+(E(G)$ are consecutive. They observe that the class of vertex-graceful graphs properly contains the super edge-magic graphs and strong vertex-graceful graphs are super edge-magic. They provide vertex-graceful and strong vertex-graceful labelings for various $(p, p+1)$-graphs of small order and their amalgamations.

Shiu and Wong [2933] proved the one-point union of an m-cycle and an n-cycle is vertex-graceful only if $m+n \equiv 0 \pmod 4$; for $k \geqslant 2$, $C(3, 4k-3)$ is strong vertex-graceful; $C(2n+3, 2n+1)$ is strong vertex-graceful for $n \geqslant 1$; and if the one-point union of two cycles is vertex-graceful, then it is also strong vertex-graceful. In [3020] Somashekara and Veena found the number of $(n, 2n-3)$ strong vertex graceful graphs. Gao, Zhang, and Xu [1016] proved that C_n, $C_n \odot K_1$ and $C_n \odot K_{1,t}$ are vertex-graceful if n is odd; C_n is super vertex-graceful if $n \neq 4, 6$; and $C_n \odot K_1$ is super vertex-graceful if n is even. They proposed two conjectures on (super)vertex-graceful labelings.

As a dual to super edge-graceful graphs Lee and Wei [1914] define a graph $G(V, E)$ to be *super vertex-graceful* if there is a bijection f from V to $\{\pm1, \pm2, \ldots, \pm(|V|-1)/2\}$ when $|V|$ is odd and from V to $\{\pm1, \pm2, \ldots, \pm|V|/2\}$ when $|V|$ is even such that the induced edge labeling f^* defined by $f^+(uv) = f(u)+f(v)$ over all edges uv is a bijection from E to $\{0, \pm1, \pm2, \ldots, \pm(|E|-1)/2\}$ when $|E|$ is odd and from E to $\{\pm1, \pm2, \ldots, \pm|E|/2\}$ when $|E|$ is even. They show: for m and $n_1, n_2, \ldots, n_m$ each at least 3, $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_m}$ is not super vertex-graceful; for n odd, books $K_{1,n} \times P_2$ are not super vertex-graceful; for $n \geqslant 3$, $P_n^2 \times P_2$ is super vertex-graceful if and only if $n = 3, 4$, or 5; and $C_m \times C_n$ is not super vertex-graceful. They conjecture that $P_n \times P_n$ is super vertex-graceful for $n \geqslant 3$.

In [1918] Lee and Wong generalize super edge-vertex graphs by defining a graph $G(V, E)$ to be *$P(a)Q(1)$-super vertex-graceful* if there is a bijection f from V to $\{0, \pm a, \pm(a+1), \ldots, \pm(a-1+(|V|-1)/2)\}$ when $|V|$ is odd and from V to $\{\pm a, \pm(a+1), \ldots, \pm(a-1+|V|/2)\}$ when $|V|$ is even such that the induced edge labeling f^* defined by $f^+(uv) = f(u)+f(v)$ over all edges uv is a bijection from E to $\{0, \pm1, \pm2, \ldots, \pm(|E|-1)/2\}$ when $|E|$ is odd and from E to $\{\pm1, \pm2, \ldots, \pm|E|/2\}$ when $|E|$ is even. They show various classes of unicyclic graphs are $P(a)Q(1)$-super vertex-graceful. In [1851] Lee, Leung, and Ng more simply refer to $P(1)Q(1)$-super vertex-graceful graphs as *super vertex-graceful* and show how to construct a variety of unicyclic graphs that are super vertex-graceful. They conjecture that every unicyclic graph is an induced subgraph of a super vertex-graceful unicyclic graph. Lee and Leung [1850] determine which trees of diameter at most 6 are super vertex-graceful graphs and propose two conjectures. Lee, Ng, and Sun [1870] found many classes of caterpillars that are super vertex-graceful. In [1025] Gao shows that the generalized butterfly graph B_n^t is super vertex-graceful when $t > 0$ is even, B_n^0 is super vertex-graceful when $n \equiv 0$ or $3 \pmod 4$, and $C_3^{(t)}$ is super vertex-graceful if and only if $t = 1, 2, 3, 5$, or 7.

In [712] Chopra and Lee define a graph $G(V, E)$ to be *$Q(a)P(b)$-super edge-graceful* if there is a bijection f from E to $\{\pm a, \pm(a+1), \ldots, \pm(a+(|E|-2)/2)\}$ when $|E|$ is even and from E to $\{0, \pm a, \pm(a+1), \ldots, \pm(a+(|E|-3)/2)\}$ when $|E|$ is odd and $f^+(u)$ is equal to the sum of $f(uv)$ over all edges uv is a bijection from V to $\{\pm b, \pm(b+1), \ldots, (|V|-2)/2\}$ when $|V|$ is even and from V to $\{0, \pm b, \pm(b+1), \ldots, \pm(|V|-3)/2\}$ when $|V|$ is odd. They say a graph is *strongly super edge-graceful* if it is $Q(a)P(b)$-super edge-graceful for all $a \geqslant 1$. Among their results are: a star with n pendent edges is strongly super edge-graceful if and only if n is even; wheels with n spokes are strongly super edge-graceful if and only if n is even; coronas $C_n \odot K_1$ are strongly super edge-graceful for all $n \geqslant 3$; and double stars $DS(m, n)$ are strongly super edge-graceful in the case that m is odd and at least 3 and n is even and at least 2 and in the case that both m and n are odd and one of them is at least 3. Lee, Song, and Valdés [1891] investigate the $Q(a)P(b)$-super edge-gracefulness of wheels W_n for $n = 3, 4, 5$, and 6.

In [1915] Lee, Wang, and Yera proved that some Eulerian graphs are super edge-graceful, but not edge-graceful, and that some are edge-graceful, but not super edge-graceful. They also showed that a Rosa-type condition for Eulerian super edge-graceful graphs does not exist and pose some conjectures, one of which was: For which n, is K_n is super edge-graceful? It was known that the complete graphs K_n for $n = 3, 5, 6, 7, 8$ are super edge-graceful and K_4 is not super edge-graceful. Khodkar, Rasi, and Sheikholeslami, [1694] answered this question by proving that all complete graphs of order $n \geqslant 3$, except 4, are super edge-graceful.

In 1997 Yilmaz and Cahit [3537] introduced a weaker version of edge-graceful called E-cordial. Let G be a graph with vertex set V and edge set E and let f a function from E to $\{0, 1\}$. Define f on V by $f(v) = \sum\{f(uv)|uv \in E\} \pmod 2$. The function f is called an *E-cordial* labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits an E-cordial labeling is called *E-cordial*. Yilmaz and Cahit prove the following graphs are E-cordial: trees with n vertices if and only if $n \not\equiv 2 \pmod 4$; K_n if and only if $n \not\equiv 2 \pmod 4$; $K_{m,n}$ if and only if $m + n \not\equiv 2 \pmod 4$; C_n if and only if $n \not\equiv 2 \pmod 4$; regular graphs of degree 1 on $2n$ vertices if and only if n is even; friendship graphs $C_3^{(n)}$ for all n (see §2.2 for the definition); fans F_n if and only if $n \not\equiv 1 \pmod 4$; and wheels W_n if and only if $n \not\equiv 1 \pmod 4$. They observe that graphs with $n \equiv 2 \pmod 4$ vertices can not be E-cordial. They generalized E-cordial labelings to E_k-cordial ($k > 1$) labelings by replacing $\{0, 1\}$ by $\{0, 1, 2, \ldots, k-1\}$. Of course, E_2-cordial is the same as E-cordial (see §3.7). Vaidya and Barasara [3249] proved that every graph can be embedded as an induced subgraph of an E-cordial graph thereby ruling out any possibility of obtaining any forbidden subgraph characterization for E-cordial graphs. They also proved that a connected graph can be embedded as an induced subgraph of an E-cordial connected graph and every planar graph can be embedded as an induced subgraph of an E-cordial planar graph.

Liu, liu, and Wu [1980] provide two necessary conditions for a graph to be E_k-cordial and prove that P_n ($n \geqslant 3$) is E_p-cordial for odd p. They also discuss the E_2-cordiality of graphs that have a subgraph that is a 1-factor. Devaraj [808] has shown that $M(m, n)$, the mirror graph of $K(m, n)$, is E-cordial when $m+n$ is even and the generalized Petersen graph $P(n, k)$ is E-cordial when n is even. (Recall that $P(n, 1)$ is $C_n \times P_2$.)

In [3324] Vaidya and Vyas prove that the following graphs are E-cordial: the mirror graphs (see §2.3 for the definition) even paths, even cycles, and the hypercube are E-cordial. In [3290] they show that the middle graph, the total graph, and the splitting graph of a path are E-cordial and the composition of P_2n with P_2. (See §2.7 for the definitions of middle, total and splitting graphs.) In [3291] Vaidya and Lekha [3291] prove the following graphs are E-cordial: the graph obtained by duplication of a vertex (see §2.7 for the definition) of a cycle; the graph obtained by duplication of an edge (see §2.7 for the definition) of a cycle; the graph obtained by joining of two copies of even cycle by an edge; the splitting graph of an even cycle; and the shadow graph (see §3.8 for the definition) of a path of even order.

Vaidya and Vyas [3325] proved the following graphs have E-cordial labelings: $K_{2n} \times P_2$; $P_{2n} \times P_2$; $W_n \times P_2$ for odd n; and $K_{1,n} \times P_2$ for odd n. Vaidya and Vyas [3326] proved that the Möbius ladders, the middle graph of C_n, and crowns $C_n \odot K_1$ are E-cordial graphs for even n while bistars $B_{n,n}$ and its square graph $B^2_{n,n}$ are E-cordial graphs for odd n. In [3328] and [3329] Vaidya and Vyas proved the following graphs are E-cordial: flowers, closed helms, double triangular snakes, gears, graphs obtained by switching of an arbitrary vertex in C_n except $n \equiv 2 \pmod 4$, switching of rim vertex in wheel W_n except $n \equiv 1 \pmod 4$, switching of an apex vertex in helms, and switching of an apex vertex in closed helms. Sugumaran and Vishnu Prakash [3109] proved that the following graphs are E-cordial: theta graphs, duplication of any vertex in theta graphs, switching of any vertex in theta graphs, the fusion of any two vertices in theta graphs, and the open star of n copies of a fixed theta graph (that is, the graph obtained by replacing each endpoint vertex of $K_{1,n}$ by copies of the theta graph).

In her PhD thesis [3338] Vanitha defines a (p,q) graph G to be *directed edge-graceful* if there exists an orientation of G and a labeling of the arcs of G with $\{1, 2, \ldots, q\}$ such that the induced mapping g on V defined by $g(v) = |f^+(v) - f^-(v)| \pmod p$ is a bijection where, $f^+(v)$ is the sum of the labels of all arcs with head v and $f^-(v)$ is the sum of the labels of all arcs with tail v. She proves that a necessary condition for a graph with p vertices to be directed edge-graceful is that p is odd. Among the numerous graphs that she proved to be directed edge-graceful are: odd paths, odd cycles, fans F_{2n} $(n \geqslant 2)$, wheels W_{2n}, nC_3-snakes, butterfly graphs B_n (two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex), $K_{1,2n}$ $(n \geqslant 2)$, odd order y-trees with at least 5 vertices, flags Fl_{2n} (the cycle C_{2n} with one pendent edge), festoon graphs $P_n \odot mK_1$, the graphs $T_{m,n,t}$ obtained from a path P_t $(t \geqslant 2)$ by appending m edges at one endpoint of P_t and n edges at the other endpoint of P_t, C_3^n, $P_3 \cup K_{1,2n+1}$, $P_5 \cup K_{1,2n+1}$, and $K_{1,2n} \cup K_{1,2m+1}$.

In [599] Boonklurb, Ruamkaew, and Singhun use $C(c \times a)$ to denote the graph obtained by identifying a vertex of c cycles C_a to a single vertex. They provide directed edge-graceful labelings for $C(c \times a)$ in the case when $a \geqslant 3$ is an odd integer and $c \geqslant 2$ and in the case where a and c are even with $a \geqslant 4$ and $c \geqslant 2$.

The table following summarizes the state of knowledge about edge-graceful labelings. In the table **EG** means edge-graceful labeling exists. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property.

Table 23: **Summary of Edge-graceful Labelings**

Graph	*Types*	*Notes*
K_n	EG	iff $n \not\equiv 2 \pmod 4$ [1830]

Continued on next page

Table 23 – *Continued from previous page*

Graph	*Types*	*Notes*
odd order trees	EG?	[1837]
$K_{n,n,\ldots,n}$ (k terms)	EG	iff n is odd or $k \not\equiv 2 \pmod 4$ [1883]
$C_n^k,\ k < \lfloor n/2 \rfloor$	EG	iff n is odd [1882]
$C_n^k,\ k \geqslant \lfloor n/2 \rfloor$	EG	iff $n \not\equiv 2 \pmod 4$ [1882]
$P_3[K_n]$	EG	n is odd [1882]
M_{4n} (Möbius ladders)	EG	[1838]
odd order dragons	EG	[1768]
odd order unicycilc graphs	EG?	[1768]
$P_{2m} \times P_{2n}$	EG	iff $m = n = 2$ [1858]
$C_n \cup P_2$	EG	n even [1886]
$C_{2n} \cup C_{2n+1}$	EG	n odd [1886]
$C_n \cup C_{2n+2}$	EG	[1886]
$C_n \cup C_{4n}$	EG	n odd [1886]
$C_{2m} \cup C_{2n+1}$	EG?	$(m,n) \neq (4,3)$ odd [1887]
$P(n,k)$ generalized Petersen graph	EG	n even, $k < n/2$ [1838]
$C_m \times C_n$	EG?	$(m,n) \neq (4,3)$ [1887]

7.4 Radio Labelings

In 2001 Chartrand, Erwin, Zhang, and Harary [680] were motivated by regulations for channel assignments of FM radio stations to introduce radio labelings of graphs. A *radio labeling* of a connected graph G is an injection c from the vertices of G to the natural numbers such that $d(u,v) + |c(u) - c(v)| \geqslant 1 + diam(G)$ for every two distinct vertices u and v of G. The *radio number* of c, $rn(c)$, is the maximum number assigned to any vertex of G. The *radio number* of G, $rn(G)$, is the minimum value of $rn(c)$ taken over all radio labelings c of G. Chartrand et al. and Zhang [3580] gave bounds for the radio numbers of cycles. The exact values for the radio numbers for paths and cycles were reported by Liu and Zhu [1969] as follows: for odd $n \geqslant 3$, $rn(P_n) = (n-1)^2/2 + 2$; for even $n \geqslant 4$, $rn(P_n) = n^2/2 - n + 1; rn(C_{4k}) = (k+2)(k-2)/2 + 1$; $rn(C_{4k+1}) = (k+1)(k-1)/2$; $rn(C_{4k+2}) = (k+2)(k-2)/2+1$; and $rn(C_{4k+3}) = (k+2)(k-1)/2$. However, Chartrand, Erwin, and Zhang [679] obtained different values than Liu and Zhu for P_4 and P_5. Chartrand, Erwin, and Zhang [679] proved: $rn(P_n) \leqslant (n-1)(n-2)/2 + n/2 + 1$ when n is even; $rn(P_n) \leqslant n(n-1)/2 + 1$ when n is odd; $rn(P_n) < rn(P_{n+1})$ $(n > 1)$; for a connected graph G of diameter d, $rn(G) \geqslant (d+1)^2/4 + 1$ when d is odd; and $rn(G) \geqslant d(d+2)/4 + 1$ when d is even. Karst, Langowitz, Oehrlein, and Troxell [1653] provide general lower bounds for $rc_k(C_n)$ for all cycles C_n when $k \geqslant \mathrm{diam}(C_n)$ and show that these bounds are exact values when $k = \mathrm{diam}(C_n) + 1$. In [581] Bloomfield, Liu, and Ramirez combine a lower bound approach with the cyclic group structure to determine the value of $\mathrm{rnk}(C_n)$ for $k \geqslant n-3$. For $d \leqslant k < n-3$, they obtain the values of $\mathrm{rnk}(C_n)$ when n and k have the same parity, and prove partial results when n and k and have different parities. These results extend the known values of $\mathrm{rn}_d(C_n)$ and $\mathrm{rn}_{d+1}(C_n)$ shown by Liu and Zhu [1969], and by Karst, Langowitz, Oehrlein, and Troxell in [1653], respectively. Benson, Porter, and Tomova [535] determined the radio numbers of all graphs of order n and diameter $n-2$. In [1965] Liu obtained lower bounds for the radio number of trees and the radio number of spiders (trees with at most one vertex of degree greater than 2) and characterized the graphs that achieve these bounds. Bantva, Vaidya, and Zhou [514] and [515] give a lower bound for the radio number of trees and a necessary and sufficient condition for their bound to be achieved. They determine the radio number for symmetric trees (that is, trees whose non-leaf vertices all have the same degree and whose leaf vertices all have the same eccentricity), banana trees, and firecracker trees. In [1724] Kola and Panigrahi provide the radio number for a class of caterpillars. In [282] Bantva and Der-Fen Liu give a lower bound for the radio number of the cartesian product of two trees. Moreover, they present three necessary and sufficient conditions, and three sufficient conditions for the product of two trees to achieve this bound. Applying these results, they determine the radio number of the cartesian product of two stars as well as a path and a star. Nazeer, Khan, Kousar, and Nazeer [2262] investigated the radio number for some families of generalized caterpillar graphs. Chakraborty, Nandi, Sen, and Supraja [664] gave the exact values of $\mathrm{rc}_k(G)$ for powers of paths where the diameter of the graph is strictly less than k. Their proof provides a linear time algorithm for providing such labelings. [664] new

Chartrand, Erwin, Zhang, and Harary [680] proved: $rn(K_{n_1,n_2,\ldots,n_k}) = n_1 + n_2 + \cdots +$

n_k+k-1; if G is a connected graph of order n and diameter 2, then $n \leqslant rn(G) \leqslant 2n-2$; and for every pair of integers k and n with $n \leqslant k \leqslant 2n-2$, there exists a connected graph of order n and diameter 2 with $rn(G) = k$. They further provide a characterization of connected graphs of order n and diameter 2 with prescribed radio number.

Fernandez, Flores, Tomova, and Wyels [926] proved $rn(K_n) = n$; $rn(W_n) = n + 2$; and the radio number of the gear graph obtained from W_n by inserting a vertex between each vertex of the rim is $4n + 2$. Morris-Rivera, Tomova, Wyels, and Yeager [2192] determine the radio number of $C_n \times C_n$. Martinez, Ortiz, Tomova, and Wyels [2097] define *generalized prisms,* denoted $Z_{n,s}$, $s \geqslant 1$, $n \geqslant s$, as the graphs with vertex set $\{(i,j) \mid i = 1,2 \text{ and } j = 1,\ldots,n\}$ and edge set $\{((i,j),(i,j\pm 1))\} \cup \{((1,i),(2,i+\sigma)) \mid \sigma = -\left\lfloor \frac{s-1}{2} \right\rfloor \ldots, 0, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\}$. They determine the radio number of $Z_{n,s}$ for $s = 1,2$ and 3. In [278] and [279] Bantva determines the radio number for three families of trees obtained by taking a graph operation on a given tree or a family of trees and the radio number for the middle graph of paths. Sun [3126] determined the radio number for the middle graph of a dandelion (the one-point union of a star and a path), which generalized Bantva's results in [279]. Zhang, Nazeer, Habib, Zia, and Ren [3578] determined the radio number for the generalized Petersen graphs $P(4k+2,2)$ and provided a lower bound for $P(4k,2)$. In [3343] [3343] new
Vasoya and Bantva gave a lower bound for the radio number for the Cartesian product of the generalized Petersen graph and a tree. They give two necessary and sufficient conditions and three other sufficient conditions to achieve the lower bound. They use these results to determine the radio number for the Cartesian product of the Petersen graph and stars.

ELrokh, Badr, Al-Shamiri, and Ramadhan [871] provided upper bounds for the radio number of triangular snakes and double triangular snakes that improved on those given by Badr and Moussa [271] and by Saha and Panigrahi in [2668] and [2671]. DeVito, Niedzialomski, and Warren [814] determined the radio number of all diameter 3 Hamming [814] new
graphs and showed that an infinite subset of them have radio graceful labelings.

In [1202] Ramyal and Sooryanarayana generalized the notion of radio labeling as follows. Let M be a subset of non-negative integers and $(M,\star)$ be a monoid with the identity e. A *radio $\star$-labeling* of graph $G(V,E)$ is a mapping $f: V \to M$ such that $|f(u)-f(v)| \star d(u,v) \geqslant diam(G)+1-e$, for all $u,v \in V$. The *radio $\star$-number $rn\star(f)$* of a radio $\star$ -labeling f of G is the maximum label assigned to a vertex of G. The radio $\star$-number of G, denoted by $rn\star(G)$, is the minimum of $rn\star(f)$ taken over all radio $\star$-labeling f of G. They completely determine $rn\star(G)$ of some transformation graphs of paths and cycles where $\star$ is the usual multiplication of integers.

Sooryanarayana and Ranghunath [3037] define a radio labeling f of a graph G to be a *consecutive radio labeling* of G if $f(V(G)) = \{1,2,\ldots,|V(G)|\}$. They call a graph for which a consecutive radio labeling exists *radio graceful.* In her Ph.D. thesis [2274] Niedzialomski (see also [2275]) investigated the existence of radio graceful labelings of Cartesian products of graphs. Among her results are: for $n \geqslant 3$ and $1 \leqslant t \leqslant n-1$ the Cartesian product of t copies of K_n is radio graceful; for $2 \leqslant p \leqslant n_2$ the Cartesian product of $p \cdot \lceil n/p \rceil$ copies of K_n is radio graceful; the Cartesian product $K_{n_1} \times K_{n_2}, \ldots, \times K_{n_s}$ is radio graceful when $n_1, n_2, \ldots, n_s$ are relatively prime; certain families of generalized

Petersen graphs are radio graceful; and the Cartesian product of $t \geqslant 1 + n(n^2 - 1)/6$ copies of K_n is not radio graceful. Locke and Niedzialomski [1985] proved that $K_n \times P$ is radio graceful where P is the Peterson graph. Wyels and Tomova [1985] proved that that $P \times P$ is radio graceful. Sooryanarayana and Raghunath [3037] determine the values of n for which C_n^3 is radio graceful.

For a simple connected graph G with at least 3 vertices the *triameter* of G, denoted by $\mathrm{tr}(G)$, is the smallest positive integer M such that $d(u,v) + d(v,w) + d(w,u) \leqslant M$ for every triplet u, v, and w in $V(G)$. Saha and Basunia [2666] proved that a graph with diameter 2 is radio graceful if and only if it contains a Hamiltonian path. They also prove that if both a graph G and its complement $\overline{G}$ are connected and $\mathrm{tr}(G) > 9$, then $\overline{G}$ is radio graceful only if G contains a Hamiltonian path and $\mathrm{tr}(\overline{G}) = 5$ or 6.

The generalized gear graph $J_{t,n}$ is obtained from a wheel W_n by introducing t-vertices between every pair (v_i, v_{i+1}) of adjacent vertices on the n-cycle of wheel. Ali, Rahim, Ali, and Farooq [158] gave an upper bound for the radio number of generalized gear graph, which coincided with the lower bound found in and [2545]. They proved for $t < n - 1$ and $n \geqslant 7$, $\mathrm{rn}(J_{t,n}) = (nt^2 + 4nt + 3n + 4)/2$. They pose the determination of the radio number of $J_{t,n}$ when $n \leqslant 7$ and $t > n - 1$ as an open problem.

Saha and Panigrahi [2669] determined the radio number of the toroidal grid $C_m \times C_n$ when at least one of m and n is an even integer and gave a lower bound for the radio number when both m and n are odd integers. Liu and Xie [1967] determined the radio numbers of squares of cycles for most values of n. In [1968] Liu and Xie proved that $rn(P_n^2)$ is $\lfloor n/2 \rfloor + 2$ if $n \equiv 1 \pmod 4$ and $n \geqslant 9$ and $rn(P_n^2)$ is $\lfloor n/2 + 1 \rfloor$ otherwise. In [1966] Liu found a lower bound for the radio number of trees and characterizes the trees that achieve the bound. She also provides a lower bound for the radio number of spiders in terms of the lengths of their legs and characterizes the spiders that achieve this bound. Sweetly and Joseph [3179] prove that the radio number of the graph obtained from the wheel W_n by subdividing each edge of the rim exactly twice is $5n-3$. Marinescu-Ghemeci [2091] determined the radio number of the caterpillar obtained from a path by attaching a new terminal vertex to each non-terminal vertex of the path and the graph obtained from a star by attaching k new terminal vertices to each terminal vertex of the star. Ahmad and Marinescu-Ghemeci [111] determined the radio numbers of Mongolian tents, diamonds, fans, and double fans.

Sooryanarayana and Raghunath [3037] determined the radio number of C_n^3, for $n \leqslant 20$ and for $n \equiv 0$ or 2 or 4 (mod 6). Sooryanarayana, Vishu Kumar, Manjula [3038] determine the radio number of P_n^3, for $n \geqslant 4$. Lo and Alegria [1983] completely determine the radio number for the fourth-power of P_n for $n \geqslant 6$, except when $n \equiv 1 \pmod 8$. Saha and Panigrahi [2670] prove that for an n-vertex simple connected graph G, the difference between the upper and lower bounds of the radio number of G^2 is at most $\lfloor (n-1)/2 \rfloor$. They also determine the radio number for square of graphs belonging to some specific class and apply this to find the radio number for square of hypercube Q_n^2 ($n \not\equiv 0 \pmod 4$), the square of toroidal grid $T_{m,n}^2$ ($m+n \equiv 1, 2, 3, 4, 6 \pmod 8$), and the square of some generalized prism graphs. Wang, Xu, Yang, Zhang, Luo, and Wang [3426] determine the radio number of ladder graphs. Jiang [1544] completely determined the radio number of the grid graph $P_m \times P_n$ $(m, n > 2)$. In [3322] Vaidya and Vihol determined upper bounds on radio numbers of cycles with chords and determined the exact radio numbers for the splitting graph and the middle graph of C_n. In [1929] Li, Mak, and Zhou determine the radio number of complete m-ary trees. Kim, Hwang, and Song [1696] determine the radio numbers of P_n with $n \geqslant 4$ and K_m with $m \geqslant 3$. Bantva [280] improved the lower bound for the radio number of graphs given by Das *et al.* in [786] and gave necessary and sufficient condition to achieve the lower bound. He also determined the radio number for cartesian product of paths P_n and the Peterson graph P and provided a short proof for the radio number of cartesian product of paths P_n and complete graphs K_m given by Kim [1696]. Bantva [281] determined the radio number of the Cartesian product of a path and a wheel. [281] new

In [2263] Nazeer, Kousar, and Nazeer give radio and radio antipodal labelings for certain circulant graphs. In [2264] Nazeer, Kousar, and Munir determined the radio number and radio antipodal number of non-bipartite cubic graphs of order 2^k. Shen, Dong, Zheng, and Guo [2878] use $C(m, t)$ to denote the caterpiller consisting of a path $x_1x_2 \cdots x_m$ with t pendent edges at each inner vertex. They determine the exact value of the radio number of $C(m, t)$ for all integers $m \geqslant 4$ and $t \geqslant 2$, and explicitly construct an optimal radio labeling. They also show that the radio number and the construction of optimal radio labelings of paths are the special cases of $C(m, t)$ with $t = 2$. An edge-joint graph G is a 1-edge connected graph having an edge uv such that eccentricity of u equuls the eccentricity of v and deletion of uv disconnects G.

Niranjan and Rao Kola [2278] determined $\mathrm{rn}(P_n \odot C_m)$ when n is even and $m \geqslant 5$ and gave an upper bound for the same when n is odd. For $m \geqslant 4$ they determined the radio number of $P_n \odot P_m$ when n is even, and gave both upper and lower bounds for $\mathrm{rn}(P_n \odot P_m)$ when n is odd.

In [655] Canales, Tomova, and Wyels investigated the question of which radio numbers of graphs of order n are achievable. They proved that the achievable radio numbers of graphs of order n must lie in the interval $[n, rn(P_n)]$, and that these bounds are the best possible. They also show that for odd n, the integer $rn(P_n) - 1 = \frac{(n-1)^2}{2} + 2$ is an unachievable radio number for any graph of order n. In [3013] Sokolowsky settled the question of exactly which radio numbers are achievable for a graph of order n.

Adefokun and Ajayi [57] investigated the radio number of $K_{1,m} \times P_{2n}$ for the case $m \geqslant 4$. They also obtained new lower and upper bounds of the radio number for the case that $m = 3$ that improve similar their results in [126]. For a graph H, Naseem, Shabbir, and Shaker [2260] use JuH to denote the graph obtained by joining two disjoint copies of a graph H by an edge between the vertices labeled with u in H. They provided a lower bound for the radio number of such graphs and show that their lower bound is optimal (i.e., equal to the radio number) for certain a subfamily of JuH. They obtained similar results for graphs obtained by contracting the edge uv of JuH. [57] new [126] new [2260] new

For any connected graph G and positive integer k Chartrand, Erwin, and Zhang, [678] define a *radio k-coloring* as an injection f from the vertices of G to the natural numbers such that $d(u, v) + |f(u) - f(v)| \geqslant 1 + k$ for every two distinct vertices u and v of G. Using $rc_k(f)$ to denote the maximum number assigned to any vertex of G by

f, the *radio k-chromatic number* of G, $rc_k(G)$, is the minimum value of $rc_k(f)$ taken over all radio k-colorings of G. Note that $rc_1(G)$ is $\chi(G)$, the chromatic number of G, and when $k = diam(G)$, $rc_k(G)$ is $rn(G)$, the radio number of G. Chartrand, Nebesky, and Zang [686] gave upper and lower bounds for $rc_k(P_n)$ for $1 \leqslant k \leqslant n-1$. Kchikech, Khennoufa, and Togni [1680] improved Chartrand et al.'s lower bound for $rc_k(P_n)$ and Kola and Panigrahi [1726] improved the upper bound for certain special cases of n. The exact value of $rc_{n-2}(P_n)$ for $n \geqslant 5$ was given by Khennoufa and Togni in [1690] and the exact value of $rc_{n-3}(P_n)$ for $n \geqslant 8$ was given by Kola and Panigrahi in [1726]. Kola and Panigrahi [1726] gave the exact value of $rc_{n-4}(P_n)$ when n is odd and $n \geqslant 11$ and an upper bound for $rc_{n-4}(P_n)$ when n is even and $n \geqslant 12$. In [2667] Saha and Panigrahi provided an upper and a lower bound for $rc_k(C_n^r)$ for all possible values of n, k and r and showed that these bounds are sharp for antipodal number of C_n^r for several values of n and r. Kchikech, Khennoufa, and Togni [1681] gave upper and lower bounds for $rc_k(G \times H)$ and $rc_k(Q_n)$. In [1680] the same authors proved that $rc_k(K_{1,n}) = n(k-1)+2$ and for any tree T and $k \geqslant 2$, $rc_k(T) \leqslant (n-1)(k-1)$.

Karst, Langowitz, Oehrlein, and Troxell [1653] provide general lower bounds for $rc_k(C_n)$ for all cycles C_n when $k \geqslant \text{diam}(C_n)$ and show that these bounds are exact values when $k = \text{diam}(C_n) + 1$.

In [638] Čada, Ekstein, Holub, and Togni [638] defined a k-labeling of a connected [638] new graph as an assignment c of non-negative integers to the vertices of the graph so that for [638] new every pair of vertices x and y, $|c(x) - c(y)| \geqslant k + 1 - d(x, y)$. The radio k-number of a graph is the largest number that has to be used in a k-labeling of the graph. A *distance graph* is a graph with integer vertices, where two vertices are adjacent if the absolute value of their difference is in some chosen set D. They established some lower and upper bounds for the radio k-number of graphs with distance sets $D(1, 2, \ldots, t)$, $D(1, t)$, and $D(t-1, t)$ for a positive t. Korze, Shao, and Vesel [1737] improved some lower and upper [1737] new bounds for the radio k-labeling number of the same three families studied by Čada et al. and conjectured that some of these upper bounds for the radio k-labeling number of $D(1, 2, \ldots, t)$ are exact radio k-labeling numbers. They obtained their main results using theoretical constructions and the mathematical optimization methods of integer linear programming. Arockiamary and Vijayalakshmi [239] investigated the existence of vertex [239] new k-labeling of various cyclic snakes and $mC_n \odot K_1$.

For a positive integer k, Arockiamary and Vijayalakshmi [226] define a *vertex k-prime* [226] new *labeling* f of a graph $G(V, E)$ as a bijective function from V to $\{k, k+1, \ldots, k+|V|-1\}$ such that $\gcd(f(u), f(v) = 1$ for every edge uv. A graph G that admits vertex k-prime labeling is called a *vertex k-prime* graph. They proved that the class of planar graphs containing the maximum number of edges possible in a graph with n vertices is vertex k-prime if and only if n is odd, $G \cup K_{1,n}$ is vertex k-prime, and if G and H are k-prime labelings, then so is $G \cup H$. They also prove that K_n $(n \geqslant 4)$ is not vertex k-prime.

A radio k-coloring of G when $k = \text{diam}(G) - 1$ is called a *radio antipodal labeling.* The minimum span of a radio antipodal labeling of G is called the *radio antipodal number* of G and is denoted by $an(G)$. Khennoufa and Togni [1686] determined the radio number and the radio antipodal number of the hypercube by using a generalization of binary Gray codes. They proved that $rn(Q_n) = (2^{n-1}-1)\lceil\frac{n+3}{2}\rceil+1$ and $an(Q_n) = (2^{n-1}-1)\lceil\frac{n}{2}\rceil+\varepsilon(n)$, with $\varepsilon(n) = 1$ if $n \equiv 0 \bmod 4$, and $\varepsilon(n) = 0$ otherwise. In [1124] Gomathi and Venugopal provided bounds for the radio antipodal number of honeycombs derived networks—triangular and rhombic honeycombs. These bounds give the optimum number of channels (bandwidth) needed for these honeycomb derived networks for effective communication without interference. [1124] new

The Dd-distance was introduced by Anto Kinsley and Siva Ananthi [218] as follows. For a connected graph G, the Dd-*length* of a connected uv path is defined as $\mathrm{Dd}(u,v) = \mathrm{D}(u,v) + \deg(u) + \deg(v)$. Letting $D^{Dd}(u,v)$ denote the Dd-distance between u and v and $\mathrm{diam}^{Dd}(G)$ denote the Dd-diameter of G, Viola and Nicholas [3404] define a *radio mean Dd-distance labeling* of a connected graph G as an injective map f from $V(G)$ to the positive integers such that for two distinct vertices u and v of G, $D^{Dd}(u,v)+\lceil(f(u)+f(v))/2\rceil \geqslant 1+\mathrm{diam}^{Dd}(G)$. The *radio mean Dd-distance number* of f, $rmn^{Dd}(f)$, is the maximum label assigned to any vertex of G. The *radio mean Dd-distance number* of G, $rmn^{Dd}(G)$, is the minimum value of f of G. They determined the radio mean Dd-distance number of complete graphs, stars, bistars, subdivisions of stars, paths, fans, and freindship graphs.

The survey article by Panigrahi [2332] includes background information and further results about radio k-colorings.

7.5 Representations of Graphs modulo n

In 1989 Erdős and Evans [894] defined a *representation* modulo n of a graph G with vertices $v_1, v_2, \ldots, v_r$ as a set $\{a_1, \ldots, a_r\}$ of distinct, nonnegative integers each less than n satisfying $\gcd(a_i - a_j, n) = 1$ if and only if v_i is adjacent to v_j. They proved that every finite graph can be represented modulo some positive integer. The *representation number*, $\mathrm{Rep}(G)$, is smallest such integer. Obviously the representation number of a graph is prime if and only if a graph is complete. Evans, Fricke, Maneri, McKee, and Perkel [909] have shown that a graph is representable modulo a product of a pair of distinct primes if and only if the graph does not contain an induced subgraph isomorphic to $K_2 \cup 2K_1$, $K_3 \cup K_1$, or the complement of a chordless cycle of length at least five. Nešetřil and Pultr [2269] showed that every graph can be represented modulo a product of some set of distinct primes. Evans et al. [909] proved that if G is representable modulo n and p is a prime divisor of n, then $p \geqslant \chi(G)$. Evans, Isaak, and Narayan [910] determined representation numbers for specific families as follows (here we use q_i to denote the ith prime and for any prime p_i we use $p_{i+1}, p_{i+2}, \ldots, p_{i+k}$ to denote the next k primes larger than p_i): $\mathrm{Rep}(P_n) = 2\cdot 3\cdot\ \cdots\ \cdot q_{\lceil \log_2(n-1)\rceil}$; $\mathrm{Rep}(C_4) = 4$ and for $n \geqslant 3$, $\mathrm{Rep}(C_{2n}) = 2\cdot 3\cdot\ \cdots\ \cdot q_{\lceil \log_2(n-1)\rceil+1}$; $\mathrm{Rep}(C_5) = 3\cdot 5\cdot 7 = 105$ and for $n \geqslant 4$ and not a power of 2, $\mathrm{Rep}(C_{2n+1}) = 3\cdot 5\cdot\ \cdots\ \cdot q_{\lceil \log_2 n\rceil+1}$; if $m \geqslant n \geqslant 3$, then $\mathrm{Rep}(K_m - P_n) = p_i p_{i+1}$ where p_i is the smallest prime greater than or equal to $m - n + \lceil n/2\rceil$; if $m \geqslant n \geqslant 4$, and p_i is the smallest prime greater than or equal to $m - n + \lceil n/2\rceil$, then $\mathrm{Rep}(K_m - C_n) = q_i q_{i+1}$ if n is even and $\mathrm{Rep}(K_m - C_n) = q_i q_{i+1} q_{i+2}$ if n is odd; if $n \leqslant m - 1$, then $\mathrm{Rep}(K_m - K_{1,n}) = p_s p_{s+1} \cdots p_{s+n-1}$ where p_s is the smallest prime greater than or equal to $m - 1$; $\mathrm{Rep}(K_m)$

is the smallest prime greater than or equal to m; $\text{Rep}(nK_2) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2 n \rceil + 1}$; if $n, m \geqslant 2$, then $\text{Rep}(nK_m) = p_i p_{i+1} \cdots p_{i+m-1}$, where p_i is the smallest prime satisfying $p_i \geqslant m$, if and only if there exists a set of $n-1$ mutually orthogonal Latin squares of order m; $\text{Rep}(mK_1) = 2m$; and if $t \leqslant (m-1)!$, then $\text{Rep}(K_m + tK_1) = p_s p_{s+1} \cdots p_{s+m-1}$ where p_s is the smallest prime greater than or equal to m. Narayan [2258] proved that for $r \geqslant 3$ the maximum value for $\text{Rep}(G)$ over all graphs of order r is $p_s p_{s+1} \cdots p_{s+r-2}$, where p_s is the smallest prime that is greater than or equal to $r-1$. Agarwal and Lopez [66] determined the representation numbers for complete graphs minus a set of stars.

Evans [908] used matrices over the additive group of a finite field to obtain various bounds for the representation number of graphs of the form nK_m. Among them are $\text{Rep}(4K_3) = 3 \cdot 5 \cdot 7 \cdot 11; \text{Rep}(7K_5) = 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$; and $\text{Rep}((3q-1)/2)K_q) \leqslant p_q p_{q+1} \cdots p_{(3q-1)/2)}$ where q is a prime power with $q \equiv 3 \pmod 4$, p_q is the smallest prime greater than or equal to q, and the remaining terms are the next consecutive $(3q-3)/2$ primes; $\text{Rep}(2q-2)K_q) \leqslant p_q p_{q+1} \cdots p_{(3q-3)/2)}$ where q is a prime power with $q \equiv 3 \bmod 4$, and p_q is the smallest prime greater than or equal to q; $\text{Rep}((2q-2)K_q) \leqslant p_q p_{q+1} \cdots p_{2q-3}$.

In [2257] Narayan asked for the values of $\text{Rep}(C_{2^k+1})$ when $k \geqslant 3$ and $\text{Rep}(G)$ when G is a complete multipartite graph or a disjoint union of complete graphs. He also asked about the behavior of the representation number for random graphs. Yahyaei and Katre [3508] gave upper and lower bounds for the representation number of a caterpillar and exact values in some cases.

Akhtar, Evans, and Pritikin [137] characterized the representation number of $K_{1,n}$ using Euler's phi function, and conjectured that this representation number is always of the form 2^a or $2^a p$, where $a \geqslant 1$ and p is a prime. They proved this conjecture for "small" n and proved that for sufficiently large n, the representation number of $K_{1,n}$ is of the form 2^a, $2^a p$, or $2^a pq$, where $a \geqslant 1$ and p and q are primes. In [138] they showed that for sufficiently large $n \geqslant m$, $\text{rep}(K_{m,n}) = 2^a, 3^a, 2^a p^b$, or $2^a pq$, where $a, b \geqslant 1$ and p and q are primes; and for sufficiently large order, $\text{rep}(K_{n_1,n_2,\ldots,n_t} = p^a, p^a q^b$, or $p^a q^b u$, where p, q, u are primes with $p, q < u$. Akhtar [139] determined the representation number of graphs of the form $K_2 \cup nK_1$ (he uses the notation $K_2 + nK_1$) and studies their prime decompositions. Using relations between representation modulo r and product representations, he determined representation number of binary trees and gave an improved lower bound for hypercubes.

7.6 Product and Divisor Cordial Labelings

Sundaram, Ponraj, and Somasundaram [3133] introduced the notion of product cordial labelings. A *product cordial labeling* of a graph G with vertex set V is a function f from V to $\{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a *product cordial* graph. In [3133] and [3142] Sundaram, Ponraj, and Somasundaram prove the following graphs are product cordial: trees; unicyclic graphs of odd order; triangular snakes; dragons; helms;

$P_m \cup P_n$; $C_m \cup P_n$; $P_m \cup K_{1,n}$; $W_m \cup F_n$ (F_n is the fan $P_n + K_1$); $K_{1,m} \cup K_{1,n}$; $W_m \cup K_{1,n}$; $W_m \cup P_n$; $W_m \cup C_n$; the total graph of P_n (the total graph of P_n has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in P_n); C_n if and only if n is odd; $C_n^{(t)}$, the one-point union of t copies of C_n, provided t is even or both t and n are even; $K_2 + mK_1$ if and only if m is odd; $C_m \cup P_n$ if and only if $m+n$ is odd; $K_{m,n} \cup P_s$ if $s > mn$; $C_{n+2} \cup K_{1,n}$; $K_n \cup K_{n,(n-1)/2}$ when n is odd; $K_n \cup K_{n-1,n/2}$ when n is even; and P_n^2 if and only if n is odd. They also prove that $K_{m,n}$ $(m, n > 2)$, $P_m \times P_n$ $(m, n > 2)$ and wheels are not product cordial and if a (p, q)-graph is product cordial graph, then $q \leqslant (p-1)(p+1)/4 + 1$.

The *bicyclic* graph $B[m, n]$ is the one-point union of C_m and C_n. Meena and Usharani [2127] provided product cordial labelings for some bicyclic relalated graphs. Among them are $B[n, n]$, the graphs obtained by adjoining the center of $K_{1,m}$ to each vertex of $B[n, n]$, $B[n, n] \odot K_2$, and $B[n, n] \odot K_3$. [2127] new

In [2776] Seoud and Helmi obtained the following results: K_n is not product cordial for all $n \geqslant 4$; C_m is product cordial if and only if m is odd; the gear graph G_m is product cordial if and only if m is odd; all web graphs are product cordial; the corona of a triangular snake with at least two triangles is product cordial; the C_4-snake is product cordial if and only if the number of 4-cycles is odd; $C_m \odot \overline{K_n}$ is product cordial; and they determine all graphs of order less than 7 that are not product cordial. Seoud and Helmi define the *conjunction* $G_1 \hat{} G_2$ of graphs G_1 and G_2 as the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2) |\ u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. They prove: $P_m \hat{} P_n$ $(m, n \geqslant 2)$ and $P_m \hat{} S_n$ $(m, n \geqslant 2)$ are product cordial. Nada, Diab, Elrokh, and Sabra [2239] proved that $P_n \odot C_m$ is product cordial if and only if $(n, m) \neq (1, 3)$ (mod 4). Gao, Lau, and Lee [1027] investigated the friendly index and product-cordial index sets of a family of Möbius-like cubic graphs. Rokad [2644] proved the following graphs are product cordial: double wheels $DW_n = 2C_n + K_1$, path unions of finite number of copies of double wheels, the graphs obtained by joining two copies of double wheels by a path of arbitrary length, $DW_n \oplus K_{1,n}$, and $DF_n \oplus K_{1,n}$ ($DF_n = P_n + \overline{K}_2$).

Vaidya and Kanani [3276] prove the following graphs are product cordial: the path union of k copies of C_n except when k is odd and n is even; the graph obtained by joining two copies of a cycle by path; the path union of an odd number copies of the shadow of a cycle (see §3.8 for the definition); and the graph obtained by joining two copies of the shadow of a cycle by a path of arbitrary length. In [3279] Vaidya and Kanani prove the following graphs are product cordial: the path union of an even number of copies of $C_n(C_n)$; the graph obtained by joining two copies of $C_n(C_n)$ by a path of arbitrary length; the path union of any number of copies of the Petersen graph; and the graph obtained by joining two copies of the Petersen graph by a path of arbitrary length.

Vaidya and Barasara [3238] prove that the following graphs are product cordial: friendship graphs; the middle graph of a path; odd cycles with one chord except when the chord joins the vertices at a diameter distance apart; and odd cycles with two chords that share a common vertex and form a triangle with an edge of the cycle and neither chord joins vertices at a diameter apart. In [3258] Vaidya and Barasara investigated the product cordial labeling of the line graph of the middle graphs of paths, triangular snakes, armed crowns, the square of paths, the splitting graphs of paths, and the total graph of paths.

In [3265] Vaidya and Dani prove the following graphs are product cordial: $< S_n^{(1)} : S_n^{(2)} : \ldots : S_n^{(k)} >$ except when k odd and n even; $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$; and $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ if and only if k is even or k is odd and n is even with $k > n$. (See §3.7 for the definitions.)

Vaidya and Barasara [3244] proved the following graphs are product cordial: closed helms, web graphs, flower graphs, double triangular snakes obtained from the path P_n if and only if n is odd, and gear graphs obtained from the wheel W_n if and only if n is odd. Vaidya and Barasara [3245] proved that the graphs obtained by the duplication of an edge of a cycle, the mutual duplication of pair of edges of a cycle, and mutual duplication of pair of vertices between two copies of C_n admit product cordial labelings. Moreover, if G and G' are the graphs such that their orders or sizes differ at most by 1 then the new graph obtained by joining G and G' by a path P_k of arbitrary length admits product cordial labeling.

Vaidya and Barasara [3246] define the *duplication of a vertex* v of a graph G *by a new edge* $u'v'$ as the graph G' obtained from G by adding the edges $u'v'$, vu' and vv' to G. They define the *duplication of an edge* uv of a graph G *by a new vertex* v' as the graph G' obtained from G by adding the edges uv' and vv' to G. They proved the following graphs have product cordial labelings: the graph obtained by duplication of an arbitrary vertex by a new edge in C_n or P_n ($n > 2$); the graph obtained by duplication of an arbitrary edge by a new vertex in C_n ($n > 3$) or P_n ($n > 3$); and the graph obtained by duplicating all the vertices by edges in path P_n. They also proved that the graph obtained by duplicating all the vertices by edges in C_n ($n > 3$) and the graph obtained by duplicating all the edges by vertices in C_n are not product cordial.

Recall (see [2444], [2431], [2432], [2433]) a *double triangular snake* DT_n consists of two triangular snakes that have a common path; a *double quadrilateral snake* DQ_n consists of two quadrilateral snakes that have a common path; an *alternate triangular snake* $A(T_n)$ is the graph obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertex v_i (that is, every alternate edge of a path is replaced by C_3); a *double alternate triangular snake* $DA(T_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i and w_i; an *alternate quadrilateral snake* $A(Q_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i and w_i respectively and then joining v_i and w_i (that is, every alternate edge of a path is replaced by a cycle C_4); a *double alternate quadrilateral snake* $DA(Q_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, x_i and w_i and y_i respectively and then joining v_i and w_i and x_i and y_i.

Vaidya and Barasara [3248] prove that the shell graph S_n is product cordial for odd n and not product cordial for even n. They also show that $D_2(C_n)$; $D_2(P_n)$; C_n^2; $M(C_n)$; $S'(C_n)$; circular ladder CL_n; Möbius ladder M_n; step ladder $S(T_n)$ and $H_{n,n}$ does not admit product cordial labeling.

Vaidya and Vyas [3332] prove the following graphs are product cordial: alternate triangular snakes $A(T_n)$ except $n \equiv 3 \pmod 4$; alternate quadrilateral snakes $A(QS_n)$ except except $n \equiv 2 \pmod 4$; double alternate triangular snakes $DA(T_n)$ and double alternate quadrilateral snakes $DA(QS_n)$. Vaidya and Vyas [3333] prove the following graphs are product cordial: the splitting graph of bistar $S'(B_{n,n})$; duplicating each edge by a vertex in bistar $B_{n,n}$ and duplicating each vertex by an edge in bistar $B_{n,n}$. They also proved that $D_2(B_{n,n})$ is not product cordial.

Ghodasara and Vaghasiya [1092] prove the following graphs admit product cordial labelings: the path union of an odd number of copies of C_n with a chord except for $n = 4$, the path union of an odd number of copies of C_n with twin chords except when $n = 6$, the path union of C_n $(n > 6)$ with three cords that form two triangles and a cycle of length $n - 3$, the graph obtained by joining two copies of the same cycle that has one chord by a path, the graph obtained by joining two copies of same cycle that has twin chords by a path, and the graph obtained by joining two copies of C_n $(n \geqslant 7)$ with three cords that form two triangles and a cycle of length $n - 3$ by a path. Ghodasara and Vaghasiya [1093] prove the following graphs are product cordial: the path union of helms, the path union of closed helms, the path union of gear graphs G_n for odd n, the graph obtained by joining two copies of the same helm by a path, the graph obtained by joining two copies of the same closed helm by a path, and the graph obtained by joining two copies of the same gear graph by a path.

In [285] Bapat proves the following graphs are product cordial: graphs obtained by identifying an endpoint of P_n with each vertex of C_3, graphs obtained by identifying an endpoint of P_n with each vertex of C_4, graphs obtained by identifying the degree m vertex of $K_{1,m}$ with each vertex of C_3, and graphs obtained by identifying the degree m vertex of $K_{1,m}$ with each vertex of the shell $C_{n,n-3)}$ (C_n with $n - 3$ chords that share a common endpoint) if and only n is even or n is odd and m is even. In [284] Bapat proves $K_5 \odot C_n$ and kayak paddles are product cordial, the one-point union of n copies of K_m is product cordial if and only if n is even, and graphs obtained by identifying one edge of K_5 with each edge of P_n is product cordial if and only if n is even.

Prajapati and Raval [2501] investigated product cordial labelings of the graphs obtained by duplication of verticies and edges of gears and graphs obtained by the vertex switching operation of gears. In [2502] Prajapati and Raval proved that the book $B_{m,n}$ is a product cordial graph if and only if m and n both are odd and $m \geqslant 3$. They showed that graphs obtained from books by duplicating or deleting vertices or edges are product cordial. For graphs with an even number of vertices they proved that the duplication of each of the vertices of a product cordial graph with an edge is a product cordial graph and that for graphs that have an odd number of vertices and even number of edges the duplication of each of the vertices of a product cordial graph with an edge is a product cordial graph.

Kwong, Lee, and Ng [1791] determine the product-cordial index sets of Möbius ladders and the graphs obtained by subdividing an edge of K_4 and an edge of a Möbius ladder that is not a rung and joining the two new vertices by an edge. They show that no Möbius ladder is product cordial. Gao, Sun, Zhang, Meng, and Lau [1023] provide sufficient conditions for a graph to admit (or not admit) a product cordial labeling. Gao, Lau, and Lee [1022] investigated the friendly index and product-cordial index sets of a family of cubic graphs known as Möbius-like graphs. Prajapati and Raval [2500] proved that windmills, barbells, the one point union at the apex of copies of a wheel (*generalized wheel*), and the one point union of copies of a wheel connected at one common rim vertex of the wheel are product cordial graphs. They also showed that duplicating all rim edges with a vertex and duplicating all the vertices with an edge of generalized wheels, and the graphs obtained by switching an apex vertex in a generalized wheel are product cordial graphs. Patel, Prajapati, and Kansagara [2363] proved that graphs obtained from the barbell graph by duplicating all vertices by edges and duplicating all edges by vertices in the path joining complete graphs are product cordial and the graphs obtained by switching a vertex of path in the barbell graph are product cordial.

In [2676] Salehi called the set $\{|e_f(0) - e_f(1)| : f \text{ is a friendly labeling of } G\}$ the *product-cordial set* of G. He determines the product-cordial sets for paths, cycles, wheels, complete graphs, bipartite complete graphs, double stars, and complete graphs with an edge deleted. Salehi and Mukhin [2685] say a graph G of size q is *fully product-cordial* if its product cordial set is $\{q - 2k : 0 \leqslant k \leqslant \lfloor q/2 \rfloor\}$. They proved: P_n $(n \geqslant 2)$ is fully product-cordial; trees with a perfect matching are fully product-cordial; and $P_2 \times P_n$ is not fully product-cordial. They determine the product-cordial sets of $P_2 \times P_n$, $P_n \times P_{2m}$, and $P_n \times P_{2m+1}$, where $m \geqslant n$. Because the product-cordial set is the multiplicative version of the friendly index set, Kwong, Lee, and Ng [1789] called it the *product-cordial index set* of G. They determined the exact values of the product-cordial index set of C_m and $C_m \times P_n$ and that $P_m \times P_n$ has the maximum product cordial-index $2mn - m - n$. In [1790] Kwong, Lee, and Ng determined the friendly index sets and product-cordial index sets of 2-regular graphs and the graphs obtained by identifying the centers of any number of wheels. In [2679] z Salehi, Churchman, Hill, and Jordan determine the product-cordial index sets of certain classes of trees.

In [2903] Shiu and Kwong define the *full product-cordial index* of G under f as FPCI(G) $= \{i_f^*(G) \mid f \text{ is a friendly labeling of } G\}$. They provide a relation between the friendly index and the product-cordial index of a regular graph. As applications, they determine the full product-cordial index sets of C_m and $C_m \times C_n$, which was asked by Kwong, Lee, and Ng in [1789]. Shiu [2894] determined the product-cordial index sets of grids $P_m \times P_n$. Recall the *twisted cylinder* graph is the permutation graph on $4n$ $(n \geqslant 2)$ vertices, $P(2n; \sigma)$, where $\sigma = (1,2)(3,4)\cdots(2n-1,2n)$ (the product of n transpositions). Shiu and Lee [2916] determined the full friendly index sets and the full product-cordial index sets of twisted cylinders.

Jeyanthi and Maheswari define a mapping $f : V(G) \to \{0,1,2\}$ to be a 3-*product cordial* labeling if $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(i) - e_f(j)| \leqslant 1$ for any $i, j \in \{0,1,2\}$, where $v_f(i)$ denotes the number of vertices labeled with $i, e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i \pmod 3$. A graph with a 3-product cordial labeling is called a 3-*product cordial* graph. In [1455] they prove that for a (p,q) 3-product cordial graph: $p \equiv 0 \pmod 3$ implies $q \leqslant \frac{p^2-3p+6}{3}$; $p \equiv 1 \pmod 3$ implies $q \leqslant \frac{p^2-2p+7}{3}$; and $p \equiv 2$ (mod 3) implies $q \leqslant \frac{p^2-p+4}{3}$. They prove the following graphs are 3-product cordial: paths; stars; C_n if and only if $n \equiv 1, 2 \pmod 3$; $C_n \cup P_n$, $C_m \odot \overline{K_n}; P_m \odot \overline{K_n}$ for $m \geqslant 3$ and $n \geqslant 1; W_n$ when $n \equiv 1 \pmod 3$; and the graph obtained by joining the centers of two identical stars to a new vertex. They also prove that K_n is not 3-product cordial for

$n \geqslant 3$ and if G_1 is a 3-product cordial graph with $3m$ vertices and $3n$ edges and G_2 is any 3-product cordial graph, then $G_1 \cup G_2$ is a 3-product cordial graph. In [1456] they prove that ladders, $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ (see §3.7 for the definition), graphs obtained by duplicating an arbitrary edge of a wheel, graphs obtained by duplicating an arbitrary vertex of a cycle or a wheel are 3-product cordial. They also prove that the graphs obtained by from the ladders $L_n = P_n \times P_2$ $(n \geqslant 2)$ by adding the edges $u_i v_{i+1}$ for $1 \leqslant i \leqslant n-1$, where the consecutive vertices of two copies of P_n are $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ and the edges are $u_i v_i$. They call these graphs *triangular ladders* . The graph $B^*_{n,n}$ is obtained from the bistar $B_{n,n}$ with $V(B_{n,n}) = \{u, v, u_i, v_i \mid 1 \leqslant i \leqslant n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i, vu_i, uv_i \mid 1 \leqslant i \leqslant n\}$ by joining u with v_i and v with u_i for $1 \leqslant i \leqslant 4$. Jeyanthi and Maheswari [1462] proved: the splitting graphs $S'(K_{1,n})$ and $S'(B_{n,n})$ are 3-product cordial graphs; $B^*_{n,n}$ is a 3-product cordial graph if and only if $n \equiv 0, 1 \pmod 3$; and the shadow graph $D_2(B_{n,n})$ is a 3-product cordial graph if and only if $n \equiv 0, 1 \mod 3$. Jeyanthi, Maheswari, and Vijaya Laksmi [1477] prove the following: graphs obtained by switching an apex vertex in a closed helm are 3-product cordial; W_n are 3-product cordial if and only if $n \equiv 2 \pmod 3$; double fans are 3-product cordial if and only if $n \equiv 0 \pmod 3$; books are 3-product cordial; and permutation graphs $P(K_2 + mK_1; T)$ are 3-product cordial if and only if $m \equiv 2 \pmod 3$. In [1480] Jeyanthi, Maheswari, and Vijayalaksmi investigated the 3-product cordial behavior of alternate triangular snakes, double alternate triangular snakes, and triangular snakes.

A *k-product cordial* labeling of a graph G is a map f from $V(G)$ to $\{0, 1, \ldots, k-1\}$ where k is a positive integer at most $|V(G)|$ such that when each edge uv is assigned the label $f(u)f(v) \pmod k$, the number of verticies (edges) labeled with i and the number of edges (verticies) labeled with j differ by at most 1. In [1418], [1415], [1420], and [1421] Jeya Daisy, Sabibha, Jeyanthi, and Youssef showed that the following graphs admit k-product cordial labelings: fans and double fans when $k = 4$ and 5; cones $(\overline{K_1} + C_m)$ and double cones $(\overline{K_2} + C_m)$ for $k = 5$. They also proved that double cones are not 4-product cordial. In [1421] they investigated the k-product cordial behavior of $G + K_t$, where G is a k-product cordial graph and found an upper bound of the size of k-product cordial graphs. In [1415] they investigated the k-product cordial behavior of union of graphs. In [1417] [1417] new
Jeya Daisy, Santrin Sabibha, Jeyanthi, and Youssef proved that the 1-cone $C_n + K_1$ and double cone $C_m + \overline{K_2}$ admit 5-product cordial labelings. They also showed that the double cone does not admit 4-product cordial labeling. In [1416] they define the *Napier bridge*, [1416] new
$P_n(t)$, as the graph obtained from the path P_n by joining the pairs of vertices u, v of P_n with $d(u, v) = t$. They provided necessary and significant conditions for $P_n(3), P_n(4)$ and $P_n(5)$ to have 3-product and 4-product cordial labelings. In [1419] Jeya Daisy, Sabibha, [1419] new
Jeyanthi and Youssef proved that fans F_n and double fans DF_n admit 4-product cordial labelings and 5-product cordial labelings. In [1422] they investigated the existence of [1422] new
k-product cordial labelings of direct products, Cartesian products, strong products, and lexicographic products of graphs.

Sundaram and Somasundaram [3137] introduced the notion of total product cordial labelings. A *total product cordial labeling* of a graph G with vertex set V is a function f from V to $\{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$ the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a *total product cordial* graph. In [3137] and [3135] Sundaram, Ponraj, and Somasundaram prove the following graphs are total product cordial: every product cordial graph of even order or odd order and even size; trees; all cycles except C_4; $K_{n,2n-1}$; C_n with m edges appended at each vertex; fans; double fans; wheels; helms; $C_n \times P_2$; $K_{2,n}$ if and only if $n \equiv 2 \pmod 4$; $P_m \times P_n$ if and only if $(m,n) \neq (2,2)$; $C_n + 2K_1$ if and only if n is even or $n \equiv 1 \pmod 3$; $\overline{K_n} \times 2K_2$ if n is odd, or $n \equiv 0$ or $2 \pmod 6$, or $n \equiv 2 \pmod 8$. Y.-L. Lai, the reviewer for MathSciNet [1793], called attention to some errors in [3135]. Pedrano and Rulete [2376] determined the total product cordial labeling of $P_m \times C_n$, $C_m \times C_n$ and the generalized Petersen graph $P(m,n)$. In [2377] Pedrano and Rulete determined the total product cordial labeling of $P_m \odot C_n$, $P_m \odot P_n$, $C_m \odot P_n$, $P_m \odot F_n$, $P_m \odot W_n$, and $P_m \odot K_n$. Villar [3368] proved $P_n \odot C_m$ $(n \geqslant 2, m \geqslant 3)$ is product cordial, $P_n \odot P_m$ $(n, m \geqslant 2)$ is product cordial except when n and m are both even, and $P_{2n+1} \odot K_m$ $(n \geqslant 1, m \geqslant 4)$ is not product cordial. Gao, Sun, Zhang, Meng, and Lau [1023] proved that P_{n+1}^m is total product cordial. Ramanjaneyulu, Venkaiah, and Kothapalli [2576] give total product cordial labeling for a family of planar graphs for which each face is a 4-cycle.

Vaidya and Vihol [3316] prove the following graphs have total product cordial labelings: a split graph; the total graph of C_n; the star of C_n (recall that the *star of a graph* G is the graph obtained from G by replacing each vertex of star $K_{1,n}$ by a graph G); the friendship graph F_n; the one point union of k copies of a cycle; and the graph obtained by the switching of an arbitrary vertex in C_n.

Sundaram, Ponraj, and Somasundaram [3140] introduced the notion of EP-*cordial labeling* (extended product cordial) labeling of a graph G as a function f from the verticies of a graph to $\{-1,0,1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, then $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(i) - e_f(j)| \leqslant 1$ where $i, j \in \{-1,0,1\}$ and $v_f(k)$ and $e_f(k)$ denote the number of vertices and edges respectively labeled with k. An EP-*cordial graph* is one that admits an EP-cordial labeling. In [3140] Sundaram, Ponraj, and Somasundaram prove the following: every graph is an induced subgraph of an EP-cordial graph, K_n is EP-cordial if and only if $n \leqslant 3$; C_n is EP-cordial if and only if $n \equiv 1, 2 \pmod 3$, W_n is EP-cordial if and only if $n \equiv 1 \pmod 3$; and caterpillars are EP-cordial. They prove that all $K_{2,n}$, paths, stars and the graphs obtained by subdividing each edge of a star exactly once are EP-cordial. They also prove that if a (p,q) graph is EP-cordial, then $q \leqslant 1 + p/3 + p^2/3$. They conjecture that every tree is EP-cordial.

Ponraj, Sivakumar, and Sundaram [2466] introduced the notion of k-product cordial labeling of graphs. Let f be a map from $V(G)$ to $\{0,1,2,\ldots,k-1\}$, where $2 \leqslant k \leqslant |V|$. For each edge uv assign the label $f(u)f(v) \pmod k$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(i) - e_f(j)| \leqslant 1$, $i, j \in \{0,1,2,\ldots,k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x. A graph with a k-product cordial labeling is called a *k-product cordial graph*. Observe that 2-product cordial labeling is simply a product cordial labeling and 3-product cordial labeling is an EP-cordial labeling. In [2466] and [2467] Ponraj et al. prove the following are 4-product cordial: P_n if and only $n \leqslant 11$, C_n if and only if $n = 5, 6, 7, 8, 9$, or 10, K_n if and only if $n \leqslant 2$, $P_n \odot K_1$,

$P_n \odot 2K_1$, $K_{2,n}$ if and only if $n \equiv 0, 3 \pmod 4$, W_n if and only if $n = 5$ or 9, $\overline{K_n} + 2K_2$ iff $n \leqslant 2$, and the subdivision graph of $K_{1,n}$. Sivakumar [2990] proved the following coronas are 4-total product cordial: $P_n \odot K_1$, $P_n \odot 2K_1$, $S(P_n \odot K_1)$, $S(P_n \odot 2K_1)$, $S(C_n \odot K_1)$ and $S(C_n \odot 2K_1)$. Jeyanthi, Maheswari, and Vijayalakshmi [1476] investigated the 3-product cordial behavior of alternate triangular snakes, double alternate triangular snakes, and triangular snake graphs. In [1478] they establish that vertex switching graphs of wheels, gears, and degree splitting of bistars are 3-product cordial graphs.

Let f be a map from $V(G)$ to $\{0, 1, 2, \ldots, k-1\}$ where $2 \leqslant k \leqslant |V|$. For each edge uv assign the label $f(u)f(v) \pmod k$. Ponraj, Sivakumar, and Sundaram [2468] define f to be a *k-total product cordial labeling* if $|f(i) - f(j)| \leqslant 1$, $i, j \in \{0, 1, 2, \ldots, k-1\}$, where $f(x)$ denote the number of vertices and edges labeled with x. A graph with a k-total product cordial labeling is called a *k-total product cordial graph* . A 2-total product cordial labeling is simply a total product cordial labeling. In [2468], [2469], [2470], [2471] and [2472], Ponraj et al. proved the following graphs are 3-total product cordial: P_n, C_n if and only if $n \neq 3$ or 6, $K_{1,n}$ if and only if $n \equiv 0, 2 \pmod 3$, $P_n \odot K_1$, $P_n \odot 2K_1$, $K_2 + mK_1$ if and only if $m \equiv 2 \pmod 3$, helms, wheels, $C_n \odot 2K_1$, $C_n \odot K_2$, dragons $C_m@P_n$ (obtained by identitying an endpoint of P_n with a vertex of C_m), $C_n \odot K_1$, bistars $B_{m,n}$, and the subdivision graphs of $K_{1,n}$, $C_n \odot K_1$, $K_{2,n}$, $P_n \odot K_1$, $P_n \odot 2K_1$, $C_n \odot K_2$, wheels and helms. They also proved that every graph is a subgraph of a connected k-total product cordial graph, $B_{m,n}$ is $(n+2)$-total product cordial, and $K_{m,n}$ is $(n+2)$-total product cordial. Ahmad, Hasni, Irfan, Naseem, and Siddiqui [104] proved that $P_m \times P_n$ for m, $n \geqslant 2$ is 3-total edge product cordial. Sharon Philomena and Thirusangu [2870] proved that the flower graph is 3-total product cordial. Ahmada, Bača, Naseemc, and Semaničová-Feňovčíková [95] described a method for obtaining a 3-total edge product cordial labeling of the hexagonal grid from a smaller hexagonal grid. In [78] Ahmad proved that the generalized Petersen graphs $P(n, m)$ are 3-total edge product cordial. In [267] Azaizeh, Hasni, Lau, and Ahmad proved that complete graphs, bipartite graphs and generalised friendship graphs have 3-total edge product cordial labelings. Ahmad, Ali, Bilal, Zafar, and Zahid [82] prove that webs, helms, gears, and $L_n \odot 2K_1$ (L_n is the ladder with $2n$ vertices) have 3-total edge product cordial labelings. Ivančo [1342] characterized graphs admitting a 2-total edge product cordial labeling and proved that dense graphs and regular graphs of degree $2(k-1)$ admit a k-total edge product cordial labeling. Javed and Jamil [1365] proved that the rhombic grid graphs R_1^m, R_2^m and R_3^m are 3-total edge product cordial for $m \geqslant 1$ and that the rhombic grid graph R_n^m is 3-total edge product cordial for $m, n \geqslant 1$. Ullah, Rahmat, Numan, Anoh Yannick, and Aslam [3235] proved that the stellation of $P_n \times P_n$ (that is, the graph obtained from $P_n \times P_n$ by adding a vertex in each square of $P_n \times P_n$ and then joining this vertex to each vertex of that square) admits a 3-total edge product cordial labeling. Bala, Krithika Devi, and Thirusangu [400] proved [400] new the existence of 3-total cordial edge magic labeling, 3-total sum cordial labeling, and total product cordial labeling for the square graph of a comb.

In [1779] Kumari and Mehra call a vertex labeling f of a graph G with 0 and 1 with the induced edge labeling f given by $f(uv) = f(u)f(v)$ a *vertex product cordial* labeling if the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. They prove the following graphs have vertex product cordial labelings: P_n^2 if and only if n is odd, the path unions of k copies of P_n^2, $P_n \odot K_1$, helms, gears G_n for odd n, graphs obtained from C_n after switching of a vertex, $C_n \odot K_1$, $C_n \odot \bar{K}_m$, and certain banana trees.

A labeling f of a graph $G(V, E)$ from $V \cup E$ to $\{0, 1\}$ is said to be a *total product cordial* labeling of G if $f(xy) = f(x)f(y)$ for all $x, y \in V$ and $xy \in E$ and the difference of the total number vertices and edges labeled 0 and the total number vertices and edges labeled 1 differ by at most 1. Cyrile, Valdehueza, and Pedrano [769] determined the total edge product cordial labeling of the corona graphs $P_n \odot P_m$ and $P_n \odot C_m$. Gondalia [1127] [1127] new
proved that the graphs obtained by vertex switching of the following graphs are Fibonacci product cordial: cycles, cycles with one chord, cycles with twin chords where the chords form two triangles and one cycle C_{n-2}, and cycles with twin chords where chords form two triangles and one cycle $C_n(1, 1, n-5)$.

In [998] Gajjar and Raval define an *E-sum cordial labeling* of a graph G as an edge [998] new
labeling $f^* : E(G) \to \{0, 1\}$, where the induced vertex labeling defined by $f(u) = \sum f(uv)$ mod 2 over all edges incident to u, has the property that the number of verticies labeled with 0 and the number of verticies labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. They intruduce two new graphs insprited by origamai models that they call the *north star* graph and the *lotus star* (see article for the defintions) and investigate the existence of cordial labelings, sum cordial labelings, signed product cordial labelings, and E-sum cordial labelings for these new graph.

In 2019 the concept of a Fibonacci product cordial labeling was introduced by Abraham and Jose [17] as follows. An injective function f from the vertices of a graph to the set [17] new
$\{F_0, F_1, F_2, \ldots, F_{n+1}\}$ where F_j is the jth Fibonacci number is said to be a *Fibonacci cordial* labeling if the induced function f^* from the edges s to the set $\{0, 1\}$ defined by $f^*(uv) = (f(u)fv))(\text{mod } 2)$ has the property that the number of edges with label 0 and the number of edges with label 1 differ by at most 1. A graph which admits a Fibonacci cordial labeling is called a *Fibonacci cordial* graph. They proved that the following graphs are Fibonacci product cordial: paths, cycles, wheels W_n except when $n = 2$ mod 3, and the Petersen graph. For $n \geqslant 2$, Sulayman and Pedrano [3120] determined Fibonacci cordial [3120] new
labelings of alternate triangular snakes $A(T_n)$, quadrilateral snakes Q_n, cycle quadrilateral snakes CQ_n, and for $n \geqslant 3$, double alternate quadrilateral snakes.

For a graph G Sundaram, Ponraj, and Somasundaram [3141] defined the *index of product cordiality*, $i_p(G)$, of G as the minimum of $\{|e_f(0) - e_f(1)|\}$ taken over all the 0-1 binary labelings f of G with $|v_f(i) - v_f(j)| \leqslant 1$ and $f(uv) = f(u)f(v)$, where $e_f(k)$ and $v_f(k)$ denote the number of edges and the number of vertices labeled with k. They established that $i_p(K_n) = \lfloor n/2 \rfloor^2$; $i_p(C_n) = 2$ if n is even; $i_p(W_n) = 2$ or 4 according as n is even or odd; $i_p(K_{2,n}) = 4$ or 2 according as n is even or odd; $i_p(K_2 + nK_1) = 3$ if n is even; $i_p(G \times P_2) \leqslant 2i_p(G)$; $i_p(G_1 \cup G_2) \leqslant i_p(G_1) + i_p(G_2) + 2 \min\{\Delta(G_1), \Delta(G_2)\}$ where G_1 and G_2 are graphs of odd order; and $i_p(G_1 \odot G_2) \leqslant i_p(G_1) + i_p(G_2) + 2\delta(G_2) + 3$ where G_1 and G_2 have odd order.

In [3200] Tenguria and Verma called a mapping f from $V(G)$ to $\{0,1,2\}$ such that each edge uv is labeled $(f(u)+f(v))$ mod 3 a 3-*total super sum cordial labeling* if $|f(i)-f(j)| \leqslant 1$ for $i,\ j \in \{0,1,2\}$, where $f(x)$ denotes the total number of vertices and edges labeled with x and for each edge uv, $|f(u)-f(v)| \leqslant 1$. A graph that has a 3-total super sum cordial labeling is called 3-*total super sum cordial* graph. They proved $P_m \cup P_n$, $C_m \cup C_n$, and $K_{1,m} \cup K_{1,n}$ are 3-total super sum cordial graphs. (These results also appeared in [3201] and [3202]).

Sridevi, Nagarajan, Nellaimurugan, and Navaneethakrishnan [3043] introduced the notion of *Fibonacci divisor cordial* labeling of a graph $G(V,E)$ as a bijection $f: V \to \{F_1, F_2, F_3, \ldots, F_p\}$, where F_i is the i^{th} Fibonacci number such that if each edge uv is assigned the label 1 if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a Fibonacci divisor cordial labeling, then it is called a *Fibonacci divisor cordial graph*. They proved P_n, C_n, $K_{2,n}$, and the subdivision of the bistar $B_{n,n}$ are Fibonacci divisor cordial graphs. They further proved that K_n $(n \geqslant 3)$ is not a Fibonacci divisor cordial graph. Ghosh and Pal [1095] proved that the graphs obtained by the switching of an arbitrary vertex of cycles and wheels, the duplication of an arbitrary vertex of cycles, the degree splitting graphs of paths, umbrella graphs, and bistars $B_{n,n}$ are also Fibonacci divisor cordial graphs. Sridevi, Palani, Navanaeethakrishnan, and Nagarajan [3044] proved that $K_{1,n}$ for $n \leqslant 9$ and $n = 11$, W_n for $n = 7,8,9,10$ and $C_m@P_n$ are Fibonacci divisor cordial graphs. They also we prove that $K_{1,n}$ for $n \geqslant 12$ and $n = 10$, and W_n for $n = 4,5,6$ and $n \geqslant 11$ are not Fibonacci divisor cordial graphs.

[3043] new

[1095] new

[3044] new

Vaidya and Vyas [3323] define the *tensor product* $G_1(T_p)G_2$ of graphs G_1 and G_2 as the graph with vertex set $V(G_1) \times V(G_2)$ and edge set
$\{(u_1,v_1)(u_2,v_2)|\ u_1u_2 \in E(G_1), v_1v_2 \in V(G_2)\}$. They proved the following graphs are product cordial: $P_m(T_p)P_n$; $C_{2m}(T_p)P_{2n}$; $C_{2m}(T_p)C_{2n}$; the graph obtained by joining two components of $P_m(T_p)P_n$ an by arbitrary path; the graph obtained by joining two components of $C_{2m}(T_p)P_{2n}$ by an arbitrary path; and and the graph obtained by joining two components of $C_{2m}(T_p)C_{2n}$ by an arbitrary path.

In [2404] Ponraj introduced the notion of an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-cordial labeling of a graph. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a finite set of distinct integers and f be a function from a vertex set $V(G)$ to S. For each edge uv of G assign the label $f(u)f(v)$. He calls f an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-*cordial labeling* of G if $|v_f(\alpha_i) - v_f(\alpha_j)| \leqslant 1$ for all $i, j \in \{1,2,\ldots,k\}$ and $|e_f(\alpha_i\alpha_j) - e_f(\alpha_r\alpha_s)| \leqslant 1$ for all $i,j,r,s \in \{1,2,\ldots,k\}$, where $v_f(t)$ and $e_f(t)$ denote the number of vertices labeled with t and the number of edges labeled with t, respectively. A graph that admits an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-cordial labeling is called an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-*cordial graph*. Note that an $(-\alpha, \alpha)$-cordial graph is simply a cordial graph and a $(0, \alpha)$-cordial graph is a product cordial graph. Ponraj proved that $K_{1,n}$ is $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-cordial if and only if $n \leqslant k$ and for $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_1 + \alpha_2 \neq 0$ proved the following: K_n is (α_1, α_2)-cordial if and only if $n \leqslant 2$; P_n is (α_1, α_2)-cordial; C_n is (α_1, α_2)-cordial if and only if $n > 3$; $K_{m,n}$ $(m, n > 2)$ is not (α_1, α_2)-cordial; the bistar $B_{n,n+1}$ is (α_1, α_2)-cordial; $B_{n+2,n}$ is (α_1, α_2)-cordial if and only if $n \equiv 1, 2 \pmod 3$; $B_{n+3,n}$ is (α_1, α_2)-cordial if and only if $n \equiv 0, 2 \pmod 3$; and $B_{n+r,n}, r > 3$ is not (α_1, α_2)-cordial. He also proved that if

G is an (α_1, α_2)-cordial graph with p vertices and q edges, then $q \leqslant 3p^2/8 - p/2 + 9/8$. In [2404] Ponraj proved that combs $P_n \odot K_1$ are (α_1, α_2)-cordial; coronas $C_n \odot K_1$ are (α_1, α_2)-cordial for $n \equiv 0, 2, 4, 5 \pmod 6$; $C_3^{(t)}$ is not (α_1, α_2)-cordial; W_n is not (α_1, α_2)-cordial; and $\overline{K_n} + 2K_2$ is (α_1, α_2)-cordial if and only if $n = 2$.

In [3340] Varatharajan, Navanaeethakrishnan Nagarajan define a *divisor cordial* labeling of a graph G with vertex set V as a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that an edge uv is assigned the label 1 if one $f(u)$ or $f(v)$ divides the other and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If graph that has a divisor cordial labeling, it is called a *divisor cordial* graph. They proved the standard graphs such as paths, cycles, wheels, stars and some complete bipartite graphs are divisor cordial. They also proved that complete graphs are not divisor cordial. In [3341] they proved dragons, coronas, wheels, and complete binary trees are divisor cordial. For t copies $S_1, S_2, \ldots, S_t$ of an n-star $K_{1,n}$ they define $\langle S_1, S_2, \ldots, S_t \rangle$ as the graph obtained by starting with $S_1, S_2, \ldots, S_t$ and joining the central vertices of S_{k-1} and S_k to a new vertex x_{k-1}. They prove that $\langle S_1, S_2 \rangle$ and $\langle S_1, S_2, S_3 \rangle$ are divisor cordial.

Vaidya and Shah [3305] proved that the splitting graphs of stars and bistars are divisor cordial and the shadow graphs and the squares of bistars are divisor cordial. In [3307] they proved that helms, flower graphs, and gears are divisor cordial graphs. They also proved that graphs obtained by switching of a vertex in a cycle, switching of a rim vertex in a wheel, and switching of an apex vertex in a helm admit divisor cordial labelings. Raj and Valli [2553] proved the following graphs divisor cordial: the duplication of a vertex of a cycle; graphs obtained by joining two wheels of the same size by a path of length at least 3; $G_v \odot K_1$, where G_v is a graph obtained by switching any vertex of a cycle of size at least 4; graphs obtained by joining the apex vertices of two shells of the same size to an isolated vertex; graphs obtained by joining the centers of two wheels of the same size to an isolated vertex; and a class of graphs obtained by removing certain edges from complete graphs. Bosmia and Kanani [602] proved that the graphs of the form $G \odot K_1$ where G any of the following admits a divisor cordial labeling: $K_{1,n}$, $K_{2,n}$, $K_{3,n}$, a wheel, a helm, a flower, a fan, a double fan, and a barycentric subdivision of a star. Bosmia and Kanani [603] prove that the following graphs admit divisor cordial labelings: bistars, the splitting graph of bistars, the degree splitting graph of bistars, the shadow graph of bistars, the restricted square graph of bistars, the barycentric subdivision of bistars, and the corona product of a bistar with K_1. Thirusangu and Madhu [3211] proved that the extended duplicate graph of star and bistar graphs are divisor cordial graphs.

In [2179] Sugumaran and Mohan proved that the following graphs are divisor cordial graphs: degree splitting graph of $K_{1,n,n}$), the splitting graph of the graph obtained from two isolated vertices are joined by n paths of length 2, the $W-$ graph (the graph obtained from two copies of $K_{1,n}$ and identifying the last pendent vertex of the first copy with the first pendent edge of the secnd copy), $B(n) \odot_{u_m} K_1$, where $B(n) = 2P_n + K_1$ (*bow* graph, the Herschel graph H_s and switching of an apex vertex in the Herschel graph (see [596, p. 53]. In [3099] Sugumaran and Suresh proved that the following graphs are divisor cordial graphs: fans, Petersen graphs, $C_m \odot K_1$, friendship graphs F_n, and the switching of a end vertex in path P_n, switching of any one of the inner vertices of Petersen graph P_e.

Barasara and Thakkar [432] investigated divisor cordial labelings of armed crowns, closed helms, webs, and the one point union of t copies of C_n). In [429] they investigated divisor cordial labeling of the combs, P_n^2, C_n^2 for $n \neq 4$, and middle graphs of paths and cycles. Barasara and Thakkar [431] proved that ladders, circular ladders, Möbius ladders, total graphs of paths, and total graphs of cycles are divisor cordial graphs.

Vaidya and Barasara [3249] proved that every graph can be embedded as an induced subgraph of a product cordial graph, thereby ruling out any possibility of obtaining any forbidden subgraph characterization for product cordial graphs. They also proved that every connected graph can be embedded as an induced subgraph of a product cordial connected graph and every planar graph can be embedded as an induced subgraph of a product cordial planar graph. Similar results for total product cordial labeling were also obtained. In [3260] they prove that every graph can be embedded as an induced subgraph of a divisor cordial graph, thereby ruling out any possibility of obtaining any forbidden subgraph characterization for divisor cordial graphs. They also proved that every connected graph can be embedded as an induced subgraph of a divisor cordial connected graph and every planar graph can be embedded as an induced subgraph of a divisor cordial planarr graph.

Gondalia [1126] proved that the Herschel graph, the fusion of any two adjacent vertices of degree 3 in a Herschel graph, the duplication of any vertex of degree 3 in a Herschel graph, the switching of a central vertex in the Herschel graph, the joint sum of two copies of a Herschel graph, and the degree splitting of Herschel graph are divisor cordial graphs.

Motivated by the concept of divisor cordial labeling, Lourdusamy and Patrick [2008] introduced a new concept of divisor cordial labeling called sum divisor cordial labeling. Let $G = (V(G), E(G))$ be a simple graph and f be a bijection from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$. For each edge uv, assign the label 1 if 2 divides $f(u)+f(v)$ and the label 0 otherwise. The function f is called a *sum divisor cordial* labeling if the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits a sum divisor cordial labeling is called a {it sum divisor cordial . They prove that paths, combs, stars, complete bipartite, K_2+mK_1, bistars, jewels, crowns, flowers, gears, subdivisions of stars, the graph obtained from $K_{1,3}$ by attaching the center of $K_{1,n}$ at each pendent vertex of $K_{1,3}$, and the square $B_{n,n}$ are sum divisor cordial graphs. In [2014] Lourdusamy and Patrick proved that every transformed tree admits a sum divisor cordial labeling. They also investigated the sum divisor cordial labelings of the graphs obtained by identifying a vertex of graphs of paths, middle graphs of paths, and the splitting graphs of cycles. [2014] new

The *zero divisor graph* of a commutative ring R, $\Gamma(R)$, is the graph whose vertex set is set of zero divisors in R in which two distinct vertices u, v are adjacent if $uv = 0$. In [2021] Lourdusamy, Jenifer Wency, and Joy Beaula investigate the existence of sum divisor cordial labelings of graphs of the form $\Gamma(Z_n), \Gamma(Z_m) + \Gamma(Z_n)$, and $\overline{\Gamma(Z_m)} + \Gamma(Z_n)$.

In [3111], [3112], [3113], and [3114] Sugumaran and Rajesh proved that the following graphs are sum divisor cordial: swastiks, path unions of finite number of copies of swastiks, cycles of k copies of swastiks, when k is odd, jelly fish, Petersen graphs, theta graphs, the fusion of any two vertices in the cycle of swastiks, duplication of any vertex in the cycle of swastiks, the switchings of a central vertex in swastiks, the path unions of two copies of a swastik, the star graph of the theta graphs, the Herschel graph, the fusion of any two adjacent vertices of degree 3 in Herschel graphs, the duplication of any vertex of degree 3 in the Herschel graph, the switching of central vertex in Herschel graph, the path union of two copies of the Herschel graph, H-graph H_n, when n is odd, $C_3@K_{1,n}$ (obtained by identifying the center of $K_{1,n}$ with a vertex of C_3), $< F_n^1 \triangle F_n^2 >$ (the graph obtained by joining the apex vertices of F_n^1 and F_n^2 by an edge and by joining the two apex vertices to a new vertex v') and open star of swastik graphs $S(t.Sw_n)$, when t is odd. In [3115], [3116] and [3117] Sugumaran and Rajesh proved that the following graphs are sum divisor cordial graphs: H-graph H_n, when n is even, duplication of all edges of the H-graph H_n, when n is even, $H_n \odot K_1$, $P(r.H_n)$, $C(r.H_n)$, plus graphs, umbrella graphs, path unions of odd cycles, kites, complete binary trees, drums graph (two copies of C_n that share exactly one vertex v and two copies of P_n that have an end point at v, twigs (graphs obtained from a path by attaching exactly two pendent edges to each internal vertices of the path), fire crackers of the form $P_n \odot S_n$, where n is even, and the double arrow graph DA_m^n, where $|m - n| \leqslant 1$ and n is even (obtained from $P_m \times P_n$ by adding two new vertices u and v such that each of the top row vertices of $P_m \times P_n$ are connected to u by an edge and the bottom row vertices of $P_m \times P_n$ are connected to v by an edge). Sugumaran and Rajesh [3098] proved that the following graphs are sum divisor cordial: $P_n + P_n$ (n is odd), $P_n@K_{1,m}$ (obtained by identifying an endpoint of P_n with the center of $K_{1,n}$), $C_n@K_{1,m}$ (n is odd), the graph obtained from W_{2n} by attaching the apex vertex of a copy of $K_{1,m}$ to each rim vertex, the graph obtained by joining the central vertices of two copies of $K_{1,n,n}$ by an edge and to a new vertex, the graph obtained by starting with C_n and, for each edge of C_n, adjoining a copy of C_n that shares an edge with the starting copy (the *flower graph* FL_n). In [54] Adalja and Ghodasara provide sum divisor cordial labelings for the graphs resulting from the duplication of graph elements in stars, cycles, and paths. In [2543] Raheem, Javaid, Numan, and Hasni investigated the existence sum divisor cordial labelings for the disjoint union of paths and subdivided stars.

Murugesan [2227] introduced a *square divisor cordial* labeling. Let G be a simple graph and $f :\rightarrow \{1, 2, \ldots, |V(G)|\}$ a bijection. For each edge uv, assign the label 1 if either $(f(u))^2$ divides $f(v)$ or $(f(v))^2$ divides $f(u)$ and the label 0 otherwise. Call f a *square divisor cordial labeling* if $|e_f(0) - e_f(1)| \leqslant 1$. A graph with a square divisor cordial labeling is called a *square divisor cordial* graph. Murugesan proved that the following are square divisor cordial graphs: P_n ($n \leqslant 12$), C_n ($3 \leqslant n \leqslant 11$), wheels, some stars, some complete bipartite graphs, and some complete graphs. Vaidya and Shah [3311] proved that the following are square divisor cordial graphs: flowers, bistars, shadow graphs of stars, splitting graphs of stars and bistars, degree splitting graphs of paths and bistars.

Kanani and Bosmia [1574] define a *cube divisor cordial* labeling f of a simple graph G as a bijection from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ such that, when each edge uv is assigned the label 1 if $(f(u))^3$ divides $f(v)$ or $(f(v))^3$ divides $f(u)$ and the label 0 otherwise, it holds that $|e_f(0) - e_f(1)| \leqslant 1$. A graph with a cube divisor cordial labeling is called a *cube divisor cordial* graph. They proved that the following graphs admit cube divisor cordial labelings: K_n if and only if $n = 1, 2, 3$; $K_{1,n}$ if and only if $n = 1, 2, 3$; $K_{2,n}$ for all n; $K_{3,n}$ if and only if $n = 1, 2$; bistars $B_{n,n}$ for all n ; and the graph obtained by joining leaves of one star of a bistar with the center of the opposite star of the bistar. Kanani and Bosmia [1574] prove: the edge deleted graph of a cube divisor cordial graph is also a cube divisor cordial graph; P_n is a cube divisor cordial graph if and only if $n = 1, 2, 3, 4, 5, 6, 8$; C_n is a cube divisor cordial graph if and only if $n = 3, 4, 5$; and wheels, flowers and fans are cube divisor cordial,

The Lucas sequence of numbers is a linear recurrence relation satisfying the conditions: $l_1 = 1, l_2 = 3$ and $l_n = l_{n-1} + l_{n-2}, n \geqslant 3$. Let $G = (V, E)$ be a simple graph and $f : V(G) \rightarrow \{l_1, l_2, \ldots, l_{|V(G)|}\}$ be a bijection such that each edge uv, assign the label 1 if either $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$ and label 0 otherwise. In [3119] Sugumaran and Rajesh call such an f a *Lucas divisor cordial labeling* if $|e_f(0) - e_f(1)| \leqslant 1$. A graph with a Lucas divisor cordial labeling is called a *Lucas divisor cordial* graph. In [3119] Sugumaran and Rajesh proved that the following graphs are Lucas divisor cordial graphs: bistars, jelly fish, square graphs of bistars, switching of a vertex in cycles, and switching of a pendent vertex in paths.

A variation of divisor cordial labeling called vertex odd divisor cordial labeling was introduced by Muthaiyan and Pugalenthi (see [2228]) as follows. Let G be a graph with p vertices and a bijection f from $V(G)$ to $\{1, 3, 5, \ldots, 2p-1\}$ such that if each edge uv is assigned the label 1 if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$, and the label 0 otherwise. The function f is called a *vertex odd divisor cordial labeling* if $|e_f(0) - e_f(1)| \leqslant 1$. A graph with vertex odd divisor cordial labeling is called a *vertex odd divisor cordial graph*. Muthaiyan and Pugalenthi (see [2228]) proved paths, cycles, $K_{2,n}, K_{1,n} \cup K_{1,m}$, helms, flowers, $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$, the switching of the apex vertex in helms, and the splitting graph of stars are vertex odd divisor cordial graphs under some conditions. In [2228] Muthaiyan and Pugalenthi proved the following graphs have vertex odd divisor cordial labelings: wheels, the switching of a pendent vertex in paths and cycles, bistars $B_{n,n}$, the subdivision graph of $K_{1,n}, B_{n,n}^2, DS(B_{n,n})$, the splitting graph of $B_{n,n}$, and $< K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)} >$.

Let G_1 and G_2 be two copies of any graph G that has an apex vertex. The graph obtained by joining the apex vertices of G_1 and G_2 by an edge and the graph obtained by joining the apex vertices of G_1 and G_2 by an edge and by joining the two apex vertices to a new vertex v', is denoted $G_1 \triangle G_2$. For any vertex u of $K_{m,n}$ the graph obtained by joining u to a new pendent vertex is denoted by $K_{m,n} \odot u(K_1)$. In [3103] Sugumaran and Suresh proved that the following graphs are vertex odd divisor cordial graphs: the shadow graph of $K_{1,n}$, $K_{2,n} \odot u(K_1)$, $K_{1,n} \triangle K_{1,n}$, the subdivision of the edge between the apex vertices of $B_{n,n}$, and the graph $K_{1,n} * P_{n+2}$ (the graph obtained by identifying an end vertex of P_{n+2} with the apex vertex of $K_{1,n}$). In [3102] they showed that the graphs $F_n \triangle F_n$, $K_{1,n} \triangle K_{1,n} \triangle K_{1,n}$, $K_{1,n} \triangle K_{1,n} \triangle K_{1,n} \triangle K_{1,n}$, theta graphs, and switching of a vertex in a Petersen graph are vertex odd divisor cordial graphs. In [3104] they proved that gears, switching of an apex vertex in $S(K_{1,n})$, $P_2 + mK_1$, $C(n, n-3)$, and $C(n, n-4)$ are vertex odd divisor cordial graphs. In [3100] they showed that the globe $Gl(n)$, jewels, $G * W_n$ (appending the central vertex of wheel W_n with any one of the vertices of G), $G * C(n, n-3)$, and wheels are vertex odd divisor cordial graphs.

Let $f : V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ be an injective map. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ or $f(v)$ is divided by $f(u)$

according as $f(u) \geqslant f(v)$ or $f(v) \geqslant f(u)$. The function f is called a *remainder cordial* labeling of G if $|e_f(0) - e_f(1)| \leqslant 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labelled with even integers and number of edges labelled with odd integers. A graph G with admits a remainder cordial labeling is called a *remainder cordial* graph. In [2419] Ponraj, Annathurai, and Kala investigated the remainder cordial behavior of $S(K_{1,n})$, $S(B_{n,n})$, $S(W_n)$ and union of some star related graphs. Ponraj, Gayathri, and Somasundaram [2421] investgated the 4-remainder cordial labeling behavior of parachutes, twigs, and kayak paddles graphs.

A *double divisor cordial labeling* of a graph $G(V, E)$ is a bijective function ψ from V to $\{1, 2, 3, \ldots, |V|\}$ such that each edge ab is given label 1 if $2\psi(a)|\psi(b)$ or $2\psi(b)|\psi(a)$ and 0 otherwise, then the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. An graph that admits a double divisor cordial labeling is said to be a *double divisor cordial graph.* In [2351] Parthiban and Sharma proved that the following graphs admit double divisor cordial labelings: full binary trees, P_n $(n \geqslant 3)$, C_n, W_n (n odd), helms, fans, stars, bistars, jelly fish, and friendship graphs. They also prove that K_n $(n \geqslant 5)$ does not admit a double divisor cordial labeling. Sharma and Parthiban [2868] proved that the following graphs admit double divisor cordial labelings: the barycentric subdivision of stars and bistars $B(n, n)$; the splitting graph of $K_{1,n}$ $(n \geqslant 2)$; the splitting graph of $B(m, n)$ $(m, n \geqslant 2)$; $\langle K_{1,n}^{(1)}, K_{1,n}^{(2)} \rangle$; $\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)} \rangle$; the graph constructed by connecting an apex vertex of one copy of $B_{n,n}$ to a new vertex r an apex vertex of another copy of $B_{n,n}$ to another vertices s; and $K_{i,n} \odot K_1$ for $i = 1, 2$, and 3.

7.7 Edge Product Cordial Labelings

Vaidya and Barasara [3249] introduced the concept of edge product cordial labeling as edge analogue of product cordial labeling. An *edge product cordial labeling* of graph G is an edge labeling function $f : E(G) \to \{0, 1\}$ that induces a vertex labeling function $f^* : V(G) \to \{0, 1\}$ defined as $f^*(u) = \prod\{f(uv) \mid uv \in E(G)\}$ such that the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1 and the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1. A graph with an edge product cordial labeling is called an *edge product cordial graph*.

In [3249], [3260], [3252], [3253], and [3256] Vaidya and Barasara proved the following graphs are edge product cordial: C_n for n odd; trees with order greater than 2; unicyclic graphs of odd order; $C_n^{(t)}$, the one point union of t copies of C_n for t even or t and n both odd; $C_n \odot K_1$; armed crowns $C_m \odot P_n$; helms; closed helms; webs; flowers; gears; shells S_n for odd n; tadpoles $C_n@P_m$ for $m + n$ even or $m + n$ odd and $m > n$ while not edge product cordial for $m + n$ odd and $m < n$; triangular snakes; for odd n, double triangular snakes DT_n, quadrilateral snakes Q_n and double quadrilateral snakes DQ_n; P_n^2 for odd n; $M(P_n)$, $T(P_n)$; $S'(P_n)$ for even n; the tensor product of P_m and P_n; and the tensor product of C_n and C_m if m and n are even. In [3257] Vaidya and Barasara investigate product and edge product cordial labelings of the degree splitting graphs of paths, shells, bistars, and gear graphs. They proved the following graphs are not edge product cordial:

C_n for n even; K_n for $n \geqslant 4$; $K_{m,n}$ for $m, n \geqslant 2$; wheels; the one point union of t copies of C_n for t odd and n even; shells S_n for even n; tadpoles $C_n@P_m$ for $m+n$ odd and $m < n$; for n even double triangular snake DT_n, quadrilateral snake Q_n and double quadrilateral snake DQ_n; double fans; C_n^2 for $n > 3$; P_n^2 for even n; $D_2(C_n)$, $D_2(P_n)$; $M(C_n)$; $T(C_n)$; $S'(C_n)$; $S'(P_n)$ for odd n; $P_m \times P_n$ and $C_m \times C_n$; the tensor product of C_n and C_m if m or n odd; and $P_n[P_2]$ and $C_n[P_2]$.

Barasara [428] proved that C_n with one chord except when n is even and the chord joins vertices that are at diameter distance, C_n with twin chords except when n is even and the chords join vertices that are at diameter distance, and $P_n \odot K_1$ are edge product cordial graphs, whereas triangular ladders are not edge product cordial. In [3250], Barasara proved that C_n with one chord, C_n with twin chords, $P_n \odot K_1$, and triangular ladders are total edge product cordial graphs. Vaidya and Barasara [3259] proved that every graph can be embedded as an induced subgraph of an edge product cordial graph, thereby ruling out any possibility of obtaining any forbidden subgraph characterization for edge product cordial graphs. They also proved that every connected graph can be embedded as an induced subgraph of an edge product cordial connected graph and every planar graph can be embedded as an induced subgraph of an edge product cordial planar graph. In [427], Barasara proved that the shadow graph of W_n and the splitting graph of W_n are not edge product cordial graphs, whereas the middle graph of W_n is edge product cordial for $n > 5$, and the total graph of W_n is edge product cordial for $n > 8$. He also investigated total edge product cordial labeling of the shadow graph, the splitting graph, the middle graph, and the total graph of W_n.

Vaidya and Barasara [3260] proved that every graph can be embedded as an induced subgraph of a total edge product cordial graph thereby ruling out any possibility of obtaining any forbidden subgraph characterization for total edge product cordial graphs. They also proved that a connected graph can be embedded as an induced subgraph of a total edge product cordial connected graph and every planar graph can be embedded as an induced subgraph of a total edge product cordial planar graph.

Prajapati and Shah [2506] proved the following graphs are edge product cordial: graphs obtained from a crown by duplication of a vertex, duplication of a vertex by an edge, or duplication of an edge by a vertex; graphs obtained from a gear graph by duplication of each of the vertices of degree three by an edge; and the graph obtained from a helm by duplication of each of the pendent vertices by a new vertex. In [2498] Prajapati and Patel provided results about the existence of edge product cordial labelings closed webs, lotus inside a circle, and sunflower graphs.

Vaidya and Barasara [3254] introduced the concept of a total edge product cordial labeling as edge analogue of total product cordial labeling. An *total edge product cordial labeling* of graph G is an edge labeling function $f : E(G) \to \{0, 1\}$ that induces a vertex labeling function $f^* : V(G) \to \{0, 1\}$ defined as $f^*(u) = \prod\{f(uv) \mid uv \in E(G)\}$ such that the number of edges and vertices labeled with 0 and the number of edges and vertices labeled with 1 differ by at most 1. A graph with total edge product cordial labeling is called a *total edge product cordial graph*.

In [3254] and [3255] Vaidya and Barasara proved the following graphs are total edge product cordial: C_n for $n \neq 4$; K_n for $n > 2$; W_n; $K_{m,n}$ except $K_{1,1}$ and $K_{2,2}$; gears; $C_n^{(t)}$, the one point union of t copies of C_n; fans; double fans; C_n^2; $M(C_n)$; $D_2(C_n)$; $T(C_n)$; $S'(C_n)$; P_n^2 for $n > 2$; $M(C_n)$; $D_2(C_n)$ for $n > 2$; $T(C_n)$; $S'(C_n)$. Moreover, they prove that every edge product cordial graph of either even order or even size admits total edge product cordial labeling. Bača, Irfan, Javad, and Semaničová-Feňočová [337] investigated the existence of total edge product cordial labeling of toroidal fullerenes and for Klein-bottle fullerenes. Prajapati and Patel [2499] proved that the one point union of t copies of a wheel with a rim vertex in common is edge product cordial if and only is t is even; all pentagonal snakes (obtained from the path by replacing every edge of a path by C_5) are edge product cordial; and a double pentagonal snakes (two pentangonal snakes that have a common path) is edge product cordial if and only is t is odd.

In [2514] Prasad and Maheswari developed a technique for coding a secret messages using sunflower graphs by subdividing edges and applying edge product cordial labeling.

7.8 Difference Cordial Labelings

Ponraj, Sathish Narayanan, and Kala [2443] introduced the notion of difference cordial labelings. A *difference cordial labeling* of a graph G is an injective function f from $V(G)$ to $\{1, \ldots, |V(G)|\}$ such that if each edge uv is assigned the label $|f(u)-f(v)|$, the number of edges labeled with 1 and the number of edges not labeled with 1 differ by at most 1. A graph with a difference cordial labeling is called a *difference cordial* graph.

The following definitions appear in [2444], [2431], [2432], and [2433]. A *double triangular snake* DT_n consists of two triangular snakes that have a common path; a *double quadrilateral snake* DQ_n consists of two quadrilateral snakes that have a common path; an *alternate triangular snake* $A\left(T_n\right)$ is the graph obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertex v_i (that is, every alternate edge of a path is replaced by C_3); a *double alternate triangular snake* $DA\left(T_n\right)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i and w_i; an *alternate quadrilateral snake* $A\left(Q_n\right)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i and w_i respectively and then joining v_i and w_i (that is, every alternate edge of a path is replaced by a cycle C_4); a *double alternate quadrilateral snake* $DA\left(Q_n\right)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, x_i and w_i and y_i respectively and then joining v_i and w_i and x_i and y_i.

In [2432] and [2433] Ponraj and Sathish Narayanan define the *irregular triangular snake* IT_n as the graph obtained from the path $P_n : u_1, u_2, \ldots, u_n$ with vertex set $V(IT_n) = V(P_n) \cup \{v_i : 1 \leqslant i \leqslant n \leqslant 2\}$ and the edge set $E(IT_n) = E(P_n) \cup \{u_i v_i, v_i u_{i+2} : 1 \leqslant i \leqslant n-2\}$. The *irregular quadrilateral snake* IQ_n is obtained from the path $P_n : u_1, u_2, \ldots, u_n$ with vertex set $V\left(IQ_n\right) = V\left(P_n\right) \cup \{v_i, w_i : 1 \leqslant i \leqslant n-2\}$ and edge set $E\left(IQ_n\right) = E\left(P_n\right) \cup \{u_i v_i, w_i u_{i+2}, v_i w_i : 1 \leqslant i \leqslant n-2\}$. They proved the following graphs are difference cordial: triangular snakes T_n, quadrilateral snakes, alternate triangular snakes, alternate quadrilateral snakes, irregular triangular snakes, irregular quadrilateral snakes, double triangular snakes DT_n if and only if $n \leqslant 6$, double quadrilateral snakes, double alternate triangular snakes $DA(T_n)$, and double alternate quadrilateral snakes.

In [2443], [2430], [2444], and [2431] Ponraj, Sathish Narayanan, and Kala proved the following graphs have difference cordial labelings: paths; cycles; wheels; fans; gears; helms; $K_{1,n}$ if and only if $n \leqslant 5$; K_n if and only if $n \leqslant 4$; $K_{2,n}$ if and only if $n \leqslant 4$; $K_{3,n}$ if and only if $n \leqslant 4$; bistar $B_{1,n}$ if and only if $n \leqslant 5$; $B_{2,n}$ if and only if $n \leqslant 6$; $B_{3,n}$ if and only if $n \leqslant 5$; $DT_n \odot K_1$; $DT_n \odot 2K_1$; $DT_n \odot K_2$; $DQ_n \odot K_1$; $DQ_n \odot 2K_1$; $DQ_n \odot K_2$; $DA(T_n) \odot K_1$; $DA(T_n) \odot 2K_1$; $DA(T_n) \odot K_2$; $DA(Q_n) \odot K_1$; $DA(Q_n) \odot 2K_1$; and $DA(Q_n) \odot K_2$. They also proved: if G is a (p, q) difference cordial graph, then $q \leqslant 2p - 1$; if G is a r-regular graph with $r \geqslant 4$, then G is not difference cordial; if $m \geqslant 4$ and $n \geqslant 4$, then $K_{m,n}$ is not difference cordial; if $m + n > 8$ then the bistar $B_{m,n}$ is not difference cordial; and every graph is a subgraph of a connected difference cordial graph. If G is a book, sunflower, lotus inside a circle, or square of a path, they prove that $G \odot mK_1$ $(m = 1, 2)$ and $G \odot K_2$ is difference cordial.

In [2445], [2447], and [2446] Ponraj, Sathish Narayanan, and Kala proved that the following graphs are difference cordial: crowns $C_n \odot K_1$; combs $P_n \odot K_1$; $P_n \odot C_m$; $C_n \odot C_m$; $W_n \odot K_2$; $W_n \odot 2K_1$; $G_n \odot K_1$ where G_n is the gear graph; $G_n \odot 2K_1$; $G_n \odot K_2$; $(C_n \times P_2) \odot K_1$; $(C_n \times P_2) \odot 2K_1$; $(C_n \times P_2) \odot K_2$; $L_n \odot K_1$; $L_n \odot 2K_1$; and $L_n \odot K_2$. Ponraj, Sathish Narayanan and Kala proved that the following subdivision graphs are difference cordial: $S(T_n)$; $S(Q_n)$; $S(DT_n)$; $S(DQ_n)$; $S(A(T_n))$; $S(DA(T_n))$; $S(AQ_n)$; $S(DAQ_n)$; $S(K_{1,n})$; $S(K_{2,n})$; $S(W_n)$; $S(P_n \odot K_1)$; $S(P_n \odot 2K_1)$; $S(LC_n)$; $S(P_n^2)$; $S(K_2 + mK_1)$; subdivision graphs of sunflowers $S(SF_n)$; subdivisions graphs flowers $S(Fl_n)$; $S(B_m)$ (B_m is a book with m pages); $S(C_n \times P_2)$; $S(B_{m,n})$; subdivisions n-cubes; $S(J(m, n))$; $S(W(t, n))$; subdivisions of Young tableaus $S(Y_{n,n})$; and if $S(G)$ is difference cordial, then $S(G \odot mK_1)$ is difference cordial. For graphs G that are a tree, a unicycle, or when $|E(G)| = |V(G)| + 1$, they proved that $G \odot P_n$ and $G \odot mK_1$ $(m = 1, 2, 3)$ are difference cordial.

In [2491] Prajapati and Gajjar define a *holiday star* as follows. Let $v_1, v_2, \ldots, v_{4n-1}, v_{4n}$ be the consecutive $4n$ vertices of C_{4n} $(n \geqslant 3)$. Let u_0 be the central vertex and $u_1, u_2, \ldots, u_{2n-1}, u_{2n}$ be end vertices of $K_{1,2n}$. Join u_0 to v_{4i-3} by an edge; for each i from 1 to n. In [2492] they define a *Kusadama flower graph* as follows. Let v_0 be the apex vertex and $v_1, v_2, v_3, \ldots, v_{2n-1}, v_{2n}$ be $2n$ consecutive rim vertices of the wheel W_{2n} $(n \geqslant 3)$. Subdivide the spoke edge $v_0 v_{2i-1}$ by a vertex w_i and at each w_i join two copies of path of length 2; $P_2^\ell = v_0, u_{2i-1}, w_i$ and $P_2^r = v_0, u_{2i}, w_i$, for each $i \in [n]$. In [2491] and [2492] Prajapati and Gajjar proved that the holiday star graph and the Kusudama flower graph admit cordial, E-cordial, difference cordial, prime, vertex prime, and total prime labelings. In [2493] Prajapati and Gajjar define a *braided star graph* as follows: Let a_0 be the apex vertex and $a_1, a_2, \ldots, a_{n-1}, a_n$ be consecutive n rim vertices of W_n $(n \geqslant 3)$; let $b_1, b_2, b_3, \ldots, b_{2n-1}, b_{2n}$ be $2n$ consecutive vertices of the cycle C_{2n}; let $c_1, c_2, \ldots, c_{2n-1}, c_{2n}$ be consecutive $2n$ vertices of C_{2n}. Join each a_i to b_{2i-1} by an edge and b_{2i} to c_{2i} by an edge. Take a new vertex d_i and join each d_i to c_{2i-1} and c_{2i+1} by an edge for each $i \in [n]$ where subscripts are taken modulo n. Prajapati and Gajjar [2493] proved that braided star graphs are cordial, E-cordial and difference cordial. In [2495] Prajapati and Gajjar investigated the existence of cordial, E-cordial, vertex prime, and difference cordial label- [2495] new ings of graphs that are inspired by the origami model called an *aboreale star* (see their paper for the definition).

Recall the splitting graph of G, $S'(G)$, is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v and the shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, G' and G'', and joining each vertex u' in G' to the neighbors of the corresponding vertex u' in G''.

Ponraj and Sathish Narayanan [2432], [2433] proved the following graphs are difference cordial: $S^{'}(P_n)$; $S^{'}(C_n)$; $S^{'}(P_n \odot K_1)$; and $S^{'}(K_{1,n})$ if and only if $n \leqslant 3$. They proved following are not difference cordial: $S^{'}(W_n)$; $S^{'}(K_n)$; $S^{'}(C_n \times P_2)$; the splitting graph of a flower graph; $DS\,(SF_n)$; $DS\,(LC_n)$; $DS\,(Fl_n)$; $D_2\,(G)$ where G is a (p,q) graph with $q \geqslant p$; and $DS\,(B_{m,n})$ $(m \neq n)$ with $m+n > 8$.

Let $G\,(V,E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of vertices having at least two vertices and having the same degree. Panraj and Sathish Narayanan [2432], [2433] define the *degree splitting graph* of G denoted by $DS\,(G)$ as the graph obtained from G by adding vertices $w_1, w_2, \ldots, w_t$ and joining w_i to each vertex of S_i $(1 \leqslant i \leqslant t)$. They proved the following graphs are difference cordial: $DS\,(P_n)$; W_n; $DS\,(C_n)$; $DS\,(K_n)$ if and only if $n \leqslant 3$; $DS\,(K_{1,n})$ if and only if $n \leqslant 4$; $DS\,(W_n)$ if and only if $n = 3$; $DS\,(K_n^c + 2K_2)$ if and only if $n = 1$; $DS\,(K_2 + mK_1)$ if and only if $n \leqslant 3$; $DS\,(K_{n,n})$ if and only if $n \leqslant 2$; $DS\,(T_n)$ if and only if $n \leqslant 5$; $DS\,(Q_n)$ if and only if $n \leqslant 5$; $DS\,(L_n)$ if and only if $n \leqslant 5$; $DS\,(B_{n,n})$ if and only if $n \leqslant 2$; $DS\,(B_{1,n})$ if and only $n \leqslant 4$; $DS\,(B_{2,n})$ if and only $n \leqslant 4$; $D_2\,(P_n)$; $D_2\,(K_n)$ if and only if $n \leqslant 2$; and $D_2\,(K_{1,m})$ if and only if $m \leqslant 2$.

In [2434], Ponraj and Sathish Narayanan proved the following graphs are difference cordial: $T_n \odot K_1$, $T_n \odot 2K_1$, $T_n \odot K_2$, $A(T_n) \odot K_1$, $A(T_n) \odot 2K_1$ and $A(T_n) \odot K_2$ where T_n and $A(T_n)$ are triangular snake and alternate triangular snake respectively. In [2448, 2449] Ponraj, Sathish Narayanan, and Kala proved the following graphs are difference cordial: $C_n \times P_2$; Möbius ladders; the n-cube; sunflower graphs; lotuses inside a circle; pyramids; books with n pentagonal pages; mongolian tents; graphs obtained from a ladder by subdividing each step exactly once; permutation graphs $P(P_{2k}, f)$ where $f = (1\,2)(3\,4)\cdots(k\,k{+}1)\cdots(2k{-}1\,2k)$; and $P(P_n, I)$, $P(C_n, I)$, $P(P_n \odot K_1, I)$, $P(P_n \odot 2K_1, I)$ where I is the identity permutation. Ponraj, Sathish Narayanan, and Kala [2448] [2449] proved the following graphs are not difference cordial: $G_1(p_1, q_1) \times G_2(p_2, q_2)$ with $q_1 \geqslant p_1$ and $q_2 \geqslant p_2$; $C_m \times C_n$; $G \times K_n$ where G connected graph and $n \geqslant 5$, $G + K_1$ where $|E(G) > |V(G) + 1$; $G_1 + G_2$ where G_1 and G_2 are connected and $|E(G_1)| > 1$ and $E(G_2)| > 3$; permutation graphs $P(G \times K_2, f)$ where $|E(G)| \geqslant |V(G)|$ and f is any permutation; $P(W_n, f)$ for any permutation f; $P(S^{'}(G), f)$ where $S^{'}(G)$ is the splitting graph of G, $|E(G)| \geqslant |V(G)|$, and f is any permutation; and $P(Fl_n, f)$ where Fl_n is a flower graph and f is any permutation. They also obtained the following necessary and sufficient conditions for difference cordiality: $K_m \times P_2$ if and only if $m \leqslant 3$; for a connected graph G, $G \times W_n$ if and only if $G = K_1$; books B_m if and only if $m \leqslant 6$; $G + G$ if and only if $|V(G)| \leqslant 3$ and $|E(G)| \leqslant 1$; $K_2 + mK_1$ if and only if $m \leqslant 4$; $\overline{K_n} + 2K_2$ if and only if $n \leqslant 2$; the double fan DF_n if and only if $n \leqslant 4$; the t-fold wheel $W_n + \overline{K_t}$ if and only if $t \leqslant 2$ and $n = 3$; cocktail party graphs $H_{n,n}$ if and only $n \leqslant 6$; $P(K_n, I)$ if and only if $n \leqslant 3$; $P(K_2 + mK_1, I)$ if and only if $m \leqslant 3$; and $P(K_{m,n}, I)$ $(m, n > 1)$ if and only if $m = n = 2$ and $n = 3, 4, 5$.

Vaidya and Barasara [3251] proved that every graph can be embedded as an induced subgraph of a difference cordial graph, thereby ruling out any possibility of obtaining any forbidden subgraph characterization for difference cordial graphs. They also proved that every connected graph can be embedded as an induced subgraph of a difference cordial connected graph and every planar graph can be embedded as an induced subgraph of a difference cordial planar graph.

In [2412], Ponraj, Maria Adaickalam, and Kala introduced a new graph labeling called a k-difference cordial labeling. Let G be a (p, q)-graph and $2 \leqslant k \leqslant |V(G)|$. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a map. For each edge uv, assign the label $|f(u) - f(v)|$. They say f is a *k-difference cordial* labeling of G if $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(0) - e_f(1)| \leqslant 1$, where $v_f(x)$ denotes the number of vertices labeled with x, $e_f(1)$ denotes the number of edges labeled with 1, and $e_f(0)$ denotes the number of edges that are not labeled with 1. A graph witha k-difference cordial labeling is called a *k-difference cordial* graph. They proved the following: every graph is a subgraph of a connected k-difference cordial graph; if k is even, then k-copies of $K_{1,p}$ is k-difference cordial; and if $n \equiv 0 \pmod{k}$ and $k \geqslant 6$, then $K_{1,n}$ is not k-difference cordial. They further prove the following are 3-difference cordial graphs: paths; C_n where $n \equiv 0, 3 \pmod 4$; $K_{m,n}$ $(m \leqslant n)$ and m is even; combs; double combs; quadrilateral snakes; bistars; subdivisions of a star; subdivisions of a bistar; $C_4^{(t)}$; K_n if and only if $n \in \{1, 2, 3, 4, 6, 7, 9, 10\}$; and $K_{1,n}$ if and only if $n \in \{1, 2, 3, 4, 5, 6, 7, 9\}$.

In [2407], [2408], [2409], Ponraj and Maria Adaickalam proved the following are 3-difference cordial graphs: $K_{1,n} \odot K_2$, $P_n \odot 3K_1$, $C_n \odot K_2$, mC_4, splitting graph of a star, fan, double fan, W_n where $n \equiv 0, 1 \pmod 3$, helms, flower, sunflower graph, lotus inside a circle, closed helm, double wheel DW_n where $V(DW_n) = V(W_n) \cup \{v_i : 1 \leqslant i \leqslant n\}$ and edge set $E(DW_n) = E(W_n) \cup \{uv_i : 1 \leqslant i \leqslant n\} \cup \{v_iv_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{v_1v_n\}$, degree splitting graph of a bistar, $spl(K_{1,n}) \cup K_{1,n}$, $spl(K_{1,n}) \cup P_n$, $K_{3,n} \cup spl(K_{1,n})$, $DF_n \cup spl(K_{1,n})$, $S(K_{1,n}) \cup S(B_{n,n})$, $K_{2,n} \cup S(K_{1,n})$, $F_n \cup S(K_{1,n})$, $W_n \cup S(K_{1,n})$, $B_{n,n} \cup S(B_{n,n})$, $K_{2,n} \cup B_{n,n}$, $(C_n \odot K_1) \cup (P_n \odot K_1)$, $F_n \cup F_n$, jelly fish, $P_n \cup K_{1,n}$, $K_{1,n} \cup K_{2,n}$, $K_{1,n} \cup S(K_{1,n})$, are Let C_n be the cycle $u_1u_2 \ldots u_nu_1$. If G is (p, q) 3-difference cordial graph with $p \equiv 0 \pmod 2$ and $q \equiv 0 \pmod 3$, then $G \cup G$ also 3-difference cordial. Let G be the graph obtained from C_n with $V(G) = V(C_n) \cup \{v_i : 1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil\}$ and $E(G) = \{u_iv_i, u_{i+1}v_i : 1 \leqslant i \leqslant n\}$. Then G is 3-difference cordial. The graph G_n with the vertex set $V(G_n) = \{u_i, v_i, w_i : 1 \leqslant i \leqslant n\}$ and $E(G_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{u_nu_1, v_1u_1\} \cup \{u_iv_i, v_iw_i : 1 \leqslant i \leqslant n\}$ is 3-difference cordial. Let C_3 be the cycle $u_1u_2u_3u_1$. Let G be a graph obtained from C_3 with $V(G) = V(C_3) \cup \{v_i, w_i, z_i : 1 \leqslant i \leqslant n\}$ and $E(G) = E(C_3) = \{u_1v_i, u_2w_i, u_3z_i : 1 \leqslant i \leqslant n\}$. Then G is 3-difference cordial if $n \equiv 0, 2, 3 \pmod 4$. If $n \equiv 0, 1 \pmod 3$, then $K_{1,n} \cup K_{1,n}$ is 3-difference cordial. Ponraj, Adaickalam, and Kala [2413] proved the following graphs have 3-difference cordial labelings: $DA(T_n) \odot K_1, DA(T_n) \odot 2K_1, DA(T_n) \odot K_2, DA(Q_n) \odot K_1$, and $DA(Q_n) \odot 2K_1$ (T_n is a triangular snake.) In [2410] Ponraj, Adaickalam, Maria Adaickalam, and Kala investigated the 3-difference cordial labeling behavior of ladders, books, dumbbell graphs, and umbrella graphs. Ponraj, Subbulakshmi, and Somasundarami [2477] investigated the 4-difference cordial labeling behavior of jelly fish, jewel graphs, combs, subdivisions of stars, subdivisions of bistars, and books with triangle pages.

For graphs G and H and a vertex v of G the graph $G \odot_v H$ is obtained by joining any particular vertex of H to vertex v. In [2180] Sugumaran and Mohan proved that the following graphs are difference cordial graphs: the path union of r copies of P_n^2 (that is, $P(r.P_n^2)$)–see Section 2.7 for the definition), the cycle union of r copies of C_n^2, (that is, $C(r.C_n^2)$), the open star of r copies the square graph P_n^2 (that is, $S(r.P_n^2)$), the graph $C_n^2 \odot_{v_n} P_k$, and the graph $C_n^2 \odot_{v_n} P_k^2$. In [2181] they proved that the plus graph Pl_n, the path union of plus graph $P(r.Pl_n)$, the cycle union of plus graph $C(r.Pl_n)$, the barycentric subdivision of Pl_n, the hanging pyramid HP_{y_n} graph, and the path union of hanging pyramid $P(r.HP_{y_n})$. In [3097] they proved that switching of a pendent vertex in path P_n, switching of an apex vertex in CH_n, the graph obtained by duplication of each vertex of path P_n by an edge, the barycentric subdivision of $C_n \odot K_1$, the path union of r copies of fan $P(r.F_n)$, the cycle union of r copies of fan $C(r.F_n)$, and the open star of r copies of fan $S(r.F_n)$ are difference cordial graphs.

7.9 Prime Cordial Labelings

Sundaram, Ponraj, and Somasundaram [3134] have introduced the notion of prime cordial labelings. A *prime cordial labeling* of a graph G with vertex set V is a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. In [3134] Sundaram, Ponraj, and Somasundram prove the following graphs are prime cordial: C_n if and only if $n \geqslant 6$; P_n if and only if $n \neq 3$ or 5; $K_{1,n}$ (n odd); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \geqslant 3$; bistars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders; $K_{1,n}$ if n is even and there exists a prime p such that $2p < n+1 < 3p$; $K_{2,n}$ if n is even and if there exists a prime p such that $3p < n+2 < 4p$; and $K_{3,n}$ if n is odd and if there exists a prime p such that $5p < n+3 < 6p$. They also prove that if G is a prime cordial graph of even size, then the graph obtained by identifying the central vertex of $K_{1,n}$ with the vertex of G labeled with 2 is prime cordial, and if G is a prime cordial graph of odd size, then the graph obtained by identifying the central vertex of $K_{1,2n}$ with the vertex of G labeled with 2 is prime cordial. They further prove that $K_{m,n}$ is not prime cordial for a number of special cases of m and n. Sundaram and Somasundaram [3137] and Youssef [3552] observed that for $n \geqslant 3$, K_n is not prime cordial provided that the inequality $\phi(2) + \phi(3) + \cdots + \phi(n) \geqslant n(n-1)/4 + 1$ is valid for $n \geqslant 3$ (ϕ is the Euler phi-function). This inequality was proved by Yufei Zhao [3592]. Haque, Lin, Yang, and Zhao [1180] show that with the exception of $P(4, 1)$, all generalized Petersen graphs are prime cordial. Haque, Lin, Yang, and Zhang [1178] show that the flower snark and related graphs are prime cordial. In [1097] Ghosh, Mohanty, and Pal gave an algorithmic approach to find cordial labelings of Cartesian product of two balanced bipartite graphs.

A *signed product cordial* labeling of graph $G(V, E)$ is a function f from V to $\{-1, 1\}$

such that when each edge uv is assigned the label $f(u)f(v)$, the number of vertices labeled -1 and the number of vertices labeled 1 differ by at most 1, and likewise for edges. A graph that admits signed product cordial labeling is called a *signed product cordial* graph. Signed product cordial labelings for fractal graphs given by Santhi and Albert in [2696] and by Raval and Prajapati in [2598]. Ghosh and Pal [1098] proved that $K_{m,m} \times P_n$ admits signed product cordial labelings and total signed product cordial labelings. Elhay, Elmshtaye, and Elrokh [816] provided necessary and sufficient conditions for which there exist a signed product cordial labeling of corona product of paths and third power of lemniscate graph. [2696] new [2598] new [1098] new [816] new

Seoud and Salim [2786] give an upper bound for the number of edges of a graph with a prime cordial labeling as a function of the number of vertices. For bipartite graphs they give a stronger bound. They prove that K_n does not have a prime cordial labeling for $2 < n < 500$ and conjecture that K_n is not prime cordial for all $n > 2$. They determine all prime cordial graphs of order at most 6. For a graph with n vertices to admit a prime cordial labeling, Seoud and Salim [2788] proved that the number of edges must be less than $n(n-1) - 6n^2/\pi^2 + 3$. As a corollary they get that K_n $(n > 2)$ is not prime cordial thereby proving their earlier conjecture.

In [1080] Ghodasara and Jena prove that the following graphs are prime cordial: C_n with one chord, C_n with twin chords (that is, two cords that form a triangle with an edge of the cycle), C_n with three cords that form two triangles and a cycle of length $n-3$ $(n \geqslant 7)$, the graph obtained by joining two copies of C_n with one chord by a path, and the graph obtained by joining two copies of the same cycle with twin chords by a path is prime cordial.

In [503] Baskar Babujee and Shobana proved sun graphs $C_n \odot K_1$; C_n with a path of length $n-3$ attached to a vertex; and P_n $(n \geqslant 6)$ with $n-3$ pendent edges attached to a pendent vertex of P_n have prime cordial labelings. Additional results on prime cordial labelings are given in [504].

In [3319] and [3320] Vaidya and Vihol prove following graphs are prime cordial: the total graph of P_n and the total graph of C_n for $n \geqslant 5$ (see §2.7 for the definition); $P_2[P_m]$ for all $m \geqslant 5$; the graph obtained by joining two copies of a fixed cycle by a path; and the graph obtained by switching of a vertex of C_n except for $n = 5$ (see §3.6 for the definition); the graph obtained by duplicating each edge by a vertex in C_n except for $n = 4$ (see §2.7 for the definition); the graph obtained by duplicating a vertex by an edge in cycle C_n (see §2.7 for the definition); the path union of any number of copies of a fixed cycle (see §3.7 for the definition); and the friendship graph F_n for $n \geqslant 3$. Vaidya and Shah [3299] prove following results: P_n^2 is prime cordial for $n = 6$ and $n \geqslant 8$; C_n^2 is prime cordial for $n \geqslant 10$; the shadow graphs of $K_{1,n}$ (see §3.8 for the definition) for $n \geqslant 4$ and the bistar $B_{n,n}$ are prime cordial graphs.

Let G_n be a simple nontrival connected cubic graph with vertex set $V(G_n) = \{a_i, b_i, c_i, d_i : 0 \leqslant i \leqslant n-1\}$, and edge set $E(G_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i a_i, d_i b_i, d_i c_i : 0 \leqslant i \leqslant n-1\}$, where the edge labels are taken modulo n. Let H_n be a graph obtained from G_n by replacing the edges $b_{n-1}b_0$ and $c_{n-1}c_0$ with $b_{n-1}c_0$ and $c_{n-1}b_0$ respectively. For odd $n \geqslant 5$, H_n is called a *flower snark* whereas G_n, H_3 and all H_n with even $n \geqslant 4$, are called the *related graphs* of a flower snark. Mominul Haque, Lin, Yang, and Zhang [2185] proved that flower snarks and related graphs are prime cordial for all $n \geqslant 3$.

In [3302] Vaidya and Shah prove that the following graphs are prime cordial: split graphs of $K_{1,n}$ and $B_{n,n}$; the square graph of $B_{n,n}$; the middle graph of P_n for $n \geqslant 4$; and W_n if and only if $n \geqslant 8$. Vaidya and Shah [3302] prove following graphs are prime cordial: the splitting graphs of $K_{1,n}$ and $B_{n,n}$; the square of $B_{n,n}$; the middle graph of P_n for $n \geqslant 4$; and wheels W_n for $n \geqslant 8$. Gajjar [999] proved that the generalized web graph $W(t,n)$ $(n \geqslant 3)$ is prime cordial for $t = 1, 2, 3,$ and 4 and that $W(t, 2p)$ is prime cordial for all t and odd primes p. [999] new

In [3306] [3308] Vaidya and Shah proved following graphs are prime cordial: gear graphs G_n for $n \geqslant 4$; helms; closed helms CH_n for $n \geqslant 5$; flower graphs Fl_n for $n \geqslant 4$; degree splitting graphs of P_n and the bistar $B_{n,n}$; double fans Df_n for $n = 8$ and $n \geqslant 10$; the graphs obtained by duplication of an arbitrary rim edge by an edge in W_n where $n \geqslant 6$; and the graphs obtained by duplication of an arbitrary spoke edge by an edge in wheel W_n where $n = 7$ and $n \geqslant 9$.

Let $G(p,q)$ with $p \geqslant 4$ be a prime cordial graph and $K_{2,n}$ be a bipartite graph with bipartition $V = V_1 \cup V_2$ with $V_1 = \{v_1, v_2\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$. If G_1 is the graph obtained by identifying the vertices v_1 and v_2 of $K_{2,n}$ with the vertices of G having labels 2 and 4 respectively, Vaidya and Prajapati [3297] proved that G_1 admits a prime cordial labeling if n is even; if n, p, q are odd and with $e_f(0) = \lfloor q/2 \rfloor$; and if n is odd, p is even and q is odd with $e_f(0) = \lceil q/2 \rceil$. Prajapati and Gajjar [2485] proved the following graphs are prime cordial: $C_n \times P_2$ except for $n = 1, 2$ and 4, $C_n \times P_4$ $(n \geqslant 3)$, $C_3 \times P_n$ $(n > 1)$, $C_5 \times P_n$ $(n > 1)$, $C_6 \times P_n$ $(n > 1)$, $C_{2p} \times P_n$ where p is an odd prime and $n > 1$, and $C_4 \times P_n$ $(n > 2)$.

In [3096] Sugumaran and Mohan proved the following graphs are prime cordial: the *cycle butterfly* graph $B_{n,m}$ (two copies of C_n that share a common vertex with m pendent vertices attached to the common vertex), $W-$ graph (obtained by starting with the two copies of $K_{1,n}$ and merging the last pendent vertex in the first copy of $K_{1,n}$ with the initial pendent vertex in the second copy of $K_{1,n}$), H_n graph (the graph obtained from two paths $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by joining the vertices $u_{(n+1)/2}$ and $v_{(n+1)/2}$ if n is odd and joining $u_{n/2}$ and $v_{n/2+1}$ if n is even), and duplication of all edges of an H_n graph. In [3095] Sugumaran and Mohan proved that the following graphs are prime cordial: $H_n \odot K_1$, the path union of r copies of H_n, the cycle union of r copies of an H_n, the open star of r copies of an H_n-graph (obtained by replacing each pendent vertex of $K_{1,n}$ by a copy of H_n.

In [3101] Sugumaran and Suresh proved that the following graphs are prime cordial graphs: the duplication of each vertex by an edge of paths, stars, jelly fish, bistars, and $C_n \odot K_1$. Sugumaran and Vishnu Prakash [3106] proved that the following graphs are prime cordial graphs: duplication of any vertex of degree 3 in theta graph , switching of any vertex of degree 3 in theta graph, fusion of any two vertices in theta graph, the path union of two copies of theta graph, and two copies of theta graph joined by a path of any length. They further proved that the theta graph is not a prime cordial labeling. In [3107] they showed that one point union of path graph $P_n^t(tn.T_\alpha)$, the open star of theta graph, and any path union of even number of theta graphs are prime cordial graphs. Also they proved [3105] the subdivision of bistar $B_{n,n}$, $P_n \bigodot K_{1,n-1}$, (that is, each i^{th} vertex of path P_n is append with the apex vertex of i^{th} copy of $K_(1, n-1)$), the disconnected graph $P_n \cup P_m$ are prime cordial graphs.

Vaidya and Barasara [3249] proved that every graph can be embedded as an induced subgraph of a prime cordial graph thereby ruling out any possibility of obtaining any forbidden subgraph characterization for prime cordial graphs. They also proved that a connected graph can be embedded as an induced subgraph of a prime cordial connected graph and every planar graph can be embedded as an induced subgraph of a prime cordial planar graph. In [1064] Gayathri, Thanjavur, Maniammai, and Sakar proved that 8-polygonal snakes containing n 8-polygons, splitting graphs of C_n for $n \geqslant 5$, and armed crowns $C_{2k} \odot P_m$ for all $k \geqslant 3$ and $m \geqslant 2$ admit prime cordial labelings. Barasara and Prajapati [430] showed that switching of a vertex of degree 1 in the path P_n is a prime cordial graph for $n \neq 2, 3, 4, 5, 7$ and not a prime cordial graph for $n = 2, 3, 4, 5, 7$, the armed crown AC_n is a prime cordial graph for all n, all web graphs are prime cordial, and $C_n^{(t)}$, one-point union of cycles, are prime cordial for $t \neq 2$ and $n \neq 3, 4$ and not a prime cordial graph for $t = 2$ and $n = 3, 4$. Babitha and Baskar Babujee [268] proved [268] new
that if G is prime cordial, then so is G with an edge deleted, $K_{1,m} \odot P_n$ $(m, n > 2)$ is prime cordial, and in certain cases, the one-point union of $K_{1,m}$ and P_n and the one-point union of a prime cordial graph G and $K_{1,n}$ are prime cordial. They further provided some characterization results.

For a graph $G(V, E)$ and the group S_3 of all permutations of $\{1, 2, 3\}$ Chandra and Kala [666] define a function $g : V(G) \to S_3$ such that $xy \in E$ if $g(x)$ and $g(y)$ have relatively prime orders. Let $n_j(g)$ denote the number of vertices of G having label j under g. Then g is called a *group S_3 cordial prime* labeling if $|n_i(g) - n_j(g)| \leqslant 1$ for every $i, j \in S_3$. A graph that admits a group S_3 cordial prime labeling is called a *group S_3 cordial prime* cordial prime graph. Chandra and Kala [666] prove that all paths, cycles, gears, ladders, and fans are group S_3 cordial prime and characterize wheels that are group S_3 cordial prime.

In [2350] Parthiban and Sharma gave a comprehensive survey on prime cordial and divisor cordial labeling of graphs.

Vaidya and Prajapati [3295] call a graph *strongly prime cordial* if for any vertex v there is a prime labeling f of G such that $f(v) = 1$. They prove the following: the graphs obtained by identifying any two vertices of $K_{1,n}$ are prime cordial; the graphs obtained by identifying any two vertices of P_n are prime cordial; C_n, P_n, and $K_{1,n}$ are strongly prime cordial; and W_n is a strongly prime cordial for every even integer $n \geqslant 4$. Prajapati and Gajjar [2497] proved that generalized prism graphs $Y_{n,2}$ is prime cordial except for $n = 1, 2$ and 4; $Y_{n,4}$ is prime cordial for $n \geqslant 3$; $Y_{3,n}, Y_{5,n}, Y_{6,n}$ and $Y_{2p,n}$ (for odd prime p) are prime cordial for $n > 1$; and $Y_{4,n}$ is prime cordial for $n > 2$. They also proved the following graphs are prime cordial: $C_n \times P_2$ except for $n = 1, 2$ and 4, $C_n \times P_4$ $(n \geqslant 3)$, $C_3 \times P_n$ $(n > 1)$, $C_5 \times P_n$ $(n > 1)$, $C_6 \times P_n$ $(n > 1)$, $C_{2p} \times P_n$ where p is an odd prime and $n > 1$, and $C_4 \times P_n$ $(n > 2)$.

In [2460] Ponraj, Singh, Kala, and Sathish Narayanan introduced a new graph labeling called k-prime cordial labeling. Let G be a (p, q)-graph and $2 \leqslant p \leqslant k$ and let $f : V(G) \to \{1, 2, \ldots, k\}$ be a map. For each edge uv, assign the label $\gcd(f(u), f(v))$. They say that f is a *k-prime cordial* labeling of G if $|v_f(i) - v_f(j)| \leqslant 1$ for $i, j \in \{1, 2, \ldots, k\}$ and $|e_f(0) - e_f(1)| \leqslant 1$, where $v_f(x)$ denotes the number of vertices labeled with x, and $e_f(1)$ and $e_f(0)$, respectively, denote the number of edges labeled with 1 and not labeled with 1. A graph with a k-prime cordial labeling is a *k-prime cordial graph*. They proved that every graph is a subgraph of a connected k-prime cordial graph; if k is even, then P_n, $n \neq 3$, is k-prime cordial; C_n, $n \neq 3$, is k-prime cordial when k is even; and the bistar $B_{n,n}$ is k-prime cordial for all even k. They studied 3-prime cordiality of paths, cycles, and olive trees. They also proved that if T is a 3-prime cordial tree, then $T \odot K_1$ is 3-prime cordial; $K_{1,n}$ is 3-prime cordial if and only if $n \leqslant 3$; K_n is 3-prime cordial if and only if $n < 3$; combs $P_n \odot K_1$ are 3-prime cordial; and $C_n \odot K_1$ is 3-prime cordial if and only if $n \neq 3$. They proved that $K_2 + mK_1$, $K_{2,n}$, and wheels are not 3-prime cordial graphs. In [2461] Ponraj, Singh, and Sathish Narayana proved if G is 3-prime cordial, then $G \cup P_n$ is a 3-prime cordial for $n > 12$, the splitting graph of a star is not a 3-prime cordial graph, and the jelly fish $J(m, n)$ is 3-prime cordial if $10m \geqslant n + 2$.

For a 4-prime cordial graph G Ponraj and Singh [2459] proved $G \cup P_n$ $(n \geqslant 5)$, $G \cup 2mK_{n,n}$, and $G \cup 2mK_{1,n}$ are 4-prime cordial. For a $(4t, q)$ 4-prime cordial graph G they prove that $G + K_1$ and $G + 2K_1$ are 4-prime cordial. Ponraj, Singh, and Kala [2462] prove that $P_m \times P_n$ and subdivisions of wheels and helms are 4-prime cordial. They also show that if G is bipartite then $G \cup G$ is 4-prime cordial; and if G is 4-prime cordial then $G \odot K_1$ is 4-prime cordial. Ponraj, Singh, and Kala [2463] proved the following graphs are 4-prime cordial: $2m(K_{n,n})$, $2m(P_n \times P_2)$, $m(C_n \oplus K_1)$, $mB_{n,n}$, and $2W_{2n+1}$.

Murugesan, Jayaraman, and Shiama (see [3108]) defined a 3-*equitable prime cordial* labeling of a graph G as a bijection f from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ such that if an edge uv is assigned the label 1 when $\gcd(f(u), f(v)) = 1$ and $\gcd(f(u)+f(v), f(u)-f(v)) = 1$, the label 2 when $\gcd(f(u), f(v)) = 1$ and $\gcd(f(u) + f(v), f(u) - f(v)) = 2$, and the label 0 otherwise, then the number of edges labeled with i and the number of edges labeled with j differ by at most 1 for $0 \leqslant i \leqslant 2$ and $0 \leqslant j \leqslant 2$. A graph that has a 3-equitable prime cordial labeling is called a 3-*equitable prime cordial* graph. Sugumaran and Vishnu Prakash [3108] proved the following graphs are 3-equitable prime cordial graphs: bistars, combs, ladders, kites, and slanting ladders. In [3110] they showed that theta graphs, the duplication of any vertex in theta graphs, switching of any vertex in theta graphs, the fusion of any two vertices in theta graphs, path unions of two copies of theta graphs, open star graphs of copies of a fixed theta graph are 3-equitable prime cordial graphs. Seoud and Jaber [2780] proved that the butterfly $BF_{n,m}$, helms H_n, the graph $\langle W_n : W_m \rangle$ obtained by joining apex vertices of two wheels with a new vertex are prime cordial, and determine the prime cordial graphs of order 7. They also gave an algorithm to calculate the maximum number of edges in a 3-equitable prime cordial graph. Murugesan, Jayaraman, and Shiama [2226] proved the following graphs are 3-equitable prime cordial: P_n, C_n, $(n \geqslant 4)$, $K_{1,n}$ if and only if $n = 2 \pmod 3$, and K_n if and only if $n \leqslant 2$. Ghodasaram and Sonchhatra [1091] proved the following graphs admit 3-equitable labeling: wheels, helms, gears, cycles with one pendant edge, and the graphs obtained by [2226] new [1091] new joining two copies of a fan by a path of arbitrary length.

7.10 Other Cordial Labelings

In [2458] Ponraj, Sathish Narayanan, and Ramasamy introduced a new graph labeling called parity combination cordial labeling. Let G be a (p, q)-graph. Let f be an injective map from $V(G)$ to $\{1, 2, \ldots, p\}$. For each edge xy, assign the label $\binom{x}{y}$ or $\binom{y}{x}$ according as $x > y$ or $y > x$. Call f a *parity combination cordial* labeling if f is a one to one map and $|e_f(0) - e_f(1)| \leqslant 1$, where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with an even number and odd number, respectively. A graph with a parity combination cordial labeling is called a *parity combination cordial graph.* They proved that the following are parity combination cordial graphs: paths, cycles, stars, triangular snakes, alternate triangular snakes, olive trees, combs, crowns, fans, umbrellas, P_n^2, helms, dragons, bistars, butterfly graphs, and graphs obtained from C_n and $K_{1,m}$ by unifying a vertex of C_n and a pendent vertex of $K_{1,m}$. They also proved that W_n admits a parity combination cordial labeling if and only if $n \geqslant 4$ and conjectured that for $n \geqslant 4$, K_n is not a parity combination cordial graph. In [2464], Ponraj, Rajpal Singh, and Sathish Narayanan proved that if G is a parity combination cordial graph, then $G \cup P_n$ is also parity combination cordial if $n \neq 2, 4$. In [2756] Seoud and Aboshady surved all graphs of order at most six with regard to whether they have a parity combination cordial labeling or not. They obtained an upper bound for the number of edges of any graph that satisfies a certain condition, and described the parity combination cordial labelings for two families of graphs.

In [2242] Naduvath defines the notion of the set-cordial labeling of a graph as follows. For a non-empty set X, a function f from the vertices of a graph G to the power set of X is said to be a *set-cordial* labeling of G if $|f(v_i)| - |f(v_j)| = \pm 1$ for all edges $v_i v_j$ of G. A graph that admits a set-cordial labeling is called a *set-cordial* graph . He proves that paths are set-cordial and that a graph G admits a set-cordial labeling if and only if G is bipartite. He defines the *glutting number* of a graph G as the minimum number of edges of G that can be removed so that the resulting graph admits a set-cordial labeling and provides the glutting number of wheels, helms, and complete graphs.

In 2011 Murugan and Selvaraj [2216] introduced the concept of V-*cordial* labeling of a graph G with vertex set V as a bijective function $\phi : V \to \{0, 1\}$ such that the induced function $\phi^* : E \to \{0, 1\}$ is defined by $\phi(uv) = 0$ if $\phi(u) = \phi(v) = 0$ and 1 otherwise with the condition that the number of verticies labeled with 0 and the number of verticies labeled with 1 differ by at most 1, and likewise for the edges. A graph that the admits a V-cordial labeling is called a V-*cordial* graph. In 2015 Murugan and Mathubala introduced the concept of homo-cordial labeling as follows. A *homo-cordial* labeling of a graph G with vertex set V is a bijection $\phi : V \to \{0, 1\}$ such that the induced function $\phi^* : E \to \{0, 1\}$ given by $\phi^*(uv) = 1$, if $\phi(u) = \phi(v)$ and 0 otherwise with the condition that the number of verticies labeled with 0 and the number of verticies labeled with 1 differ by at most 1, and likewise for the edges. A graph that the admits a homo-cordial labeling is called a *homo-cordial* graph. In [2217] Murugan and Vidhya say that a *hetro-cordial* labeling of a graph G with vertex set V is a bijection $\phi : V \to \{0, 1\}$ such that the induced function $\phi^* : E \to \{0, 1\}$ given by $\phi^*(uv) = 0$, if $\phi(u) = \phi(v)$ and 1, otherwise with the condition

[2216] new

[2217] new

that the number of verticies labeled with 0 and the number of verticies labeled with 1 differ by at most 1, and likewise for the edges. A graph that the admits a hetro-cordial labeling is called a *hetro-cordial* graph. Murugan, and Mathubala [2215] proved that odd cycles, shadow graphs of cycles, graphs obtained by identifying the end points of n paths of length 2 (globe graphs), and double triangular snakes are homo-cordial graphs. Murugan, and Selva Vidhya [2217] proved that paths, combs, fans, double fans, and ladders are hetro-cordial graphs. Murugan and Selvaraj [2216] proved that paths, fans double fans, and combs are V cordial graphs. Bala, Sundarraj, and Thirusangu [401] proved existence of a homo-cordial labeling, a hetro-cordial labeling, and a V-cordial labeling for the extended triplicate graph of a comb. [2215] new [2217] new [2216] new [401] new

In [2273] Nicholas and Maya introduced the concept of an *integer edge cordial* labeling of a graph G with edge set E as an injective map f from E to $[-q/2,\dots,q/2]$ or $[\lfloor -q/2\rfloor,\dots,\lfloor q/2\rfloor]$ as q is even or odd, which induces a vertex labeling $f^*: V \to \{0,1\}$ such that, a vertex u is assigned the label 1 if $\sum(f^*(e_i) \geqslant 0$ taken over all i, and 0 otherwise, and the number of vertices labeled with 1 and the number of vertices labeled with 0 differ by at most by 1. A graph that has integer edge cordial labeling is called an *integer edge cordial graph*. They proved that P_n $(n \geqslant 3)$, cycles, W_n $(n > 3)$, helms, closed helms, $K_{1,2n}$, and flower graphs are integer edge cordial graphs. They also proved that $K_{n,n}$ is not integer edge cordial and that $K_{n,n} \setminus M$ is integer edge cordial if n is even, where M is a perfect matching of $K_{n,n}$. [2273] new

For a planar graph $G(p,q)$ suppose that $g: E(G) \to [\frac{-q}{2},\dots,\frac{q}{2}]$ or $[-\lfloor\frac{q}{2}\rfloor,\dots,\lfloor\frac{q}{2}\rfloor]$ as q is even or odd, is an injective map that induces a vertex labeling $g^*: V(G) \to \{0,1\}$ defined by $g^*(v) = 1$ if the sum of all $g(e)$ over all edges e of G adjacent to v is nonnegative and $g^*(v) = 0$ otherwise, and the face labeling function g^{**} from the faces of G to $\{0,1\}$ defined by $g^{**}(f) = 1$, if the sum of $g(e)$ over all edges e of f is nonnegative, and $g^{**}(f) = 0$ otherwise. Then g is called a *face integer edge cordial labeling* if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of faces labeled 0 and the number of faces labeled 1 differ by at most 1. A graph that admits a face integer edge cordial labeling is called *face integer edge cordial*. Sheriff and Abbas [2884] gave face integer edge cordial labelings of the duplication of each vertex by an edge in nontrivial fans, the planar graph G' obtained from joining the outer vertex of the two copies of a planar graph G by a path of arbitrary length, and the splitting graph of nontrival stars. [2884] new

For a simple, finite and planar graph G of order p and size q. Sumathi and J. Suresh Kumar [3125] introduced the concept of fuzzy quotient-3 cordial labeling as follows. Let $\sigma: V(G) \to [0,1]$ be a function defined by $\sigma(v) = \frac{r}{10}$, $r \in Z_4 - \{0\}$. For each edge uv define $\mu: E(G) \to [0,1]$ by $\mu(uv) = \frac{1}{10}\frac{3\sigma(u)}{\sigma(v)}$ where $\sigma(u) \leqslant \sigma(v)$. The function σ is called *fuzzy quotient-3 cordial* labeling of G if the number of vertices labeled with i and the number of vertices labeled with j differ by at most 1, the number of edges labeled with i and the number of edges labeled with j differ by at most 1 where $i \neq j$ have the form $\frac{r}{10}$, where $r \in \{1,2,3\}$. They proved that stars and star related graphs are fuzzy quotient-3 cordial. [3125] new

7.11 Mean Labelings

Somasundaram and Ponraj [3023] introduced the notion of mean labelings of graphs. A graph G with p vertices and q edges is called a *mean graph* if there is an injective function f from the vertices of G to $\{0, 1, 2, \ldots, q\}$ such that when each edge uv is labeled with $(f(u) + f(v))/2$ if $f(u) + f(v)$ is even, and $(f(u) + f(v) + 1)/2$ if $f(u) + f(v)$ is odd, then the resulting edge labels are distinct. In [3023], [3024], [3025], [3026], [2474], and [2475] they prove the following graphs are mean graphs: $P_n,\ C_n,\ K_{2,n},\ K_2 + mK_1,\ \overline{K_n} + 2K_2,\ C_m \cup P_n,\ P_m \times P_n,\ P_m \times C_n,\ C_m \odot K_1,\ P_m \odot K_1$, triangular snakes, quadrilateral snakes, K_n if and only if $n < 3$, $K_{1,n}$ if and only if $n < 3$, bistars $B_{m,n}$ $(m > n)$ if and only if $m < n + 2$, the subdivision graph of the star $K_{1,n}$ if and only if $n < 4$, the friendship graph $C_3^{(t)}$ if and only if $t < 2$, the one point union of two copies a fixed cycle, dragons (the one point union of C_m and P_n, where the chosen vertex of the path is an end vertex), the one point union of a cycle and $K_{1,n}$ for small values of n, and the arbitrary super subdivision of a path, which is obtained by replacing each edge of a path by $K_{2,m}$. They also prove that W_n is not a mean graph for $n > 3$ and enumerate all mean graphs of order less than 5.

Gayathri and Gopi [1050] prove the following are mean graphs: double triangular snakes; double quadrilateral snakes; generalized antiprisms; graphs obtained by joining the 2 vertices of $K_{2,n}$ of degree n with an edge; and graphs obtained from C_n with consecutive vertices $v_1, v_2, \ldots, v_n$ by adding the chords joining v_i and v_{n-i+2} for $2 \leqslant i \leqslant \lfloor n/2 \rfloor$. In [1048] Gayathri and Gopi gave various necessary conditions for mean labelings. Kannan, Manivannan, Loganathan, and Gyeltshen [1635] investigated the existence of mean labelings for the graphs obtained by duplicating an edge or vertex of paths, cycles, combs, and the splitting graph of paths.

Lourdusamy and Seenivasan [2017] prove that kC_n-snakes are means graphs and every cycle has a super subdivision that is a mean graph. They define a generalized kC_n-*snake* in the same way as a C_n-snake except that the sizes of the cycle blocks can vary (see Section 2.2). They prove that generalized kC_n-snakes are mean graphs. Recall that $P_{a,b}$ denotes the graph obtained by identifying the endpoints of b internally disjoint paths each of length a. Vasuki and Nagarajan [3356] proved that the following graphs admit mean labelings: $P_{r,2m+1}$ for all r and m; $P_{r,2m}$ for all m and $2 \leqslant r \leqslant 6$; P_r^{2m+1} for all r and m; and P_r^{2m} for all m and $2 \leqslant r \leqslant 6$. Anusa, Sandhya, and Somasundaram [186] proved that triangular ladders, triangular snakes, double triangular snakes, quadrilateral snakes, and double quadrilateral snakes are mean graphs.

Lourdusamy and Seenivasan [2018] define an *edge linked cyclic snake,* $EL(kC_n)$, as the connected graph obtained from k copies of C_n $(n \geqslant 4)$ by identifying an edge of the $(i+1)^{th}$ copy to an edge of the i^{th} copy for $i = 1, 2, \ldots, k-1$ in such a way that the consecutive edges so chosen are not adjacent. They proved that all $EL(kC_{2n})$ are mean graphs and some cases of $EL(C_{2n-1})$ are mean graphs. They also define a *generalized edge linked cyclic snake* in the same way but allow the cycle lengths (at least 4) to vary. They prove that certain cases of generalized edge linked cyclic snakes are mean graphs.

Barrientos and Krop [454] proved that there exist $n!$ graphs of size n that admit mean labelings. They give two necessary conditions for the existence of a mean labeling of a graph G with m vertices and n edges: if G is a mean graph, then $n+1 \geqslant m$; if G is a mean graph with n edges and maximum degree $\Delta(G)$, then $\Delta(G) \leqslant \frac{n+3}{2}$ when n is odd and $\Delta(G) \leqslant \frac{n+2}{2}$ when n is even. They proved that the disjoint union of n copies of C_3 is a mean graph and if a mean r-regular graph has n vertices, then $r < n-2$. They established a connection between α-labelings and mean labelings by proving that every tree that admits an α-labeling is a mean graph when the size of its stable sets differ by at most one. When the tree is a caterpillar, this difference can be up to two. Barrientos and Krop call a mean labeling of a bipartite graph an *α-mean labeling* if the labels assigned to vertices of the same color have the same parity. They show that the complementary labeling of a α-mean labeling is also an α-mean labeling. They use graphs with α-mean labelings to construct new mean graphs. One construction consists of connecting a pair of corresponding vertices of two copies of an α-mean graph by an edge. The other construction identifies a pair of suitable vertices from two α-mean graphs. Barrientos and Krop also proved that every quadrilateral snake admits an α-mean labeling. They conjecture that all trees of size n and maximum degree at most $\lceil (n+1)/2 \rceil$ are mean graphs and state some open problems. In [449] Barrientos proves that all trees with up to four end-vertices except $K_{1,4}$ are mean graphs. Bailey and Barrientos [397] prove the following are mean graphs: $C_n \cup C_m,\ C_n \cup P_m,\ K_2 + nK_1,\ 2K_2 + nK_1, C_n \times K_2$.

In [397] Bailey and Barrientos study several operations with mean graphs. They prove that the coronas $G \odot K_1$ and $G \odot K_2$ are mean graphs when G is an α-mean graph. Also, if G and H are mean graphs with n vertices and $n-1$ edges and H is an α-mean graph, then $G \times H$ is a mean graph. They prove that given two mean graphs G and H, there exists a mean graph obtained by identifying an edge from G with an edge from H and uses this result to prove that the graphs R_n ($n \geqslant 2$) of order $2n$ and size $4n-3$ with vertex set $V(R_n) = \{v_1, v_2, \ldots, v_{2n}\}$ and edge set $E(R_n) = \{v_i v_{i+1} \mid 1 \leqslant i \leqslant n-1 \text{ and } n+1 \leqslant i \leqslant 2n-1\} \cup \{v_i v_{n+i} \mid 1 \leqslant i \leqslant n\} \cup \{v_i v_{n+i-1} \mid 2 \leqslant i \leqslant n\}$ (*rigid ladders*) are mean graphs.

Barrientos, Abdel-Aal, Minion, and Williams [450] use A_n to denote the set of all α-mean labeled graphs of size n such that the difference of the cardinalities of the bipartite sets of the verticies of the graphs is at most one.They prove that the class A_n is equivalent to the class of α-labeled graphs of size n with bipartite sets that differ by at most one. They also prove that when $G \in A_n$, the coronas $G \odot mK_1,\ G \odot P_2$, and $G \odot P_3$ admit mean labelings.

In [3262] Vaidya and Bijukumar define two methods of creating new graphs from cycles as follows. For two copies of a cycle C_n the *mutual duplication* of a pair of vertices v_k and v'_k respectively from each copy of C_n is the new graph G such that $N(v_k) = N(v'_k)$. For two copies of a cycle C_n and an edge $e_k = v_k v_{k+1}$ from one copy of C_n with incident edges $e_{k-1} = v_{k-1} v_k$ and $e_{k+1} = v_{k+1} v_{k+2}$ and an edge $e'_m = u_m u_{m+1}$ in the second copy of C_n with incident edges $e'_{m-1} = u_{m-1} u_m$ and $e'_{m+1} = u_{m+1} u_{m+2}$, the *mutual duplication* of a pair of edges e_k and e'_m respectively from two copies of C_n is the new graph G such that $N(v_k) - v_{k+1} = N(u_m) - u_{m+1} = \{v_{k-1}, u_{m-1}\}$ and $N(v_{k+1}) - v_k = N(u_{m+1}) - u_m = \{v_{k+2}, u_{m+2}\}$. They proved that the graph obtained by mutual duplication of a pair of vertices each from each copy of a cycle and the mutual duplication of a pair of edges from each copy of a cycle are mean graphs. Moreover, they proved that the shadow graphs of the stars $K_{1,n}$ and bistars $B_{n,n}$ are mean graphs.

Vasuki and Nagarajan [3358] proved the following graphs are admit mean labelings: the splitting graphs of paths and even cycles; $C_m \odot P_n$; $C_m \odot 2P_n$; $C_n \cup C_n$; disjoint unions of any number of copies of the hypercube Q_3; and the graphs obtained from by starting with m copies of C_n and identifying one vertex of one copy of C_n with the corresponding vertex in the next copy of C_n.) Jeyanthi and Ramya [1499] define the jewel graph J_n as the graph with vertex set $\{u, x, v, y, u_i : 1 \leqslant i \leqslant n\}$ and edge set $\{ux, vx, uy, vy, xy, uu_i, vu_i : 1 \leqslant i \leqslant n\}$. They proved that the jewel graphs, jelly fish graphs, and the graph obtained by joining any number of isolated vertices to the two endpoints of P_3 are mean graphs. Ramya and Jeyanthi [2584] proved several families of graphs constructed from T_p-tree are mean graphs. Ahmad, Imran, and Semaničová-Feňovčiková [106] studied the relation between mean labelings and (a, d)-edge-antimagic vertex labelings. They show that two classes of caterpillars admit mean labelings. Revathi [2616] proved that the shadow graphs of bistars, combs, and the splitting graph of combs have mean labelings.

Recall from Section 2.7 that given connected graphs $G_1, G_2, \ldots, G_n$, Kaneria, Makadia, and Jariya [1602] define a *cycle of graphs* $C(G_1, G_2, \ldots, G_n)$ as the graph obtained by adding an edge joining G_i to G_{i+1} for $i = 1, \ldots, n-1$ and an edge joining G_n to G_1. (The resulting graph can vary depending on which vertices of the G_i are chosen.) When the n graphs are isomorphic to G the notation $C(n \cdot G)$ is used. Also recall Kanneria and Makadia [1595] define a *step grid graph* St_n as the graph obtained by starting with paths $P_n, P_n, P_{n-1}, \ldots, P_2$ $(n \geqslant 3)$ arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. In [1626], [1612], and [1615], Kaneria, Viradia, and Makadia proved the following graphs are mean graphs: the path union of any number of copies of a mean graph; $C(2t \cdot P_n)$; $C(2t \cdot C_n)$; $C(2t \cdot P_n \times P_m)$; $C(2r \cdot B^2_{n,n})$ ($B^2_{n,n}$ is the square of the bistar $B_{n,n}$); $C(2r \cdot M(C_n))$ ($M(C_n)$ is the middle graph of C_n); $C(2r \cdot (P_{2n} + 2K_1))$; step grid graphs; the path union of finitely copies of the step grid graphs; cycles of step grid graphs $C(2r \cdot St_n)$; and $C(2t \cdot K_{2,m})$.

For a fixed vertex v of C_m Avadayappan and Vasuki [265] use $(P_m; C_n)$ to denote the graph obtained from m copies of C_n and the path $P_m : u_1u_2 \cdots u_m$ by joining u_i with v of the ith copy of C_n with an edge for $1 \leqslant i \leqslant m$. They define $(P_m; Q_3)$, $(P_{2n}; S_m)$, $(P_n; S_1)$ and $(P_n; S_2)$, where v is a fixed vertex of the cube Q_3 and v is the center of the star S_k, in an analogous way. For $C_n : v_1v_2 \ldots v_nv_1$ they use $[P_m; C_n]$ to denote the graph obtained from m copies of C_n with vertices $v_{1_1}, v_{1_2}, \ldots, v_{1_n}, v_{2_1}, \ldots, v_{2_n}, \ldots, v_{m_1}, \ldots, v_{m_n}$ by joining v_{i_j} and $v_{(i+1)_j}$ with an edge, for some j and $1 \leqslant i \leqslant m-1$. They define $[P_m; Q_3]$ and $[P_m; C^{(2)}_m]$, where $C^{(2)}_m$ is the friendship graph, similarly. In [265] they prove these families are mean graphs.

Maheswari, Hariprabakaran, and Balaji [2064] introduced a coding technique for converting a text message using a sub super mean labeling on two and three star graphs to a picture coding message. For more on coding a text message using a graph labelings see [1187], [2061], and [2057]. [2064] new [1187] new [2061] new [2057] new

Ramya, Ponraj, and Jeyanthi [2587] called a mean graph *super mean* if vertex labels and the edge labels are $\{1, 2, \ldots, p+q\}$. They prove following graphs are super mean: paths, combs, odd cycles, P_n^2, $L_n \odot K_1$, $C_m \cup P_n$ $(n \geqslant 2)$, the bistars $B_{n,n}$ and $B_{n+1,n}$. They also prove that unions of super mean graphs are super mean and K_n and $K_{1,n}$ are not super mean when $n > 3$. In [1504] Jeyanthi, Ramya, and Thangavelu prove the following are super mean: $nK_{1,4}$; the graphs obtained by identifying an endpoint of P_m $(m \geqslant 2)$ with each vertex of C_n; the graphs obtained by identifying an endpoint of two copies of P_m $(m \geqslant 2)$ with each vertex of C_n; the graphs obtained by identifying an endpoint of three copies of P_m $(m \geqslant 2)$; and the graphs obtained by identifying an endpoint of four copies of P_m $(m \geqslant 2)$. In [1500] Jeyanthi and Ramya prove the following graphs have super mean labelings: the graph obtained by identifying the endpoints of two or more copies of P_5; the graph obtained from C_n $(n \geqslant 4)$ by joining two vertices of C_n distance 2 apart with a path of length 2 or 3; Jeyanthi and Rama [1502] use $S(G)$ to denote the graph obtained from a graph G by subdividing each edge of G by inserting a vertex. They prove the following graphs have super mean labelings: $S(P_n \odot K_1), S(B_{n,n}), C_n \odot \overline{K_2}$; the graphs obtained by joining the central vertices of two copies of $K_{1,m}$ by a path P_n (denoted by $\langle B_{m,m} : P_n \rangle$); generalized antiprisms (see §6.2 for the definition), and the graphs obtained from the paths $v_1, v_2, v_3, \ldots, v_n$ by joining each v_i and v_{i+1} to two new vertices u_i and w_i (double triangular snakes).

Lourdusamy and Seenivasan [2019] introduced the notion of super vertex mean labeling as follows. For a (p, q)-graph and an injective function f from the edges to the set $\{1, 2, 3, \ldots, p+q\}$ that induces for each vertex v the label defined by $f^*(v) =$ (Round $\sum_{e \in E_v} f(e))/d(v)$, where E_v denotes the set of edges in G that are incident to the vertex v, $d(v)$ is the degree of v, and Round(x) is the integer nearest to x, such that the set of all edge labels and the induced vertex labels is $\{1, 2, 3, \ldots, p+q\}$ is called a *super vertex mean* labeling of G and G is called a *super vertex mean graph.* In [2004] they investigated the all graphs of order up to 5 and regular graphs of order up to 7 for the property of being super vertex mean and proved that all linear triangular snakes are super vertex mean. Lourdusamy, George, and Seenivasan [2006] proved that all cycles except C_4 are super vertex mean and Lourdusamy and George [2005] proved that linear C_n snakes with at least 2 blocks are super vertex mean graphs for the following cases: $n = 4, 5, 6$, and 7; $n \geqslant 8$ even; $n \geqslant 9$ and $n \equiv 1 \bmod 4$; and $n \geqslant 11$ and $n \equiv 3 \bmod 4$. Inayah, Sudarsana, Musdalifah, and Mangesa [1325] have showed that the total graphs of paths and cycles are super mean graphs.

In [3067] Sudarsana, Suryanto, Lusianti, and Putri show how super mean graph labelings can be used to increase the security level of encrypted text on social medias.

A graph G with q edges is called a *k-mean graph* if there is an injective function f from the vertices of G to $\{0, 1, 2, \ldots, k+q-1\}$ such that when each edge uv is labeled with $(f(u)+f(v))/2$ if $f(u)+f(v)$ is even, and $(f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd, the resulting edge labels are $\{k, k+1, k+2, \ldots, k+q-1\}$. A graph G with q edges is said to have a *restricted k-mean* labeling if there is an injective function f from the vertices of G to $\{k-1, k, k+1, \ldots, k+q-1\}$ such that when each edge uv is labeled with $\{k, k+1, k+2, \ldots, k+q-1\}$, the resulting edge labels $\{k, k+1, k+2, \ldots, k+q-1\}$ are distinct where k is a positive integer. A graph that admits a restricted k-mean labeling is called a *restricted k-mean* graph. Gayathri and Gopi proved some properties of k-mean labelings in [1051]. In [1052] they proved that if G_1 and G_2 are restricted k-mean graphs for all k, then $G_1 \cup G_2$ is restricted k-mean for all k, and if G_1 is a restricted k-mean graph for all $k \geqslant k_1$ and G_2 is a restricted k-mean graphs for all k, then $G_1 \cup G_2$ is restricted k-mean for all $k \geqslant k_1$.

A mean graph is called *k-super mean* if vertex labels and the edge labels are $\{k, k+1, k+2, \dots, p+q+k-1\}$. Jeyanthi, Ramya, Thangavelu [1505] give super mean labelings for $C_m \cup C_n$ and k-super mean labelings for a variety of graphs. Tamilselvi, Akilandeswari, and Suguna [3186] proved that the following graphs admit k-super mean labelings: the graph obtained by subdividing the central edge of the bistar $B_{n,n}$ $(n \geqslant 2)$, the subdivision graph of $B_{n,n}$, and the corona product of a triangular snake and K_1.

Vasuki and Nagarajan [3357] define H_n, called the H-graph of a path P_n, as the graph obtained from two copies of P_n with vertices $v_1, v_2, \dots, v_n$ and $u_1, u_2, \dots, u_n$ by joining the vertices $v_{(n+1)/2}$ and $u_{(n+1)/2}$ if n is odd, and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even, and a cyclic snake mC_n as the graph obtained from m copies of C_n by identifying the vertex $v_{(k+2)_j}$ in the jth copy of the vertex $v_{1_{j+1}}$ in the $(j+1)$th copy if $n = 2k+1$ and identifying the vertex $v_{(k+1)_j}$ in the jth copy with the vertex $v_{1_{j+1}}$ in the $(j+1)$th copy if $n = 2k$. They establish the super meanness of even cycles, H-graphs, the coronas of H-graphs, 2-coronas of H-graphs, coronas of cycles, mC_n-snakes $(n \neq 4)$, dragons $P_n(C_m)$ for $m \neq 4$, and $C_m \times P_n$ for $m = 3$ and 5. Vasuki, Sugirtha, and Venkateswari [3366] proved that the subdivision of the following graphs are super mean graphs: H_n, $H_n \odot K_1$, H_n with two pendent edges attached to each vertex, $C_n \odot K_1$ $(n \geqslant 3)$, slanting ladders, triangular snakes with a pendent edge at each vertex, and $C_m@C_n$ (the graph obtained by attaching paths P_n to C_m by identifying the endpoints of the paths with each successive pairs of vertices of C_m).

Let $G(V, E)$ be a simple graph of order p and size q. Then G is said to be a *relaxed mean graph* if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $\{0, 1, 2 \dots, q-1, q+1\}$ in such a way that when each edge uv is labeled with $(f(u)+f(v))/2$ if $f(u)+f(v)$ is even and $(f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd, then the resulting edge labels $\{1, 2, 3, \dots, q\}$ are distinct. Such an f is called a *relaxed mean labeling* of G. Balaji, Ramesh, and Sudhaker [404] prove that the disjoint union of any path with $n-1$ edges joining the pendent vertices of distinct paths is a relaxed mean graph and $K_{1,m}$ is not a relaxed mean graph for $m \geqslant 5$. They also prove that the graph consisting of two stars $K_{1,m}$ and $K_{n,1}$ with an edge in common is a relaxed mean graph if and only if $|m-n| \leqslant 5$. Balaji and Maheswari [399] proved the following graphs are relaxed mean graphs C_n $(n > 4)$; $K_{2,n}$; triangular snakes; quadrilateral snakes; ${P_n}^2$; $(P_n \times P_2) \odot K_1$; $\overline{K_n} + 2K_2$; $K_2 + mK_1$; $C_3^{(t)}$; the union of any two trees; $P_m \times P_n$ $(m > 1, n > 1)$; and $P_m \times C_n$ $(m > 1)$. They also prove that K_n is not a relaxed mean graph. Maheswari, Ramesh, and Balaji [2062] proved the following graphs are relaxed mean graphs: P_n $(n > 5)$; bistars $B_{m,n}$ if and only if $|m-n| \leqslant 3$; combs; and $C_3 \cup P_n$ $(n > 1)$. They also proved that $K_{1,n}$ is not a relaxed mean graph for $n > 5$. Francis and Balaji proved that W_n $(n > 4)$ are relaxed mean graphs.

In [236] Arockiaraj, Rajesh Kannan, and Durai Baskar introduced the F-centroidal mean labeling of graphs by defining a function f to be an *F-centroidal mean* labeling of a graph $G(V,E)$ with q edges if $f : V(G) \to \{1,2,3,\dots,q+1\}$ is injective and the induced function $f^* : E(G) \to \{1,2,3,\dots,q\}$ defined as $f^*(uv) = \left\lfloor \frac{2\ [f(u)^2+f(u)f(v)+f(v)^2]}{3\ [f(u)+f(v)]} \right\rfloor$ for all $uv \in E(G)$ is bijective. A graph that admits an F-centroidal mean labeling is called an *F-centroidal mean* graph. They discussed the F-centroidal meanness of the tree $P_n(X_1, X_2, \dots X_n)$ obtained from a path on n vertices by attaching X_i pendent vertices at each i^{th} vertex of the path for $1 \leqslant i \leqslant n$, the twig graph $TW(P_n)$, the graph $P_n \circ S_m$ for $m \leqslant 4$, $P_m \times P_n$ for $m \leqslant 3$, ladders, $P_n \circ K_2$, P_a^b for $a \geqslant 2$ and $b \leqslant 3$, the middle graphs and splitting graphs of paths, the total graphs of paths, P_n^2, and $P(1,2,\dots,n-1)$ the graph obtained by replacing each i^{th} edge of P_n by identifying its end vertices with the vertices of the two element component of $K_{2,i}$. In the same article they introduced super F-centroidal mean graphs [236] as follows. Let G be a graph and $f : V(G) \to \{1,2,3,\dots,p+q\}$ be an injection. With f^* defined as for the F-centroidal case f is called a *super F-centroidal mean* labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1,2,3,\dots,p+q\}$. A graph that admits a super F-centroidal mean labeling is called a *super F-centroidal mean* graph. They proved that the following graphs are super F-centroidal mean graphs: paths, cycles, the union of any number of paths, the mirror graph of P_n, $P_n \circ S_m$, $TW(P_n)$, $P_n \cup C_m$, P_n^2, and dragons $P_n(C_m)$ the graph obtained from C_m by identifying an end vertex of P_n at a vertex of C_m.

In [1138] Gopi and Kumar proved the following graphs are are F-centroidal mean graphs: the mirror graph $M(P_n)$ $(n \geqslant 3)$, Spl(P_n) $(n \geqslant 3)$, P_n^2 $(n \geqslant 3)$, $VD(G)$, the *tortoise* graph T_n $(n \geqslant 4)$ (obtained from a path $v_1, v_2, \dots, v_n$ by attaching an edge between v_i and v_{n-i+1} for $1 \leqslant i \leqslant \lfloor n/2 \rfloor$. $(n = 1, 3 \pmod 4))$, and PC_n $(n \geqslant 5)$ (obtained from $C_n = v_1, v_2, \dots, v_n$ by adding chords joining v_i and v_{n-i+2} for $2 \leqslant i \leqslant t$, where $t = \lfloor n/2 \rfloor$), In [1137] they proved $D(T_n)$ $(n \geqslant 3)$, $AD(T_n)$ $(n \geqslant 4)$, Q_n $(n \geqslant 2)$, $A(Q_n)$ $(n \geqslant 2)$ $AD(Q_n)$ $(n \geqslant 2)$, and SL_n $(n \geqslant 2)$ are F-centrodial mean graphs. [1138] new [1137] new

Arockiaraj, Kannan, Manivannan, and Durai Baskar [238] investigated the F-centroidal mean property for paths, cycles, stars, K_n, $P_n \circ S_1$, the triangular snakes, arbitrary subdivisions of $K_{1,3}$, and some line graphs.

In [406] and [407] Balaji, Ramesh, and Subramanian use the term "Skolem mean" labeling for super mean labeling. They prove: P_n is Skolem mean; $K_{1,m}$ is not Skolem mean if $m \geqslant 4$; $K_{1,m} \cup K_{1,n}$ is Skolem mean if and only if $|m-n| \leqslant 4$; $K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is Skolem mean if $|m-n| = 4+l$ for $l = 1,2,3,\dots, m = 1,2,3,\dots,$ and $l \leqslant m < n$; $K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is not Skolem mean if $|m-n| > 4+l$ for $l = 1,2,3,\dots, m = 1,2,3,\dots, n \geqslant l+m+5$ and $l \leqslant m < n$; $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is Skolem mean if $|m-n| = 4+2l$ for $l = 2,\dots, m = 2,3,4\dots, n = 2l+m+4$ and $l \leqslant m < n$; $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is not Skolem mean if $|m-n| > 4+l$ for $l = 1,2,3,\dots, m = 1,2,3,\dots, n \geqslant l+m+5$ and $l \leqslant m < n$; $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is not Skolem mean if $|m-n| > 4+2l$ for $l = 2,\dots, m = 2,3,4\dots, n \geqslant 2l+m+5$ and $l \leqslant m < n$; $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is Skolem mean if $|m-n| = 7$ for $m = 1,2,3,\dots, n = m+7$ and $1 \leqslant m < n$; and $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is not Skolem mean if $|m-n| > 7$ for $m = 1,2,3,\dots, n \geqslant m+8$ and $1 \leqslant m < n$. Balaji [405] proved that $K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is Skolem mean if $|m-n| < 4+l$

for integers $1, m \geqslant 1$ and $l \leqslant m < n$. In [2850] Shainy and Balaji determined necessary and sufficient conditions for the disjoint union of three stars to be Skolem mean.

In [2880] Shendra Shainy, Hariprabakaran, Swathy, and Balaji provided a technique for coding a secret messages by applying a Skolem mean-like labeling on a graph obtained from the two stars of the form $K_{1,n}$ and $K_{1,n+1}$ that are joined by an edge from one vertex of degree 1 in $K_{1,n}$ to one vertex of degree 1 in $K_{1,n+1}$.

In [1526] Jeyanthi, Selvi, and Ramya prove that $C_m \cup C_n$, $(P_n + K_1) \cup (n-2)K_2$ $(n > 2)$, $(P_n + K_2) \cup (2n-3)K_2$ $(n \geqslant 2)$ and $W_n \cup (n-1)K_2$ $(n \geqslant 3)$ are Skolem difference mean graphs. In [1527] they show that the union of any finite number of paths, the union of any finite number of stars, $G \cup nK_2$ where G is Skolem difference mean and all the vertex labels are odd, $C_m \cup P_m$ $(m \geqslant 2)$, $K_{m,n} \cup (m-1)(n-1)K_2$, and $K_{1,1,n} \cup (n-1)K_2$. are skolem difference mean graphs.

In [1506] Jeyanthi, Ramya, and Thangavelu proved the following graphs have super mean labelings: the one point union of any two cycles, graphs obtained by joining any two cycles by an edge (dumbbell graphs), $C_{2n+1} \odot C_{2m+1}$, graphs obtained by identifying a copy of an odd cycle C_m with each vertex of C_n, the quadrilateral snake Q_n, where n is odd, and the graphs obtained from an odd cycle $u_1, u_2, \ldots, u_n$ by joining the vertices u_i and u_{i+1} by the path P_m (m is odd) for $1 \leqslant i \leqslant n-1$ and joining vertices u_n and u_1 by the path P_m. Jeyanthi, Ramya, Thangavelu, and Aditanar [1504] give super mean labelings of $C_m \cup C_n$ and T_p-trees. Vasuki and Arockiaraj [3355] proved that nC_4, $n > 1$, triangular grid graphs, the edge mC_n-snakes, and the braid graphs are super mean graphs. They further proved that the graphs obtained by identifying an edge of two cycles C_m and C_n is a super mean graph.

In [1498] Jeyanthi and Ramya define $S_{m,n}$ as the graph obtained by identifying one endpoint of each of n copies of P_m and $< S_{m,n} : P_m >$ as a graph obtained by identifying one end point of a path P_m with the vertex of degree n of a copy of $S_{m,n}$ and the other endpoint of the same path to the vertex of degree n of another copy of $S_{m,n}$. They prove the following graphs have super mean labelings: caterpillars, $< S_{m,n} : P_{m+1} >$, and the graphs obtained from P_{2m} and $2m$ copies of $K_{1,n}$ by identifying a leaf of ith copy of $K_{1,n}$ with ith vertex of P_{2m}. They further establish that if T is a T_p-tree, then $T \odot K_1$, $T \odot \overline{K_2}$, and, when T has an even number of vertices, $T \odot \overline{K_n}$ $(n \geqslant 3)$ are super mean graphs.

In [824] Dhanalakshmi and Parvathi define a *mean square cordial* labeling of a graph $G(V, E)$ with p vertices and q edges as one for which there is a bijection from V to $\{0, 1\}$ such that when each edge uv is assigned the label $\lceil (f(u)^2 + f(v)^2)/2 \rceil$ the condition that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled 0 and the number of edges labeled with 1 differ by at most 1 is satisfied. In [824] Dhanalakshmi and Parvathi proved that helms, closed helms, gears, sunlet graphs, fans, and $C_p \odot nK_1$ admit mean square cordial labelings. In [825] Dhanalakshmi and Parvathi showed that the paths, combs, n-centipedes, and stars admit mean square cordial labelings. They also proved that the induced subgraph obtained by the upper approximation of any subgraph H of the above acyclic graphs admits a mean square cordial labeling. In [826] Dhanalakshmi and Parvathi prove that bistars, subdivisions of stars, coconut trees, banana trees, the one-point union of C_n and $K_{1,n}$, and $P_m \odot K_{1,n}$ admit mean square cordial labelings. Dhanalakshmim [827] proved that shell-butterfly graphs with shell orders $m > 2$ and $n > 2$ admit mean square cordial labelings. Dhanalakshmi and Thirunavukkarasu [828] z proved pentagonal snakes, subdivision of a pentagonal snakes, double pentagonal snakes, and alternate pentagonal snakes admit mean square cordial labelings.

Arockiaraj, Durai Baskar, and Rajesh Kannan [227] calls a graph G with p vertices and q edges a *F-root square mean graph* if there is an injective function f from the vertices of $V(G)$ to $\{1, 2, \dots, q+1\}$ such that for each edge uv the induced function $f^*(uv) = \lfloor\sqrt{(f(u)^2 + f(v)^2)/2}\rfloor$ is bijective. They proved that the following are F-root square mean graphs: paths, the graph $P_n \circ S_m$ obtained from the path P_n by attaching m pendant vertices to each vertex of P_n, twigs $TW(P_n)$, the graph $[P_n; S_m]$ obtained from n copies of S_m and the path P_n by joining u_i with the central vertex $v_1^{(i)}$ of the i^{th} copy of S_m with of an edge for $i \leqslant n$, the mirror graph of P_n, the total graph of P_n, P_n^2, ladders, and slanting ladders. In [228], Arockiaraj, Durai Baskar, and Rajesh Kannan analyzed that the line graph operation preserves the F-root square meanness of line graph of the path, cycle, star, $P_n \circ S_1, P_n \circ S_2, [P_n; S_1], S(P_n \circ S_1)$, ladder, slanting ladder, the crown graph $C_n \circ S_1$ and the arbitrary subdivision of S_3. Gopi [1133] proved that triangular snakes T_n $(n \geqslant 2)$, $A(T_n)$ $(n \geqslant 3), D(T_n)$ $(n \geqslant 2)$, quadrilateral snakes, $A(Q_n), D(Q_n)$ $(n \geqslant 3)$ are F-root square mean graphs.

Rajesh Kannan, Vikrama Prasad, and Gopi [1636] call a graph G with p vertices and q edges a *super root mean graph* if there is an injective function f from the vertices of G to $\{1, 2, \dots, p+q\}$ such that for each edge uv the induced function $f^*(uv) = \lfloor\sqrt{(f(u)^2 + f(v)^2)/2}\rfloor$ or $f^*(uv) = \lceil\sqrt{(f(u)^2 + f(v)^2)/2}\rceil$ yields the set of vertex labels and edge labels $\{1, 2, \dots, p+q\}$. They proved the following are super root square mean graphs: $P_m \cup P_m$ $(m, n \geqslant 3)$; $P_m \cup (P_n \odot K_1)$ $(m, n \geqslant 3)$; $(P_m \odot K_1) \cup (P_n \odot K_1)$ $(m, n \geqslant 3)$; the union of a path and a triangular snake; and the union of $P_n \odot K_1$ and a triangular snake. Gopi and Kalaiyarasi [1135] prove that the following graphs have a super root square mean labeling: P_n^2 $(n \geqslant 4)$, slanting ladders $SL_n(n \geqslant 3)$, triangular snakes with a pendent edge attached to each vertex, and quadrilateral snakes with a pendent edge attached to each vertex.

Chitra devi and Saravana Kumar [705] proved that $nP_m, nK_3, P_n \odot \overline{K_2}$, the middle graph of path P_n (graphs obtained by starting with a path and joining every consecutive pair of vertices excluding the two end vertices of the path to a new isolated vertex), and $C_m \odot P_n$ admits super root square mean labelings. Venkatesan and Thirugnanasambandam [3373] proved that the following graphs admit super root square mean labelings: P_n^2 $(n \geqslant 4)$, slanting ladders, $T_n \odot K_1$ $(n \geqslant 3)$ (T_n is a triangular snake), and $Q_n \odot K_1$ (Q_n is a quadrilateral snake). Orias and Pedrano [2322] determined the super root square mean labelings of $C_n \odot K_1$, middle cycles (graphs obtained by starting with a cycle and joining every consecutive pair of verticies of the cycle to a new isolated vertex), polygonal chains, alternate polygonal chains, and kayak paddles.

[705] new
[3373] new
[2322] new

In [133] Akilandeswari calls a graph is a *k-super root square mean* graph if it is possible to label the vertices with distinct elements from $\{k, k+1, k+2, \dots, p+q+k-1\}$ in such a way that when each edge uv is labeled with $\lfloor\sqrt{(f(u)^2 + f(v)^2)/2}\rfloor$ or

$\lceil \sqrt{(f(u)^2+f(v)^2)/2} \rceil$, the union of the vertex and edge labels is the set $\{k, k+1, k+2, \ldots, p+q+k-1\}$. He proved that $P_n \odot K_{1,2}, P_n \odot K_{1,3}$, the corona product of the quadrilateral snake and K_1, double triangular snakes, and the corona product of the double triangular snake and K_1 admit k-super root square mean labelings.

A *radio mean square* labeling of a connnected graph is an injective map h from the set of vertices of the graph G to the set of positive integers $\mathbb{N}$, such that for any two distinct vertices x, y, the inequality $d(x,y) + \lceil (h(x))^2 + (h(y))^2/2 \rceil \geqslant \dim(G) + 1$ holds. For a particular radio mean square labeling h, the maximum number of $h(v)$ taken over all vertices of G is called its spam, denoted by $\mathrm{rmsn}(h)$, and the minimum value of $\mathrm{rmsn}(h)$ taking over all radio mean square labeling h of G is called the *radio mean square number* of G, denoted by $\mathrm{rmsn}(G)$. In [273] Badr, Nada, Al-Shamiri, Abdel-Hay, and Elrokh investigated the radio mean square numbersfor paths and cycles. For a graph G, they present an approximate algorithm to determine $\mathrm{rmsn}(G)$. Finally, they introduced a new mathematical model to find the upper bound of $\mathrm{rmsn}(G)$ for graph G.

In [2700] Sandhya, Somasundaram, and Anusa say a graph with q edges is a *root square mean* graph if it is possible to label the vertices with distinct elements from $\{1, 2, \ldots, q+1\}$ in such a way that when each edge uv is labeled with $\lfloor \sqrt{(f(u)^2+f(v)^2)/2} \rfloor$ or $\lceil \sqrt{(f(u)^2+f(v)^2)/2} \rceil$, the resulting edge labels are distinct. They prove that paths, cycles, combs, ladders, triangular snakes, quadrilateral snakes, $K_{1,n)}$ $(1 \leqslant 6)$, and K_n for $n = 1, 2$ and 3 are root square mean graphs. In [2701] they proved that the following graphs admit root square mean labelings: double triangular snakes, alternate double triangular snakes, double quadrilateral snakes, alternate double quadrilateral snakes, and polygonal chains.

In [1771] Kulandhai Therese and Romila introduced the notion of cube root cube mean labeling as follows. For a graph $G(V, E)$ with $|(V(G)| = p$ and $|E(G)| = q$, let $f : V(G) \to \{1, 2, \ldots, q+1\}$ be an injective function. The induced edge labeling f^* for a vertex labeling f is defined by $f^*(e) = \lfloor \sqrt[3]{(f(u)^3+f(v)^3)/2} \rfloor$ or $\lceil \sqrt[3]{(f(u)^3+f(v)^3)/2} \rceil$ for all $e = uv \in E(G)$. If f is a bijection it is called a *cube root cube mean labeling.* If such labeling exists, G is said to be a *k-cube root cube mean* graph. They proved that paths, combs, ladders, and quadrilateral snakes admit cube root cube mean labelings.

In [2522] Princy Kala introduced the notion of k-super cube root cube mean labeling as follows. For a graph $G(V, E)$ with $|(V(G)| = p$ and $|E(G)| = q$ let $f : V(G) \to \{k, k+1, k+2, \ldots, p+q+k-1\}$ be an injective function. The induced edge labeling f^* for a vertex labeling f is defined by $f^*(e) = \lfloor \sqrt[3]{(f(u)^3+f(v)^3)/2} \rfloor$ or $\lceil \sqrt[3]{(f(u)^3+f(v)^3)/2} \rceil$ for all $e = uv \in E(G)$. If f is a bijection and $f(V(G)) \cup f^*(E(G)) = \{k, k+1, k+2, \ldots, p+q+k-1\}$, f is called a *k-super cube root cube mean labeling.* If such labeling exists, then G is said to be a *k-super cube root cube mean* graph. He proved the existence k-super cube root mean labelings for triangular snakes, double triangular snakes, quadrilateral snakes, double quadrilateral snakes, alternate triangular snakes, alternate double triangular snakes, alternate quadrilateral snakes, and alternate double quadrilateral snakes. In [2523] Princy Kala proved that the corona products of a triangular snake and K_1, an alternate triangular snake and K_1, an alternate triangular snake and $2K_1$, an alternate quadrilateral snake and K_1, $P_n \odot K_{1,2}$, and $P_n \odot K_{1,3}$ are k-super cube root cube mean graphs.

Let G be a graph and let $f : V(G) \rightarrow \{1, 2, \ldots, n\}$ be a function such that the label of the edge uv is $(f(u) + f(v))/2$ or $(f(u) + f(v) + 1)/2$ according as $f(u) + f(v)$ is even or odd and $f(V(G)) \cup \{f^*(e) : e \in E(G)\} \subseteq \{1, 2, \ldots, n\}$. If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex and edge labels are distinct and there is no common vertex and edge labels, then n is called the *super mean number* of a graph G and it is denoted by $S_m(G)$. Nagarajan, Vasuki, and Arockiaraj [2252] proved that for any graph of order p, $S_m(G) \leqslant 2^p - 2$ and provided an upper bound of the super mean number of the graphs: $K_{1,n}$ $n \geqslant 7$; $tK_{1,n},\ n \geqslant 5,\ t > 1$; the bistar $B(p, n),\ p > n$; the graphs obtained by identifying a vertex of C_m and the center of $K_{1,n},\ n \geqslant 5$; and the graphs obtained by identifying a vertex of C_m and the vertex of degree 1 of $K_{1,n}$. They also gave the super mean number for the graphs $C_n, tK_{1,4}$, and $B(p, n)$ for $p = n$ and $p = n + 1$.

Manickam and Marudai [2072] defined a graph G with q edges to be an *odd mean graph* if there is an injective function f from the vertices of G to $\{1, 3, 5, \ldots, 2q - 1\}$ such that when each edge uv is labeled with $(f(u) + f(v))/2$ if $f(u) + f(v)$ is even, and $(f(u) + f(v) + 1)/2$ if $f(u) + f(v)$ is odd, then the resulting edge labels are distinct. Such a function is called a *odd mean* labeling. For integers a and b at least 2, Vasuki and Nagarajan [3359] use P_a^b to denote the graph obtained by starting with verticies $y_1, y_2, \ldots, y_a$ and connecting y_i to y_{i+1} with b internally disjoint paths of length $i + 1$ for $i = 1, 2, \ldots, a-1$ and $j = 1, 2, \ldots, b$. For integers $a \geqslant 1$ and $b \geqslant 2$ they use $P_{\langle 2a\rangle}^b$ to denote the graph obtained by starting with verticies $y_1, y_2, \ldots, y_{a+1}$ and connecting y_i to y_{i+1} with b internally disjoint paths of length $2i$ for $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, b$. They proved that the graphs $P_{2r,m}, P_{2r+1,2m+1}$, and $P_{\langle 2r\rangle}^m$ are odd mean graphs for all values of r and m. Jeyanthi and Gomathi [1433] proved the edge linked cyclic snake $EL(kC_n)$ $(n \geqslant 6)$ is an odd mean graph.

For a T_p-tree T with m vertices $T@P_n$ is the graph obtained from T and m copies of P_n by identifying one pendent vertex of ith copy of P_n with ith vertex of T. For a T_p-tree T with m vertices $T@2P_n$ is the graph obtained from T by identifying the pendent vertices of two vertex disjoint paths of equal lengths $n - 1$ at each vertex of T. Ramya, Selvi and Jeyanthi [2589] prove that $P_m \odot \overline{K_n}$ $(m \geqslant 2, n \geqslant 1)$ is an odd mean graph, T_p trees are odd mean graphs, and, for any T_p tree T, the graphs $T@P_n$, $T@2P_n$, $\langle T\tilde{o}K_{1,n}\rangle$ are odd mean graphs.

For a T_p-tree T with m vertices let $T\hat{o}C_n$ denote the graph obtained from T and m copies of C_n by identifying a vertex of i^{th} copy of C_n with i^{th} vertex of T and $T\tilde{o}C_n$ denote the graph obtained from T and m copies of C_n by joining a vertex of i^{th} copy of C_n with i^{th} vertex of T by an edge. In [1529] Selvi, Ramya, and Jeyanthi prove that for a T_p tree T the graphs $T\hat{o}C_n$ $(n > 3, n \neq 6)$ and $T\tilde{o}C_n,\ (n > 3,\ n \neq 6)$ are odd mean graphs.

Ramya, Selvi, and Jeyanthi [2588] prove that for a T_p-tree T the following graphs are odd mean graphs: $T@P_n,\ T@2P_n,\ P_m \odot \overline{K_n}$, and the graph obtained from T and m copies of $K_{1,n}$ by joining the central vertex of ith copy of $K_{1,n}$ with ith vertex of T by an edge.

For a graph G and some fixed vertex v of G, Pooranam, Vaski, and Suganthi [2478] proved the following graphs have odd mean labelings: graphs obtained from a path P_m :

$u_1u_2\cdots u_m$ and G by joining u_i to v to the ith copy of G. Their results include the cases where $G = C_{4n}$, a star, or the cube Q_3. For a graph G and some fixed vertex v of G they also proved the existence of odd mean labelings for graphs obtained from a path $P_m : u_1u_2\cdots u_m$ and G by identitying u_i with v in the ith copy of G, where G is Q_3 or $C_{4n}^{(2)}$ and v is the vertex of $C_{4n}^{(2)}$ of degree 4.

A graph G is said to be *vertex odd mean* graphvertex odd mean if there exist an injective function $f : V(G)$ to $\{1, 3, 5, \ldots, 2|E(G)| - 1\}$ such that the induced mapping $f^* : E(G)$ to the set of positive integers defined by $f^*(uv) = (f(u) + f(v))/2$ is injective. Such a function is called a *vertex odd mean* labeling. A graph G is called a *vertex even mean* graph if there exist an injective function $f : V(G)$ to $\{2, 4, 6, \ldots, 2|E(G)|\}$ such that the induced mapping $f^* : E(G)$ to the set of positive integers defined by $f^*(uv) = (f(u)+f(v))/2$ is injective. Such a function is called a *vertex even mean* labeling. Revathi [2617] gave vertex odd and even mean labelings for nontrivial umbrella graphs, mongolian [2617] new
tents, and $K_1 + C_n$. Anitha, Selvam, and Thirusangu [214] proved that the extended [214] new
duplicate graph of kite graph admits mean, even mean, and odd mean labelings, Prajapati and Raval [2503] and [2504] proved that cyclic snakes for $n = 3, 4, 5$ are vertex even mean [2503] new
and vertex odd mean graphs. They have proved that double cyclic snake and alternating [2504] new
cyclic snake for $n = 3, 4, 5$ are vertex odd mean and vertex even mean graph. Prajapati and Raval [2505] proved that every vertex even mean graph is vertex odd mean graph and [2505] new
vice versa. They also proved that triangular snakes TS_n, $TS_n \odot P_2$, quadrilateral snakes QS_n, $QS_n \odot P_2$, double quadrilateral snakes, alternating double quadrilateral snakes, jelly fish, crowns, shells, and fans are vertex even mean and vertex odd mean graphs. Raval and Prajapati [2597] proved that cyclic snakes (obtained by replacing each vertex of a [2597] new
path by a cycle), quadrilateral snakes, pentagonal snakes, and alternating quadrilateral snakes admit vertex even and odd mean labelings. They also proved that even vertex odd mean graphs are even mean graphs.

In [184] Amuthavalli and S. Dineshkumar say a (p, q) graph G has a *k-odd edge mean* [184] new
labeling if there exists an injection f from the edges of G to $\{0, 1, 2, 3, \ldots, 2k+2p-3\}$ such that the induced map f^* defined on $V(G)$ by $f^*(v) = \lceil\sum(f(vu)/\deg(v)\rceil$ is a bijection from V to $\{2k-1, 2k+1, 2k+3, \ldots, 2k+2p-3\}$. A graph that admits a k-odd edge mean labeling is called a *k-odd edge mean* graph. They proved that P_n $(n \neq 4)$, $K_{1,2n}$ $(n \geqslant 2)$, and C_n $(\neq 6, 7)$ are k-odd edge mean graphs for all k.

Thirugnanasambandam, Chitra, and Vishnupriya [3209] introduced a notion of prime [3209] new
odd mean graphs as follows. A graph G with p vertices and q edges for which there exists an injective function $f : V(G) \to \{1, 3, 5, \ldots, 2q + 1\}$ such that gcd $(f(u), f(v)) = 1$ and the induced edge labeling $f^* : E(G) \to \{2, 4, 6, \ldots, 2p - 2\}$ defined by $f^*(uv) = \lceil(f(u) + f(v))/2\rceil$ is injective is called a *prime odd mean* graph. They proved that caterpillars, cycles, kites, ladders, spiders, and stars are prime odd mean graphs.

In [704] Chitra, Priya, and Vishnupriya [704] introduced the notion of *odd Fibonacci* [704] new
mean labeling of graph G as an injective f from the verticies of G to the odd Fibonacci [704] new
numbers such that the induced edge labeling f^* defined by $f^*(uv) = (f(u) + f(v))/2$ is injective. A graph that admits a odd Fibonacci mean labeling is called an *odd Fibonacci mean* graph. They proved the following graphs admit odd Fibonacci mean labelings:

wheels, helms, gears, cycles, banana trees, tadpoles, fire crackers, fans, stars, ladders, complete tripartite grahs, and Dutch windmills.

Uma Devi, Kamaraj, and Arockiaraj [811] determined the odd Fibonacci edge irregularity strength for P_n $(n \geqslant 2), K_{1,n}, P_n \odot K_1$ $(n \geqslant 2)$, bistars $B_{(m,n)}$ and proved the nonexistence of an odd Fibonacci edge irregular labeling for K_n $(n \leqslant 3)$ and $K_{m,n}$ $(m \geqslant 2, n \geqslant 4)$. In [813] Devi, Kamaraj, and Arockiaraj determined the odd Fibonacci edge irregularity strength for P_n $(n \geqslant 2)$, stars, subdivison of stars, subdivision of fans, $P_n \odot mK_1$ $(m, n \geqslant 2)$, and the graphs obtained by joining i pendant vertices to the ith vertex of P_n $(n \geqslant 2)$. [811] new [813] new

Gayathri and Amuthavalli [1034] (see also [182]) say a (p, q)-graph G has a *(k, d)-odd mean* labeling if there exists an injection f from the vertices of G to $\{0, 1, 2, \ldots, 2k - 1 + 2(q-1)d\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = \lceil (f(u)+f(v))/2 \rceil$ is a bijection from edges of G to $\{2k-1, 2k-1+2d, 2k-1+4d, \ldots, 2k-1+2(q-1)d\}$. When $d = 1$ a (k, d)-odd mean labeling is called *k-odd mean*. For $n \geqslant 2$ they prove the following graphs are k-odd mean for all k: P_n; combs $P_n \odot K_1$; crowns $C_n \odot K_1$ $(n \geqslant 4)$; bistars $B_{n,n}$; $P_m \odot \overline{K_n}$ $(m \geqslant 2)$; $C_m \odot \overline{K_n}$; $K_{2,n}$; C_n except for $n = 3$ or 6; the one-point union of C_n $(n \geqslant 4)$ and an endpoint of any path; grids $P_m \times P_n$ $(m \geqslant 2)$; $(P_n \times P_2) \odot K_1$; arbitrary unions of paths; arbitrary unions of stars; arbitrary unions of cycles; the graphs obtained by joining two copies of C_n $(n \geqslant 4)$ by any path; and the graph obtained from $P_m \times P_n$ by replacing each edge by a path of length 2. They prove the following graphs are not k-odd mean for any k: K_n; K_n with an edge deleted; $K_{3,n}$ $(n \geqslant 3)$; wheels; fans; friendship graphs; triangular snakes; Möbius ladders; books $K_{1,m} \times P_2$ $(m \geqslant 4)$; and webs. For $n \geqslant 3$ they prove $K_{1,n}$ is k-odd mean if and only if $k \geqslant n-1$. Gayathri and Amuthavalli [1035] prove that the graph obtained by joining the centers of stars $K_{1,m}$ and $K_{1,n}$ are k-odd mean for $m = n, n+1, n+2$ and not k-odd mean for $m > n+2$. For $n \geqslant 2$ the following graphs have a (k, d)-mean labeling [1063]: $C_m \cup P_n$ $(m \geqslant 4)$ for all k; arbitrary unions of cycles for all k; P_{2m}; P_{2m+1} for $k \geqslant d$; (P_{2m+1} is not (k, d)-mean when $k < d$); combs $P_n \odot K_1$ for all k; $K_{1,n}$ for $k \geqslant d$; $K_{2,n}$ for $k \geqslant d$; bistars for all k; nC_4 for all k; and quadrilateral snakes for $k \geqslant d$.

In [2788] Seoud and Salim [2789] proved that a graph has a k-odd mean labeling if and only if it has a mean labeling. In [2788] Seoud and Salim give upper bounds of the number of edges of graphs with a (k, d)-odd mean labeling

Pricilla [2519] defines an *even mean labeling* of a graph G as an injective function f from the verticies of G to $\{2, 4, \ldots, 2|E(G)|\}$ such that the edge labels given by $(f(u)+f(v))/2$ are distinct. Vaidya and Vyas [3330] proved that $D_2(P_n)$, $M(P_n)$, $T(P_n)$, $S'(P_n)$, P_n^2, P_n^3, switching of pendent vertex in P_n, $S'(B_{n,n})$, double fans, and duplicating each vertex by an edge in paths are even mean graphs.

Gayathri and Gopi [1043] defined a graph G with q edges to be an *k-even mean graph* if there is an injective function f from the vertices of G to $\{0, 1, 2, \ldots, 2k + 2(q-1)\}$ such that when each edge uv is labeled with $(f(u) + f(v))/2$ if $f(u) + f(v)$ is even, and $(f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd, then the resulting edge labels are $\{2k, 2k+2, 2k+4, \ldots, 2k + 2(q-1)\}$. Such a function is called a *k-even mean* labeling. In [1043] they proved that the graphs obtained by joining two copies of C_n with a path P_m are k-even mean for all k and all $m, n \geqslant 3$ when $n \equiv 0, 1 \pmod 4$ and for all $k \geqslant 1,\ m \geqslant 7$, and $n \geqslant 3$. In [1045] Gayathri and Gopi proved that various graphs obtained by joining two copies of stars $K_{1,m}$ and $K_{1,n}$ with a path by identifying the one endpoint of the path with the center of one star and the other endpoint of the path with the center of the other star are k-even mean. In [1044] they proved that various graphs obtained by appending a path to a vertex of a cycle are k-even mean. In [1046] they proved that $C_n \cup P_m$, $n \geqslant 4,\ m \geqslant 2$, is k-even mean for all k. Gayathri and Gopi [1049] proved the following are k-even mean graphs: shadow graphs of stars with at least 3 vertices; edge duplication graphs of cycles with at least 4 vertices; and vertex duplication graphs of paths and cycles with at least 4 vertices.

Gayathri and Gopi [1047] say graph G with q edges has a (k, d)-*even mean* labeling if there exists an injection f from the vertices of G to $\{0, 1, 2, \ldots, 2k + 2(q-1)d\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = (f(u) + f(v))/2$ if $f(u) + f(v)$ is even and $f^*(uv) = (f(u) + f(v) + 1)2$ if $f(u) + f(v)$ is odd is a bijection from edges of G to $\{2k, 2k + 2d, 2k + 4d, \ldots, 2k + 2(q-1)d\}$. A graph that has a (k, d)-even mean labeling is called a (k, d)-*even mean graph.* They proved that $P_m \oplus nK_1 (m \geqslant 3, n \geqslant 2)$ has a (k, d)-even mean labeling in the following cases: all (k, d) when m is even; all (k, d) when m is odd and n is odd; and m is odd, n is even and $k \geqslant d$.

Kalaimathy [1562] investigated conditions under which a mean labeling for a graph G will yield a (k, d)-even mean labeling for G and vice versa. He also gave conditions under which two graphs that have $(1, 1)$-mean labelings can be joined by an single edge to obtain a new graph that has a $(1, 1)$-even mean labeling. Gopi's Ph. D. thesis [1130] has a large number of results about mean, k-mean, k-odd mean, k-even mean, (k, d)-odd mean , and (k, d)-mean labelings.

In [3363] Vasuki, Nagarajan, and Arockiaraj introduced the notion of even vertex odd mean graphs as follows. A (p, q)-graph is said to have an *even vertex odd mean* labeling if there exists an injective function f from $V(G)$ to $\{0, 2, 4, \ldots, 2q-2, 2q\}$ such that the induced map $f^* : E(G)$ to $\{1, 3, 5, \ldots, 2q-1\}$ defined by $f^*(uv) = (f(u) + f(v))/2$ is a bijection. A graph that admits an even vertex odd mean labeling is called an *even vertex odd mean* graph. They proved that paths, C_{4n}, $K_{2,n}$, bistars $B_{m,n}$ for $n = m, m+1$, quadrilateral snakes Q_n, combs, $P_m \times P_n$, $C_{4m} \times P_n$, ladders, and dragons are even vertex odd mean graphs. Rajesh Kannan, Vikrama Prasad, Gopi [1637] proved the following graphs have an even vertex odd mean labeling: slanting ladders $SL_n (n \geqslant 3)$; double triangular snakes; alternative double triangular snakes; graphs obtained by starting with a tree G with at least 3 vertices and a mean labeling and a copy G' of G by joining each vertex of G to its corresponding vertex in G' with an edge; graphs obtained by starting with a path $v_1v_2 \cdots v_n$ $(n \geqslant 4)$ and joining v_1 and v_3 to an isolated vertex; graphs obtained by starting with a path $v_1v_2 \cdots v_n$ $(n \geqslant 4)$ and appending two edges to each of $v_2, v_3, \ldots, v_{n-1}$; and graphs obtained from a quadrilateral snake and appending an edge at each vertex. The *H-graph of a path* P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{n/2+1}$ and $u_{n/2+1}$ by an edge if n is odd and the vertices $v_{(n+1)/2}$ and $u_{(n/2}$ by an edge if n is even. Rajesh Kannan, Vikrama Prasad, and Gopi [1638] prove that the H-graph of P_n $(n \geqslant 3)$ and the graph $H \odot K_1$ have even vertex odd mean labelings where H is the H graph of P_n ($n \geqslant 3$). In [1639] and [1640] Rajesh Kannan, Vikrama Prasad, and Gopi proved the following are even vertex odd mean graphs: graphs obtanied by joining the centers of two stars $K_{1,m}$ and $K_{1,n}$ by a path P_t ($m, n, t \geqslant 2$), graphs obtained by duplicating an edge of C_n ($n \geqslant 4$), graphs obtained by joining each endpoint of P_3 to n isolated vertices, shadow graphs of stars, shadow graphs of bistars $B(n, n)$, mirror graphs of paths, and the graphs obtained taking two copies of $P_n \times P_2$ and joining each vertex of one with the matching vertex in the other with an edge. Kannan, Vikrama Prasad, and Gopi [1641] proved that the following graphs admit even vertex odd mean labelings: slanting ladders SL_n ($n \geqslant 3$), twigs $\mathrm{TW}(n)$ ($n \geqslant 4$, double triangular snakes, alternative double quadrilateral snakes, and $Q_n \cdot K_1$. Prasad, Rejesh Kannan, and Gopi [2515] proved that $C_{4m} \odot K_{1,4n}$, $P_m \odot P_n$, and $\overline{K_2} + \overline{K_n}$ have even vertex odd mean labelings. In [2503] Prajapati and Raval proved that quadrilateral snakes, pentagonal snakes, and alternating quadrilateral snakes are vertex even and odd mean graphs. They also proved that even vertex odd mean graphs are even mean graphs. Jeyanthi, Ramya, and Selvi [1503] prove that T_P-trees (transformed trees), $T@P_n$, $T@2P_n$, and $\langle T\hat{o}K_{1,n}\rangle$ (where T is a T_P tree) are even vertex odd mean graphs.

Basher [477] investigated an even vertex labeling for the calendula graphs and introduced an arbitrary *calendula graph* as one in which if every edge from C_m is attached by an edge from arbitrary C_{n_i}, where n_i may vary for each $1 \leqslant i \leqslant m$. He proved that these graphs are also even vertex odd mean graphs. In [479] Basher proved several cycle related graphs are even vertex odd mean graphs. Basher and Kamran Siddiqui [481] proved that paths, cycles, combs, crowns, and planar grid super subdivisions are even vertex odd mean graphs.

For a graph $G(V, E)$ a bijection f from $V(G) \cup E(G)$ onto $\{1, 2, \ldots, |V(G)| + |V(E)|\}$ is said to be a *total mean labeling* if the values of $f^*(uv) = \lceil (f(u) + f(v) + f(uv))/3 \rceil$ taken over all edges are distinct. A graph G is said to be a *total mean labeling* graph if it admits a total mean labeling. Karuppasamy and Kaleeswari [1659] proved that $P_n, P_n^+, K_{1,n}, K_{2,n}, C_n, B_{m,n}$, triangular snakes, and alternate triangular snakes are total mean labeling graphs.

Murugan and Subramanian [2218] say a (p, q)-graph G has a *Skolem difference mean* labeling if there exists an injection f from the vertices of G to $\{1, 2, \ldots, p + q\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = (|f(u) - f(v)|)/2$ if $|f(u) - f(v)|$ is even and $f^*(uv) = (|f(u) - f(v)| + 1)/2$ if $|f(u) + f(v)|$ is odd is a bijection from edges of G to $\{1, 2, \ldots, q\}$. A graph that has a Skolem difference mean labeling is called a *Skolem difference mean* graph. They show that the graphs obtained by starting with two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ and joining the vertices $v_{(n+1)/2}$ and $u_{(n+1)/2}$ if n is odd and the vertices $v_{n/2+1}$ and $u_{n/2}$ if n is even are Skolem difference mean. Parmar and Vaghela [2348] proved the following graphs have Skolem difference mean labelings: brooms $B_{n,d}$ ($n \geqslant 4, d \geqslant 2$) (the graph with n vertices obtained from P_d by appending $n-d$ edges at an endpoint), combs $P_n \odot K_1$ ($n \geqslant 2$), $K_{1,m} \cup K_{1,n}$ ($m, n \geqslant 2$), and $K_{1,3} * K_{1,n}$ ($n \geqslant 2$) obtained from $K_{1,3}$ by attaching root of a star $K_{1,n}$ at each pendent vertex of $K_{1,3}$. Jeyanthi [1424] proved $\cup P_{n_i}$ ($n_i \geqslant 2$); $\cup K_{1,n_i}$ ($n_i \geqslant 2$); $C_n \cup P_m$ ($n \geqslant$

$3, m \geqslant 2)$; $K_{m,n} \cup (m-1)(n-1)K_2$; and $K_{1,1,n} \cup (n-1)K_2$ are Skolem difference mean graphs. She also proved that if G is a Skolem difference mean graph, then $G \cup nK_2$ is a Skolem difference mean graph.

Let $L_0, L_1, \ldots$ denote the sequence of Lucas numbers. In [2402] Ponmoni, Navaneetha Krishnan, and Nagarajan introduce the following graph labeling method. A graph G with p vertices and q edges is said to have a *Skolem difference Lucas mean* labeling if there is an injective function f from the vertices to $\{1, 2, \ldots, L_{p+q}\}$ such that when the edge uv is labeled with $|f(u) - f(v)|/2$ if $|f(u) - f(v)|$ is even, and $(|f(u) - f(v)| + 1)/2$ if $|f(u) - f(v)|$ is odd, then the resulting edge labels are distinct and belong to $\{L_1, L_2, \ldots L_q\}$. A graph that admits a Skolem difference Lucas mean labeling is called a *Skolem difference Lucas mean* graph. They proved the graphs obtained from $K_{1,m}$ by identifying the center of $K_{1,n}$ with the endpoint of each non-center vextex of $K_{1,m}$, bistars, $K_{1,m} \odot 2P_n$ and $\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(m)} \rangle$ are Skolem difference Lucas mean graphs.

Selvi, Ramya, and Jeyanthi [2747] prove that $C_n@P_n$ $(n \geqslant 3,\ m \geqslant 1)$, $K_n (n \leqslant 3)$, the shrub $St(n_1, n_2, \cdots, n_m)$, and the banana tree $Bt(n, n, \ldots, n)$ are Skolem difference mean graphs. They show that if G is a (p, q) graph with $q > p$ then G is not a Skolem difference mean graph and prove that K_n $(n \geqslant 4)$ is not a Skolem difference mean graph. A skolem difference mean labeling for which all the labels are odd is called an *extra Skolem difference mean* labeling. They also prove that the graph $T\langle K_{1,n_1} : K_{1,n_2} : \cdots : K_{1,n_m} \rangle$, obtained from the stars K_{1,n_1}, K_{1,n_2}, …, K_{1,n_m} by joining the central vertex of K_{1,n_j} and $K_{1,n_{j+1}}$ to a new vertex w_j for $1 \leqslant j \leqslant m-1$ and the graph $T\langle K_{1,n_1} \circ K_{1,n_2} \circ \cdots \circ K_{1,n_m} \rangle$, obtained from K_{1,n_1}, K_{1,n_2}, …, K_{1,n_m} by joining a leaf of $K_{1,n_{j+1}}$ to a new vertex w_j for $1 \leqslant j \leqslant m-1$ by an edge are extra Skolem difference mean graphs. Jeyanthi, Selvi, and Ramya [1527] proved that the union of any number of paths, any number of stars, $G \cup nK_2$ where G is an extra Skolem difference mean tree, $C_n \cup P_m$ $(n \geqslant 3, m \geqslant 2)$, $K_{m,n} \cup (m-1)(n-1)K_2$, and $K_{1,1,n} \cup (n-1)K_2$ have Skolem difference mean labelings. Vaghela and Parmar [3242] provided an extra Skolem difference mean labeling for combs, twigs of P_n, H graphs of P_n, graphs obtained from $K_{1,2}$ by attaching the root of $K_{1,n}$ at each pendant vertex of $K_{1,2}$, graphs obtained from $K_{1,3}$ by attaching the root of $K_{1,n}$ at each pendant vertex of $K_{1,3}$, and related graphs. Gross and Yellen [1161] proved that T_p-trees and caterpillars are extra Skolem difference mean graphs.

Let $G(V, E)$ be a graph with p vertices and q edges. Ramya, Kalaiyarasi, and Jeyanthi [2586] say G is a *Skolem odd difference mean* if there exists an injective function $f : V(G) \to \{0, 1, 2, 3, \ldots, p+3q-3\}$ such that the induced map $f^* : E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ denoted by $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$ is a bijection. A graph that admits a Skolem odd difference mean labeling is called a *odd difference mean* graph. They prove that P_n, C_n $(n \geqslant 4)$, $K_{1,n}$, $P_n \odot K_{1,n}$, coconut trees $T(n, m)$ obtained by identifying the central vertex of the star $K_{1,m}$ with a pendent vertex of P_n, $B_{m,n}$, caterpillars $S(n_1, n_2, \ldots, n_m)$, $P_m@P_n$ and $P_m@2P_n$ are Skolem odd difference mean graphs. ($P_m@P_n$ is obtained from P_m and m copies of P_n by identifying one pendent vertex of the i-th copy of P_n with the i-th vertex of P_m; $P_m@2P_n$ is defined analogously.) They establish that K_n, $n > 3$ and $K_{2,n}$ $(n \geqslant 3)$ are not Skolem odd difference mean graphs. They also prove that $K_{2,n}$ is a Skolem odd difference mean graph if $n \leqslant 2$. In [1452] Jeyanthi, Kalaiyarasi, Ramya, and

Saratha Devi prove that bistars $B(m,n)$, mP_n, $mP_n \cup tP_s$, $mK_{1,n} \cup tK_{1,s}$ and the graph $\langle P_m \tilde{o} S_n \rangle$ obtained from P_m and m copies of $K_{1,n}$ by joining the central vertex of i^{th} copy of $K_{1,n}$ with i^{th} vertex of P_m by an edge admit Skolem odd difference mean labelings. They also prove that if $G(p,q)$ is a Skolem odd differences mean graph then $p \geqslant q$ and that wheels, umbrellas, books, and ladders are not Skolem odd difference mean graphs. They call a Skolem odd difference mean labeling a *Skolem even vertex odd difference mean* labeling if all the vertex labels are even. They prove that P_n, $K_{1,n}$, $P_n \odot K_1$, the coconut tree $T(n,m)$ obtained by identifying the central vertex of $K_{1,m}$ with a pendent vertex of a path P_n, $B(m,n)$, caterpillars $S(n_1, n_2, \ldots, n_m)$, $P_m@P_n$ are $P_m@2P_n$ are even vertex odd difference mean and C_n is not a Skolem even vertex odd difference mean graph. In [1566] Kalaiyarasi, Ramya, and Jeyanthi prove the following graphs have Skolem odd difference mean labelings: graphs obtained from a T_p tree with m vertices and m copies of $K_{1,n}$ by identifying the central vertex of ith copy of $K_{1,n}$, with ith vertex of T; graphs obtained by connecting an isolated vertex to central vertex of each of a number of stars; the banana trees obtained by connecting an isolated vertex to one leaf of each of any number of $K_{1,n}$; graphs obtained from $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_m}$ by joining the central vertices of K_{1,n_j} and $K_{1,n_{j+1}}$ to a new vertex w_j for $1 \leqslant j \leqslant m-1$; graphs obtained from $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_m}$ by joining a leaf of K_{1,n_j} and a leaf of $K_{1,n_{j+1}}$ to a new vertex w_j for $1 \leqslant j \leqslant m-1$.

Lau, Jeyanthi, Ramya, and Kalaiyarasi [1805] say a (p,q)-graph $G(V,E)$ is a *Skolem even difference mean* if there exists an injective function $f : V(G) \to \{0, 1, 2, 3, \ldots, p + 3q - 1\}$ such that the induced map $f^* : E(G) \to \{2, 4, \ldots, 2q\}$ defined by $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$ is a bijection. A graph that admits a Skolem even difference mean labeling is called a *even difference mean*. They prove: the disjoint union of paths of length at least 2 and $K_{2,n} \cup (n-1)K_2$ $(n \geqslant 2)$ are Skolem even vertex odd difference mean graphs; if G is a Skolem even vertex odd difference mean $(q+1,q)$-graph, then $G \cup nK_2$, $G \cup P_n$, and $G \cup K_{1,n}$ are Skolem odd difference mean graphs; $C_m \cup P_n$ $(n \geqslant 2)$ is a Skolem odd difference mean graph for $m = 4$ and 6; the caterpillar $S(n_1, n_2, \ldots, n_m)$ is a Skolem even vertex even difference mean graph; $P_m@P_n$, mP_n, $K_{m,n} \cup (m-1)(n-1)K_2$ $(m, n \geqslant 2)$, $K_{1,n} \cup nK_2$, and $K_{1,1,n} \cup nK_2$ are Skolem even difference mean graphs; and if G is a Skolem even vertex even difference mean $(q+1,q)$-graph, then $G \cup nK_2$ is a Skolem even difference mean graph. They conclude with the following open problem: Establish that $G \cup nK_2$ where G is a (complete) multipartite graph is a Skolem even difference mean graph.

Kalaiyarasi, Ramya, and Jeyanthi [1451] say a graph $G(V,E)$ with p vertices and q edges has a *centered triangular mean* labeling if it is possible to label the vertices with distinct elements $f(x)$ from S, where S is a set of non-negative integers in such a way that for each edge $e = uv$, $f^*(e) = \lceil (f(u) + f(v))/2 \rceil$ and the resulting edge labels are the first q centered triangular numbers. A graph that admits a centered triangular mean labeling is called a *centered triangular mean* graph. They prove that P_n, $K_{1,n}$, bistars $B_{m,n}$, coconut trees, caterpillars $S(n_1, n_2,\ n_3, \ldots, n_m)$, $St(n_1, n_2,\ n_3, \ldots, n_m)$, banana trees $Bt(n, n, \ldots, n)$ and $P_m@P_n$ are centered triangular mean graphs.

Selvi, Ramya, and Jeyanthi [2746] define a *triangular difference mean* labeling of a graph $G(p,q)$ as an injection $f : V \longrightarrow Z^+$, such that when the edge labels are defined as $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$ the values of the edges are the first q triangular numbers.

A graph that admits a triangular difference mean labeling is called a *triangular difference mean* graph. They prove that the following are triangular difference mean graphs: P_n, $K_{1,n}$, $P_n \odot K_1$, bistars $B_{m,n}$, graphs obtained by joining the roots of different stars to the new vertex, trees $T(n,m)$ obtained by identifying a central vertex of a star with a pendent vertex of a path, the caterpillar $S(n_1, n_2, \ldots, n_m)$ and the graph $C_n@P_m$.

A graph $G(V,E)$ with p vertices and q edges is said to have *centered triangular difference mean* labeling if there is an injective mapping f from V to Z^+ such that the edge labels induced by $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$ are the first q centered triangular numbers. A graph that admits a centered triangular difference mean labeling is called a *centered triangular difference mean* graph. Ramya, Selvi, and Jeyanthi [1530] prove that P_n, $K_{1,n}$, $C_n \odot K_1$, bistars $B_{m,n}$, C_n $(n > 4)$, coconut trees, caterpillars $S(n_1, n_2, n_3, \ldots, n_m)$, $C_n@P_m$ $(n > 4)$ and $S_{m,n}$ are centered triangular difference mean graphs.

Gayathri and Tamilselvi [1063] say a (p,q)-graph G has a (k,d)-*super mean* labeling if there exists an injection f from the vertices of G to $\{k, k+d, \ldots, k+(p+q)d\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = \lceil (f(u)+f(v))/2 \rceil$ has the property that the vertex labels and the edge labels together are the integers from k to $k+(p+q)d$. When $d = 1$ a (k,d)-super mean labeling is called k-*super mean*. For $n \geqslant 2$ they prove the following graphs are k-super mean for all k: odd cycles; P_n; $C_m \cup P_n$; the one-point union of a cycle and the endpoint of P_n; the union of any two cycles excluding C_4; and triangular snakes. For $n \geqslant 2$ they prove the following graphs are (k,d)-super mean for all k and d: P_n; odd cycles; combs $P_n \odot K_1$; and bistars. In [1506] Jeyanthi, Ramya, and Thangavelu proved the following graphs have k-super mean labelings: C_{2n}, $C_{2n+1} \times P_m$, grids $P_m \times P_n$ with one arbitrary crossing edge in every square, and antiprisms on $2n$ vertices $(n > 4)$. (Recall an antiprism on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ where subscripts are taken modulo n). Jeyanthi, Ramya, Thangavelu [1505] give k-super mean labelings for a variety of graphs. Jeyanthi, Ramya, Thangavelu, and Aditanar [1504] show how to construct k-super mean graphs from existing ones. For $n \geqslant 3$ Gayathri and Tamilselvi [1063] proved the following graphs are k-super edge mean for all k: paths; cycles; combs $P_n \odot K_1$; triangular snakes; crowns $C_n \odot K_1$; the one-point union of C_3 and an endpoint of P_n; and $P_n \odot K_2$.

In [2704] Sandhya, Somasundaram, and Ponraj call a graph with q edges a *harmonic mean* graph if there is an injective function f from the vertices of the graph to the integers from 1 to $q+1$ such that when each edge uv is labeled with $\lceil 2f(u)f(v)/(f(u)+f(v)) \rceil$ or $\lfloor 2f(u)f(v)/(f(u)+f(v)) \rfloor$ the edge labels are distinct. They prove the following graphs have such a labeling: paths, ladders, triangular snakes, quadrilateral snakes, $C_m \cup P_n$ $(n > 1)$; $C_m \cup C_n$; nK_3; $mK_3 \cup P_n$ $(n > 1)$; mC_4; $mC_4 \cup P_n$; $mK_3 \cup nC_4$; and $C_n \odot K_1$ (crowns). They also prove that wheels, prisms, and K_n $(n > 4)$ with an edge deleted are not harmonic mean graphs. In [2702] Sandhya, Somasundaram, and Ponraj investigated the harmonic mean labeling for a polygonal chain, square of the path and dragon and enumerate all harmonic mean graph of order at most 5. In [1370] Jayasekaran and David Raj prove that some disconnected graphs are harmonic mean graphs. In [2551] Raj, Jayasekaran, and Sandhya investigate some new families of harmonic mean graphs. Seoud and Salim [2793] provided upper bounds of the number of edges of graphs of given orders with harmonic mean labelings and showed that all graphs of order at most 9 have have harmonic mean labelings using the floor function portion of the definition. Meena and Sivasakth [2126] prove that subdivision graphs of $P_n \odot K_1, P_n \odot \overline{K_2}$, H-graphs, $C_n \odot K_1, C_n \odot \overline{K_2}$, quadrilateral snakes, and triangular snakes are harmonic mean.

Sandhya, Somasundaram, and Ponraj [2703] proved that the following graphs have harmonic mean labelings: graphs obtained by duplicating an arbitrary vertex or an arbitrary edge of a cycle; graphs obtained by joining two copies of a fixed cycle by an edge; the one-point union of two copies of a fixed cycle; and the graphs obtained by starting with a path and replacing every other edge by a triangle or replacing every other edge by a quadrilateral.

Vaidya and Barasara [3247] proved that the following graphs have harmonic mean labelings: graphs obtained by the duplication of an arbitrary vertex or arbitrary edge of a path or a cycle; the graphs obtained by the duplication of an arbitrary vertex of a path or cycle by a new edge; and the graphs obtained by the duplication of an arbitrary edge of a path or cycle by a new vertex.

In [2259] Narasimhan and Sampathkumar called a graph with p vertices a *contra harmonic mean* graph if there is an injective function f from the vertices of the graph to the integers from 1 to p such that when each edge uv is labeled with $f(uv) = \lceil (f(u))^2 + (f(v))^2/(f(u) + f(v)) \rceil$ or $f(uv) = \lfloor (f(u))^2 + (f(v))^2/(f(u) + f(v)) \rfloor$ the edge labels are distinct. They prove the following graphs have such a labeling: paths, cycles, $C_m \cup P_n, C_m \cup C_n, nK_3, nK_3 \cup P_m$, and $nK_3 \cup C_m$. Gopi [1134] called a graph with q edges a *k-contra harmonic mean* graph if there is an bijective function f from the edges of the graph to the integers from $k-1$ to $k+q+1$ such that each edge uv is labeled with $f(uv) = \lceil (f(u))^2 + (f(v))^2/(f(u)+f(v)) \rceil$ or $f(uv) = \lfloor (f(u))^2 + (f(v))^2/(f(u)+f(v)) \rfloor$. He proves that triangular snakes, double triangular snakes, quadrilateral snakes, and double quadrilateral snakes have k-contra harmonic mean labelings.

Gopi and Suba [1140] say a graph G with p verticies and q edges is a *super Lehmer-3 mean graph* if there is an injective function f from the vertices of G to $\{1, 2, \ldots, q+1\}$ such that for each edge uv the induced function $f^*(uv) = \lfloor (f(u)^3 + f(v)^3)/(f(u)^2 + f(v)^2) \rfloor$ or $f^*(uv) = \lceil (f(u)^3 + f(v)^3)/(f(u)^2 + f(v^2)) \rceil$ yields that the set of vertex labels and edge labels is $\{1, 2, \ldots, p\}$. They prove that $P_m \odot K_{1,n}$ and the graph obtained by identifying each endpoint of a path with an endpoint of the star $K_{1,n}$ have a super Lehmer-3 labeling. In [1139] Gopi and Nirmala provide Lehmer-3 mean labelings for $P_m \odot C_n$ $(m, n \geqslant 3)$ and $P_m \odot K_1 \odot C_n$ $(m, n \geqslant 3)$.

In [2859] Shalini and Meena introduced the notion of Lehmer-4 mean labelings as follows. A graph $G(V, E)$ with p vertices and q edges is called a *Lehmer*-4 *mean graph*, if there is an injection from V to $\{2, 4, 6, \ldots, 2p\}$ and the induced function $f^*(uv) = \lfloor (f(u)^4 + f(v)^4)/(f(u)^2 + f(v)^2) \rfloor$ or $f^*(uv) = \lceil (f(u)^4 + f(v)^4)/(f(u)^2 + f(v^2)) \rceil$ in an injection. In this case, f is called a *Lehmer*-4 *mean labeling* mean of G. They proved paths, combs, $P_n \odot K_{1,2}$ and $P_n \odot K_{1,3}$ are Lehmer-4 mean graphs. [2859] new

An *F-geometric mean* labeling of a graph G with q edges, is an injective function from the vertex set of G to $\{1, 2, \ldots, q+1\}$ such that the edge labels obtained from the floor function of geometric mean of the vertex labels of the end vertices of each edge, are all distinct and the set of edge labels is $\{1, 2, \ldots, q\}$. Durai Baskar, Arockiaraj, and Rajendran [859] proved that the following graphs are F-geometric mean: graphs obtained by identifying a vertex of consecutive cycles (not necessarily of the same length) in a particular way; graphs obtained by identifying an edge of consecutive cycles (not necessarily of the same length) in a particular way; graphs obtained by joining consecutive cycles (not necessarily of the same length) by paths (not necessarily of the same length) in a particular way; $C_n \odot K_1$; $P_n \odot K_1$; $L_n \odot K_1$; $G \odot K_1$ where G is the graph obtained by joining two copies of P_n by an edge in a particular way; graphs obtained by appending two edges at each vertex of graphs obtained by joining two copies of P_n by an edge in a particular way; graphs obtained from C_n by appending two edges at each vertex of C_n; graphs obtained from ladders by appending two edges at each vertex of the ladders; graphs obtained from P_n by appending an end point of the star S_2 to each vertex of P_n; and graphs obtained from P_n by appending an end point of the star S_3 to each vertex of P_n.

A C-*geometric mean* labeling of a graph G with q edges, is an injective function from the vertex set of G to $\{1, 2, 3, \ldots, q + 1\}$ such that the edge labels obtained from the ceiling function of the geometric mean of the vertex labels of the end vertices of each edge are all distinct and the set of edge labels is $\{2, 3, 4, \ldots, q + 1\}$. A graph is said to be a C-*geometric mean graph* if it admits a C-geometric mean labeling. In [861] Durai Baskar and Arockiaraj study the C-geometric meanness of some cycle related graphs such as cycle, union of a path and a cycle, unions of two cycles, the graphs $C_3 \times P_n$, corona of cycle, the graphs $P_{a,b}, P_b^a$ and some chain graphs.

A geometric mean labeling f of $G(V, E)$ is called a *super geometric mean* labeling if $f(V) \cup f(E) = \{1, 2, \ldots, |V| + |E|\}$. Sandhya, Merly, and Shiny [2697] [2698] prove that the subdivision graphs of the following graphs have super geometric mean labelings: alternate quadrilateral snakes, double quadrilateral snakes, alternate double quadrilateral snakes, triple quadrilateral snakes, and subdivisions of alternate triple quadrilateral snakes. In [2699] they prove that the following graphs have super geometric mean labelings: triangular ladders, triangular snakes, alternate triangular snakes, quadrilateral snakes, and alternate quadrilateral snakes. Hemalatha and Selvi [1241] prove that following graphs have super geometric mean labelings: flags, kayak paddles, dumbells, polygonal snakes, and graphs obtained by connecting any number of copies of C_n where each joined to the next with an edge.

Durai Basker and Arockiaraj [858] study the F-geometric meanness of cycles, stars, complete graphs, combs, ladders, triangular ladders, middle graphs of paths, graphs obtained from duplicating arbitrary vertex by a vertex as well as arbitrary edge by an edge in cycles, and subdivisions of combs and stars.

In 2017 Hemalatha, Mohanaselvi, and Amuthavalli [1243] defined a *radio geometric mean* labeling of a graph G as a mapping from the vertex set to the set of natural numbers such that for two distinct vertices u and v of G, $d(u, v) + \lceil f(u)f(v) \rceil \geqslant 1 + diam(G)$. The *radio geometric mean number* of f is the maximum number assigned to any vertex of G. The *radio geometric mean number* of G is the minimum value taken over all radio [1243] new geometric mean labeling f of G. A *radio antipodal geometric mean* labeling of a graph G is a mapping f from the vertex set of G to the set of natural numbers such that for two distinct vertices u and v of G, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geqslant diam(G)$. If $d(u,v) = diam(G)$, the vertices u and v can be given the same label and if $d(u,v) \neq diam(G)$, the vertices u and v are assigned different labels. The *radio antipodal geometric mean number* of f is the maximum number assigned to any vertex of G. The *radio antipodal geometric mean number* of G is the minimum value taken over all radio geometric mean labeling f of G. Hemalatha, Mohanaselvi, and Amuthavalli [1243] provided the radio geometric mean numbers for stars, bistars, compelte k-ary trees with 3 levels, and the graph lotus inside a circle. In [1242] Hemalatha and Amuthavalli determined the radio geometric mean number of splitting graphs of stars and bistars. Hemalatha and Mohanaselvi [1244] found the radio geometric mean numbers of the subdivision graphs of complete graphs, complete bipartite graphs, books, and windmills. In [1102] Giridaran, Jose, and Jeony provided upper bounds for the radio antipodal geometric mean number of ladders, triangular ladders, circular ladders ($C_n \times K_2$), and pagoda graphs (graphs obtained ny joining two adjacent endpoints of a ladder to new vertex.

[1243] new
[1242] new
[1244] new
[1102] new

Arockiaraj and Meena Kumari introduced the F-face magic mean labeling of graphs in [859]. This motivated Meena Kumari and Arockiaraj [2121] to introduce the (1,0,0)-F-face magic mean labeling of graphs as follows. A bijection ϕ from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ is called a (1,0,0)-F-*face magic mean* labeling of G if the induced face labeling $\phi^*(f_i) = \lfloor$(sum of the labels of the vertices in the boundary of $f_i)/\deg(f_i)\rfloor$ is a constant for each face f_i, including the exterior face of G, where deg f_i is the number of edges that bound the face. A graph that admits an (1,0,0)-F-face magic mean labeling is called (1,0,0)-F-*face magic mean*. In [2121] Arockiaraj and Meena Kumar showed that $P_n + K_1$ ($n \geqslant 2$), cycles with certain cords, $C_m \times P_n$ where m and n are even, and graphs obtained by duplicating every edge of a cycle by a vertex admit (1,0,0)-F-face magic mean labelings.

In [3139] Sundaram, Ponraj, and Somasundaram introduced a new labeling parameter called the *mean number* of a graph. Let f be a function from the vertices of a graph to the set $\{0, 1, 2, \ldots, n\}$ such that the label of any edge uv is $(f(u)+f(v))/2$ if $f(u)+f(v)$ is even and $(f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd. The smallest integer n for which the edge labels are distinct is called the *mean number* of a graph G and is denoted by $m(G)$. They proved that for a graph G with p vertices $m(tK_{1,n}) \leqslant t(n+1)+n-4$; $m(G) \leqslant 2^{p-1}-1$; $m(K_{1,n}) = 2n-3$ if $n > 3$; $m(B(p,n)) = 2p-1$ if $p > n+2$ where $B(p,n)$ is a bistar; $m(kT) = kp-1$ for a mean tree T, $m(W_n) \leqslant 3n-1$, and $m(C_3^{(t)}) \leqslant 4t-1$.

Let f be a function from $V(G)$ to $\{0, 1, 2\}$. For each edge uv of G, assign the label $\lceil (f(u)+f(v))/2 \rceil$. Ponraj, Sivakumar, and Sundaram [2473] say that f is a *mean cordial labeling* of G if $|v_{f(i)} - v_{f(j)}| \leqslant 1$ for i and j in $\{0, 1, 2\}$ where $v_{f(x)}$ and $e_{f(x)}$ denote the number of vertices and edges labeled with x, respectively. A graph with a mean cordial labeling is called a *mean cordial graph*. Observe that if the range set of f is restricted to $\{0, 1\}$, a mean cordial labeling coincides with that of a product cordial labeling. Ponraj, Sivakumar, and Sundaram [2473] prove the following: every graph is a subgraph of a connected mean cordial graph; $K_{1,n}$ is mean cordial if and only $n \leqslant 2$; C_n is mean cordial if and only $n \equiv 1, 2 \pmod 3$; K_n is mean cordial if and only $n \leqslant 2$; W_n is not mean cordial for all $n \geqslant 3$; the subdivision graph of $K_{1,n}$ is mean cordial; the comb $P_n \odot K_1$ is mean cordial; $P_n \odot 2K_1$ is mean cordial; and $K_{2,n}$ is a mean cordial if and only $n \leqslant 2$. Seoud and Salim [2793] provided upper bounds of the number of edges of graphs of given orders with mean cordial labelings and proved that $P_{2t} \times P_2$ is mean cordial if and only if $t \equiv 2 \bmod 3$ and $C_n \odot K_1$ is mean cordial if and only if $n \equiv 1$ or $2 \bmod 3$.

In [2465] Ponraj and Sivakumar proved the following graphs are mean cordial: mG where $m \equiv 0 \pmod 3$; $C_m \cup P_n$; $P_m \cup P_n$; $K_{1,n} \cup P_m$; $S(P_n \odot K_1)$; $S(P_n \odot 2K_1)$; P_n^2 if and only if $n \equiv 1 \pmod 3$ and $n \geqslant 7$; and the triangular snake T_n $(n > 1)$ if and only if $n \equiv 0 \pmod 3$. They also proved that if G is mean cordial then mG, $m \equiv 1 \pmod 3$ is mean cordial. Deshmukh and Shaikh [807] prove the graph $\langle K_{1,n} : 2 \rangle$ and the path union of n copies of $K_{1,m}$ are mean cordial graphs.

In [2436] Ponraj and Sathish Narayanan proved double triangular snakes $D(T_n)$ are mean cordial if and only if $n > 3$ and obtained partial results on mean cordial labelings of alternate triangular snakes, double alternate triangular snakes. Shendra shainy, Vinothkumar, and Balaji [2881] proved that for $\beta_1 < \beta_2 < \beta_3$, the three star graph $K_{1,\beta_1} \wedge K_{1,\beta_2} \wedge K_{1,\beta_3}$ is a mean cordial graph if and only if $|\beta_2 - \beta_3| \leqslant 3\beta_1 + 7$. Further, in [2063] Maheshwari, Vinothkumar and Balaji proved that for $\beta_1 = \beta_2 < \beta_3$ the three star graph $K_{1,\beta_1} \wedge K_{1,\beta_2} \wedge K_{1,\beta_3}$ is a mean cordial graph if and only if $|\beta_2 - \beta_3| \leqslant 3\beta_1 + 6$. [2881] new [2063] new

In [2453] Ponraj, Sathish Narayanan, and Ramasamy introduced the notion of total mean cordial labeling. A *total mean cordial* labeling of a graph $G(V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that when each edge xy is assigned the label $\lceil (f(x) + f(y))/2 \rceil$ we have $|ev_f(i) - ev_f(j)| \leqslant 1$, $i, j \in \{0, 1, 2\}$, where $ev_f(x)$ denotes the total number of vertices and edges labeled with x. A graph with a total mean cordial labeling is called *total mean cordial*. In [2453], [2454], and [2455], Ponraj, Sathish Narayanan, and Ramasamy determined the total mean cordiality of the following graphs: P_n; C_n; $K_{1,n}$; W_n; $K_2 + mK_1$; combs $P_n \odot K_1$; double combs $P_n \odot 2K_1$; crowns; flowers; lotuses inside a circle; bistars; quadrilateral snakes; $K_{2,n}$; olive trees; $S(P_n \odot K_1)$; $S(K_{1,n})$ ($S(G)$ denotes the subdivision of G); triangular snakes; P_n^2; fans F_n; umbrellas; butterflies; and dumbbells. In [2435], [2437], and [2438], Ponraj and Sathish Narayanan determined the total mean cordiality of $K_n^c + 2K_2$; prisms; gears; helms; $P_1 \cup P_2 \cup \cdots \cup P_n$; $L_n \odot K_1$; $S(W_n)$; $S(P_n \odot 2K_1)$; and graphs obtained by subdividing each step of a ladder exactly once.

Let G be a (p, q)-graph. Ponraj and Sathish Narayanan [2440] and [2441] proved the following. If G satisfies any one of the following three conditions then $G \odot 2K$ is total mean cordial: (1) G is a tree, (2) G is a unicycle, (3) $q = p + 1$. If G satisfies any one of the following three conditions then the shadow graph of G is total mean cordial: (1) G is a tree, (2) G is a unicycle, (3) $q = p + 1$. They also proved that the following are total mean cordial graphs: $C_n \odot K_2$, $C_n^{(2)}$, dragons, splitting graphs of stars, splitting graphs of combs, books, ladders, $P_n \odot K_2$ if and only if $n \neq 1$, and $G \cup P_n$ $(n \neq 3)$.

In [2002] Lourdusamy and Joy Beaula define the notion of S_3 (all permutations of $\{1, 2, 3\}$) mean cordial graphs as follows. Let $G(V, E)$ be a graph and $g : V(G) \to S_3$ be a function. For each edge xy assign the label 1 if $\lceil (|g(x)| + |g(y)|)/2 \rceil$ is odd, and 0 otherwise. Such a g is said to be a *group S_3 mean cordial* labeling if the number of vertices labeled with i and j differ by at most 1 where i, j are the elements in S_3 and the number of edges labeled with 0 and 1 differ by at most 1. A graph with a group S_3 mean cordial labeling is called a *group S_3 mean cordial* graph. They prove that $K_{2,n}$, subdivisions of fans, double fans, wheels, and helms are S_3 mean cordial graphs.

Ponraj, Sathish Narayanan, and Kala introduced the concept of radio mean labeling in [2450]. A *radio mean* labeling of a connected graph G is a one-to-one map f from $V(G)$ to the set of natural numbers such that for each pair of distinct vertices u and v of G, $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geqslant 1 + \text{diam }(G)$. The *radio mean number* of f, $rmn(f)$, is the maximum number assigned to any vertex of G. The radio mean number of G, $rmn(G)$, is the minimum value of $rmn(f)$ taken over all radio mean labelings f of G. They proved $rmn(G) \geqslant |V(G)|$; if G is a (p,q)-graph with diameter $d \geqslant 2$, then $rmn(G) \leqslant p + d - 2$; and if G is a (p,q)-connected graph with diameter 2 or 3, then $rmn(G) = p$. They also determine the radio mean number of K_n, $K_{m,n}$, sunflowers, helms, gears, lotuses inside a circle, and graphs obtained by identifying any two vertices of two wheels of the same size, Aasi, Asif, Iqbal, and Ibrahim [1] determine the radio number and the radio mean number for the lexicographic product of a path with a path, a path with a cycle, and a cycle with a cycle. Moreover, they present computer programs for finding such radio labelings of these families of graphs.

In [2451] and [2452] Ponraj, Sathish Narayanan, and Kala determine the radio mean numbers of $S(K_{m,n})$ $(m > 1, n > 1)$; $K_{m,n} \odot P_t$; $C_6^{(t)}$; $W_n \odot P_m$; graphs obtained by joining the rim vertices of the two wheels with an edge; and graphs obtained from a wheel by subdividing each spoke by a vertex. In [2456] Ponraj, Sathish Narayanan, and Kala give the radio mean number of graphs with diameter three, lotuses inside a circle, helms, and sunflower graphs.

In [2457] and [2442] Ponraj and Sathish Narayanan give the radio mean number of the following graphs: subdivisions of stars, subdivisions of wheels, subdivisions of $K_2 + mK_1$, subdivisions of bistars, jelly fish, subdivisions of jelly fish, books with pentagonal pages, graphs obtained by taking m disjoint copies of $K_{1,n}$ and joining a new vertex to the centers of the m copies of $K_{1,n}$.

A *radio mean D-distance* labeling of a connected graph G is an injective map f from $V(G)$ to the natural numbers such that for two distinct vertices u and v of G, $d^D(u,v) + \lceil (f(u) + f(v))/2 \rceil \geqslant 1 + diam^D(G)$, where $d^D(u,v)$ denotes the distance D between u and v and $diam^D(G)$ denotes the D-diameter of G. The *radio mean D-distance number* of f, $rmn^D(f)$, is the maximum label assigned to any vertex of G. The *radio mean D-distance number* of G, $rmn^D(G)$, is the minimum value of $rmn^D(f)$ taken over all radio mean D-distance labeling f of G. Nicholas and Bosco [2272] determined the radio mean D-distance number of cycles, wheels, gears, helms, fans, and friendship graphs.

In [2439] Ponraj and Sathish Narayanan proved that the following graphs are not mean cordial: $K_2 + \overline{K}_m$; $\overline{K}_n + 2K_2$; $P_n \times K_2$; flower graphs; sunflower graphs; $C_n \odot K_2$. Also they proved the following: the Mongolian tent $MT_{m,n}$ is mean cordial if and only if $m \equiv 0 \pmod 3$ or $n \equiv 0 \pmod 3$ ($MT_{m,n}$ is the graph obtained from $P_m \times P_n$, n odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex); the book B_m is mean cordial if and only if $m = 1$; books with n pentagonal pages are mean cordial if and only if $n \equiv 1 \pmod 3$; $P_n \odot K_2$ is mean cordial if and only if $n \equiv 0 \pmod 3$; quadrilateral snakes are mean cordial; alternate quadrilateral snakes $A(Q_n)$ are mean cordial if and only if the square starts from second vertex of the path P_n, ends with $(n-1)^{th}$ vertex and $n \equiv 0, 2 \pmod 3$, or the square starts from first vertex, ends with n^{th} vertex and $n \equiv 0, 2 \pmod 3$, or the square starts from second vertex, ends with n^{th} vertex and $n \equiv 0, 1 \pmod 3$.

Let G be a graph and let $f : V(G) \to \{0, 1, \ldots, k-1\}$ $k > 1$. For each edge uv, assign the label $f(uv) = \lceil \frac{f(u)+f(v)}{2} \rceil$. Ponraj, Subbulakshmi, and Somasundaram [2476] say that f is a *k-total mean cordial* labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leqslant 1$, for all $i, j \in \{0, 1, \ldots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labeled with $x \in \{0, 1, \ldots, k-1\}$. A graph with admit a k-total mean cordial labeling is called *k-total mean cordial graph.* Ponraj, Subbulakshmi, and Somasundaram [2476] investigated the 4-total mean cordial labeling behavior of fans, wheels, jelly fish, jewel graphs, ladders, and triangular snakes.

A *balanced mean cordial* labeling is a mean cordial labeling f with the property that the number of verticies labeled with i is the same as the number labeled with j for $i, j \in \{0, 1, 2\}$ and likewise for the edgess. Kaneria, Khoda, and Karavadiya [1591] prove: the path union of n copies of a graph G is a mean cordial when $n \equiv 0 \pmod 3$; if G is balanced mean cordial, then $P_n \times G$ and $C_n \times G$ are balanced mean cordial; and if $f : V(G) \longrightarrow \{0, 1, 2\}$ is a balanced mean cordial labeling for G, then G^* is also a balanced mean cordial graph.

In [1459] Jeyanthi and Maheswari define a *one modulo three mean* labeling of a graph G with q edges as an injective function ϕ from the vertices of G to $\{a \mid 0 \leqslant a \leqslant 3q-2$ where $a \equiv 0 \pmod 3$ or $a \equiv 1 \pmod 3\}$ and ϕ induces a bijection ϕ^* from the edges of G to $\{a \mid 1 \leqslant a \leqslant 3q-2$ where $a \equiv 1 \pmod 3\}$ given by $\phi^*(uv) = \lceil (\phi(u) + \phi(v))/2 \rceil$. They proved that P_{2n}, combs, bistars $B_{n,n}$, T_p-trees with an even number of vertices, C_{4n+1}, ladders, $K_{1,2n} \times P_2$ are one modulo three mean graphs. They also proved that bistars $B_{m,n}$ $(m \neq n), K_{1,n}$ $(n > 3)$, and K_n, $(n > 3)$ are not one modulo three mean graphs. In [1466] Jeyanthi, Maheswari, and Pandiaraj [1466] proved that $DA(Q_n), DA(Q_2) \odot nK_1, DA(Q_m) \odot nK_1, DA(T_2) \odot nK_1, DA(T_m) \odot nK_1, \overline{S}(DA(T_n)), \overline{S}(DA(Q_n))$, and mP_n are one modulo three mean graphs.

Jeyanthi, Maheswari, and Pandiaraj [1465] prove that following graphs have one modulo three mean labelings: books $K_{1,2n} \times P_2$; splitting graphs $S'(P_{2n})$; vertex duplication graphs $D(G, v')$; edge duplication graphs $D(G, e')$; nth alternate quadrilateral snake graphs $NA(Q_m)$; graphs obtained by joining the endpoints of paths P_{4m} to n isolated verticies; and extended jewel graphs EJ_n with vertex set $\{u, v, x, y, w, z, u_i : 1 \leqslant i \leqslant n\}$ and edge set $\{uv, ux, xy, yz, vw, wz, vu_i, zu_i : 1 \leqslant i \leqslant n\}$. For graphs G_1 and G_2, $G_1 \widehat{\circ} G_2$ is the graph obtained from G_1 and $|V(G_1)|$ copies of G_2 by joining a vertex of ith copy of G_2 with the ith vertex of G_1 by an edge. Jeyanthi, Maheswari, and Pandiaraj [1468] prove that the graphs $T \odot \overline{K_n}$, $T \,\widehat{\circ}\, K_{1,n}$, $T \,\widehat{\circ}\, P_n$, and $T \,\widehat{\circ}\, 2P_n$ are one modulo three mean graphs. Sudarvizhi and Balasangu [3068] provide one modulo three mean labelings for triangular books, duplication subdivisions of the central edge of bistars, and slanting ladders (graphs obtained from two paths $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ by joining each u_i with v_{i+1} for $1 \leqslant i \leqslant n_1$).

In [1053] and [1059] Gayathri and Prakash proved the following graphs admit one modulo three mean graphs: the mirror graph of P_n, Spl(P_n), $P_m \times P_n, (P_m \times P_2) \odot K_{1,n}$, bistars, $C_n@P_m, C_n \odot P_m$, and $H \odot \overline{K_m}$, where H is the graph is obtained from two copies of P_n with verticies $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by joining $u_{(n+1)/2}$ to $v_{(n+1)/2}$ with an edge when $n > 1$ is odd, and $u_{n/2}$ to $v_{n/2+1}$ with an edge when n is even. In [1056] Gayathri and Prakash investigated the existence of one modulo three mean labelings of disconnected graphs. In [1054] they provided a necessary condition for a graph to admit a one modulo three mean labeling. In [1055] Gayathri and Prakash provided one modulo three mean labelings for various trees. In [1057] and [1058] they proved the following graphs admit are k-one modulo three mean labelings: $P_n, K_{1,n}$, combs, $P_m \odot K_{1,n}, B_{m,n}, L_n = P_n \times P_2, K_{1,n} \times P_2$, $A(Q_n), D(Q_n), A(D(Q_n)), A(D(Q_n)) \odot \overline{K_m}, A(Q_n) \odot \overline{K_m}$, and the graphs obtained by joining the end points of P_m to n isolated points. [1053] new [1059] new [1056] new [1054] new [1055] new [1057] new [1058] new

A graph G is said to be *one modulo three root square mean* graph if there is an injective function ϕ from the vertex set of G to the set $\{0, 1, 3, \ldots, 3q-2, 3q\}$ where q is the number of edges of G and ϕ induces a bijection ϕ^* from the edge set of G to $\{1, 4, \ldots, 3q-2\}$ given by $\phi^*(uv) = \left\lceil \sqrt{\frac{[\phi(u)]^2+[\phi(v)]^2}{2}} \right\rceil$ or $\left\lfloor \sqrt{\frac{[\phi(u)]^2+[\phi(v)]^2}{2}} \right\rfloor$. The function ϕ is called a *one modulo three root square mean* labeling of G. In [1372] Jayasekaran and Jaslin Melbha investigated some path related graphs that have one modulo three root square mean labelings.

Somasundaram, Vidhyarani, and Ponraj [3027] introduced the concept of a *geometric mean* labeling of a graph G with p vertices and q edges as an injective function $f : V(G) \to \{1, 2, \ldots, q+1\}$ such that the induced edge labeling $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined as $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$ or $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ is bijective. Among their results are: paths, cycles, combs, ladders are geometric mean graphs and K_n $(n > 4)$ and $K_{1,n}$ $(n > 5)$ are not geometric mean graphs. Somasundaram, Vidhyarani, and Sandhya [3028] proved $C_m \cup P_n$, $C_m \cup C_n$, nK_3, $nK_3 \cup P_n$, $nK_3 \cup C_m$, P_n^2, and crowns are geometric mean graphs. Vaidya and Barasara [3250] investigated geometric mean labelings in context of duplication of graph elements in cycle C_n and path P_n. Durai Baskar, Arockiyaraj, and Rajendran investigate the geometric meanness of some graphs obtained from paths.

In Jeyanthi, Maheswari, and Pandiaraj [1467] define a graph G to be a one modulo three geometric mean graph if there is an injective function ϕ from the vertex set of G to the set $\{a|1 \leqslant a \leqslant 3q-2$ and either $a \equiv 0 \pmod 3$ or $a \equiv 1 \pmod 3\}$ where q is the number of edges of G and ϕ induces a bijection ϕ^* from the edge set of G to $\{a|1 \leqslant a \leqslant 3q-2$ and $a \equiv 1 \pmod 3\}$ given by $\phi^*(uv) = \left\lceil \sqrt{\phi(u)\phi(v)} \right\rceil$ or $\left\lfloor \sqrt{\phi(u)\phi(v)} \right\rfloor$ the function ϕ is called one modulo three geometric mean labeling of G. They proved paths, cycles with length at least 5, ladders, $P_n \odot K_1, P_n \odot P_2, P_n \odot \overline{P_2}$, subdivision graphs $S(P_n \odot K_1)$, and subdivision graphs $S(P_n \odot K_2)$ are one modulo three geometric graphs. They also prove that $K_{1,n}$ $(n \geqslant 3)$ and graphs in which every edge lies on a triangle are not one modulo three geometric mean graph.

In [1533] Jeyanthi, Selvi and Ramya introduced a new graph labeling as follows. A [1533] new graph $G(p,q)$ is said to be a *one modulo* N-*difference mean* graph if there is an injection f from the vertex set of G to the set $\{0 \leqslant a \leqslant 2(q-1)N+1\}$ and either $a = 0$ (mod N) or $a = 1$ (mod N), where N is a positive integer and f induces a bijection f^* from the edge set of G to $\{1 \leqslant a \leqslant (q-1)N+1\}$ and $a = 1$ (mod N) given by $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$. Such a function f is called a *one modulo* N-*difference mean labeling* of G. They establish that $B_{m,n}$, $S_{m,n}$, $P_n@P_m$, $B(l,m,n)$, $T(n,m)$, shrubs, caterpillars, and $K_{1,n}$ are one modulo N-difference mean graph. In addition, they show that C_3 is not a one modulo N-difference mean graph.

Jeyanthi, Selvi, and Ramya [1528] define a *restricted triangular difference mean* labeling of a graph G with p vertices and q edges as an injection $f : V \to \{1, 2, 3, \ldots, pq\}$ such that for each edge uv, the edge labels defined by $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$ are the first q triangular numbers. A graph that admits a restricted triangular difference mean labeling is called a *restricted triangular difference mean graph*. Jeyanthi, Selvi, and Ramya [1528] investigate the restricted triangular difference mean behaviors of the paths, combs, K_n, bistars $B_{m,n}$, caterpillars $S(n_1, n_2, \ldots, n_m)$, $K_{m,n}$, wheels, and graphs obtained by joining the centers of different stars to the new vertex. They also give a necessary condition for a graph to be a restricted triangular difference mean graph.

A (p,q) graph $G(V,E)$ is said to be an *analytic mean* graph if it is possible to injectively label the vertices with $\{0, 1, 2, \ldots, p-1\}$ in such a way that when each edge uv is labeled with $|(f(v)^2 - (f(u))^2|/2$ when $|(f(v)^2 - (f(u))^2|$ is even and $|(f(v)^2 - (f(u))^2 + 1|/2$ when $|(f(v)^2 - (f(u))^2| + 1$ is odd and the edge labels are distinct. In this case, f is called an *analytic mean* labeling of G. Raj and Vivek [2554] proved that $P_m \cup C_n \cup K_{1,s}$ $(m, s \geqslant 2,\ n \geqslant 3)$, $(P_m + K_1) \cup K_{1,n}$ $(m, n \geqslant 2)$, graphs obtained by identifying the apex vertices of $K_{1,m}$ and $K_{1,n}$ and one vertex of two copies of C_s where $m, n \geqslant 2,\ c \geqslant 3$ are analytic mean graphs.

Let $G = (V,E)$ be a graph with p vertices and q edges. A graph G is *analytic odd mean* if there exist an injective function $f : V \to \{0, 1, 3, 5 \ldots, 2q-1\}$ with an induce edge labeling $f^* : E \to Z$ such that for each edge uv with $f(u) < f(v)$, $f^*(uv) = \left\lceil \frac{f(v)^2 - (f(u)+1)^2}{2} \right\rceil$ if $f(u) \neq 0$, and $f^*(uv) = \left\lceil \frac{f(v)^2}{2} \right\rceil$ if $f(u) = 0$ is injective. In this case we say that f is an *analytic odd mean labeling* of G. Jeyanthi, Gomathi, and Lau [1449] proved that fans, double fans, double wheels, closed helms, total graphs of cycles, total graphs of paths, armed crowns $C_n \Theta P_m$, generalized Petersen graphs $GP(n,2)$ are analytic odd mean graphs. In [1437] they prove that P_n, C_n, $P_n \odot K_1$, bistars, fans, $C_n \odot K_1$, $L_n \odot K_1$, $C_m \cup S_m$, two copies of C_n sharing a common edge, and $C_m \cup C_n$ are analytic odd mean graphs. In [1436] they prove that wheels, flower graphs, some splitting graphs, and multiples of graphs are analytic odd mean graphs. In [1438] they prove that quadrilateral snakes, double quadrilateral snakes, coconut trees, fire cracker graphs, some star graphs, splitting graphs, complete bipartite graphs, unicyclic graphs, and the graphs obtained from a path of vertices $v_1, v_2, v_3, \ldots, v_n$ by joining i pendent vertices at each of ith vertex $1 \leqslant i \leqslant n$ (denoted $P_n(1, 2, \ldots, n)$) are analytic odd mean graphs. Jeyanthi, Gomathi, and Lau [1450] proved the square graphs of $P_n, C_n, B_{n,n}$ H-graphs, and $H \odot mK_1$ admit analytic odd mean labelings.

In [1534] Jeyanthi, Selvi, and Rama proved that the following graphs admit ana- [1534] new lytic odd mean labeling: arbitrary union of paths, $M_2(P_n)(n \geqslant 2)$, ladders, slanting slanting ladders, diamond snakes, quadrilateral snakes, alternately quadrilateral snakes, $Jl_n(P_3)(n \geqslant 1)$, $C_4 \odot K_{1,n}(n \geqslant 1)$, $DUP_2(K_{1,n})$, $DUP_2(B_{n,n})$, friendship graphs, and $nC_4(n \geqslant 1)$. Jeyanthi, Gomathi, Lau, and Shiu [1440] proved that the following graphs admit analytic odd mean labeling: $P_n + C_m$, tadpoles $T(m,n)$, $C_{m_1} + C_{m_2}$, $JC(m_1, m_2)$, edge linked cycle snakes $EL(kC_n)$, triangular snakes T_n, double triangular snakes $D(T_n)$, $P_n(C_m)$, nC_m, $P_n(T(m,1))$, shell graphs Sh_n. They propose the conjecture that a (p,q)-graph is analytic odd mean when $p \leqslant q+1$. [1440] new

In [1434] Jeyanthi and Gomathi proved that the subdivision and super subdivision of the following graphs are analytic odd mean: cycles, stars, combs, and graphs obtained from $P_n \odot K_1$ by subdivding the each edge of P_n. Jeyanthi, Gomathi, and Lau [1438] proved that quadrilateral snakes, double quadrilateral snakes, coconut trees, fire cracker graphs, and $P_n(1,2,\ldots,n)$ are analytic odd mean graphs. They also proved that $C_n \odot K_1$, prisms, helms, banana trees, perfect binary trees, unicyclic graphs, certain caterpillars, and spiders are analytic odd mean graphs. Jeyanthi and Gomathi [1435] proved that the graphs TL_n, (the triangular ladder obtained from L_n), $TL_n \odot K_1$, $T_n \odot K_1$, and $Q_n \odot K_1$ admit an analytic odd mean labelings. Simaringa and Thirunavukkarasu [2963] proved that triangular books, double triangular books, triangular snakes, double triangular snakes, butterflies, drums, $C_c \bigodot P_c$, $F_c \bigodot K_{1,d}$, $1 \leqslant d \leqslant 2c-1$, $W_c \odot K_{1,d}$, $1 \leqslant d \leqslant 2c$, $P_c^2 \odot K_{1,d}$, $c \geqslant 3$ and $1 \leqslant d \leqslant 2c-3$ are analytic odd mean graphs.

Let G be a (p,q) graph and f a injective function from $V(G)$ to $\{k, k+1, \ldots, p+q+k-1\}$ For each edge uv, let $f^* = \lceil (2f(u)f(v)/(f(u)+f(v)\rceil$ or $\lfloor (2f(u)f(v)/(f(u)+f(v)\rfloor$. We say f is a *k-super harmonic mean* if $f(V) \cup \{f^*(uv) \mid uv \in E(G)\} = \{k, k+1, \ldots, p+q+k-1\}$. A graph that admits a k-super harmonic mean labeling is called a *k-super harmonic mean* graph. In the case that $k=1$ the labeling is called a *super harmonic mean* labeling. For all $n>1$ Tamilselvi and Revathi [3187] prove that the following graphs have k-super harmonic mean labelings: P_n, nP_m $(m>1)$, $P_n \odot K_1, P_n \odot \overline{K_2}, P_n \odot \overline{K_3}, P_n^2$ $(n \geqslant 4)$, the subdivision graph of $P_n \odot K_1$, and the middle graph of P_n.

A graph $G = (V,E)$ with p vertices and q edges is said to be a (k,d)-*Heronian mean graph* if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $k, k+d, k+2d, \ldots, k+qd$ in such a way that when each edge uv is labeled with $f^*(uv) = \lfloor (f(u)+f(v)+\sqrt{f(u)f(v)})/3 \rfloor$ or $\lceil (f(u)+f(v)+\sqrt{f(u)f(v)})/3 \rceil$, then the resulting edge labels are distinct. In this case f is called a (k,d)-*Heronian mean labeling* of G. In the case $k=1$ and $d=1$, the labeling is called *Heronian mean* labeling . Akilandeswari and Tamilselvi [134] proved that paths, ladders, and $P_n \odot mK_1$ for $n \geqslant 2$, $1 \leqslant m \leqslant 4$, are k-Heronian mean graphs. In [134] Akilandeswari and Tamilselvi proved that the following graphs have (k,d)-Heronian mean labelings: paths, $(P_n \times P_2) \odot K_1$, $T_n \odot K_1$ (T_n is the triangular snake obtained from P_n), $Q_n \odot K_1$, $TL_n \odot K_1$ (TL_n is the triangular ladder obtained from L_n), Peterson graphs, and the graphs obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $u_{(n+1)/2}$ and $v_{(n+1)/2}$ if n is odd and $u_{n/2+1}$ and $v_{n/2+1}$ if n is even. Sampath, Narasimhan, and Nagaraja [2691] proved cycles, $K_{1,n}$ if and only if $n \leqslant 4$, $C_m \cup P_n$, $C_m \cup C_n$, nK_3, $nK_3 \cup P_m$, $nK_3 \cup C_m$, mC_4, crowns $C_n \odot K_1$, dragons $C_n@P_m$, and P_n^2 admit $(1,1)$-Heronian mean labelings. Anitha, Selvam, and Thirusangu [215] provide Heronian mean labelings for the extended duplicate graph of the kite graph.

A *pronic number* is one of the form $n(n+1)$, where n is a positive integer. In [2480] [2480] new Porchelvi and Devi defined a *pronic Heron mean* labeling of a graph G with n vertices as a bijection $f: V(G) \to \{0, 2, 6, 12, \ldots, n(n+1)\}$ such that the resulting edge labels obtained by $f^*(uv) = \lceil (f(u)+f(v)+\sqrt{f(u)f(v)}/3 \rceil$ for every edge uv are distinct. In [812] [812] new Devi and Porchevi gave pronic Heron mean labelings for the generalized Peterson graphs $P(6,2)$, $P(8,3)$,$P(10,2)$, $P(10,3)$, and $P(12,5)$. They also investigated the existence of pronic Heron labelings for mK_3, the unions of cycles and paths, and the union of combs and paths. They described an algorithm to label the vertices for the pronic Heron mean labeling for certain disconnected graphs. They further investigated the existence of proniceron labelings for mK_3, the unions of cycles and paths, and the union of combs and paths.

Arockiaraj and Meena [234] say a planar graph has an F-*face magic mean* labeling it there exists an assignment of labels to the edges that induces an assignment of labels to the faces of the graph such that the mean weight of each face is constant. They proved that the following graphs have F-face magic mean labelings: $P_{2n} + K_1$, the one-point union of m copies of C_n, mC_n-snakes, and graphs obtained identifying the endpoints of any number of copies P_n. In [235] and [2122] Arockiaraj and Meena Kumari investigated the existence [235] new of $(1,0,0)$ F-face magic mean labelings for paths, butterflies, and middle graphs of cycles, [2122] new $P_n + K_1$, $C_m \times P_n$, lattitude graphs, cyclic ladders, graphs obtained by duplicating every edge of P_n $(n \geqslant 2)$, even cycles, the middle graphs of cycles, and the middle graphs of butterflies. In [2123] they investigated $(0,1,0), (1,1,0)$-F-face magic mean labelings of [2123] new slanting ladders, $P_m \times P_n$, $K_{2,n}$, and certain graphs obtained by applying graph operations. Vani Shree and Dhanalakshmi [3337] investigated the existence of (1, 0, 0)-F-face magic [3337] new mean labelings of ladders, tortoises, and the middle graph of paths. They also provided (1,0,0)- and (1,1,0)-F-face magic mean labelings for ortho chain square cactus graphs, para chain square cactus graphs, triangular snakes, and quadrilateral snakes.

Amara Jothi, Baskar Babujee, and David [173] investigated face magic labeling of planar graphs of types $(1,0,1), (1,1,0)$, $(0,1,1)$ and $(1,\ 1,\ 1)$ on duplication graphs. For a graph $G(V,E)$, Rajesh Kannan, Manivannan, and Durai Basker [1643] defined an *exponential* labeling as injective function χ from V to $\{1, 2, 3, \ldots, |E|+1\}$ such that the induced function χ^* from E to $\{1, 2, \ldots, |E|\}$ given by $\chi(uv)^* = \lfloor (1/e)(\chi(v)^{\chi(v)}/\chi(u)^{\chi(u)})^{(1/\chi(v)-\chi(u))} \rfloor$ for all $uv \in E$ is a bijection. A graph that allows an exponential mean labeling is called an *exponential*. They investigated the exponential mean labeling of graphs obtained by duplication of an edge or vertex of paths and path related graphs.

In [509] Baskaran and Ganapathy introduced a new variant of mean labeling as follows. A function h is said to be a C-*exponential mean* labeling of a graph $G(V,E)$ if $h: V \to \{1, 2, \ldots, |E|+1\}$ is injective and the induced function $h^*: E \to \{2, 3, \ldots, |E|+1\}$ defined by $h^*(uv) = \lceil 1/((h(v)^{h(v)}/(h(u)^{h(u)}))^{1/(h(v)-h(u))} \rceil$ is a bijection. They investigate the exponential meanness of paths, triangular trees (see Section 3.1), $C_m \odot P_n$, $P_m \times P_n$, one-sided step graphs (the graph obtained by starting with the path P_n with cconsecutive edges $e_1, e_2, \ldots, e_{n-1}$ and erecting a ladder that has that has $n-i-1$ steps with e_i at the bottom) double-sided step graphs (the graph obtained by starting with the path P_{2n} with $2n-1$ edges $e_1, e_2, \ldots e_{2n-1}$, where on every edge e_i we erect a ladder that has $i+1$ steps for $i = 1, 2, \ldots n$, and on every edge and for $i = n+1, n+2, \ldots, 2n-1$ erect a ladder on edge e_i that has $2n+1-i$ steps), one-sided arrow graphs (the graphs obtained from joining the vertices of the bottom row of $P_m \times P_n$ to a new vertex), double-sided arrow graphs, graphs (the graphs obtained from joining the vertices of the bottom row of $P_m \times P_n$ to a new vertex and by joining the vertices of the right column of $P_m \times P_n$ to a new vertex), and subdivision of ladders.

A function f is said to be an *absolute mean graceful* of a graph G, if $f : V(G) \to \{0, \pm1, \pm2, \ldots, \pm|E|\}$ is injective and the edge labeling function $f^* : E(G) \to \{1, 2, \ldots, |E|\}$ defined as $f^*(uv) = \lceil (f(u) - f(v))/2 \rceil$ is bijective. A graph that admits such labeling is called *absolute mean graceful* graph. In [1583] and [1584] Kaneria, Chudasama, and Andharia proved that the path unions of a finite number of copies of trees, paths, cycles, complete bipertite graphs, grid graphs, step grid graphs, and double step grid graphs are absolute mean graceful graphs. In [132] Akbari1, Kaneria, and Parmar prove that jewel graphs, jelly fish graphs, and the extended jewel graphs admit absolute mean graceful labelings.

In [2325] Palani and Shunmugapriya defined the concept of near mean labeling in digraphs as follows. Let $D(V, A)$ be a digraph where V is the vertex is set and A is the arc set. Let $f : V \to \{0,\ 1,\ 2,\ \ldots, q\}$ be a 1-1 map. Define $f^* : A \to \{1,\ 2,\ \ldots, q\}$ by $f^*(e = \vec{uv}) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. Let $f^*(v) = \left| \sum_{w \in V} f^*(\vec{vw}) - \sum_{w \in V} f^*(\vec{wv}) \right|$. If $f^*(v) \leqslant 2$ for all $v \in A(D)$, f is said to be a *near mean* labeling of D and D is said to be a *near mean digraph*. They investigated the existence of near mean labeling in dragon digraphs (the one-point union of cycle and a path where the edges of a cycle are directed clockwise or anti-clockwise and the edges of the path edges are directed towards the cycle). In [2326] they investigated the existence of near mean labelings for various dicyclic snakes.

[2325] new

[2326] new

7.12 Pair Sum and Pair Mean Graphs

For a (p, q) graph G Ponraj and Parthipan [2422] define an injective map f from $V(G)$ to $\{\pm1, \pm2, \ldots, \pm p\}$ to be a *pair sum* labeling if the induced edge function f_{em} from $E(G)$ to the nonzero integers defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\} \cup \{k_{\frac{q+1}{2}}\}$, according as q is even or odd. A graph with a pair sum labeling is called *pair sum graph*. In [2422] and [2424] they proved the following are pair sum graphs: $P_n,\ C_n,\ K_n$ if and only if $n \leqslant 4,\ K_{1,n},\ K_{2,n}$, bistars $B_{m,n}$, combs $P_n \odot K_1,\ P_n \odot 2K_1$, and all trees of order up to 9. Also they proved that $K_{m,n}$ is not pair sum graph if $m, n \geqslant 8$ and enumerated all pair sum graphs of order at most 5.

In [2426], [2427], [2428], and [2429] Ponraj, Parthipan, and Kala proved the following are pair sum graphs: $K_{1,n} \cup K_{1,m}$, $C_n \cup C_n$, mK_n if $n \leqslant 4$, $(P_n \times K_1) \odot K_1$, $C_n \odot K_2$, dragons $D_{m,n}$ for n even, $\overline{K_n} + 2K_2$ for n even, $P_n \times P_n$ for n even, $C_n \times P_2$ for n even, $(P_n \times P_2) \odot K_1$, $C_n \odot K_2$ and the subdivision graphs of $P_n \times P_2, C_n \odot K_1, P_n \odot K_1$, triangular snakes, and quadrilateral snakes. Ponraj, Parthipan, and Kala [2423] prove the following graphs are pair sum graphs: the union of two stars, the union of a path and a star, ladders, $C_n \odot K_1$, $C_m \cup C_m, mK_n$ if and only if $n \leqslant 4$, quadrilateral snake $Q_n (n$ odd), and triangular snakes.

A (p, q)-graph G is said to be a *super pair sum* if there exists a bijection f from $V(G) \cup E(G)$ to $\{0, \pm 1, \pm 2, \ldots, \pm(\frac{p+q-1}{2})\}$ when $p+q$ is odd and from $V(G) \cup E(G)$ to $\{\pm 1, \pm 2, \ldots, \pm(\frac{p+q}{2})\}$ when $p+q$ is even such that $f(uv) = f(u) + f(v)$. A graph that admits a super pair sum labeling is called a *super pair sum graph*. Vasuki, Velmurugan, and Sugirtha [3367] prove that the graphs $H_n \odot mK_1$, (H_n is obtained from two copies of P_n ($n \geqslant 3$) with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining $v_{(n+1)/2}$ and $u_{(n+1)/2}$ if n is odd and $v_{n/2}$ and $u_{(n+2)/2}$ if n is even); $(P_{2n}; S_m)$, $S'(P_{2n})$, $< B_{m,n} : P_k >$ for $m \geqslant 1$, $n \geqslant 1$, $k \equiv 2 \pmod 4$, $< B(m) : P_k >$ for $m \geqslant 1$ $k \equiv 0, 2 \pmod 4$ and $2B_{m,n}$ ($m \geqslant 1, n \geqslant 1$) are super pair sum graphs.

Jeyanthi and Sarada Devi [1507] define an injective map f from $E(G)$ to $\{\pm 1, \pm 2, \ldots, \pm q\}$ as an *edge pair sum labeling* of a graph $G(p, q)$ if the induced function of f^* from $V(G)$ to $Z - \{0\}$ defined by $f^*(v) = \sum f(e)$ taken over all edges e incident to v is one-one and $f^*(V(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{p/2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{(p-1)/2}\} \cup \{k_{p/2}\}$ according as p is even or odd. A graph with an edge pair sum labeling is called an *edge pair sum graph*. They proved that P_n, C_n, triangular snakes, $P_m \cup K_{1,n}$, and $C_n \odot \overline{K_m}$ are edge pair sum graphs.

Jeyanthi, Sarada Devi, and Lau [1517] proved that the following graphs have edge pair sum labelings: triangular snakes T_n, $C_n \cup C_n$, $K_{1,n} \cup K_{1,m}$, and bistars $B_{m,n}$. They also proved that every graph is a subgraph of a connected edge pair sum graph. Jeyanthi and Sarada Devi [1508] showed that $P_{2n} \times P_2$ and the graphs $P_n(+)N_m$ obtained from a path P_n by joining its endpoints to m isolated vertices are edge pair sum graphs. Jeyanthi and Sarada Devi [1510] proved that the following graphs have edge pair sum labeling: shadow graphs $S_2(P_n)$, $S_2(K_{1,n})$, total graphs $T(C_{2n})$ and $T(P_n)$, the one-point union of any number of copies of C_n, the one-point union of C_m and C_n, P^2_{2n-1}, and full binary trees in which all leaves are at the same level and every parent has two children. Jeyanthi and Sarada Devi [1509] proved the spiders $SP(1^m, 2^t)$, $SP(1^m, 2^t, 3)$, $SP(1^m, 2^t, 4)$, and for t even $SP(1^m, 3^t, 3)$ are edge pair sum graphs. In [1508] Jeyanthi and Sarada Devi prove some cycle related graphs are edge pair sum graphs. In [1510] they prove that the one point union of cycles, perfect binary trees, shadow graphs, total graphs, and P^2_n admit edge pair sum graph. In [1516] Jeyanthi and Sarada provide edge pair sum labelings for jewel graphs, gears, triangular ladders, balanced lobsters, and double wheels $2C_n + K_1$.

The tree $WT(n)$ is obtained from $K_{1,n+2}$ with central vertex c_1 and end vertices x_i : $1 \leqslant i \leqslant n+2$ and another $K_{1,n+2}$ with central vertex c_2 and end vertices $y_j : 1 \leqslant j \leqslant n+2$ by identifying vertex x_{n+2} and y_{n+2} and denoting the identified vertices by w. A w-tree $WT(n : k)$ is obtained from k copies of $WT(n)$ by joining a new vertex a to vertex w of each copy of $WT(n)$. Jeyanthi, Sarada Devi, and Lau [1518] proved that the graphs $WT(n : k)$ trees have edge pair sum labelings (see also [1519]).

In [1512], [1518], [1511], [1515] Jeyanthi and Sarada Devi prove the following graphs are edge pair sum graphs: shell graphs; some butterfly graphs; jelly fish; Y-trees; theta graphs; wheels with subdivided spokes, $P_m + 2K_1$; $C_4 \times P_m$; $P_n \odot \overline{K}_m$; $(P_2 \times P_m) \odot \overline{K_n}$; $P_m \times C_3$; books; graphs obtained from the path P_n having an even fixed even number quadrilaterals on each edge of the path; K_2+mK_1; graphs obtained by identifying one end point from each of m copies of P_n; closed helms; graphs that are two copies of generalized Petersen graphs joined by a path P_n, $n \geqslant 5$; and graphs that two copies of fan $P_n \odot K_1$ joined by a path P_n, $n \geqslant 5$.

In [1513] Jeyanthi and Sarada Devi prove the following graphs admit edge pair sum labelings: $K_{2,n}$, double triangular snakes, wheels, flowers, $\langle C_m, K_{1,n}\rangle$ ($m \geqslant 4$, n odd) obtained from C_m and $K_{1,n}$ by identifying any vertex of C_m with the central vertex of $K_{1,n}$, and $\langle C_m * K_1\rangle$ ($m \geqslant 4$) the graphs obtained from C_m and $K_{1,n}$ by identifying any vertex of C_m with an endpoint vertex of $K_{1,n}$. In [1514] they prove that the subdivision of graph of bistars $B_{m,n}$, $P_n \odot K_1$, triangular snakes when the path has an odd number of verticies, double triangular snakes, double quadrilateral snakes, double alternative triangular snakes, and double alternative quadrilateral snakes are edge pair sum graph.

In [177] Amudha and Jayapriya introduced the notion of labeling pair sum modulo labelings as follows. An undirected simple graph G with p vertices and q edges is said to be a *pair sum modulo* graph if there is an injective function f from $V(G)$ to $\{\pm 1, \pm 2, \ldots, \pm p\}$ such that the induced edge labeling g from $E(G)$ to $\{0, 1, 2, \ldots, q-1\}$ defined by $g(uv) = (f(u) + f(v))(\bmod q)$ is injective. They proved the Petersen graphs, a certain family of coconut trees, $K_{2,n}$, and bistars have pair sum modulo labelings. Jayapriya [1368] proved spider graphs with at most five legs admit pair sum modulo labelings. Ponraj, Parthipan, and Kala [2423] prove the following graphs are pair sum graphs: the union of two stars, the union of a path and a star, ladders, $C_n \odot K_1$, $C_m \cup C_m, mK_n$ if and only if $n \leqslant 4$, quadrilateral snake $Q_n(n$ odd), and triangular snakes.

For a (p, q) graph G Ponraj and Parthipan [2425] define an injective map f from $V(G)$ to $\{\pm 1, \pm 2, \ldots, \pm p\}$ to be a *pair mean* labeling if the induced edge function f_{em} from $E(G)$ to the nonzero integers defined by $f_{em}(uv) = (f(u) + f(v))/2$ if $f(u) + f(v)$ is even and $f_{em}(uv) = (f(u) + f(v) + 1)/2$ if $f(u) + f(v)$ is odd is one-one and $f_{em}(E(G)) = \{\pm k_1, \pm k_2, \ldots, \pm k_{q/2}\}$ or
$f_{em}(E(G)) = \{\pm k_1, \pm k_2, \ldots, \pm k_{(q-1)/2}\} \cup \{k_{(q+1)/2}\}$, according as q is even or odd. A graph with a pair mean labeling is called a *pair mean* graph. They proved the following graphs have pair mean labelings: P_n, C_n if and only if $n \leqslant 3$, K_n if and only if $n \leqslant 2$, $K_{2,n}$, bistars $B_{m,n}$, $P_n \odot K_1$, $P_n \odot 2K_1$, and the subdivision graph of $K_{1,n}$. Also they found the relation between pair sum labelings and pair mean labelings.

The graph $G@P_n$ is obtained by identifying an end vertex of a path P_n with any vertex of G. A graph $G(V, E)$ with q edges is called a $(k + 1)$-*equitable mean* graph if there is a function f from V to $\{0, 1, 2, \ldots, k\}$ $(1 \leqslant k \leqslant q)$ such that the induced edge that labeling f^* from E to $\{0, 1, 2, \ldots, k\}$ given by $f^*(uv) - \lceil (f(u) + f(v))/2\rceil$ has the properties $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leqslant 1$ for $i, j = 0, 1, 2, \ldots, k$ where $v_f(x)$ and $e_{f^*}(x)$ are the number of vertices and edges of G respectively with the label x. In [1423] Jeyanthi proved the following: a connected graph with q edges is a $(q+1)$-equitable mean graph if and only if it is a mean graph; a graph is 2-equitable mean graph if and only if it is a product cordial graph; for every graph G, the graph $3mG$ is a 3-equitable mean graph; for every 3-equitable mean graph G, the graph $(3m+1)G$ is a 3-equitable mean graph; C_n is a 3-equitable mean graph if and only if $n \not\equiv 0 \pmod 3$; P_n is a 3-equitable mean graph for all $n \geqslant 2$; if G is a 3-equitable mean graph then $G@P_n$ is a 3-equitable mean graph for $n \equiv 1 \pmod 3$; the bistar $B(m,n)$ with $m \geqslant n$ is a 3-equitable mean graph if and only if $n \geqslant \lfloor q/3 \rfloor$; $K_{1,n}$ is a 3-equitable mean graph if and only if $n \leqslant 2$; and for any graph H and $3m$ copies $H_1, H_2, \ldots, H_{3m}$ of H, the graph obtained by identifying a vertex of H_i with a vertex of H_{i+1} for $1 \leqslant i \leqslant 3m-1$ is a 3-equitable mean graph.

In [1794] Lakshmi and Nagarajan introduced the notion of geometric mean cordial labeling of graphs as follows. Let $G = (V, E)$ be a graph and f be a mapping from $V(G)$ to $\{0, 1, 2\}$. The graph G is called *geometric mean cordial* if each edge uv can be assigned the label $\lceil \sqrt{f(u)f(v)} \rceil$ in such a way that and $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(i) - ef_(j)| \leqslant 1$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x and $x \in \{0, 1, 2\}$ They proved that P_n, C_n $(n \equiv 1, 2 \pmod 3)$ and $K_{1,n}$ are geometric mean cordial graphs and $K_n(n > 2)$, $K_{2,n}$ $(n > 2)$, $K_{n,n}$ $(n > 2)$ and wheels are not geometric mean cordial graphs. In [1611] Kaneria, Meera, and Maulik call these graphs *geometric mean 3-equitable*. They proved: K_{mn} $(m, n \geqslant 4)$ is not a geometric mean 3-equitable graph, caterpillars $S(x_1, x_2, \ldots, x_t)$ and $C_n \odot K_1$ $(t \geqslant 2)$ are geometric mean 3-equitable graphs, and $C_n \odot K_1$ is a geometric mean 3-equitable graph if and only if $n \equiv 1, 2 \pmod 3$.

7.13 Irregular Total Labelings

In 1988 Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba [682] defined an *irregular labeling* of a graph G with no isolated vertices as an assignment of positive integer weights to the edges of G in such a way that the sums of the weights of the edges at each vertex are distinct. The minimum of the largest weight of an edge over all irregular labelings is called the *irregularity strength* $s(G)$ of G. If no such weight exists, $s(G) = \infty$. Chartrand et al. gave a lower bound for $s(mK_n)$. Faudree and Lehel [918] obtained $s(G) \leqslant \lceil n/2 \rceil + 9$ for regular graphs. For every graph G of order $n \geqslant 4$ and with finite irregularity strength, Nierhoff [2276] proved $s(G) \leqslant n-1$. Using deterministic and mostly probabilistic techniques, Frieze, Gould, Karonski, and Pfender [961] obtained the following bounds: $s(G) \leqslant c_1 \cdot n/\delta)$ if $\Delta \leqslant n^{1/2}$; and $s(G) \leqslant c_2 \cdot (\log n)n/\delta$ if $\Delta > n^{1/2}$, where c_1 and c_2 are explicit constants. Bohman and Kravitz found an infinite sequence of trees with strength converging to $(11 - \sqrt{5})/8$. Faudree, Jacobson, and Lehel [920] gave an upper bound for $s(mK_n)$ when $n \geqslant 5$ and proved that for graphs G with $\delta(G) \geqslant n-2 \geqslant 1$, $s(G) \leqslant 3$. They also proved that if G has order n and $\delta(G) = n-t$ and $1 \leqslant t \leqslant \sqrt{n/18}$, $s(G) \leqslant 3$. Aigner and Triesch proved $s(G) \leqslant n+1$ for any graph G with $n \geqslant 4$ vertices for which $s(G)$ is finite. In [2526] Przybylo proved that $s(G) < 112n/\delta + 28$, where δ is the minimum degree of G and G has n vertices. The best bound of this form is currently due to Kalkowski, Karońki, and Pfender, who showed in [1567] that $s(G) \leqslant 6\lceil n/\delta \rceil < 6n/\delta + 6$. In Przybylo [2524] showed that for d-regular graphs $s(G) < 16n/d + 6$. In 1991 Cammack, Schelp, and Schrag [654] proved that the irregularity strength of a full d-ary tree $(d = 2, 3)$ is its number of pendent vertices and conjectures that the irregularity strength of a tree with no vertices of degree two is its number of pendent vertices. This conjecture was proved by Amar and Togni [172] in

1998. Muthu Guru Packiam, Manimaran, and Thuraiswamy [2232] prove the following: $s(C_n \odot mK_1) = mn, s(P_n \odot K_2) = n+1, s(C_n \odot K_2) = n+1, s(P_n \odot K_3) = n+1$, and $sC_n \odot K_3) = n+1$. In [1550] Jinnah and Kumar determined the irregularity strength of triangular snakes and double triangular snakes. Hasni, Tarawneh, Siddiqui, Raheem, and Asim [1195] determined the exact value of edge irregularity strength of disjoint union of cycles and generalized prisms. These results provide an upper bound Hashi, Tarawneh, and Husin [1194] determined the edge irregularity strength of $P_n \odot P_m$ for $n \geqslant 2$ and [1194] new $m = 3, 4$, and 5. Imran, Aslam, Zafar, and Nazeer [1317] determined the exact value [1317] new of the edge irregularity strength of caterpillars, n-star graphs, (n,t)-kite graphs, cycle chains, and friendship graphs. Tarawneh, Hasni, Ahmad, and Asim [3192] determined the exact value of the edge irregularity strength for some classes of plane graphs. In [109] Ahmad, Khan, Ahmad, Nadeem, and Siddiqui determined the exact value of the edge irregularity strength for categorical product of two paths.

For R be a commutative ring R with the set of all zero divisor set $Z(R)$ of R the zero divisor graph of $R, \Gamma(R)$, has vertex set $Z(R)$ and (x,y) is an edge if and only if $xy = 0$. Ahmad [79] determined the total vertex irregularity strength of zero divisor graphs associated with the commutative rings $Z_{p^2} \times Z_q$ where p and q are distinct primes.

Motivated by the notion of the irregularity strength of a graph and various kinds of other total labelings, Bača, Jendroľ, Miller, and Ryan [341] introduced the *total edge irregularity strength* of a graph as follows. For a graph $G(V,E)$ a labeling $\partial : V \cup E \to \{1, 2, \ldots, k\}$ is called an *edge irregular total k-labeling* if for every pair of distinct edges uv and xy, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. Similarly, ∂ is called an *vertex irregular total k-labeling* if for every pair of distinct vertices u and v, $\partial(u) + \sum \partial(e)$ over all edges e incident to $u \neq \partial(v) + \sum \partial(e)$ over all edges e incident to v. The minimum k for which G has an edge (vertex) irregular total k-labeling is called the *total edge (vertex) irregularity strength* of G. The total edge (vertex) irregular strength of G is denoted by tes(G) (tvs(G)). They prove: for $G(V,E)$, E not empty, $\lceil (|E|+2)/3 \rceil \leqslant$ tes(G)$\leqslant |E|$; tes(G)$\geqslant \lceil (\Delta(G)+1)/2 \rceil$ and tes(G)$\leqslant |E| - \Delta(G)$, if $\Delta(G) \leqslant (|E|-1)/2$; tes($P_n$) = tes($C_n$)= $\lceil (n+2)/3 \rceil$; tes(W_n)= $\lceil (2n+2)/3 \rceil$; tes(C_3^n) (friendship graph) $= \lceil (3n+2)/3 \rceil$; tvs(C_n) $= \lceil (n+2)/3 \rceil$; for $n \geqslant 2$, tvs(K_n)= 2; tvs($K_{1,n}$) $= \lceil (n+1)/2 \rceil$; and tvs($C_n \times P_2$)= $\lceil (2n+3)/4 \rceil$. Ahmad, Nurdin, and Baskoro [113] determined the exact value of the total edge (vertex) irregularity strength of generalized Halin graphs. Al-Mushayt, Ahmad, and Siddiqui [163] determined the exact values of the total edge irregular strength of hexagonal grid graphs. The (m,n)-lollipop graph denoted by $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a path graph P_n with a bridge. Ni'mah and Indriati [2277] determined tvs($L_{m,n}$) for $m \geqslant 3$ and $n \geqslant 1$. Aftiana and Indriati [61] proved that for $n \geqslant 3$ the total edge irregularity strength of the graph obtained by joining two copies of K_n (barbell graph) with an edge is $\lceil (n^2 - n + 3)/3 \rceil$. In [2308] Nurdin and Hye consider the splitting graph of stars as a land transportation system and give the exact value of their total vertex irregularity strength. For $m, n \geqslant 3$ Indriati, Widodo, and Sugeng [1332] determined the exact value of the total vertex irregularity strength for generalized helm graphs H_n^m (obtained from W_n by attaching P_m vertices at each vertex of the n-cycle) and for prisms with outer pendent edges. Nurdin [2309] determined the [2309] new total vertex irregularity strength of banana trees and quad trees. Susanti, Puspitasari and Khotimah [3151] determined the exact value of total edge irregularity strength for staircase graphs, double staircase graphs, and mirror-staircase graphs. In [3152] Susanti, Wahyuni, Sutjijana, Sutopo, and Ernanto provide the total edge irregularity strength of generalized arithmetic staircase graphs and generalized double-staircase graphs.

In [105] Ahmad, Ibrahim, and Siddiqui determined the total irregularity strength of generalized Petersen graphs. In [165] Al-Mushayt, Ahmad, and Siddiqui [165] determined the total edge (vertex) irregularity strength for convex polytope graphs having the same diameter. In [2950] Siddiqui determined the irregularity strength of six classes of convex polytope graphs with pendent edges. Ramdani, Salman, Assiyatum, and Semaničová-Feňovčíková [2582] establish upper bounds for the total vertex (edge) irregularity strength and total irregularity strength for disjoint union of arbitrary graphs. Naeem and Siddiqui [2244] determined the total irregularity strength of disjoint union of isomorphic copies of the generalized Petersen graph. In [3517] Yang, Siddiqui, Ibrahim, Ahmad, and Ahmad determined the exact value of the total irregularity strength of three planar graphs. In [2673] Salama introduced three kinds of snake graphs and investigated their total irregularity strength. Ibrahim, Khan, Asim, and Waseem [1281] determined the exact value of the total irregularity strength of cubic graphs. Tilukay, Tomasouw, and Rumlawang [3215] determined the total irregularity strength of complete graphs and complete bipartite graphs.

Jendroľ, Miškul, and Soták [1375] (see also [1376]) proved: $\text{tes}(K_5) = 5$; for $n \geqslant 6$, $\text{tes}(K_n)= \lceil (n^2 - n + 4)/6 \rceil$; and that $\text{tes}(K_{m,n})= \lceil (mn + 2)/3 \rceil$. They conjecture that for any graph G other than K_5, $\text{tes}(G) = \max\{\lceil (\Delta(G) + 1)/2 \rceil, \lceil (|E| + 2)/3 \rceil\}$. Ivančo and Jendroľ [1343] proved that this conjecture is true for all trees. Jendroľ, Miškuf, and Soták [1375] prove the conjecture for complete graphs and complete bipartite graphs. The conjecture has been proven for the categorical product of two paths [90], the categorical product of a cycle and a path [2948], the categorical product of two cycles [98], the Cartesian product of a cycle and a path [387], the subdivision of a star [2949], and the toroidal polyhexes [348]. In [116] Ahmad, Siddiqui, and Afzal proved the conjecture is true for graphs obtained by starting with m vertex disjoint copies of P_n $(m, n \geqslant 2)$ arranged in m horizontal rows with the jth vertex of row $i + 1$ directly below the jth vertex row i for $1 = 1, 2, \ldots, m-1$ and joining the jth vertex of row i to the $j+1$th vertex of row $i+1$ for $1 = 1, 2, \ldots, m - 1$ and $j = 1, 2, \ldots, n - 1$ (the *zigzag* graph). Siddiqui, Ahmad, Nadeem, and Bashir [2952] proved the conjecture for the disjoint union of p isomorphic sun graphs (i. e., $C_n \odot K_1$) and the disjoint union of p sun graphs in which the orders of the n-cycles are consecutive integers. They pose as an open problem the determination of the total edge irregularity strength of disjoint union of any number of sun graphs. Brandt, Misškuf, and Rautenbach [607] proved the conjecture for large graphs whose maximum degree is not too large relative to its order and size. In particular, using the probabilistic method they prove that if $G(V, E)$ is a multigraph without loops and with nonzero maximum degree less than $|E|/10^3\sqrt{8|V|}$, then $\text{tes}(G) = (\lceil |E| + 2)/3 \rceil$. As corollaries they have: if $G(V, E)$ satisfies $|E| \geqslant 3 \cdot 10^3 |V|^{3/2}$, then $\text{tes}(G) = \lceil (|E|+2)/3 \rceil$; if $G(V, E)$ has minimum degree $\delta > 0$ and maximum degree Δ such that $\Delta < \delta\sqrt{|V|}/10^3 \cdot 4\sqrt{2}$ then $\text{tes}(G)$

$= \lceil (|E|+2)/3 \rceil$; and for every positive integer Δ there is some $n(\Delta)$ such that every graph $G(V,E)$ without isolated vertices with $|V| \geqslant n(\Delta)$ and maximum degree at most Δ satisfies $\text{tes}(G) = \lceil (|E|+2)/3 \rceil$. Notice that this last result includes d-regular graphs of large order. They also prove that if $G(V,E)$ has maximum degree $\Delta \geqslant 2|E|/3$, then G has an edge irregular total k-labeling with $k = \lceil (\Delta+1)/2 \rceil$. Pfender [2388] proved the conjecture for graphs with at least 7×10^{10} edges and proved for graphs $G(V,E)$ with $\Delta(G) \leqslant E(G)/4350$ we have $\text{tes}(G) = (\lceil |E|+2)/3 \rceil$. Susanti and Haq [3148] determined the minimum k such that an odd staircase graph can be labeled by an edge irregular total k-labeling. Murhu Guru Packiam, Manimaran, and Thuraiswamy [2210] investigate how the addition of a new edge affects the total edge irregularity strength of a graph. Laurence and Kathiresan [1818] determined the total edge irregular strength of path union of cycles. Sivakumar, Vidyanandini, Sreedevi, Nayak, and Bhoi [2991] determined the total edge irregularity strength of complete tripartite graphs.

In [1532] Jeyanthi and Sudha investigated the total edge irregularity strength of the disjoint union of wheels. They proved the following: $\text{tes}(2W_n) = \lceil (4n+2)/3 \rceil, n \geqslant 3$; for $n \geqslant 3$ and $p \geqslant 3$ the total edge irregularity strength of the disjoint union of p isomorphic wheels is $\lceil (2(pn+1)/3 \rceil$; for $n_1 \geqslant 3$ and $n_2 = n_1 + 1$, $\text{tes}(W_{n_1} \cup W_{n_2}) = \lceil (2(n_1+n_2+1)/3) \rceil$; for n_1, n_2, n_3 where $n_1 \geqslant 3$ and $n_{i+1} = n_1 + i$ for $i = 1, 2$, $\text{tes}(W_{n_1} \cup W_{n_2} \cup W_{n_3}) = \lceil (2(n_1+n_2+n_3+1)/3) \rceil$; the total edge irregularity strength of the disjoint union of $p \geqslant 4$ wheels $W_{n_1} \cup W_{n_2} \cup \cdots \cup W_{n_p}$ with $n_{i+1} = n_1 + i$ and $N = \sum_{j=1}^{p} n_j + 1$ is $\lceil 2N/3) \rceil$; and the total edge irregularity strength of $p \geqslant 3$ disjoint union of wheels $W_{n_1} \cup W_{n_2} \cup \cdots \cup W_{n_p}$ and $N = \sum_{j=1}^{p} n_j + 1$ is $\lceil (2N/3 \rceil$ if $\max\{n_i \mid 1 \leqslant i \leqslant p\} \leqslant \frac{1}{2} \lceil (2N/3 \rceil$.

In [2201] Mughal and Jamil determined the tight lower bounds for the total face irregular strength of type $(1,1,0)$ of grids and wheels.

A k-labeling ϕ of a planar graph G is defined to be a face irregular k-labeling of type (α, β, γ) if for every two different faces f and g of G we have $wt_{\phi(\alpha,\beta,\gamma)}(f) \neq wt_{\phi(\alpha,\beta,\gamma)}(g)$. The face irregularity strength of type (α, β, γ) of a planar graph G, denoted $\text{fs}(\alpha, \beta, \gamma)(G)$, is the smallest integer k such that G admits a face irregular k-labeling of type (α, β, γ). In [380] Bača, Ovais, Semaničová-Feňovčíková, and Nengah Suparta, estimated the lower bounds and the upper bounds of the face irregularity strength of type (α, β, γ) for 2-connected planar graphs, where $\alpha, \beta, \gamma \in \{0, 1\}$, and determined the precise values of these parameters for ladders and fan graphs and proved the sharpness of the lower bounds.

The *complete star* of a graph G is the graph obtained from $p+1$ copies of the graph G by joining each vertex of $G^{(0)}$ with all corresponding vertices of all the copies $G^{1)}, \ldots, G^{(p)}$. Susanti, Khotimah, Hidayati, and Wahyujati [3149] deterimined the total edge irregularity strength of snowflake graphs, water bears graphs, the complete star of C_n, and two other families of ladder related graphs.

In [1535], [1537], [1538], and [1536] Jeyanthi and Sudha determine the total edge irregularity strength of fans, helms, closed helms, webs, flowers, gears, sun flowers, tadpoles, armed crowns, split graphs of cycles, split graph of paths, disjoint unions of isomorphic double wheels, and disjoint unions of consecutive non-isomorphic double wheels. Bokhary, Ali, and Maqbool [592] determined the exact values for the total vertex and edge irregularity strength of three wheel related families of graphs. Bokhary, Imran, and Ali [594] determined the exact value of convex polytopes generated by prisms and antiprisms and determined their total vertex irregularity strength and total edge irregularity strength. Ibrahim, Asif, Ahmad, and Siddiqui [1279] investigated the total irregularity strength of fans, helms, closed helms, webs, flower graphs, gears, and sunflowers. In [2607] Ratnasari1, Wahyuni1, Susanti1, and Palupi1 determined the total edge irregularity strength of book graphs, double and triple book graphs, and gave the exact value of the total edge irregularity strength of quadruplet book graphs and quintuplet book graphs. Nurdini, Rosyida, and Mulyono [2313] determined the total edge irregularity strength of the chain graph that consists of tadpole graph $T(6, n)$ on each block and constructed an algorithm to find it.

A generalized helm H_n^m is a graph obtained by inserting m vertices in every pendent edge of a helm H_n. Indriati, Widodo, and Sugeng [1330] proved that for $n \geqslant 3$, $\mathrm{tes}(H_n^1) = \lceil (4n+2)/3 \rceil$, $\mathrm{tes}(H_n^2) = \lceil (5n+2)/3 \rceil$, and $\mathrm{tes}(H_n^m) = \lceil ((m+3)n+2))/3 \rceil$ for $m \equiv 0 \bmod 3$. They conjecture that $\mathrm{tes}(H_n^m) = \lceil ((m+3)n+2))/3 \rceil$, for all $n \geqslant 3$ and $m \geqslant 10$.

A *cactus* graph G is a connected graph where no edge lies in more than one cycle. A cactus graph consisting of some blocks where each block is C_n with same n is called an *n-uniform cactus graph*. If each cycle of the cactus graph has no more than two cut-vertices and each cut-vertex is shared by exactly two cycles, then G is called *n-uniform cactus chain* graph. Rosyida and Indriati [2657] determined the tes of n-uniform cactus chain graphs of length r for some $n \equiv 0 \bmod 3$. They also investigated the tes of tadpole chain graphs. Rosyida and Indriati [2656] determined the total edge irregularity strength of the triangular cactus chain with length r and $r+1$ pendant vertices (TC_r^{r+1}) is $\lceil (4r+3)/3 \rceil$. A *para-squares cactus* is a graph each of whose blocks is a 4-cycle and two or more squares share a cut vertex. The *para-squares cactus chain graph* is a cactus graph such that each of the two squares has one common cut vertex. Rosyida and Indriati determined that total edge irregularity strength of para square cactus chain graph with length r and r pendant vertices (Q_r^r) is $\lceil (5r+2)/3 \rceil$. Rosyida, Ningrum, Mulyono, and Indriati [2658] determined the total edge irregularity strength of general uniform cactus chain graphs having $(n-2)r$ pendant vertices and length r.

Nurdin, Baskoro, Salman, and Gaos [2311] determine the total vertex irregularity strength of trees with no vertices of degree 2 or 3; improve some of the bounds given in [341]; and show that $\mathrm{tvs}(P_n) = \lceil (n+1)/3) \rceil$. In [2314] Nurdin, Salman, Gaos, and Baskoro prove that for $t \geqslant 2$, $\mathrm{tvs}(tP_1)= t$; $\mathrm{tvs}(tP_2)= t+1$; $\mathrm{tvs}(tP_3)= t+1$; and for $n \geqslant 4$, $\mathrm{tvs}(tP_n)= \lceil (nt+1)/3) \rceil$. Ahmad, Bača, and Bashir [91] proved that for $n \geqslant 3$ and $t \geqslant 1$, $\mathrm{tvs}((n,t)\text{-kite}) = \lceil (n+t)/3 \rceil$, where the (n,t)-kite is a cycle of length n with a t-edge path (the tail) attached to one vertex. In [1163] Guo, Chen, Wang, and Yao give the total vertex irregularity strength of certain complete m-partite graphs. In [3159] Susilawati, Baskaro, and Simanjuntak determined the total vertex irregularity strength of a particular subdivision of any tree. In [3161], [3162], and [3160] Susilawati, Baskoro and Simanjuntak determined the total vertex irregularity strength for trees with maximum degree four or five, for subdivision of several classes of caterpillars, for the subdivision of fire crackers, and for the subdivision of an amalgamation of stars. In [3163] they studied the vertex

[3159] new
[3161] new
[3162] new
[3160] new
[3163] new

irregularity strength for trees having many vertices of degree 2. Nurdin, Baskoro, Salman and Gaos [2312] determined the total vertex irregularity strength for firecrackers and banana trees. [2312] new

Anholcer, Kalkowski, and Przybylo [211] prove that for every graph with $\delta(G) > 0$, tvs$(G) \leqslant \lceil 3n/\delta \rceil + 1$. Majerski and Przybylo [2068] prove that the total vertex irregularity strength of graphs with n vertices and minimum degree $\delta \geqslant n^{0.5} \ln n$ is bounded from above by $(2 + o(1))n/\delta + 4$. Their proof employs a random ordering of the vertices generated by order statistics. Anholcer, Karonński, and Pfender [210] prove that for every forest F with no vertices of degree 2 and no isolated vertices tvs$(F) = \lceil (n_1 + 1)/2 \rceil$, where n_1 is the number of vertices in F of degree 1. They also prove that for every forest with no isolated vertices and at most one vertex of degree 2, tvs$(F) = \lceil (n_1 + 1)/2 \rceil$. Anholcer and Palmer [212] determined the total vertex irregularity strength C_n^k, which is a generalization of the circulant graphs $C_n(1, 2, \ldots, k)$. They prove that for $k \geqslant 2$ and $n \geqslant 2k + 1$, tvs$(C_n^k = \lceil (n + 2k)/(2k + 1) \rceil$. Przybylo [2526] obtained a variety of upper bounds for the total irregularity strength of graphs as a function of the order and minimum degree of the graph.

In [3220] Tong, Lin, Yang, and Wang give the exact values of the total edge irregularity strength and total vertex irregularity strength of the toroidal grid $C_m \times C_n$. In [2953] Siddiqui, Miller, and Ryan determine the exact values of the total edge irregularity strength of octagonal grid graph. In [99] Ahmad, Bača, and Siddiqui gave the exact value of the total edge and total vertex irregularity strength for disjoint union of prisms and for disjoint union of cycles. In [96] Ahmad, Bača, and Numan showed that tes$(\bigcup_{j=1}^m F_{n_j}) = 1 + \sum_{j=1}^m n_j$ and tvs$(\bigcup_{j=1}^m F_{n_j}) = \lceil (2 + 2\sum_{j=1}^m n_j)/3 \rceil$, where $\bigcup_{j=1}^m F_{n_j}$ denotes the disjoint union of friendship graphs. Chunling, Xiaohui, Yuansheng, and Liping, [727] showed tvs$(K_p) = 2$ $(p \geqslant 2)$ and for the generalized Petersen graph $P(n, k)$ they proved tvs$(P(n, k)) = \lceil n/2 \rceil + 1$ if $k \leqslant n/2$ and tvs$(P(n, n/2)) = n/2 + 1$. They also obtained the exact values for the total vertex strengths for ladders, Möbius ladders, and Knödel graphs. For graphs with no isolated vertices, Przybylo [2524] gave bounds for tvs(G) in terms of the order and minimum and maximum degrees of G. For d-regular $(d > 0)$ graphs, Przybylo [2526] gave bounds for tvs(G) in terms d and the order of G.

Ahmad, Ahtsham, Imran, and Gaig [81] determined the exact values of the total vertex irregularity strength for five families of cubic plane graphs. In [88] Ahmad and Bača determine that the total edge-irregular strength of the categorical product of C_n and P_m where $m \geqslant 2$, $n \geqslant 4$ and n and m are even is $\lceil (2n(m-1)+2)/3 \rceil$. They leave the case where at least one of n and m is odd as an open problem. In [98] and [99] Ahmad, Bača, and Siddiqui determine the exact values of the total edge irregularity strength of the categorical product of two cycles, the total edge (vertex) irregularity strength for the disjoint union of prisms, and the total edge (vertex) irregularity strength for the disjoint union of cycles. In [87] Ahmad, Awan, Javaid, and Slamin study the total vertex irregularity strength of flowers, helms, generalized friendship graphs, and web graphs. Indriati, Widodo, Wijayanti, Sugeng, and Bača [1329] determine the exact value of the total edge irregularity strength of the generalized web graph $W(n, m)$ and two families of related graphs. Ahmad, Bača, and Numan [96] determined the exact values of the total vertex irregularity strength and the total edge irregularity strength of a disjoint union of friendship graphs. Bokhary, Ahmad, and Imran [591] determined the exact value of the total vertex irregularity strength of cartesian and categorical product of two paths. Koam and Ahmad [1708] determined the total vertex irregularity strength for all theta graphs and certain values of the total vertex irregularity strength of the centralized uniform theta graphs. They provide a conjecture for the lower bound of total vertex irregularity strength of the centralized uniform theta graphs. In [593] Bokhary and Faheem proved the conjecture of Bokhary, Ahmam, and Imran [591] that the $tvs(P_m \Box P_n) = \lfloor \frac{mn+2}{5} \rfloor$ for $m, n \geqslant 2$ for $5 \leqslant m \leqslant 10$ and $n \geqslant 1$. the state graph for Tower of Hanoi problems with three towers. Farida and Indriati [917] determined the total edge irregularity strength of the state graph for Tower of Hanoi problems with three towers.

In [2310] Nurdin, Salman, and Baskoro determine the total edge-irregular strength of the following graphs: for any integers $m \geqslant 2,\ n \geqslant 2$, tes($P_m \odot P_n$)= $\lceil(2mn+1)/3\rceil$; for any integers $m \geqslant 2,\ n \geqslant 3$, tes($P_m \odot C_n$)= $\lceil((2n+1)m+1)/3\rceil$; for any integers $m \geqslant 2,\ n \geqslant 2$, tes($P_m \odot K_{1,n}$)= $\lceil(2m(n+1)+1)/3\rceil$; for any integers $m \geqslant 2$ and $n \geqslant 3$, tes($P_m \odot G_n$)= $\lceil(m(5n+2)+1)/3\rceil$ where G_n is the gear graph obtained from the wheel W_n by subdividing every edge on the n-cycle of the wheel; for any integers $m \geqslant 2,\ n \geqslant 2$, tes($P_m \odot F_n$)= $\lceil m(5n+2)+1\rceil$, where F_n is the friendship graph obtained from W_{2m} by subdividing every other rim edge; for any integers $m \geqslant 2$ and $n \geqslant 3$; and tes($P_m \odot W_n$)= $\lceil((3n+2)m+1)/3\rceil$.

In [2560], [2561], and [2559] Rajasingh, Rajan, Teresa Arockiamary, and Quadras provide the total edge irregularity strengths of honeycomb mesh networks, hexagonal networks, butterfly networks, benes networks, and series compositions of uniform theta graphs. Ratnasari and Susanti [2596] determined the exact value of the total edge irregularity strength of triangular ladders, diagonal ladders, and circular triangular ladders. In [2312] Nurdin, Baskoro, Salman, and Gaos proved: the total vertex-irregular strength of the complete k-ary tree ($k \geqslant 2$) with depth $d \geqslant 1$ is $\lceil(k^d+1)/2\rceil$ and the total vertex-irregular strength of the subdivision of $K_{1,n}$ for $n \geqslant 3$ is $\lceil(n+1)/3\rceil$. They also determined that if G is isomorphic to the caterpillar obtained by starting with P_m and m copies of P_n denoted by $P_{n,1}, P_{n,2}, \ldots, P_{n,m}$, where $m \geqslant 2,\ n \geqslant 2$, then joining the i-th vertex of P_m to an end vertex of the path $P_{n,i}$, tvs(G)= $\lceil(mn+3)/3\rceil$. They conjectured that the total vertex irregularity strength of any tree T is determined only by the number of vertices of degrees 1, 2 and 3 in T. This conjecture was confirmed by Susilawati, Baskoro, and Simanjuntak [3158] by considering all trees with maximum degree five. They also characterized all such trees having the total vertex irregularity strength either t_1, t_2, or t_3, where $t_i = \lceil(1+\sum_{j=1}^{i} n_j)/(i+1)\rceil$ and n_i is the number of vertices of degree i.

Ahmad and Bača [89] proved tvs($J_{n,2}$)= $\lceil(n+1)/2)\rceil$ ($n \geqslant 4$) and conjectured that for $n \geqslant 3$ and $m \geqslant 3$, tvs($J_{n,m}$)= $\max\{\lceil(n(m-1)+2)/3\rceil, \lceil(nm+2)/4\rceil\}$. They also proved that for the circulant graph (see §5.1 for the definition) $C_n(1,2),\ n \geqslant 5$, tvs($C_n(1,2)$)= $\lceil(n+4)/5\rceil$. They conjecture that for the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ with degree r at least 5 and $n \geqslant 5,\ 1 \leqslant a_i \leqslant \lfloor n/2\rfloor$, tvs($C_n(a_1, a_2, \ldots, a_m)$)= $\lceil(n+r)/(1+r)\rceil$. Ahmad, Arshadb, and Ižaríková [86] determine tes(G) where G is the generalized helm and tvs(G) where G is the generalized sun graph.

Slamin, Dafik, and Winnona [2997] consider the total vertex irregularity strengths of the disjoint union of isomorphic sun graphs, the disjoint union of consecutive nonisomorphic sun graphs, $\text{tvs}(\cup_{i=1}^{t} S_{i+2})$, and disjoint union of any two nonisomorphic sun graphs. (Recall $S_n = C_n \odot K_1$.) Rajasingh and Annamma [2558] determine the total vertex irregularity strength of 1-fault tolerant Hamiltonian graphs $CH(n), H(n)$, and $W(m)$. Indriati, Widodo, Wijayanti, Sugeng, Bača, and Semaničová-Feňovčíková [1331] determine the exact value of the total vertex irregularity strength for generalized helm graphs and for prisms with outer pendent edges. In [260] Asim and Hasni provided an upper bound for $\text{es}(K_n)$ that is far better than the previously known upper bound.

In [76] Ahmad shows that the total vertex irregularity strength of the antiprism graph A_n ($n \geqslant 3$) is $\lceil (2n+4)/5 \rceil$ (see §5.7 for the definition) and gives the vertex irregularity strength of three other families convex polytope graphs. Al-Mushayt, Arshad, and Siddiqui [164] determined an exact value of the total vertex irregularity strength of some convex polytope graphs. Ahmad, Baskoro, and Imran [102] determined the exact value of the total vertex irregularity strength of disjoint union of helm graphs.

For $n \geqslant 3, m \geqslant 2$ Jeyanthi and Sudha [1539] determine the total vertex irregularity strength of $P_n \odot K_1, P_n \odot K_2,\ C_n \odot K_2,\ L_n \odot K_1,\ P_2 \odot C_n,\ P_n \odot \overline{K_m},\ (C_n \times P_2) \odot K_1$, and $C_n \odot \overline{K_m}$. In [1540] they determine the total vertex irregularity strength for the graph obtained from a cycle by identifying the endpoint of a path and the vertex of a cycle, $C_n \odot P_m$, the split graph of a cycle, and split graph of a path. In [1540] they determine the total vertex irregularity strength for quadrilateral snakes, sunflowers, double wheels, triangular books, quadrilateral books, and graphs obtained from the wheel W_n and attaching n pendent edges to the center. In [1542] Jeyanthi and Sudha determined the total irregularity strength of the n-crossed prism, m copies of crossed prism, necklace and m copies of necklace graph and that these graphs admit totally irregular total k-labeling.

Tilukay, Tomasouw, Rumlawang, and Salman in [3216] proved that K_n and $K_{n,n}$ are both totally irregular total graphs with their ts equal to their tes. Tilukay, Taihuttu, Salman, Rumlawang, and Leleury [3214] proved that $K_{m,n}$ is a totally irregular total graph for any positive integer m and n.

A total edge Fibonacci irregular labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ of a graph G is a labeling of vertices and edges of G in such a way that for disinct edges xy and $x'y'$ their weights $f(x) + f(xy) + f(y)$ and $f(x') + f(x'y') + f(y')$ and are distinct Fibonacci numbers. A graph that has a total edge Fibonacci irregular labeling is called a *total edge Fibonacci irregular* graph. Karthikeyan, Navanaeethakrishnan, and Sridevi [1658] proved that stars, bistars $B_{n,n}$ ($n \geqslant 2$), and two particular families of star related and bistar related graphs are total edge Fibonacci irregular graphs. The total edge Fibonacci irregularity strength, $\text{tefs}(G)$ is the minimum k for which G has total edge Fibonacci irregular labeling. Amutha and Uma Devi [181] determined the exact values of the total edge Fibonacci irregularity strength of fans, double fans, umbrella, and wheels.

[1658] new

[181] new

In [70] Agustin, Dafik, Marsidi, and Albirri introduced a natural extension of the notation of the total H-irregularity strength of graphs by considering the evaluation of the weight that is not only for each edge but also the weight on each subgraph H of G. They say a total α-labeling is a *total H-irregular α-labeling* of the graph G if for a subgroup H of G, the total H-weights $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ are distinct. The minimum α for which the graph G has a total H-irregular α-labeling is called the *total H-irregularity strength* of G, denoted by $tHs(G)$. They study the tHs of shackles and amalgamations of any graphs and their bounds. In [257] Ashraf, Bača, Semaničová-Feňovčíková, and Siddiqui determined the exact value of the total (vertex, edge) H-irregularity strengths for the ladders and fan graphs.

The notion of an irregular labeling of an Abelian group Γ was introduced Anholcer, Cichacz and Milanič in [204]. They defined a Γ-*irregular labeling* of a graph G with no isolated vertices as an assignment of elements of an Abelian group Γ to the edges of G in such a way that the sums of the weights of the edges at each vertex are distinct. The *group irregularity strength* of G, denoted $s_g(G)$, is the smallest integer s such that for every Abelian group Γ of order s there exists Γ-irregular labeling of G. They proved that if G is connected, then $s_g(G) = n+2$ when $\cong K_{1,3^{2q+1}-2}$ for some integer $q \geqslant 1$; $s_g(G) = n+1$ when $n \equiv 2 \pmod 4$ and $G \ncong K_{1,3^{2q+1}-2}$ for any integer $q \geqslant 1$; and $s_g(G) = n$ otherwise. Moreover, Anholcer and Cichacz [200] showed that if G is a graph of order n with no component of order less than 3 and with all the bipartite components having both color classes of even order. Then $s_g(G) = n$ if $n \equiv 1 \pmod 2$; $s_g(G) = n+1$ if $n \equiv 2 \pmod 4$; and $s_g(G) \leqslant n+1$ if $n \equiv 0 \pmod 4$. In [203] Anholcer, Cichacz, Jura, Marczyk presented some upper bound on group the irregularity strength for all graphs. Moreover, they gave the exact values and bounds on $s_g(G)$ for disconnected graphs without a star as a component. Anholcer, Cichacz, and Przybyło [208] proved that that $s_g(G) \leqslant 2n$. They also considered locally irregular labelings where only sums of adjacent vertices are required to be distinct. For the corresponding graph invariant they proved the general upper bound: $\Delta(G) + \mathrm{col}(G) - 1$ (where $\mathrm{col}(G)$ is the coloring number of G) in the case when the identity element is not used as an edge label, and a slightly worse one if additionally the identity is forbidden as the sum of labels around a vertex. In the both cases they provided a sharp upper bound for trees and a constant upper bound for planar graphs. In [746] Cichacz and Krupińska gave a bound and exact values of the group irregular strength for graphs without small stars as components. [746] new

Marzuki, Salman, and Miller [2109] introduced a new irregular total k-labeling of a graph G called *total irregular total k-labeling*, denoted by $ts(G)$, which is required to be at the same time both vertex and edge irregular. They gave an upper bound and a lower bound of $ts(G)$; determined the total irregularity strength of cycles and paths; and proved $ts(G) \geqslant \max\{tes(G), tvs(G)\}$. For $n \geqslant 3$, Ramdani and Salman [2578] proved $ts(S_n \times P_2) = n+1$; $ts((P_n + P_1) \times P_2) = \lceil (5n+1)/3 \rceil$, $ts(P_n \times P_2) = n$; and $ts(C_n \times P_2) = n$. In [2579] Ramdani, Salman, and Assiyatun prove that for a regular graph G $ts(mG) \leqslant m(ts(G)) - \lfloor (m-1)/2 \lfloor$, $ts(mC_n) = \lceil (mn+2)/3 \rfloor$ for $n \equiv 3$ mod 3, and $ts(m(C_n \times P_2) = mn+1$. In [2580] Ramdani, Salman, Assiyatun, Semaničová-Feňovčíková, and Bača estimate the upper bound of the total irregularity strength of graphs and determine the exact value of the total irregularity strength for three families of graphs. In [1471] Jeyanthi and Sudha determined the total irregularity strength of double fans DF_n $(n \geqslant 3)$, double triangular snakes DT_p $(p \geqslant 3)$, joint-wheel graphs WH_n $(n \geqslant 3)$, and $P_m + \overline{K_m}$ $(m \geqslant 3)$. In addition, they show that these graphs admit totally irregular total k-labeling and they determined the exact ts value for each. In [1328] Indriati, Widodo, Wijayanti, and Sugeng provided the total irregularity strength of some caterpillars.

In [3213] Tilukay, Salman, and Persulessy proved that fans, wheels, triangular books, friendship graphs, double fans DF_n, $(n \geqslant 3)$, double triangular snakes DT_p $(p \geqslant 3)$, joint-wheel graphs, $P_m + K_m$ $(m \geqslant 3)$, stars, double-stars, and caterpillars (see also [2581]) are totally irregular total graphs. Ramdani, Salman, Assiyatun, Semaničová-Feňovčíková, and Bača [2580] proved that for any positive integer $n \geqslant 2$, $\mathrm{ts}(K_{n,n}) = \lceil (n^2+2)/3 \rceil$. Tilukay, Taihuttu, Salman, Rumlawang, and Leleury [3214] proved that $K_{m,n}$ is a totally irregular total graph for any positive integer m and n. In [3213] Tilukay, Salman, and Persulessy proved that fans, wheels, triangular books, friendship graphs are totally irregular total graphs, double fans DF_n, $(n \geqslant 3)$, double triangular snakes DT_p, $(p \geqslant 3)$, joint-wheel graphs, $P_m + K_m$ $(m \geqslant 3)$, stars, double-stars, and caterpillars (see also [2581]).

In [2231] Muthgu Guru Packiam defines a *face irregular total k-labeling f* from $V \cup E \cup F$ to $\{1, 2, \ldots, k\}$ of a 2-connected plane graph $G(V, E, F)$ as a labeling of vertices and edges such that different faces have different weigths. The minimum k for which a plane graph G has a face irregular total k-labeling is called *total face irregularity strength* of G and is denoted by $tfs(G)$. He provides a bound on this parameter and the exact values for shell graphs and a family of planar graphs consisting of an even number of 5-sided faces and one external infinite face. In [346] Bača, Lascsḱová, Naseem, and Semaničová-Feňovčíková estimate the lower and upper bounds of the entire face irregularity strength for the disjoint union of multiple copies of a plane graph and prove the sharpness of the lower bound in two cases. Tilukay, Salman, Ilwaru, and Rumlawang [3217] estimated the bounds of $tfs(G)$ and prove that the lower bound is sharp for cycles, books with m polygonal pages, and wheels.

For a graph G, Tanna, Ryan, and Semaničová-Feňovčíková [3188] define a k-labeling ρ as a labeling such that the edges of G are labeled with $\{1, 2, \ldots, k_e\}$ and the vertices of G are labeled with $\{0, 2, \ldots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. The labeling ρ is called an *edge irregular reflexive k-labeling* if distinct edges have distinct weights, where the edge weight is defined as the sum of the label of that edge and the labels of its endpoints. The smallest k for which such a labeling exists is called the *reflexive edge strength* of G. The authors give exact values for the reflexive edge strength for prisms, wheels, baskets (graphs obtained by removing a spoke from a wheel), and fans. Guirao, Ahmad, Siddiqui, and Ibrahim [1162] investigated the exact value of the reflexive edge strength for disjoint union of s isomorphic copies of generalized Peterson graphs. Zhang, Ibrahim, ul Haq Bokhary, and Siddiqui [3581] provided exact value of the reflexive edge strength for disjoint union of gears and prisms. Bača, Irfan, Ryan, Semaničová-Feňovčíková, and Tanna [339] determined the exact value of the reflexive edge strength for cycles, the Cartesian product of two cycles, and for join graphs of the path and cycle with $2K_2$. Wang, Khan, Ibrahim, Bonyah, Siddiqui, and Khalid [3412] determined the reflexive edge irregularity strength for the Cartesian product of a path and a cycle. In [3538] Yoong, Hasni, Lau, Asim, and Ahmad obtained the exact reflexive edge strength for antiprisms, two kinds of convex polytopes, and $C_n \odot P_m$ $(n \geqslant 3, m \geqslant 2)$. Yoong, Hasni, Lau, and Irfan [3539] provided the exact value of the reflexive edge strength of three clssess of plane graphs. In [290] Basher examined two types of eight-sided and four-faced or six-sided and four-faced planar maps that have an edge irregular reflexive k-labeling. He gave the precise value of the reflexive edge strength for these two classes. In [338] Bača, Irfan, Ryan, Semaničová-Feňovčíková, and Tanna provided the precise values of the reflexive edge strength for the generalized friendship graphs $f_{n,m}$ for $n = 3, 4, 5, m \geqslant 1$ and made a conjecture of the value for the remaining cases. In [339] they determined the exact value of the reflexive edge strength for cycles, the Cartesian product of two cycles, and for join graphs of the path and cycle with $2K_2$. Bača, Kovář, and Semaničová-Feňovčíková [1334] provided the precise value of the reflexive edge strength of $C_n \times P_m$ where $n \geqslant 3$ and $m \geqslant 2$. Basher [480] determined the exact value of reflexive edge strength of toroidal polyhexes. [3539] new [290] new [338] new [339] new [1334] new [480] new

A *vertex irregular reflexive k-labeling* of a graph G is total k-labeling such that for every two distinct vertices the sums of labels of edges that are incident to each vertex and the vertex label itself are distinct. The *reflexive vertex strength* of a graph G is a minimum k such that G has a vertex irregular reflexive k-labeling. Agustin, Iman Utoyo, Dafik, and Venkatachalam [74] determined the exact value of reflexive vertex strength of ladders and $K_{2,n}$.

A total k-labeling of a graph is a function f_e from the edge set to $\{1, 2, \ldots, k_e\}$ and a function f_v from the vertex set to $\{0, 2, 4, \ldots, 2k_v\}$, where $k = \max k_e, 2k_v$. A *distance irregular reflexive k-labeling* of graph G is a total k-labeling if for every two distinct vertices u and u' of G, $wt(u) \neq wt(u')$, where $wt(u) = f_v(u) +$ the sum of $f_e(uv)$ over all edges uv of G. The minimum k for graph G that has a distance irregular reflexive k-labeling is called the *distance reflexive strength* of the graph G. In [71] Agustin, Dafik, Mohanapriya, Marsidi, and Cangul determine the lower bound of distance reflexive strength of any graph and the exact value of distance reflexive strength of paths, stars, and friendship graphs. [71] new

Recall that an *edge-covering* of G is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i, i = 1, 2, \ldots, t$. In this case we say that G admits an $(H_1, H_2, \ldots, H_t)$-(edge) covering. If every subgraph H_i is isomorphic to a given graph H, we say that G admits an *H-covering*. Motivated by the irregularity strength and the edge irregularity strength of a graph G, Ashraf, Bača, Kimáková, and Semaničová-Feňovčíková [253] introduced two new parameters, edge (vertex) H-irregularity strengths, as the natural extensions of the parameters $\mathrm{s}(G)$ and $\mathrm{es}(G)$ as follows. Let G be a graph admitting an H-covering. For the subgraph H of G under the edge k-labeling β from $E(G)$ to $\{1, 2, \ldots, k\}$, the associated H-weight is defined as $\mathrm{wt}_\beta(H) = \sum \beta(e)$ over all edges e. An edge k-labeling β is called an *H-irregular edge k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H we have $wt_\beta(H') \neq wt_\beta(H'')$. The edge *$H$-irregularity strength* of a graph G, denoted by $\mathrm{ehs}(G, H)$, is the smallest integer k such that G has an H-irregular edge k-labeling. Ibrahim, Gulzar, Fazil, and Azhar [1280] compute the exact value of edge H-irregularity strength of hexagonal and octagonal grid graphs. Ashraf et al. define the *vertex H-irregularity strength* of a graph G, $\mathrm{vhs}(G, H)$, analogously. They estimate the bounds of the parameters $\mathrm{ehs}(G, H)$ and $\mathrm{vhs}(G, H)$ and determine the exact values of the edge (vertex) H-irregularity strength for paths, ladders, and fans in order to prove the sharpness of lower bounds of these parameters. Nisviasari, Dafik, and Agustin [2280] determined the total H-irregularity strength of triangular ladders when H is a windmill or triangular ladder.

In [255] Ashraf, Bača, Semaničová-Feňovčíková, and Saputro determined the exact value of the cycle-irregularity strength of ladders and fan graphs. Ashraf, Bača, Lascsáková, and Semaničová-Feňovčíková [254] estimated the bounds for the total H-irregularity strength of a graph and determind the exact values of the total H-irregularity strength for paths ladders and fans. Ashrafa, Bača, Semaničová-Feňovčíková, and Shabbirc [256] investigated the total (respectively, edge and vertex) G-irregularity strengths of the graphs that contains exactly n subgraphs isomorphic to G. Ahmad, Bača, and Semaničová-Feňovčíková [97] determined the exact values of $\mathrm{ehs}(G, C_4)$ for grids and generalized prisms.

In [84] Ahmad, Al-Mushayt, and Bača define a vertex k-labeling ϕ of a graph G from $V(G)$ to $\{1, 2, \ldots, k\}$ to be *edge irregular k-labeling* if for every two distinct edges e and f, there is $w_\phi(e) \neq w_\phi(f)$, where the weight of an edge $e = xy$ is $w_\phi(xy) = \phi(x) + \phi(y)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, denoted by $es(G)$. They estimated the bounds of the edge irregularity and determined its exact values for paths, cycles, stars, double stars and $P_m \times P_n$. Tarawneh, Hasni, and Ahmad [3190] determined the exact value of the edge irregularity strength of the corona product of graphs with paths. Tarawneh, Hasni, and Ahmad [3191] determine the exact value of edge irregularity strength of corona graphs $C_n \odot mK_1$ $(m \geqslant 2)$. Ahmad [77] determined the exact value of $\mathrm{es}(C_n \odot K_1)$. In [94] Ahmad, Bača, and Nadeen determine the exact value of the edge irregularity strength for several classes of Toeplitz graphs. Tarawneh, Hasni, Siddiqui, and Asim [3194] determined the exact value of edge irregularity strength of disjoint union of zigzag graphs, grids, and generalized sun graphs. Ahmed, Omar, and Bača [114] obtained estimations of the edge [114] new
irregularity strength of graphs and determined the precise values for paths, stars, double stars, and the Cartesian product of two paths. Suparta and Suharta [3146] determined [3146] new
the edge irregularity strength of the chain graph mK_3-path for $m \geqslant 3 \pmod 4$ and the chain graph $C[C_n^{(m)}]$ for $n = 0 \pmod 4$ and provided bounds for the edge irregularity strength of $P_m + \overline{K_n}$ for $m, n \geqslant 3$. In [3584] Zhang, Mehmood, Rehman, Hussain, and [3584] new
Zhang provided the exact values of the edge irregularity strengths of C_4-snakes, complete m-partite graphs, and the middle graphs of paths and cycles.

A *chain graph* $C[B_1, B_2, \ldots, B_n]$ is a graph with blocks $B_1, B_2, \ldots, B_n$ such that B_i and B_{i+1} have a common vertex in such a way that the block-cut vertex graph is a path. Ahmad, Gupta, and Simanjuntak [103] prove the following: $es(C[C_4^{(n)}]) = 2n + 1$; if H_m is an mK_3-path, then $es(H_m)$ has lower bound $\lceil \frac{3m+3}{2} \rceil$ and upper bound $2m + 1$; $es(mK_4\text{-path}) = 3m + 2$; and $es(K_{1,n} + \overline{K_1}) = n + 2$ for $n \geqslant 3$. They obtained bounds for $es(P_m + \overline{K_n})$ and determined that the edge irregularity strength of a graph obtained by joining the vertex of degree m in $K_{1,m}$ to each vertex in $K_{1,n}$, and the vertex of degree n in $K_{1,n}$ to each vertex in $K_{1,m}$ is $m+n+2$. They posed the open problems of determining $es(mK_3\text{-path})$, $es(mK_n\text{-path})$ $(m \geqslant 2)$ and $n \geqslant 5$, and $es(P_m + \overline{K_n})$ for $n \geqslant 1$ and

$m \geqslant 7$. Tarawneh, Hasni, and Asim [3193] determined the exact value of edge irregularity strength for disjoint union of a star graph and the subdivision of a star graph.

The *strong product* of graphs G_1 and G_2 has as vertices the pairs (x, y) where $x \in V(G_1)$ and $y \in V(G_2)$. The vertices (x_1, y_1) and (x_2, y_2) are adjacent if either x_1x_2 is an edge of G_1 and $y_1 = y_2$ or if $x_1 = x_2$ and y_1y_2 is an edge of G_2. For $m, n \geqslant 2$ Ahmad, Bača, Bashir, Siddiqui [92] proved that the total edge irregular strength of the strong product of P_m and P_n is $\lceil 4(mn+1)/3 \rceil - (m+n)$. Al-Mushayt [162] determined the edge irregularity strength of cartesian product of a star and P_2 and a cycle and P_2, and the strong product of path P_n with P_2. Conjectures for the exact value of $K_{1,n} \times P_m$ and $C_n \times P_m$ are stated. Bača and Siddiqui [388] determine the exact value of the total edge irregularity strength of the strong product of any two cycles.

An edge $e \in \overline{G}$ is called a *total positive edge* or *total negative edge* or *total stable edge* of G if $\mathrm{tvs}(G + e) > \mathrm{tvs}(G)$ or $\mathrm{tvs}(G + e) < \mathrm{tvs}(G)$ or $\mathrm{tvs}(G + e) = \mathrm{tvs}(G)$, respectively. If all edges of $\overline{G}$ are total stable (total negative) edges of G, then G is called a *total stable* (*total negative*) graph. Otherwise G is called a *total mixed* graph.

Muthu Guru Packiam and Kathiresan [2324] showed that $K_{1,n}$ $n \geqslant 4$, and the disjoint union of $t \geqslant 2$ copies of K_3 are total negative graphs and that the disjoint union of $t \geqslant 2$ copies of P_3 is a total mixed graph.

For a simple graph G with no isolated edges and at most one isolated vertex Anholcer [198] calls a labeling $w : E(G) \to \{1, 2, \dots, m\}$ *product-irregular*, if all product degrees $pd_G(v) = \prod_{e \ni v} w(e)$ are distinct. Analogous to the notion of irregularity strength the goal is to find a product-irregular labeling that minimizes the maximum label. This minimum value is called the *product irregularity strength* of G and is denoted by $ps(G)$. He provides bounds for the product irregularity strength of paths, cycles, cartesian products of paths, and cartesian products of cycles. In [199] Anholcer gives the exact values of $ps(G)$ for $K_{m,n}$ where $2 \leqslant m \leqslant n \leqslant (m + 2)(m + 1)/2$, some families of forests including complete d-ary trees, and other graphs with $d(G) = 1$. Darda and Hujdurović [783] proved that $ps(X) \leqslant |V(X)| - 1$ for any graph X with more than 3 vertices and gave a connection between the product irregularity strength and the multidimensional multiplication table problem. Skowronek-Kaziów [2994] proves that for the complete graphs $ps(K_n) = 3$. In 2023 Bensmail, Hocquard, Lajou, and Sopena [534] proved that the product version of the 1-2-3 conjecture, raised by Skowronek-Kazió [2994] in 2012, that for every connected graph with order at least 3, there is an assignment of the labels 1, 2, 3 to the edges in such a way that no two adjacent vertices are incident to the same product of labels. [534] new

In [6] Abdo and Dimitrov introduced the total irregularity of a graph. For a graph G, they define $\mathrm{irr}_t(G) = (1/2) \sum_{u,v \in V} |d_G(u) - d_G(v)|$, where $d_G(w)$ denotes the vertex degree of the vertex w. For G with n vertices they proved $\mathrm{irr}_t(G) \leqslant (1/12)(2n^3 - 3n^2 - 2n + 3)$. For a tree G with n vertices they prove $\mathrm{irr}_t(G) \leqslant (n - 1)(n - 2)$ and equality holds if and only if $G \approx S_n$. You, Yang, and You [3540] determined the graph with the maximal total irregularity among all unicyclic graphs.

Inspired by the concept of distant chromatic numbers Przybylo [2527] calls a labeling f from the edges of a graph G to $\{1, 2, 3, \dots, k\}$ *r-distant irregular*, if for every vertex v, the weights of the set of all vertices that are at distance less than or equal to r from v

are pairwise distinct, where the weight of the vertex is the sum of the labels of the edges that are incident with that vertex. The minimum k for which there exists an r-distant irregular labeling of G is called *r-distant irregularity strength* of G and is denoted by $s_r(G)$. Muthu Guru Packiam, Manimaran, and Thuraiswamy [2233] proved the following: $s_1(P_n) = 2$ for $n = 3, 4, 5$; $s_1(P_n) = 3$ if $n > 5$; $s_1(C_n) = 3$; $s_1(K_{m,n}) = s(K_{m,n})$; $s_1(F_n) = s(F_n) = \lceil (n+1)/3 \rceil$ for $n > 2$; $s_1(K_{m,n}) = 3$ when $1 < n/2 \leqslant m < n$; $s_1(P_n \times K_2) = 3$; $s_1(C_n \times K_2) = 3$; $s_1(K_{m,n}) = 3$ when $1 < n/2 \leqslant m < n$; and provide the exact value for $s_1(P_m \odot \overline{K_n})$ for all m and n. They also prove that if G is d-regular with n vertices, then $s_1(G) = s(G) \leqslant \lceil n/2 \rceil + 1$ for $d \geqslant n/2$. Susanto, Wijaya, Purnama, and Slamin [3155] derived a new lower bound of distance irregularity strength for graphs with pendant vertices. They also determined the distance irregularity strength of some families of disjoint unions of paths, suns, helms and friendships graphs. In [533] Bensmail proved that determining the distant irregularity strengths are NP-hard problems.

In a preprint [3473] Wijayanti, Noor, Indriati, Alghofari, and Slamin defined a *distance vertex irregular total k-labeling* of a simple undirected graph $G(V, E)$, as a function $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that for every pair of distinct vertices $u, v \in V(G)$ the weights of u and v are distinct. The *weight* of vertex $v \in V(G)$ is the sum of the labels of vertices in neighborhood of v and the labels of all incident edges to v. The *total distance vertex irregularity strength* of G (denoted by $tdis(G)$) is the minimum k for which G has a distance vertex irregular total k-labeling. In [3474] Wijayanti et al. generalized to distance D where D is any subset of $\{1, 2, \ldots, \text{diam}(G)\}$. Wijayanti, Noor, Indriati, Alghofari [3474] determined the exact total distance vertex irregularity strength of fan and wheel graphs for $D = \{1\}$ In [3472] Wijayanti, Noor, Indriati, Alghofari, and Slamin determined the exact value of the total distance vertex irregularity strength of $\overline{K_m} \odot \overline{K_n}, C_n \odot \overline{K_1}$, and $P_n \odot \overline{K_1}$. [3473] new [3474] new [3474] new [3472] new

An *inclusive distance vertex irregular labeling* of a graph G is an assignment of elements of the set $\{1, 2, \ldots, k\}$ to the vertices of G such that the sums of numbers assigned to the closed neighborhoods of all verticies are distinct. The minimum number k for which there exists an inclusive distance vertex irregular labeling of G is denoted by $\widehat{dis}(G)$. Bača, Semaničová-Feňovčíková, Slamin, and Sugeng [383] establish a lower bound for the inclusive distance vertex irregularity strength for any graph and determine the exact value of this parameter for several families of graphs. Cichacz, Gömrlich, and Semaničová-Feňovčíková [736] prove that for a simple graph G on n vertices in which no two vertices have the same closed neighborhood is $\widehat{dis}(G) \leqslant n^2$.

A mapping $\phi : V(G) \to \{1, 2, \ldots, k\}$ of a simple graph G is said to be an *non-inclusive distance vertex irregular k-labeling* of G if the sums of labels of vertices in the open neighborhood of every vertex are distinct. The minimum k for which G has a non-inclusive distance vertex irregular k-labeling is called a *non-inclusive*distance irregularity strength and is denoted by dis(G)). Susanto, Wijaya, Slamin, and Semaničová-Feňovčíková [3156] developed new lower bounds for non-inclusive distance irregularity strength of graphs and proved that such bounds are sharp for some classes of graphs with pendant vertices. They also discussed some properties for non-inclusive distance irregularity strength of trees.

Susanto, Wijaya, Sudarsana, and Slamin [3157] studied non-inclusive and inclusive distance irregularity strength for the join product of two given graphs. They gave bounds on non-inclusive and inclusive distance irregularity strength for $G+H$ and showed that in some conditions, the bounds are achievable. They determined the exact value of dis$(G+K_1)$ for any given graph G and developed an algorithm for calculating the upper bound on inclusive distance irregularity strength of complete multipartite graphs.

Susanto, Betistiyan, Halikin, and Wijaya [3154] derived new lower bounds on inclusive distance irregularity strength for arbitrary graph and showed that the bounds are sharp for two stars $2K_{1,n}$ and three stars $3K_{1,n}$. For a integer $m \geqslant 2$ and non-trivial connected graph G of order p and size q, the *m-shadow of a graph* G, denoted by $D_m(G)$, is the graph obtained by taking m copies of G, say G_1, G_2,..., G_m, and joining each vertex u in G_i, $i = 1, 2, \ldots, m-1$, to the neighbors of the corresponding vertex u' in G_{i+1}.

For a graph G and an integer $m \geqslant 3$, the *closed m-shadow of a graph* G, denoted by $CD_m(G)$, is a graph obtained from the m-shadow of G by joining an edge from every vertex u in G_1 to the neighbors of the corresponding vertex u' in G_m.

Inayah, Susanto and Semaničová-Feňovčíková [1326] proved the following: for every non-trivial connected graph G of order p and size q, and integers m, t with $2 \leqslant t \leqslant m-1$, $D_m(G)$ is super (a, d)-$D_t(G)$-antimagic for $d \in \{0, 2, \ldots, p+3q\}$ when $p+q$ is even and for $d \in \{1, 3, \ldots, p+3q\}$ when $p+q$ is odd; for every non-trivial connected graph G of order p and size q, and integers m, t with $2 \leqslant t \leqslant m-1$ and $\gcd(m, t) = 1$, $CD_m(G)$ is super (a, d)-$D_t(G)$-antimagic for $d \in \{0, 2, \ldots, p+q\}$ when $p+q$ is even and for $d \in \{1, 3, \ldots, p+q\}$ when $p+q$ is odd; for every non-trivial connected graph G of order p and size q, and for any integer $m \geqslant 3$, $CD_m(G)$ is super (a, d)-$D_m(G)$-antimagic for $d \in \{0, 2, \ldots, 2q\}$.

A natural modification of an irregular labeling is a modular irregular labeling introduced by Muthugurupackiam and Ramya [2211] in 2018. They call an edge k-labeling $\phi : E(G) \to \{1, 2, \ldots, k\}$ of positive integers to the edges of a graph G of order n a *modular irregular* labeling of G if the weight function $\theta : V(G) \to Z_n$ defined by $\theta(v) = wt\phi(v) = \sum \phi(uv)$ over all $u \in N(v)$ is bijective. The *modular irregularity strength* is defined as the minimum k for which G has a modular irregular labeling. If there is no such labeling for a graph then its modular irregularity strength is defined as ∞. They proved the modular irregularity of the tadpole graph and double-cycle graph. Bača, Muthugurupackiam, Kathiresan, and Ramya in [374] gave a lower bound of the modular irregularity strength, and the exact values for paths, cycles, stars, triangular graphs, and gear graphs are determined. In [343] Bača, Kimáková, Lascsáková, and Semaničová-Feňovčikovă determine the exact value of the irregularity strength and the modular irregularity strength of fan graphs. Bačca, Imran, and Semaničovă-Feňovčcíkovă showed the existence of an irregular labeling scheme that proves the exact value of the irregularity strength of wheels. By modifying this irregular mapping in six cases, they obtained labelings that determine the exact value of the modular irregularity strength of wheels as a natural modification of the irregularity strength. Ali, Bača, Lascsaková, Semaničová-Feňovčiková, ALoqaily, and Mlaiki [151] obtained estimations on the modular total vertex irregularity strength, and gave the precise values of this invariant for certain graphs. In [3079] Sugeng, Barack, Hinding, and Simanjuntak constructed a modular irregular labeling and determined its modular irregularity strength of regular double- [151] new star graphs and friendship graphs. Haryeni, Awanis, Bača, and Semaničová-Feňovčiková [1190] estimated the bounds of the (modular) edge irregularity strength for $P_n + \overline{K_m}$ and $C_n + \overline{K_m}$ and determined the corresponding exact value of the (modular) edge irregularity strength for some fans and wheels in order to prove the sharpness of the presented bounds. [1190] new

7.14 Geometric Labelings

If a and r are positive integers at least 2, we say a (p, q)-graph G is (a, r)-*geometric* if its vertices can be assigned distinct positive integers such that the value of the edges obtained as the product of the endpoints of each edge is $\{a, ar, ar^2, \ldots, ar^{q-1}\}$. Hegde [1214] has shown the following: no connected bipartite graph, except the star, is (a, a)-geometric where a is a prime number or square of a prime number; any connected (a, a)-geometric graph where a is a prime number or square of a prime number, is either a star or has a triangle; $K_{a,b},\ 2 \leqslant a \leqslant b$ is (k, k)-geometric if and only if k is neither a prime number nor the square of a prime number; a caterpillar is (k, k)-geometric if and only if k is neither a prime number nor the square of a prime number; $K_{a,b,1}$ is (k, k)-geometric for all integers $k \geqslant 2$; C_{4t} is (a, a)-geometric if and only if a is neither a prime number nor the square of a prime number; for any positive integers t and $r \geqslant 2,\ C_{4t+1}$ is (r^{2t}, r)-geometric; for any positive integer $t,\ C_{4t+2}$ is not geometric for any values of a and r; and for any positive integers t and $r \geqslant 2,\ C_{4t+3}$ is (r^{2t+1}, r)-geometric. Hegde [1216] has also shown that every T_p-tree and the subdivision graph of every T_p-tree are (a, r)-geometric for some values of a and r (see Section 3.2 for the definition of a T_p-tree). He conjectures that all trees are (a, r)-geometric for some values of a and r.

Hegde and Shankaran [1226] prove: a graph with an α-labeling (see §3.1 for the definition) where m is the fixed integer that is between the endpoints of each edge has an (a^{m+1}, a)-geometric for any $a > 1$; for any integers m and n both greater than 1 and m odd, mP_n is (a^r, a)-geometric where $r = (mn + 3)/2$ if n is odd and (a^r, a)-geometric where $r = (m(n + 1) + 3)/2$ if n is even; for positive integers $k > 1, d \geqslant 1$, and odd n, the generalized closed helm (see §5.3 for the definition) $CH(t, n)$ is (k^r, k^d)-geometric where $r = (n - 1)d/2$; for positive integers $k > 1, d \geqslant 1$, and odd n, the generalized web graph (see §5.3 for the definition) $W(t, n)$ is (k^r, a)-geometric where $a = k^d$ and $r = (n - 1)d/2$; for positive integers $k > 1, d \geqslant 1$, the generalized n-crown $(P_m \times K_3) \odot K_{1,n}$ is (a, a)-geometric where $a = k^d$; and $n = 2r + 1,\ C_n \odot P_3$ is (k^r, k)-geometric.

If a and r are positive integers and r is at least 2 Arumugan, Germina, and Anadavally [243] say a (p, q)-graph G is *additively* (a, r)-*geometric* if its vertices can be assigned distinct integers such that the value of the edges obtained as the sum of the endpoints of each edge is $\{a, ar, ar^2, \ldots, ar^{q-1}\}$. In the case that the vertex labels are nonnegative integers the labeling is called *additively* $(a, r)*$-*geometric*. They prove: for all a and r every tree is additively $(a, r)*$-geometric; a connected additively (a, r)-geometric graph is either a tree or unicyclic graph with the cycle having odd size; if G is a connected unicyclic graph and not a cycle, then G is additively (a, r)-geometric if and only if either a is even or a is odd and r is even; connected unicyclic graphs are not additively $(a, r)*$-geometric; if a disconnected graph is additively (a, r)-geometric, then each component is a tree or a unicyclic graph with an odd cycle; and for all even a at least 4, every disconnected graph for which every component is a tree or unicyclic with an odd cycle has an additively (a, r)-geometric labeling.

Vijayakumar [3382] calls a graph G (not necessarily finite) *arithmetic* if its vertices can be assigned distinct natural numbers such that the value of the edges obtained as the sum of the endpoints of each edge is an arithmetic progression. He proves [3381] and [3382] that a graph is arithmetic if and only if it is (a, r)-geometric for some a and r.

7.15 Strongly Multiplicative Graphs

Beineke and Hegde [528] call a graph with p vertices *strongly multiplicative* if the vertices of G can be labeled with distinct integers $1, 2, \ldots, p$ such that the labels induced on the edges by the product of the end vertices are distinct. They prove the following graphs are strongly multiplicative: trees; cycles; wheels; K_n if and only if $n \leqslant 5$; $K_{r,r}$ if and only if $r \leqslant 4$; and $P_m \times P_n$. They then consider the maximum number of edges a strongly multiplicative graph on n vertices can have. Denoting this number by $\lambda(n)$, they show: $\lambda(4r) \leqslant 6r^2; \lambda(4r+1) \leqslant 6r^2+4r; \lambda(4r+2) \leqslant 6r^2+6r+1$; and $\lambda(4r+3) \leqslant 6r^2+10r+3$. Adiga, Ramaswamy, and Somashekara [58] give the bound $\lambda(n) \leqslant n(n+1)/2 + n - 2 - \lfloor (n+2)/4 \rfloor - \sum_{i=2}^{n} i/p(i)$ where $p(i)$ is the smallest prime dividing i. For large values of n this is a better upper bound for $\lambda(n)$ than the one given by Beineke and Hegde. It remains an open problem to find a nontrivial lower bound for $\lambda(n)$.

Seoud and Zid [2804] prove the following graphs are strongly multiplicative: wheels; rK_n for all r and n at most 5; rK_n for $r \geqslant 2$ and $n = 6$ or 7; rK_n for $r \geqslant 3$ and $n = 8$ or 9; $K_{4,r}$ for all r; and the corona of P_n and K_m for all n and $2 \leqslant m \leqslant 8$. In [2782] Seoud and Mahran [2782] give some necessary conditions for a graph to be strongly multiplicative.

In Kanani and Chhaya [1575] and [1576] prove the following graphs are strongly multiplicative: the total graph, splitting graph, and shadow graph of paths; triangular snakes; splitting graphs of stars and bistars, the degree splitting graph of the bistars $B_{n,n}$, and restricted square graph $B^2_{m,n}$. In [1579] and [1580] Kanani and Chhaya prove the following graphs are strongly multiplicative: helms, flowers, fans, friendship graphs, bistars, gears, double triangular snakes, double fans, double wheels, snakes, double alternate quadrilateral snakes, double quadrilateral snakes, braid graphs, and triangular ladders.

Germina and Ajitha [1071] (see also [37]) prove that $K_2 + \overline{K_t}$, quadrilateral snakes, Petersen graphs, ladders, and unicyclic graphs are strongly multiplicative. Acharya, Germina, and Ajitha [37] have shown that $C_k^{(n)}$ (see §2.2 for the definition) is strongly multiplicative and that every graph can be embedded as an induced subgraph of a strongly multiplicative graph. Germina and Ajitha [1071] define a graph with q edges and a strongly multiplicative labeling to be *hyper strongly multiplicative* if the induced edge labels are $\{2, 3, \ldots, q+1\}$. They show that every hyper strongly multiplicative graph has exactly one nontrivial component that is either a star or has a triangle and every graph can be embedded as an induced subgraph of a hyper strongly multiplicative graph.

Vaidya, Dani, Vihol, and Kanani [3271] prove that the arbitrary supersubdivisions of tree, K_{mn}, $P_n \times P_m$, $C_n \odot P_m$, and C_n^m are strongly multiplicative. Vaidya and Kanani [3277] prove that the following graphs are strongly multiplicative: a cycle with one chord; a cycle with twin chords (that is, two chords that share an endpoint and with opposite endpoints that join two consecutive vertices of the cycle; the cycle C_n with three chords that form a triangle and whose edges are the edges of two 3-cycles and a $n-3$-cycle. duplication of an vertex in cycle (see §2.7 for the definition); and the graphs obtained from C_n by identifying of two vertices v_i and v_j where $d(v_i, v_j) \geqslant 3$. In [3280] the same authors prove that the graph obtained by an arbitrary supersubdivision of path, a star, a cycle, and a tadpole (that is, a cycle with a path appended to a vertex of the cycle.

Krawec [1732] calls a graph G on n edges *modular multiplicative* if the vertices of G can be labeled with distinct integers $0, 1, \ldots, n-1$ (with one exception if G is a tree) such that the labels induced on the edges by the product of the end vertices modulo n are distinct. He proves that every graph can be embedded as an induced subgraph of a modular multiplicative graph on prime number of edges. He also shows that if G is a modular multiplicative graph on prime number of edges p then for every integer $k \geqslant 2$ there exist modular multiplicative graphs on p^k and kp edges that contain G as a subgraph. In the same paper, Krawec also calls a graph G on n edges *k-modular multiplicative* if the vertices of G can be labeled with distinct integers $0, 1, \ldots, n+k-1$ such that the labels induced on the edges by the product of the end vertices modulo $n+k$ are distinct. He proves that every graph is k-modular multiplicative for some k and also shows that if $p = 2n+1$ is prime then the path on n edges is $(n+1)$-modular multiplicative. He also shows that if $p = 2n+1$ is prime then the cycle on n edges is $(n+1)$-modular multiplicative if there does not exist $\alpha \in \{2, 3, \ldots, n\}$ such that $\alpha^2 + \alpha - 1 \equiv 0 \bmod p$. He concludes with four open problems. In [1733] Krawec shows that every graph is a subgraph of a modular multiplicative graph. He also defines k-modular multiplicative graphs and proves that certain families of paths and cycles admit such a labeling.

In [2261] Nasir, Idrees, Sadiq, Farooq, Kanwal, and Imran show that the join of K_1 and a triangular ladder TL_n ($n \geqslant 3$), umbrella graphs, and generalized Petersen graphs $GP(n, k)$ for ($n \leqslant 3$) and $1 \leqslant k \leqslant n/2$, double combs, and sunflower planar graphs (obtained by appending one edge to each vertex of the rim of a wheel) admit strongly multiplicative labelings.

7.16 Line-graceful Labelings

Gnanajothi [1104] has defined a concept similar to edge-graceful. She calls a graph with n vertices *line-graceful* if it is possible to label its edges with $0, 1, 2, \ldots, n$ such that when each vertex is assigned the sum modulo n of all the edge labels incident with that vertex the resulting vertex labels are $0, 1, \ldots, n-1$. A necessary condition for the line-gracefulness of a graph is that its order is not congruent to 2 (mod 4). Among line-graceful graphs are (see [pp. 132–181][1104]) P_n if and only if $n \not\equiv 2 \pmod 4$; C_n if and only if $n \not\equiv 2 \pmod 4$; $K_{1,n}$ if and only if $n \not\equiv 1 \pmod 4$; $P_n \odot K_1$ (combs) if and only if n is even; $(P_n \odot K_1) \odot K_1$ if and only if $n \not\equiv 2 \pmod 4$; (in general, if G has order n, $G \odot H$ is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H); mC_n when mn is odd; $C_n \odot K_1$ (crowns) if and only if n is even; mC_4 for all m; complete n-ary trees when n is even; $K_{1,n} \cup K_{1,n}$ if and only if n is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete n-ary tree when n is even; trees for which exactly one vertex has even degree. She conjectures that all trees with $p \not\equiv 2 \pmod 4$ vertices are line-graceful and proved this conjecture for $p \leqslant 9$.

Gnanajothi [1104] has investigated the line-gracefulness of several graphs obtained from stars. In particular, the graph obtained from $K_{1,4}$ by subdividing one spoke to form a path of even order (counting the center of the star) is line-graceful; the graph obtained from a star by inserting one vertex in a single spoke is line-graceful if and only if the star has $p \not\equiv 2 \pmod 4$ vertices; the graph obtained from $K_{1,n}$ by replacing each spoke with a path of length m (counting the center vertex) is line-graceful in the following cases: $n = 2$; $n = 3$ and $m \not\equiv 3 \pmod 4$; and m is even and $mn + 1 \equiv 0 \pmod 4$.

Gnanajothi studied graphs obtained by joining disjoint graphs G and H with an edge. She proved such graphs are line-graceful in the following circumstances: $G = H$; $G = P_n, H = P_m$ and $m+n \not\equiv 0 \pmod 4$; and $G = P_n \odot K_1$, $H = P_m \odot K_1$ and $m+n \not\equiv 0 \pmod 4$.

In [3284] and [3285] Vaidya and Kothari proved following graphs are line graceful: fans F_n for $n \not\equiv 1 \pmod 4$; W_n for $n \not\equiv 1 \pmod 4$; bistars $B_{n,n}$ if and only if for $n \equiv 1, 3 \pmod 4$; helms H_n for all n; $S'(P_n)$ for $n \equiv 0, 2 \pmod 4$; $D_2(P_n)$ for $n \equiv 0, 2 \pmod 4$; $T(P_n)$, $M(P_n)$, alternate triangular snakes, and graphs obtained by duplication of each edge of P_n by a vertex are line graceful graphs.

7.17 k-sequential Labelings

In 1981 Bange, Barkauskas, and Slater [419] defined a *k-sequential* labeling f of a graph $G(V, E)$ as one for which f is a bijection from $V \cup E$ to $\{k, k+1, \ldots, |V \cup E|+k-1\}$ such that for each edge xy in E, $f(xy) = |f(x) - f(y)|$. This generalized the notion of *simply sequential* where $k = 1$ introduced by Slater. Bange, Barkauskas, and Slater showed that cycles are 1-sequential and if G is 1-sequential, then $G + K_1$ is graceful. Hegde and Shetty [1225] have shown that every T_p-tree (see §4.4 for the definition) is 1-sequential. In [3000], Slater proved: K_n is 1-sequential if and only if $n \leqslant 3$; for $n \geqslant 2$, K_n is not k-sequential for any $k \geqslant 2$; and $K_{1,n}$ is k-sequential if and only if k divides n. Acharya and Hegde [42] proved: if G is k-sequential, then k is at most the independence number of G; P_{2n} is n-sequential for all n and P_{2n+1} is both n-sequential and $(n+1)$-sequential for all n; $K_{m,n}$ is k-sequential for $k = 1, m$, and n; $K_{m,n,1}$ is 1-sequential; and the join of any caterpillar and $\overline{K_t}$ is 1-sequential. Acharya [28] showed that if $G(E, V)$ is an odd graph with $|E| + |V| \equiv 1$ or $2 \pmod 4$ when k is odd or $|E| + |V| \equiv 2$ or $3 \pmod 4$ when k is even, then G is not k-sequential. Acharya also observed that as a consequence of results of Bermond, Kotzig, and Turgeon [547] we have: mK_4 is not k-sequential for any k when m is odd and mK_2 is not k-sequential for any odd k when $m \equiv 2$ or $3 \pmod 4$ or for any even k when $m \equiv 1$ or $2 \pmod 4$. He further noted that $K_{m,n}$ is not k-sequential when k is even and m and n are odd, whereas $K_{m,k}$ is k-sequential for all k. Acharya points out that the following result of Slater's [3001] for $k = 1$ linking k-graceful graphs and k-sequential graphs holds in general: A graph is k-sequential if and only if $G + v$ has a k-graceful labeling f with $f(v) = 0$. Slater [3000] also proved that a k-sequential graph with p vertices and $q > 0$ edges must satisfy $k \leqslant p-1$. Hegde [1211] proved that every graph can be embedded as an induced subgraph of a simply sequential graph. In [28] Acharya conjectured that if G is a connected k-sequential graph of order p with $k > \lfloor p/2 \rfloor$, then $k = p-1$ and $G = K_{1,p-1}$ and that, except for $K_{1,p-1}$, every tree in which all vertices are odd is k-sequential for all odd positive integers $k \leqslant p/2$. In [1211] Hegde gave counterexamples for both of these conjectures.

In [1223] Hegde and Miller prove the following: for $n > 1$, K_n is k-sequentially additive if and only if $(n,k) = (2,1), (3,1)$ or (3,2); $K_{1,n}$ is k-sequentially additive if and only if k divides n; caterpillars with bipartition sets of sizes m and n are k-sequentially additive for $k = m$ and $k = n$; and if an odd-degree (p,q)-graph is k-sequentially additive, then $(p+q)(2k+p+q-1) \equiv 0 \pmod 4$. As corollaries of the last result they observe that when m and n are odd and k is even $K_{m,n}$ is not k-sequentially additive and if an odd-degree tree is k-sequentially additive then k is odd.

In [2779] Seoud and Jaber proved the following graphs are 1-sequentially additive: graphs obtained by joining the centers of two identical stars with an edge; $S_n \cup S_m$ if and only if nm is even; $C_n \odot \overline{K_m}$; $P_n \odot \overline{K_m}$; kP_3; graphs obtained by joining the centers of k copies of P_3 to each vertex in $\overline{K_m}$; and trees obtained from K by replacing each edge by a path of length 2 when $n \equiv 0, 1 \pmod 4$. They also determined all 1-sequentially additive graphs of order 6.

7.18 IC-colorings

For a subgraph H of a graph G with vertex set V and a coloring f from V to the natural numbers define $f_s(H) = \Sigma f(v)$ over all $v \in H$. The coloring f is called an IC-*coloring* if for any integer k between 1 and $f_s(G)$ there is a connected subgraph H of G such that $f_s(H) = k$. The IC-*index* of a graph G, $M(G)$, is $\max\{f_s |\ f_s$ is an IC-coloring of $G\}$. Salehi, Lee, and Khatirinejad [2683] obtained the following: $M(K_n) = 2^n - 1$; for $n \geqslant 2$, $M(K_{1,n}) = 2^n + 2$; if Δ is the maximum degree of a connected graph G, then $M(G) \geqslant 2^{\Delta} + 2$; if $ST(n; 3^n)$ is the graph obtained by identifying the end points of n paths of length 3, then $ST(n; 3^n)$ is at least $3^n + 3$ (they conjecture that equality holds for $n \geqslant 4$); for $n \geqslant 2$, $M(K_{2,n}) = 3 \cdot 2^n + 1$; $M(P_n) \geqslant (2 + \lfloor n/2 \rfloor)(n - \lfloor n/2 \rfloor) + \lfloor n/2 \rfloor - 1$; for $m, n \geqslant 2$, the IC-index of the double star $DS(m,n)$ is at least $(2^{m-1}+1)(2^{n-1}+1)$ (they conjecture that equality holds); for $n \geqslant 3$, $n(n+1)/2 \leqslant M(C_n) \leqslant n(n-1)+1$; and for $n \geqslant 3$, $2^n + 2 \leqslant M(W_n) \leqslant 2^n + n(n-1) + 1$. They pose the following open problems: find the IC-index of the graph obtained by identifying the endpoints of n paths of length b; find the IC-index of the graph obtained by identifying the endpoints of n paths; and find the IC-index of $K_{m,n}$. Shiue and Fu [2936] completed the partial results by Penrice [2379] Salehi, Lee, and Khatirinejad [2683] by proving $M(K_{m,n}) = 3 \cdot 2^{m+n-2} - 2^{m-2} + 2$ for any $2 \leqslant m \leqslant n$.

7.19 Minimal k-rankings

A k-*ranking* of a graph is a labeling of the vertices with the integers 1 to k inclusively such that any path between vertices of the same label contains a vertex of greater label.

The *rank number of a graph* G, $\chi_r(G)$, is the smallest possible number of labels in a ranking. A k-ranking is *minimal* if no label can be replaced by a smaller label and still be a k-ranking. The concept of the rank number arose in the study of the design of very large scale integration (VLSI) layouts and parallel processing (see [788], [1922] and [2754]). Ghoshal, Laskar, and Pillone [1099] were the first to investigate minimal k-rankings from a mathematical perspective. Laskar and Pillone [1800] proved that the decision problem corresponding to minimal k-rankings is NP-complete. It is HP-hard even for bipartite graphs [803]. Bodlaender, Deogun, Jansen, Kloks, Kratsch, Müller, Tuza [582] proved that the rank number of P_n is $\chi_r(P_n) = \lfloor \log_2(n) \rfloor + 1$ and satisfies the recursion $\chi_r(P_n) = 1 + \chi_r(P_{\lceil (n-1)/2 \rceil})$ for $n > 1$. The following results are given in [803]: $\chi_r(S_n) = 2$; $\chi_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2$; $\chi_r(W_n) = \lfloor \log_2(n-3) \rfloor + 3 (n > 3)$; $\chi_r(K_n) = n$; the complete t-partite graph with n vertices has ranking number $n+1$ - the cardinality of the largest partite set; and a split graph with n vertices has ranking number $n+1$ - the cardinality of the largest independent set (a *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set.) Wang proved that for any graphs G and H $\chi_r(G+H) = \min\{|V(G)| + \chi_r(H), |V(H) + \chi_r(G)\}$.

In 2009 Novotny, Ortiz, and Narayan [2305] determined the rank number of P_n^2 from the recursion $\chi_r(P_n^2) = 2 + \chi_r(P(_{\lceil (n-2)/2 \rceil})$ for $n > 2$. They posed the problem of determining $\chi_r(P_m \times P_n)$ and $\chi(P_n^k)$. In 2009 [170] and [169] Alpert determined the rank numbers of $P_n^k,\ C_n^k,\ P_2 \times C_n,\ K_m \times P_n,\ P_3 \times P_n$, Möbius ladders and found bounds for rank numbers of general grid graphs $P_m \times P_n$. About the same time as Alpert and independently, Chang, Kuo, and Lin [667] determined the rank numbers of $P_n^k,\ C_n^k,\ P_2 \times P_n, P_2 \times C_n$. Chang et al. also determined the rank numbers of caterpillars and proved that for any graphs G and H $\chi_r(G[H]) = \chi_r(H) + |V(H)|(\chi_r(G) - 1)$.

In 2010 Jacob, Narayan, Sergel, Richter, and Tran [1350] investigated k-rankings of paths and cycles with pendent paths of length 1 or 2. Among their results are: for any caterpillar G $\chi_r(P_n) \leqslant \chi_r(G) \leqslant \chi_r(P_n) + 1$ and both cases occur; if $2^m \leqslant n \leqslant 2^{m+1}$ then for any graph G obtained by appending edges to an n-cycle we have $m+2 \leqslant \chi_r(G) \leqslant m+3$ and both cases occur; if G is a lobster with spine P_n then $\chi_r(P_n) \leqslant \chi_r(G) \leqslant \chi_r(P_n) + 2$ and all three cases occur; if G a graph obtained from the cycle C_n by appending paths of length 1 or 2 to any number of the vertices of the cycle then $\chi_r(P_n) \leqslant \chi(G) \leqslant \chi(P_n) + 2$ and all three cases occur; and if G the graph obtained from the comb obtained from P_n by appending one path of length m to each vertex of P_n then $\chi_r(G) = \chi_r(P_n) + \chi_r(P_{m+1}) - 1$.

Sergel, Richter, Tran, Curran, Jacob, and Narayan [2806] investigated the rank number of a cycle C_n with pendent edges, which they denote by CC_n, and call a *caterpillar cycle*. They proved that $\chi(CC_n) = \chi_r(C_n))$ or $\chi(CC_n) = \chi_r(C_n))+1$ and showed that both cases occur. A *comb tree*, denoted by $C(n,m)$, is a tree that has a path P_n such that every vertex of P_n is adjacent to an end vertex of a path P_m. In the comb tree $C(n,m)$ $(n \geqslant 3)$ there are 2 pendent paths P_{m+2} and $n-2$ paths P_{m+1}. They proved $\chi_r(C(n,m)) = \chi_r(P_{m+1}) - 1$. They define a *circular lobster* as a graph where each vertex is either on a cycle C_n or at most distance two from a vertex on C_n. They proved that if G is a lobster with longest path P_n, then $\chi_r(P_n) \leqslant \chi_r(G) \leqslant \chi_r(P_n) + 2$ and determined the conditions under which each true case occurs. If G is circular lobster with cycle C_n, they showed that

$\chi_r(C_n) \leqslant \chi_r(G) \leqslant \chi_r(C_n) + 2$ and determined the conditions under which each true case occurs. An *icicle graph* I_n $(n \geqslant 3)$ has three pendent paths P_2 and is comprised of a path P_n with vertices $v_1, v_2, \dots, v_n$ where a path P_{i-1} is appended to vertex v_i. They determine the rank number for icicle graphs.

Richter, Leven, Tran, Ek, Jacob, and Narayan [2623] define a *reduction* of a graph G as a graph G_S^* such that $V(G_S^*) = V(G) \setminus S$ and, for vertices u and v, uv is an edge of G_S^* if and only if there exists a $u - v$ path in G with all internal vertices belonging to S. A vertex *separating set* of a connected graph G is a set of vertices whose removal disconnects G. They define a *bent ladder* $BL_n(a,b)$ as the union of ladders L_a and L_b (where $L_n = P_n \times P_2$) that are joined at a right angle with a single L_2 so that $n = a+b+2$. A staircase ladder SL_n is a graph with $n-1$ subgraphs $G_1, G_2, \dots, G_{n-1}$ each of which is isomorphic to C_4. (They are ladders with a maximum number of bends.) Richter et al. [2623] prove: $\chi_r(BL_n(a,b)) = \chi_r(L_n) - 1$ if $n = 2^k - 1$ and $a \equiv 2$ or $3 \pmod 4$ and is equal to $\chi_r(L_n)$ otherwise; $\chi_r(SL_n) = \chi_r(L_{n+1})$ if $n = 2^k + 2^{k-1} - 2$ for some $k \geqslant 3$ and is equal to $\chi_r(L_n)$ otherwise; and for any ladder L_n with multiple bends, the rank number is either $\chi_r(L_n)$ or $\chi_r(L_n) + 1$).

The *arank number* of a graph G is the maximum value of k such that G has a minimal k-ranking. Eyabi, Jacob, Laskar, Narayan, and Pillone [912] determine the arank number of $K_n \times K_n$, and investigated the arank number of $K_m \times K_n$.

7.20 Set Graceful and Set Sequential Graphs

The notions of set graceful and set sequential graphs were introduced by Acharaya in 1983 [29]. A graph is called *set graceful* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph such that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge, the assignment of sets to the vertices is injective, and the assignment to the edges is bijective. A graph is called *set sequential* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph such that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge and the the assignment of sets to the vertices and the edges is bijective. The following has been shown: P_n $(n > 3)$ is not set graceful [1215]; C_n is not set sequential [43]; C_n is set graceful if and only if $n = 2^m - 1$ [1217] and [29]; K_n is set graceful if and only if $n = 2, 3$ or 6 [2184]; K_n $(n \geqslant 2)$ is set sequential if and only if $n = 2$ or 5 [1217]; $K_{a,b}$ is set sequential if and only if $(a+1)(b+1)$ is a positive power of 2 [1217]; a necessary condition for $K_{a,b,c}$ to be set sequential is that a, b, and c cannot have the same parity [1215]; $K_{1,b,c}$ is not set sequential when b and c even [1217]; $K_{2,b,c}$ is not set sequential when b and c are odd [1215]; no theta graph is set graceful [1215]; the complete nontrivial n-ary tree is set sequential if and only if $n+1$ is a power of 2 and the number of levels is 1 [1215]; a tree is set sequential if and only if it is set graceful [1215]; the nontrivial plane triangular grid graph G_n is set graceful if and only if $n = 2$ [1217]; every graph can be embedded as an induced subgraph of a connected set sequential graph [1215]; every graph can be embedded as an induced subgraph of a connected set graceful graph [1215], every planar graph can be embedded as an induced subgraph of a set sequential planar graph [1217]; every tree can be embedded as an induced subgraph of a set sequential tree [1217]; and every tree can be embedded as an induced subgraph of a set graceful tree [1217]. Hegde conjectures [1217] that no path is set sequential. Hegde's conjecture [1218] that every complete bipartite graph that has a set graceful labeling is a star was proved by Vijayakumar [3383]. Shahida and Sunitha [2849] prove that the concept of set-gracefulness is equivalent to topologically set-gracefulness in trees and almost all finite trees are not set-graceful. Using this they characterize topologically set-graceful stars and topologically set-graceful paths. In [34] Acharya and Germina survey results on set-valuations of graphs and give open problems and conjectures.

Germina, Kumar, and Princy [1070] prove: if a (p,q)-graph is set-sequential with respect to a set with n elements, then the maximum degree of any vertex is $2^{n-1}-1$; if G is set-sequential with respect to a set with n elements other than K_5, then for every edge uv with $d(u)=d(v)$ one has $d(u)+d(v)<2^{n-1}-1$; $K_{1,p}$ is set-sequential if and only if p has the form $2^{n-1}-1$ for some $n\geqslant 2$; binary trees are not set-sequential; hypercubes Q_n are not set-sequential for $n>1$; wheels are not set-sequential; and uniform binary trees with an extra edge appended at the root are set-graceful and set graceful.

Vijayakumar [3383] and Gyri, Balister, and Schelp [328] proved that if a complete bipartite graph G has a set-graceful labeling, then it is a star. Abhishek [8] described a method for constructing a set-graceful bipartite graph of arbitrarily large order and size beginning with a set-graceful bipartite graph. Acharya, Germina, Princy, and Rao [39] proved that $K_{1,m,n}$ is set-graceful if and only if $m=2^s-1$ and $n=2^t-1$ and almost all graphs are not set-graceful. In [9] Abhishek surveys results on set-valued graphs. Many open problems and conjectures are included.

Acharya [29] has shown: a connected set graceful graph with q edges and $q+1$ vertices is a tree of order $p=2^m$ and for every positive integer m such a tree exists; if G is a connected set sequential graph, then $G+K_1$ is set graceful; and if a graph with p vertices and q edges is set sequential, then $p+q=2^m-1$. Acharya, Germina, Princy, and Rao [39] proved: if G is set graceful, then $G\cup\overline{K_t}$ is set sequential for some t; if G is a set graceful graph with n edges and $n+1$ vertices, then $G+\overline{K_t}$ is set graceful if and only if m has the form 2^t-1; $P_n+\overline{K_m}$ is set graceful if $n=1$ or 2 and m has the form 2^t-1; $K_{1,m,n}$ is set graceful if and only if m has the form 2^t-1 and n has the form 2^s-1; $P_4+\overline{K_m}$ is not set graceful when m has the form 2^t-1 ($t\geqslant 1$); $K_{3,5}$ is not set graceful; if G is set graceful, then graph obtained from G by adding for each vertex v in G a new vertex v' that is adjacent to every vertex adjacent to v is not set graceful; and $K_{3,5}$ is not set graceful.

Acharya, Germina, Abhishek, and Slater [36] prove C_m is set-graceful if and only if $m=(4^n-1)/3$; mK_2 is set-sequential if and only if $m=(4^n-1)/3$; and, for $r+s=2^{n-1}$ the bistar $B(r,s)$ is set-sequential if and only if r and s are odd. They also prove that connected planar graphs with an even number of faces, regular polyhedrons, and cacti containing an odd number of cycles are not set-sequential.

Abhishek [8] proved that if G is a set-sequential bipartite graph and H is $2k$-set-sequential, then $4^kG\cup H$ is set-sequential. As a corollary, he gets mP_3 is set-sequential if and only if $m=(16^n-1)/5$. Abhishek and Agustine [11] characterized the set-sequential caterpillars of diameter four and give a necessary condition for a graph to be set-sequential. Abhishek [10] characterized the set-sequential caterpillars of diameter five. Golowich and Kim [1117] showed that all caterpillars T of diameter k such that $k \leqslant 18$ or $|V(T)| \geqslant 2^{k-1}$ are set-sequential, where T has only odd-degree vertices and $|V(T)| = 2^{n-1}$ for some n. They also provided a new method of recursively constructing set-sequential trees.

In [2130] Mehra and Puneet introduce a topological integer additive set-labeling of signed graphs as follows. Let $S = (V, E, s)$ be a signed graph with corresponding graph $G = (V, E)$ and the signature function s. Here, G is an integer additive set-labeled graph having an injective function $f : V(G) \to P(X) - \{\emptyset\}$ that produces another injective function $g_f : E(G) \to P(X) - \{\emptyset, \{0\}\}$ defined by $g_f(uv) = f(u) + f(v)$ for every edge uv, where X is the subset of non-negative integers, $P(X)$ is its power set, and the signature function defined as $s : E(G) \to \{+, -\}$ is such that $s(uv) = -1^{|f(u)+f(v)|}$ for all edges uv. If $f(V(G)) \cup \{\emptyset\}$ forms a topology on X then the signed graph S is called a *topological integer additive set-labeled signed graph* (T-IASL). They proved the following graphs have T-IASL labelings: paths, stars, double stars, tadpoles, and graphs obtained by identifying an end of a path with the center of a star.

7.21 Vertex Equitable Graphs

Given a graph G with q edges and a labeling f from the vertices of G to the set $\{0, 1, 2, \ldots, \lceil q/2 \rceil\}$ define a labeling f^* on the edges by $f^*(uv) = f(u) + f(v)$. If for all i and j and each vertex the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the edge labels induced by f^* are $1, 2, \ldots, q$, Lourdusamy and Seenivasan [2016] call a f a *vertex equitable* labeling of G. They proved the following graphs are vertex equitable: paths, bistars, combs, n-cycles for $n \equiv 0$ or 3 (mod 4), $K_{2,n}$, ${C_3}^t$ for $t \geqslant 2$, quadrilateral snakes, $K_2 + mK_1$, $K_{1,n} \cup K_{1,n+k}$ if and only if $1 \leqslant k \leqslant 3$, ladders, arbitrary super divisions of paths, and n-cycles with $n \equiv 0$ or 3 (mod 4). They further proved that $K_{1,n}$ for $n \geqslant 4$, Eulerian graphs with n edges where $n \equiv 1$ or 2 (mod 4), wheels, K_n for $n > 3$, triangular cacti with $q \equiv 0$ or 6 or 9 (mod 12), and graphs with p vertices and q edges, where q is even and $p < \lceil q/2 \rceil + 2$ are not vertex equitable. Lourdusamy and Patrick [2010] prove that triangular ladders TL_n, $L_n \odot mK_1$, $Q_n \odot K_1$, $TL_n \odot K_1$, and alternate triangular snakes $A(T_n)$ are vertex equitable graphs. In [52] Acharya, Jain, and Kansal introduced vertex equitable labelings of signed graphs and studied vertex equitable behavior of signed paths, signed stars, and signed complete bipartite graphs $K_{2,n}$.

Jeyanthi and Maheswari [1463] proved that the following graphs have vertex equitable labelings: the square of the bistar $B_{n,n}$; the splitting graph of the bistar $B_{n,n}$; C_4-snakes; connected graphs for in which each block is a cycle of order divisible by 4 (they need not be the same order) and whose block-cut point graph is a path; $C_m \odot P_n$; tadpoles; the one-point union of two cycles; and the graph obtained by starting friendship graphs, $C_{n_1}^{(2)}, C_{n_2}^{(2)}, \ldots, C_{n_k}^{(2)}$ where each $n_i \equiv 0$ (mod 4) and joining the center of $C_{n_i}^{(2)}$ to the center of $C_{i+1}^{(2)}$ with an edge for $i = 1, 2, \ldots, k-1$. In [1454] Jeyanthi and Maheswari prove that T_p trees, bistars $B(n, n+1)$, $C_n \odot K_m$, ${P_n}^2$, tadpoles, certain classes of caterpillars, and

$T \odot \overline{K_n}$ where T is a T_p tree with an even number of vertices are vertex equitable. Jeyanthi and Maheswari [1457] gave vertex equitable labelings for graphs constructed from T_p trees by appending paths or cycles.

In [1453] Jeyanthi and Maheswari show a number of families of graphs have vertex equitable labelings. Their results include: armed crowns $C_m \odot P_n$, shadow graphs $D_2(K_{1,n})$; the graph $C_m * C_n$ obtained by identifying a single vertex of a cycle graph C_m with a single vertex of a cycle graph C_n if and only if $m+n \equiv 0, 3 \pmod 4$; for $n \equiv 0 \pmod 4$ the graph obtained from m copies of $C_n * C_n$ and P_m by joining each vertex of P_m with the cut vertex in one copy of $C_n * C_n$; and graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle; the total graph of P_n; the splitting graph of P_n; and the fusion of two edges of C_n.

For a graph H with vertices $v_1, v_2, \ldots, v_n$ and n copies of a graph G, $H \ \widehat{o}\ G$ is a graph obtained by identifying a vertex u_i of the ith copy of G with a vertex v_i of H for $1 \leqslant i \leqslant n$. The graph $H \ \tilde{o}\ G$ is a graph obtained by joining a vertex u_i of the ith copy of G with a vertex v_i of H by an edge for $1 \leqslant i \leqslant n$. Jeyanthi, Maheswari, and Laksmi prove [1479] that the graphs $L_m \ \hat{o}\ nC_4$, $L_m \ \tilde{o}\ nC_4$, $C_m \ \tilde{o}\ nC_4$, and $P_m \ \tilde{o}\ nC_4$ are vertex equitable graphs. The graph $S^*(G)$ is obtained from a graph G by replacing every edge e of G with $K_{2,m}$ $(m \geqslant 2)$ with the endpoints of e merged with the two vertices of the 2-vertices part of $K_{2,m}$ after removing the edge e from G. Jeyanthi, Maheswari, and Vijaya Laksmi [1475] prove the graphs $S^*(P_n \cdot K_1)$, $S^*(B(n,n))$, $S^*(P_n \times P_2)$, and $S^*(Q_n)$ of the quadrilateral snake are vertex equitable.

In [1461] Jeyanthi and Maheswari proved the double alternate triangular snake $DA(T_n)$ obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i and w_i is vertex equitable; the double alternate quadrilateral snake $DA(Q_n)$ obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i, x_i and w_i, y_i respectively and then joining v_i, w_i and x_i, y_i is vertex equitable; and $NQ(m)$ the n^{th} quadrilateral snake obtained from the path $u_1, u_2, \ldots, u_m$ by joining u_i, u_{i+1} with $2n$ new vertices v_j^i and $w_j^i, 1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant n$ is vertex equitable. Jeyanthi and Maheswari [1473] prove $DA(T_n) \odot K_1$, $DA(T_n) \odot 2K_1$, $DA(T_n)$, $DA(Q_n) \odot K_1$, $DA(Q_n) \odot 2K_1$, and $DA(Q_n)$ are vertex equitable.

Jeyanthi, Maheswari, and Vijayalaksmi [1474] proved the following graphs are vertex equitable: jewel graphs J_n with vertex set $\{u, v, x, y, u_i : 1 \leqslant i \leqslant n\}$ and edge set $\{ux, uy, xy, xv, yv, uu_i, vu_i : 1 \leqslant i \leqslant n\}$; jelly fish graphs $(JF)_n$ with vertex set $\{u, v, u_i, v_j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-2\}$ and edge set $\{uu_i : 1 \leqslant i \leqslant n\} \cup \{vv_j : 1 \leqslant j \leqslant n-2\} \cup \{u_{n-1}u_n, vu_n, vu_{n-1}\}$; lobsters constructed from the path $a_1, a_2, \ldots, a_n$ with verticies a_{i1} and a_{i2} adjacent to a_i for $1 \leqslant i \leqslant n$ and pendent vertices $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^k$ joining a_{ij} for $1 \leqslant i \leqslant n$ and $j = 1, 2$; $L_n \odot \overline{K_m}$; and the graph obtained from ladder a L_n and $2n$ copies of $K_{1,m}$ by identifying a non-central vertex of ith copy of $K_{1,m}$ with ith vertex of L_n.

Jeyanthi, Mahewari, and Vijaya Laksmi [1470] prove the following graphs are vertex equitable: graphs obtained by joining a vertex of a cycle to a degree 2 vertex of a comb $(P_n \odot K_1)$ with an edge; path unions of quadrilateral snakes; cycle unions of n copies of mC_4-snakes where $n \equiv 0, 3 \bmod 4$; the graphs obtained from a path $u_1, u_2, \ldots, u_m$ by joining the end points of each edge u_iu_{i+1} to $2n$ isolated vertices v_j^i, w_j^i for $1 \leqslant m-1,\ 1 \leqslant j \leqslant n$, where n is even (the nth quadrilateral snake).

Jeyanthi, Maheswari, and Vijaya Laksmi [1470] prove that subdivisions of double triangular snakes $S(D(Tn))$, subdivisions of double quadrilateral snakes $S(D(Q_n))$, subdivisions of double alternate triangular snakes $S(DA(T_n))$, subdivisions of double alternate quadrilateral snakes $S(DA(Q_n))$, $DA(Q_m) \odot nK_1$, and $DA(T_m) \odot nK_1$ admit vertex equitable labelings.

The *super subdivision* graph $S^*(G)$ of a graph G is the graph obtained from G by replacing every edge uv of G by $K_{2,m}$ (m may vary for each edge) and identifying u and v with the two vertices in $K_{2,m}$ that form the partite set with exactly two members. Jeyanthi, Maheswari, and Vijayalaksmi [1475] prove that super subdivision graphs of $P_n \odot K_1$, bistars $B(n,n)$, $P_n \times P_2$, and quadrilateral snakes are vertex equitable.

For a graph H with vertices $v_1, v_2, \ldots, v_n$ and n copies of a graph G, $H \ \widehat{o}\ G$ is a graph obtained by identifying a vertex u_i of the ith copy of G with a vertex v_i of H for $1 \leqslant i \leqslant n$. The graph $H \ \widetilde{o}\ G$ is a graph obtained by joining a vertex u_i of the ith copy of G with a vertex v_i of H by an edge for $1 \leqslant i \leqslant n$. Jeyanthi, Maheswari, and Laksmi [1479] prove that the graphs $L_m\ \hat{o}\ nC_4,\ L_m\ \tilde{o}\ nC_4,\ C_m\ \tilde{o}\ nC_4$ and $P_m\ \tilde{o}\ nC_4$ are vertex equitable graphs.

For a graph G with p vertices and q edges and $A = \{1, 3, \ldots, q\}$ if q is odd or $A = \{1, 3, \ldots, q+1\}$ if q is even, Jeyanthi, Maheswari and Vijaya Laksmi [1469] say a vertex labeling f from $V(G)$ to A is an *odd vertex equitable even labeling* if the induced edge labeling f^* defined by $f^*(uv) = f(u)+f(v)$ for all edges uv has the property that for all u and v in A the number of vertices labeled with u and the number of vertices labeled with v differ by at most 1 and the induced edge labels are $2, 4, \ldots, 2q$. A graph that admits odd vertex equitable even labeling is called an *odd vertex equitable even* graph. They show that the following graphs have odd vertex equitable even lableings: paths, graphs obtained by identifying an endpoint of P_m with each vertex of P_n, $K_{1,n}$ if and only if $n = 1$ or 2, $K_{1,n} \cup K_{1,n-2}$ $(n \geqslant 3)$, $K_{2,n}$, T_p-trees, C_n when $n \equiv 0$ or 1 mod 4, quadrilateral snakes, ladders L_n, $L_n \odot K_1$, and arbitrary super subdivision of paths. They prove that if every edge of a graph G is an edge of a triangle, then G is not an odd vertex equitable even graph. As a corollary of this they get that the following are not odd vertex equitable even graphs: K_n $(n \geqslant 3)$, wheels, triangular snakes, double triangular snakes, triangular ladders, flower graphs, fans $P_n \odot K_1$ $(n \geqslant 2)$, double fans $P_n \odot K_2$, $(n \geqslant 2)$, friendship graphs C_n^3, windmills K_m^n $(m > 3)$, $K_2 + mK_1$, $B_{n,n}^2$, total graphs $T(P_n)$, and composition graphs $P_n[P_2]$. They also show that if G is a (p,q) graph with $p \leqslant \lceil q/2 \rceil + 1$, then G is not an odd vertex equitable even graph. Jeyanthi and Maheswari [1464] proved that the subdivision of double triangular snakes and the subdivision of double quadrilateral snakes are odd vertex equitable even graphs.

Lourdusamy and Patrick [2009] proved that $P_n \odot mK_1$, the quadrilateral snake attached to each vertex of path P_n, the super splitting graph $S^*(P_n \odot K_1)$, the super splitting graphs of ladders and the bistars $B_{n,n}$, $B_{n,n}^2$, and the splitting $S'(B_{n,n})$ admit even vertex equitable even labelings. Maheswari and Jeyanthi [2058] have shown the following graphs admit odd vertex equitable even labelings: duplicate graphs of ladders; duplicate graphs of the subdivision of ladders; duplicate graphs of a quadrilaterals; and $P_3 \times P_n$ for odd $n > 2$. Lourdusamy, Wency, and Patrick [2024] prove that $S(D(Q_n)), S(D(T_n)), DA(Q_m) \odot nK_1, DA(T_m) \odot nK_1, S(DA(Q_n))$ and $S(DA(T_n))$ are an even vertex equitable even graphs.

For graphs G_1 and G_2 that graph $G_1 \hat{O} G_2$ is obtained from G_1 and $|VG_1)|$ copies of G_2 by identifying one vertex of ith copy of G_2 with ith vertex of G_1. Jeyanthi, Maheswari, and Vijayalakshmi [1481] proved the following graphs have odd vertex equitable even labelings: subdivision graphs of ladders, $L_m \hat{O} P_n$, $L_n \odot \overline{K_m}$ $(m > 1)$, C_n if and only if $n \equiv 0$ or 1 (mod 4), $K_{1,n+k} \cup K_{1,n}$ i if and only if $k = 1, 2$, and $\langle L_n \hat{O} K_{1,m} \rangle$.

Motivated by the concept of vertex equitable labeling first defined by Lourdusamy and Seenivasan in [2016], Lourdusamy, Mary, and Patrick [2007] introduced the concept of even vertex equitable even labeling as follows. Let G be a graph with p vertices and q edges and $A = \{0, 2, 4, \ldots, q+1\}$ if q is odd or $A = \{0, 2, 4, \ldots, q\}$ if q is even. A graph G is said to be an *even vertex equitable even* labeling if there exists a vertex labeling f from $V(G)$ to A that induces an edge labeling f defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leqslant 1$ and the induced edge labels are $2, 4, \ldots, 2q$, where $v_f(a)$ is the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits even vertex equitable even labeling is called an *even vertex equitable even* graph. They proved that paths, combs, complete bipartite graphs, cycles, $K_2 + mK_1$, bistars, ladders, $(P_n \times P_2) \odot K_1$, and the subdivision graphs of ladders and bistars $B_{n,n}$ admit an even vertex equitable even labeling. In [2013] Lourdusamy and Patrick proved that $C_m \odot P_n$, C_{4n} and C_{4n+3} with a quadrilateral snake attached to each vertex of the cycle, the graphs obtained by indentifying an edge of C_m and C_n, and the graphs obtained by duplicating an arbitrary vertex and edge of a cycle admit an even vertex equitable even labeling. Lourdusamy, Shobana Mary, and Patrick [2015] proved P_n^2, $S(P_n \odot K_1)$, $S'(P_n)$, $T(P_n)$, graphs obtained by duplication of an egde of a path, quadrilateral snakes, $D(Q_n)$, $A(T_n)$, and $DA(T_n)$ have even vertex equitable even labelings. Lourdusamy and Patrick [2009] proved that $P_n \odot mK_1$, the quadrilateral snake attached to each vertex of path P_n, the super splitting graph $S^*(P_n \odot K_1)$, the super splitting graphs of ladders and the bistars $B_{n,n}$, $B^2_{n,n}$, and the splitting $S'(B_{n,n})$ admit even vertex equitable even labelings.

7.22 Sequentially Additive Graphs

Bange, Barkauskas, and Slater [420] defined a *k-sequentially additive labeling* f of a graph $G(V, E)$ to be a bijection from $V \cup E$ to $\{k, \ldots, k + |V \cup E| - 1\}$ such that for each edge xy, $f(xy) = f(x) + f(y)$. They proved: K_n is 1-sequentially additive if and only if $n \leqslant 3$; C_{3n+1} is not k-sequentially additive for $k \equiv 0$ or 2 (mod 3); C_{3n+2} is not k-sequentially additive for $k \equiv 1$ or 2 (mod 3); C_n is 1-sequentially additive if and only if $n \equiv 0$ or 1 (mod 3); and P_n is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive. Hegde [1213] proved that $K_{1,n}$ is k-sequentially additive if and only if k divides n.

Hajnal and Nagy [1171] investigated 1-sequentially additive labelings of 2-regular graphs. They prove: kC_3 is 1-sequentially additive for all k; kC_4 is 1-sequentially additive if and only if $k \equiv 0$ or 1 (mod 3); $C_{6n} \cup C_{6n}$ and $C_{6n} \cup C_{6n} \cup C_3$ are 1-sequentially additive for all n; C_{12n} and $C_{12n} \cup C_3$ are 1-sequentially additive for all n. They conjecture that every 2-regular simple graph on n vertices is 1-sequentially additive where $n \equiv 0$ or $1 \pmod 3$.

Acharya and Hegde [44] have generalized k-sequentially additive labelings by allowing the image of the bijection to be $\{k, k+d, \ldots, (k+|V \cup E|-1)d\}$. They call such a labeling *additively* (k,d)*-sequential*.

7.23 Difference Graphs

Analogous to a sum graph, Harary [1183] calls a graph a *difference graph* if there is an bijection f from V to a set of positive integers S such that $xy \in E$ if and only if $|f(x) - f(y)| \in S$. Bloom, Hell, and Taylor [576] have shown that the following graphs are difference graphs: trees, C_n, K_n, $K_{n,n}$, $K_{n,n-1}$, pyramids, and n-prisms. Gervacio [1075] proved that wheels W_n are difference graphs if and only if $n = 3, 4$, or 6. Sonntag [3034] proved that cacti (that is, graphs in which every edge is contained in at most one cycle) with girth at least 6 are difference graphs and he conjectures that all cacti are difference graphs. Sugeng and Ryan [3089] provided difference labelings for cycles; fans; cycles with chords; graphs obtained by the one-point union of K_n and P_m; and graphs made from any number of copies of a given graph G that has a difference labeling by identifying one vertex the first with a vertex of the second, a different vertex of the second with the third and so on. Vaithilingam [3334] proved that gears, ladders, fans, [3334] new friendship graphs, helms, and wheels admit differnce labelings.

Hegde and Vasudeva [1238] call a simple digraph a *mod difference digraph* if there is a positive integer m and a labeling L from the vertices to $\{1, 2, \ldots, m\}$ such that for any vertices u and v, (u, v) is an edge if and only if there is a vertex w such that $L(v) - L(u) \equiv L(w) \pmod m$. They prove that the complete symmetric digraph and unidirectional cycles and paths are mod difference digraphs.

In [2778] Seoud and Helmi provided a survey of all graphs of order at most 5 and showed the following graphs are difference graphs: K_n, $(n \geqslant 4)$ with two deleted edges having no vertex in common; K_n, $(n \geqslant 6)$ with three deleted edges having no vertex in common; gear graphs G_n for $n \geqslant 3$; $P_m \times P_n$ $(m, n \geqslant 2)$; triangular snakes; C_4-snakes; dragons (that is, graphs formed by identifying the end vertex of a path and any vertex in a cycle); graphs consisting of two cycles of the same order joined by an edge; and graphs obtained by identifying the center of a star with a vertex of a cycle.

7.24 Square Sum Labelings and Square Difference Labelings

A bijective mapping $f : V(G)$ to $\{0, 1, 2, \ldots, |V(G)| - 1\}$ is said to be a *square sum labeling* if the induced function f^* from $E(G)$ to the positive integers defined by $f^*(xy) = (f(x)^2 + (f(y))^2$ is injective. A graph that has a square sum labeling is called a *square sum* graph. A *square differnce* graph is defined the same way with $f^*(xy) = (f(x)^2 + (f(y))^2$ replaced with $f^*(xy) = |(f(x)^2 - (f(y))^2|$. Maheswari and Srividya [2060] proved that every cycle C_n $(n \geqslant 6)$ with parallel P_3 chords admit a vertex odd mean labeling, a vertex even mean labeling, and a square sum labeling. Maheswari, Azhagarasi, and Samuvel

[2055] proved cycles with at least 6 vertices with parallel P_4 chords are vertex odd mean graphs and vertex even mean graphs and they admit square sum labelings, and square difference labelings. In [2059] they proved that the following graphs are square sum graphs: cycles with parallel chords, graphs obtained by attaching an arbitrary number of pendant edges at a vertex of degree 2 of a cycle with parallel chords, duplication of a vertex of degree 2 of a cycle with parallel chords, crowns with parallel chords, chains of even cycles with parallel chords, and graphs obtained from copies of C_n by joining a vertex from each copy of C_n to a common vertex. Patel and Ghodasara [2359] proved that the graph obtained by joining two copies of a specific graph by a path of arbitrary length admits a square sum labeling. They gave some results about the square sum graphs of arbitrary super subdivisions. Zhang, Naeem, Tariq, and Zhao [3585] investigated the square sum labeling of generalized Petersen graphs and double generalized Petersen graphs. Shiama [2887] proved that the total graph of paths and cycles, and the middle graphs of paths and cycles, admit square sum labelings. Parameswari and Margret [2338] proved that [2338] new
the octopus graph and Vanessa graph admit square sum labelings. Kulli [1773] defined a [1773] new
semitotal-block graph $T_b(G)$ of a graph G as the graph whose set of vertices is the union of the set of vertices and blocks of G and in which two points are adjacent if and only the corresponding vertices of G are adjacent or the corresponding blocks are incident. Mirajkar and Sthavarmath [2154] proved the following graphs admit square sum and [2154] new
square difference labelings: $T_b(P_n \odot K_1)$ $(n \geqslant 3)$; $T_b(C_n \odot K_1)$ $(n \geqslant 3)$ except for $n = 0 \pmod 8$; and $T_b(F_n \odot K_1)$ $(n \geqslant 4)$.

Ajitha, Arumugam, and Germina [161] prove the following graphs have square sum labelings: trees; cycles; $K_2 + mK_1$; K_n if and only if $n \leqslant 5$; $C_n^{(t)}$ (the one-point union of t copies of C_n); grids $P_m \times P_n$; and $K_{m,n}$ if $m \leqslant 4$. They also prove that every strongly square sum graph except K_1, K_2, and K_3 contains a triangle.

In [2886] Shiama proved that the total graphs of paths and cycles and the middle graphs of paths and cycles are square sum graphs. A *lilly* graph is otained by identifying one endpoint of each of two copies of P_n with the center of $K_{1,2n}$ $(n > 1)$. Samuvel and Kalaivani proved [2692] the following graphs are square sum graphs: the graph obtained by the duplication of the center vertex and another vertex of a lilly graph; the graph obtained by identifying any two verticies of a lilly graph, and the graph obtained by the switching of the center vertex and another vertex of a lilly graph. Ghodasara and Patel [1084] proved the following graphs are square sum graphs: restricted square graphs; splitting graphs and shadow graphs of the bistar $B_{n,n}$; the restricted total, the restricted middle, and the degree splitting graph of $B_{n,n}$; and the duplication of a vertex and arbitrary super subdivision of $B_{n,n}$. Subhashini, Ramanathan, and Manimekalai [3062] proved that pyramids with at least 3 rows, hanging pyramids, graphs obtained from starting with r copies of a hanging pyramid and joining each copy with the next one with an edge, and the one point union of r copies of a pyramid and hanging pyramids admit square sum labelings. Maheswari, Azhagarasi, and Samuvel [2056] showed that the following graphs are square sum graphs: the corona product of P_m $(m \geqslant 2)$ and C_{2n} $(n \geqslant 3)$ with P_3 parallel chords; the corona product P_m $(m \geqslant 2)$ and C_{2n+1} $(n \geqslant 3)$ with P_3 parallel chords; $C_{2n} \odot K_1$ $(n \geqslant 3)$ with P_3 parallel chords; $C_{2n+1} \odot K_1$ with P_3 parallel chords; the chain of cycle $C_{2n,m}$ $(n \geqslant 3)$

with P_3 parallel chords (a chain of m copies of C_{2n} with P_3 parallel chords where each copy shares exactly one vertex of the next copy); and edge connected cycle C_{2n} $(n \geqslant 3)$ with P_3 chords.

In [67] Agasthi and Narvathi provided ways to construct square sum, square difference, root mean square, strongly multiplicative, even mean and odd mean labelings for triangular snakes, and the central graph of triangular snake graphs. [67] new

In [1085] Ghodasara and Patel gave a counterexample to the conjecture by Germina and Sebastian[1074] that if G_1 and G_2 are square sum graphs then $G_1 \cup G_2$ is a square sum graph. They proved that the duplication graphs of any vertex of the following graphs are square sum graphs: K_n if and only if $n \leqslant 7$, the Petersen graph $P(5, 2)$, $K_{1,n}$, and C_n They also proved that cycle C_n with $[\frac{n}{2}]$ concurrent chords is a square sum graph.

In [1081] Ghodasara and Patel proved that the following constructions based on the bistar $B_{n,n}$ are square sum graphs: the restricted square, the splitting graph, the shadow graph, the degree splitting graph, the arbitrary super subdivision graph, and the duplication of any vertex of $B_{n,n}$. They defined restricted total graph of $B_{n,n}$ as a graph with vertex set $= V(B_{n,n}) \cup E(B_{n,n}) = \{u, v, w, u_i, v_i, u_i', v_i'/1 \leqslant i \leqslant n\}$, where u and v are apex vertices, u_i and v_i are pendent vertices, w, u_i' and v_i' are vertices corresponding to the edges of $B_{n,n}$ and edge set $= E(B_{n,n}) \cup \{uw, vw, wu_i', wv_i', uu_i', vv_i', u_iu_i', v_iv_i', /1 \leqslant i \leqslant n\}$. They also defined restricted middle graph of $B_{n,n}$ as a graph with vertex set $= V(B_{n,n}) \cup E(B_{n,n}) = \{u, v, w, u_i, v_i, u_i', v_i'/1 \leqslant i \leqslant n\}$, where u and v are apex vertices, u_i and v_i are pendent vertices, w, u_i' and v_i' are vertices corresponding to the edges of $B_{n,n}$ and edge set $= \{uw, vw, wu_i', wv_i', uu_i', vv_i', u_iu_i', v_iv_i', /1 \leqslant i \leqslant n\}$. They proved that restricted total graph and restricted middle graph of $B_{n,n}$ are square sum graphs.

Germina and Sebastian [1073] proved that the following graphs are square sum graphs: trees; unicyclic graphs; mC_n; cycles with a chord; the graphs obtained by joining two copies of cycle C_n by a path P_k; and graphs that are a path union of k copies of C_n and the path is P_2. In [2766] Seoud and Al-Harere give several necessary conditions for a graph to be a square sum graph and show that $2C_n, P_{2n}$, and C_{2n} are square sum graphs. Huilgol and Sriram [1274] prove that if G_1 and G_2 are square sum, then $G_1 \cup G_2 \cup G_3$ is also square sum, where G_3 is a set of isolated vertices.

In [3021] Somashekara and Veena used the term "square sum labeling" to mean "strongly square sum labeling." They proved that the following graphs have strongly square sum labelings: paths, $K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$, complete n-ary trees, and lobsters obtained by joining centers of any number of copies of a star to a new vertex. They observed that that if every edge of a graph is an edge of a triangle then the graph does not have strongly square sum labeling. As a consequence, the following graphs do not have a strongly square sum labelings: $K_n, n \geqslant 3$; wheels; fans $P_n + K_1$ $(n \geqslant 2)$; double fans $P_n + K_2$ $(n \geqslant 2)$; friendship graphs $C_3^{(n)}$; windmills $K_m^{(n)}$ $(m > 3)$; triangular ladders; triangular snakes; double triangular snakes; and flowers. They also proved that helms are not strongly square sum graphs and the graphs obtained by joining the centers of two wheels to a new vertex are not strongly square sum graphs.

Govindan, Pinelas, and Dhivya [1142] introduced the notion a *cube sum* graph as a graph $G(V, E)$ with p vertices and q edges such that if there exists a bijection f : [1142] new

$V \to \{0, 1, 2, \ldots, p-1\}$, then the induced function $f^* : E \to N$, defined by $f(^*uv) = (f(u))^3 + (f(v))^3$, is injective. They proved that paths, cycle, stars, wheels, and fans are cube sum graphs. Pate and Ghodasara [2356] proved that the following graphs are cube sum graphs: trees, gears, shell graphs, helms, K_n if and only if n is at most 11, and $K_{2,n}$ for all n. [2356] new

In [3030] Sonchhatra and Ghodasara call a (p, q)-graph $G = (V, E)$ *sum perfect square* if there exists a bijection f from V to $\{0, 1, 2, \ldots, p-1\}$ such that the function f^* from E defined by $f^*(uv) = (f(u)) + (f(v))^2$ for all edges uv is an injection. Such an f is called a *sum perfect square* labeling of G. In a series of four papers the following graphs are proved to be sum perfect square graphs: cycles, cycles with one chord, cycles with twin chords, trees [3030]; several snake related graphs [3031]; $K_{1,n} + K_1$, $K_2 + mK_1$, $C_n \odot K_1$, graphs obtained from $K_{1,n}$ with endpoint verticies $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} with an edge for $i = 1, 2, \ldots, v_{\lfloor n/2 \rfloor}$ ("half wheel"), the middle graphs of paths, the total graphs of paths [3032]; P^2 $(n > 1)$, $mK_{1,n}$, mC_n,, and the splitting graph and the shadow graph of a star [3029]. In [3032] they prove that the union of two stars and that for any sum perfect square graph G, $G \cup P_n$ is sum perfect square. They conjecture that the union of any two sum perfect square graphs is sum perfect square.

Ajitha, Princy, Lokesha, and Ranjini [127] defined a graph $G(p, q)$ to be a *square difference graph* if there exist a bijection f from $V(G)$ to $\{0, 1, 2, \ldots, p-1\}$ such that the induced function f^* from $E(G)$ to the natural numbers given by $f^*(uv) = |(f(u))^2 - (f(v))^2|$ for every edge uv of G is a bijection. Such a the function is called a *square difference labeling* of the graph G. They proved that following graphs have square difference labelings: paths, stars, cycles, K_n if and only if $n \leqslant 5$, $K_{m,n}$ if $m \leqslant 4$, friendship graphs $C_3^{(n)}$, triangular snakes, and $K_2 + mK_1$. They also prove that every graph can be embedded as a subgraph of a connected square difference graph and conjecture that trees, complete bipartite graphs and $C_k^{(n)}$ are square difference graphs.

Tharmaraj and Sarasija [3204] proved that following graphs have square difference labelings: fans F_n $(n \geqslant 2)$; $P_n + \overline{K_2}$; the middle graphs of paths and cycles; the total graph of a path; the graphs obtained from m copies of an odd cycle and the path P_m with consecutive vertices $v_1, v_2, \ldots, v_m$ by joining the vertex v_i to a vertex of the i^{th} copy of the odd cycle; and the graphs obtained from m copies of the star S_n and the path P_m by joining the vertex v_i of P_m to the center of the i^{th} copy of S_n. Sebastian and Germina [2731] proved that certain planar graphs and higher order level joined planar grid admit square sum labeling. They also study square sum properties of several classes of graphs with many odd cycles.

In [1066] Geetha and Kalamani showed that the following graphs have square difference labelings: two copies of the same star whose centers are joined by a path; and two copies of the same cycles whose centers are joined by a path, the restricted square of bistar $B_{n,n}$; the restricted total graph of bistar $B_{n,n}$; the restricted middle graph of $B_{n,n}$ Shiama [2889] showed that cycles, complete graphs, cycle cacti, ladders, lattice grids, quadrilateral snakes, $K_2 + mK_1$ admit square difference labelings.

Vaghela and Parmar [3239] say a graph G admits a *difference perfect square cordial* labeling if there is a bijection $f : V(G) \to \{1, 2, \ldots, |V(G|\}$ such that for each edge

uv the induced map $f^* : E(G) \to \{0, 1\}$ defined by $f^*(uv) = 1$ if $u^2 - 2uv + v^2 = 1$, and 0 otherwise, has the property that the number of edges labeled with 0 and the number of edges labeled with 1 difference by at most 1. A graph that admits a difference perfect square cordial labeling is said to be a *difference perfect square cordial* graph. They obtained difference perfect square cordial labelings for paths, cycles, wheels, fans, combs, crowns, $(C_m \odot K_1) \cup (P_n \odot K_1)$, $D_2(P_n)$, P_n^2, $K_2 \odot C_n$, graphs consisting of two copies of C_n that share a common edge, the vertex switching of C_n, and the graph obtained by starting with P_n $(n \geqslant 6)$ and two new vertices u and v of P_n and joining v to first two vertices and last two vertices of P_n and joining u to the remaining vertices of P_n (called the *shipping graph*) In [3240] Vaghela and Parmar provided difference perfect square cordial labelings of the H-graphs of paths, some corona graphs, total graphs of paths, and graphs obtained from $P_n \times P_2$ where P_n has conscutive vertices $v_1, v_2, \ldots, v_n$ by joining v_i in left P_n to v_{i+1} in the right copy of P_n with an edge for $i = 1, 2, \ldots, n-1$. In [3241] Vaghela and Parmar obtain difference perfect square cordial labeling of triangular snake graphs, quadrilateral snake graphs, alternate triangular snake graphs, alternate quadrilateral snake graphs, irregular triangular snake graphs, irregular quadrilateral snake graphs, double triangular snake graphs, double quadrilateral snake graphs, double alternate triangular snake graphs, and double alternate quadrilateral snake graphs.

A graph G is said to admit a *square difference labeling* if there exists a injection $f : V(G) \to \{0, 1, 2, \ldots, n-1\}$ for some n such that when each edge of G is assigned the absolute square difference of its end-vertices, the resulting labels are distinct. *Cube difference labeling* are defined analogously.

Shiana [2888] proved that paths, cycles, stars, fans, wheels, crowns, helms, dragons, coconut trees, and shell graphs admit cube difference labelings. Sharon Philomena and Thirusangu [2871] proved the cycle cactus graph $C_n^{(3)}$, the tree of diameter 4 obtained from the bistar $B_{n,n}$ by subdividing the middle edge with a new vertex, and the graph obtained by joining one vertex of a cycle and one vertex of degree 2 of a comb by an edge have square and cube difference labelings (that is, the absolute cube difference of end-vertices of the edges are distinct). Sherman [2885] proved the path union of nC_3 and the disjoint union of m stars $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_m}$ are square difference graphs

Subashini, Bhuvaneswari, and Manimekalai [3059] proved the following graphs have square difference labelings: theta graphs, the duplication of any vertex of degree 3 in the cycle of a theta graph, the one point union of any number of theta graphs, the path union of any number of copies of a theta graph, the fusion of any two vertices in the cycle of a theta graph, and the switching of a central vertex of a theta graph.

7.25 Permutation and Combination Graphs

Hegde and Shetty [1232] define a graph G with p vertices to be a *permutation graph* if there exists a injection f from the vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f defined by $g_f(uv) = f(u)!/|f(u) - f(v)|!$ is injective. They say a graph G with p vertices is a *combination graph* if there exists a injection f from the vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f defined as $g_f(uv) = f(u)!/|f(u) - f(v)|!f(v)!$ is injective. They prove: K_n is a permutation graph if and only if $n \leqslant 5$; K_n is a combination graph if and only if $n \leqslant 5$; C_n is a combination graph for $n > 3$; $K_{n,n}$ is a combination graph if and only if $n \leqslant 2$; W_n is a not a combination graph for $n \leqslant 6$; and a necessary condition for a (p, q)-graph to be a combination graph is that $4q \leqslant p^2$ if p is even and $4q \leqslant p^2 - 1$ if p is odd. They strongly believe that W_n is a combination graph for $n \geqslant 7$ and all trees are combinations graphs. Baskar Babujee and Vishnupriya [507] prove the following graphs are permutation graphs: P_n; C_n; stars; graphs obtained adding a pendent edge to each edge of a star; graphs obtained by joining the centers of two identical stars with an edge or a path of length 2); and complete binary trees with at least three vertices. Seoud and Salim [2790] determine all permutation graphs of order at most 9 and prove that every bipartite graph of order at most 50 is a permutation graph. Seoud and Mahran [2781] give an upper bound on the number of edges of a permutation graph and introduce some necessary conditions for a graph to be a permutation graph. They show that these conditions are not sufficient for a graph to be a permutation graph.

Ghodasara and Patel [1083] proved that the following graphs are permutation graphs: the Petersen graph $P(5, 2)$, trees, $K_{3,n}$ ($n \geqslant 1$) for $n+3$ prime, W_n ($n \geqslant 3$) for $n+1$ prime, shell graph S_n ($n \geqslant 3$) for prime n, dumbbell graph $D_{n,k,2}$ ($n, k \geqslant 3$), $C_n \odot K_1$ ($n \geqslant 3$), and the one point union $C_n^{(k)}$ ($k \geqslant 2, n \geqslant 3$) of k copies of cycle C_n. A t-ply $P_t(u, v)$ is a graph with t paths, each of length at least two and such that no two paths have a vertex in common except for the end vertices u and v. Ghodasara and Patel defined t^*-ply $P_{t^*}(u, v)$ as a special case of t-ply $P_t(u, v)$ graph with every t path have same length and proved that t^*-ply $P_{t^*}(u, v)$ is a permutation graph.

The graph obtained from two copies of an (m, n) kite graph by connecting the degree 1 vertex of one copy to the vertex of degree 3 and the second coppy is called the 1-*join (m,n) kite*. The graph ontained by repeating this construction with t copies of an (m, n) kite is called a is called the *t-join (m,n) kite*. Sriramr and Govindarajan [3047] proved t-join (m, n) kites are permutation graphs.

Ghodasara and Patel [1082] proved that the following graphs are combination graphs: $C_n \times P_2$ for $n \geqslant 6$, umbrella graph $U(m, n)$ for $m, n > 2$, armed crown $C_n \oplus P_m$ for $n \geqslant 4$ and $m \geqslant 1$, the graphs obtained by joining C_{2m} ($m \geqslant 2$) to each pendent vertex of $K_{1,n}(n \geqslant 2)$, the duplication of any rim vertex of W_n for $n \geqslant 7$, C_n with $[\frac{n-4}{2}]$ concurrent chords for $n \geqslant 6$, and the duplication of vertex in C_n for $n \geqslant 5$.

Hegde and Shetty [1232] say a graph G with p vertices and q edges is a *strong k-combination graph* if there exists a bijection f from the vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f from the edges to $\{k, k+1, \ldots, k+q-1\}$ defined by $g_f(uv) = f(u)!/|f(u) - f(v)|!f(v)!$ is a bijection. They say a graph G with p vertices and q edges is a *strong k-permutation graph* if there exists a bijection f from the vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f from the edges to $\{k, k + 1, \ldots, k+q-1\}$ defined by $g_f(uv) = f(u)!/|f(u)-f(v)|!$ is a bijection. Seoud and Anwar [2768] provided necessary conditions for combination graphs, permutation graphs, strong k-combination graphs, and strong k-permutation graphs.

Seoud and Al-Harere [2767] showed that the following families are combination graphs: graphs that are two copies of C_n sharing a common edge; graphs consisting of two cycles of the same order joined by a path; graphs that are the union of three cycles of the same order; wheels W_n ($n \geqslant 7$); coronas $T_n \odot K_1$, where T_n is the triangular snake; and the graphs obtained from the gear G_m by attaching n pendent vertices to each vertex which is not joined to the center of the gear. They proved that a graph $G(n, q)$ having at least 6 vertices such that 3 vertices are of degree $1, n-1, n-2$ is not a combination graph, and a graph $G(n, q)$ having at least 6 vertices such that there exist 2 vertices of degree $n-3$, two vertices of degree 1 and one vertex of degree $n-1$ is not a combination graph.

Seoud and Al-Harere [2765] proved that the following families are combination graphs: unions of four cycles of the same order; double triangular snakes; fans F_n if and only if $n \geqslant 6$; caterpillars; complete binary trees; ternary trees with at least 4 vertices; and graphs obtained by identifying the pendent vertices of stars S_m with the paths P_{n_i}, for $1 \leqslant n_i \leqslant m$. They include a survey of trees of order at most 10 that are combination graphs and proved the following graphs are not combination graphs: bipartite graphs with two partite sets with $n \geqslant 6$ elements such that $n/2$ elements of each set have degree n; the splitting graph of $K_{n,n}$ ($n \geqslant 3$); and certain chains of two and three complete graphs. Seoud and Anwar [2768] proved the following graphs are combination graphs: dragon graphs (the graphs obtained from by joining the endpoint of a path to a vertex of a cycle); triangular snakes T_n ($n \geqslant 3$); wheels; and the graphs obtained by adding k pendent edges to every vertex of C_n for certain values of k.

In [2764] and [2765] Seoud and Al-Harere proved the following graphs are non-combination graphs: $G_1 + G_2$ if $|V(G_1)|, |V(G_2)| \geqslant 2$ and at least one of $|V(G_1)|$ and $|V(G_2)|$ is greater than 2; the double fan $\overline{K_2} + P_n$; $K_{l,m,n}$; $K_{k,l,m,n}$; $P_2[G]$; $P_3[G]$; $C_3[G]$; $C_4[G]$; $K_m[G]$; $W_m[G]$; the splitting graph of K_n ($n \geqslant 3$); K_n ($n \geqslant 4$) with an edge deleted; K_n ($n \geqslant 5$) with three edges deleted; and $K_{n,n}$ ($n \geqslant 3$) with an edge deleted. They also proved that a graph $G(n, q)$ ($n \geqslant 3$) is not a combination graph if it has more than one vertex of degree $n-1$.

In [3206] and [3205] Tharmaraj and Sarasija defined a graph $G(V, E)$ with p vertices to be a *beta combination* graph if there exist a bijection f from $V(G)$ to $\{1, 2, \ldots, p\}$ such that the induced function B_f from $E(G)$ to the natural numbers given by $B_f(uv) = (f(u) + f(v))!/f(u)!f(v)!$ for every edge uv of G is injective. Such a function is called a *beta combination* labeling. They prove the following graphs have beta combination labelings: K_n if and only if $n \leqslant 8$; ladders L_n ($n \geqslant 2$); fans F_n ($n \geqslant 2$); wheels; paths; cycles; friendship graphs; $K_{n,n}$ ($n \geqslant 2$); trees; bistars; $K_{1,n}$ ($n > 1$); triangular snakes; quadrilateral snakes; double triangular snakes; alternate triangular snakes (graphs obtained from a path $v_1, v_2, \ldots, v_n$, where for each odd $i \leqslant n-1$, v_i and v_{i+1} are joined to a new vertex $u_{i,i+1}$; alternate quadrilateral snakes (graphs obtained from a path $v_1, v_2, \ldots, v_n$, where for each odd $i \leqslant n-1$, v_i and v_{i+1} are joined to two new vertices $u_{i,i+1,1}$ and $u_{i,i+1,2}$); helms; gears; combs $P_n \odot K_1$; and coronas $C_n \odot K_1$.

7.26 Strongly *-graphs

A variation of strong multiplicity of graphs is a strongly *-graph. A graph of order n is said to be a *strongly *-graph* if its vertices can be assigned the values $1, 2, \ldots, n$ in such a way that, when an edge whose vertices are labeled i and j is labeled with the value $i + j + ij$, all edges have different labels. Adiga and Somashekara [59] have shown that all trees, cycles, and grids are strongly *-graphs. They further consider the problem of determining the maximum number of edges in any strongly *-graph of given order and relate it to the corresponding problem for strongly multiplicative graphs. In [2783] and [2784] Seoud and Mahan give some technical necessary conditions for a graph to be strongly *-graph,

Baskar Babujee and Vishnupriya [507] have proved the following are strongly *-graphs: $C_n \times P_2, (P_2 \cup \overline{K}_m) + \overline{K}_2$, windmills $K_3^{(n)}$, and *jelly fish* graphs $J(m, n)$ obtained from a 4-cycle v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4.

Baskar Babujee and Beaula [489] prove that cycles and complete bipartite graphs are vertex strongly *-graphs. Baskar Babujee, Rajesh Kannan, and Vishnupriya [500] prove that wheels, paths, fans, crowns, $(P_2 \cup mK_1) + \overline{K_2}$, and umbrellas (graphs obtained by appending a path to the central vertex of a fan) are vertex strongly *-graphs.

In [2785] Seoud, Roshdy, and AboShady gave an upper bound for the number of edges of any graph in terms of the number of vertices to be a strongly *-graph and some new families to be strongly*- graphs. They also provided an algorithm for checking if a graph is a strongly *-graph or not.

7.27 Triangular Sum Graphs

Hegde and Shankaran [1227] call a labeling of graph with q edges a *triangular sum labeling* if the vertices can be assigned distinct non-negative integers in such a way that, when an edge whose vertices are labeled i and j is labeled with the value $i + j$, the edges labels are $\{k(k + 1)/2|\ k = 1, 2, \ldots, q\}$. They prove the following graphs have triangular sum labelings: paths, stars, complete n-ary trees, and trees obtained from a star by replacing each edge of the star by a path. They also prove that K_n has a triangular sum labeling if and only if n is 1 or 2 and the friendship graphs $C_3^{(t)}$ do not have a triangular sum labeling. They conjecture that K_n $(n \geqslant 5)$ are forbidden subgraphs of graph with triangular sum labelings. They conjectured that every tree admits a triangular sum labeling. They show that some families of graphs can be embedded as induced subgraphs of triangular sum graphs. They conclude saying "as every graph cannot be embedded as an induced subgraph of a triangular sum graph, it is interesting to embed families of graphs as an induced subgraph of a triangular sum graph". In response, Seoud and Salim [2787] showed the following graphs can be embedded as an induced subgraph of a triangular sum graph: trees, cycles, nC_4, and the one-point union of any number of copies of C_4 (friendship graphs).

Vaidya, Prajapati, and Vihol [3298] showed that cycles, cycles with exactly one chord, and cycles with exactly two chords that form a triangle with an edge of the cycle can be embedded as an induced subgraph of a graph with a triangular sum labeling. They proved that several classes of graphs do not have triangular sum labelings. Among them are: helms, graphs obtained by joining the centers of two wheels to a new vertex, and graphs in which every edge is an edge of a triangle. As a corollary of the latter result they have that $P_m + \overline{K_n}$, $W_m + \overline{K_n}$, wheels, friendship graphs, flowers, triangular ladders, triangular snakes, double triangular snakes, and flowers. do not have triangular sum labelings.

Seoud and Salim [2787] proved the following are triangular sum graphs: $P_m \cup P_n,\ m \geqslant 4$; the union of any number of copies of $P_n,\ n \geqslant 5$; $P_n \odot \overline{K_m}$; symmetrical trees; the graph obtained from a path by attaching an arbitrary number of edges to each vertex of the path; the graph obtained by identifying the centers of any number of stars; and all trees of order at most 9.

For a positive integer i the ith *pentagonal number* is $i(3i-1)/2$. Somashekara and Veena [3022] define a *pentagonal sum labeling* of a graph $G(V,E)$ as one for which there is a one-to-one function f from $V(G)$ to the set of nonnegative integers that induces a bijection f^+ from $E(G)$ to the set of the first $|E|$ pentagonal numbers. A graph that admits such a labeling is called a *pentagonal sum graph*. Somashekara and Veena [3022] proved that the following graphs have pentagonal sum labelings: paths, $K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$, complete n-ary trees, and lobsters obtained by joining centers of any number of copies of a star to a new vertex. They conjecture that every tree has a pentagonal sum labeling and as an open problem they ask for a proof or disprove that cycles have pentagonal labelings. They observed that if every edge of a graph is an edge of a triangle then the graph does not have pentagonal sum labeling. As was the case for triangular sum labelings the following graphs do not have a pentagonal sum labeling: $P_m + \overline{K_n}$, and $W_m + \overline{K_n}$ wheels, friendship graphs, flowers, triangular ladders, triangular snakes, double triangular snakes, and flowers. Somashekara and Veena [3022] also proved that helms and the graphs obtained by joining the centers of two wheels to a new vertex are not pentagonal sum graphs.

7.28 Divisor Graphs

Santhosh and Singh [2715] call a graph $G(V,E)$ a *divisor graph* if V is a set of integers and $uv \in E$ if and only if u divides v or vice versa. They prove the following are divisor graphs: trees; mK_n; induced subgraphs of divisor graphs; cocktail party graphs $H_{m,n}$ (see Section 7.1) for the definition); the one-point union of complete graphs of different orders; complete bipartite graphs; W_n for n even and $n > 2$; and $P_n + \overline{K_t}$. They also prove that C_n $(n \geqslant 4)$ is a divisor graph if and only if n is even and if G is a divisor graph then for all n so is $G + K_n$.

Chartrand, Muntean, Saenpholphat, and Zhang [685] proved complete graphs, bipartite graphs, complete multipartite graphs, and joins of divisor graphs are divisor graphs. They also proved if G is a divisor graph, then $G \times K_2$ is a divisor graph if and only if G is a bipartite graph; a triangle-free graph is a divisor graph if and only if it is bipartite; no divisor graph contains an induced odd cycle of length 5 or more; and that a graph G is divisor graph if and only if there is an orientation D of G such that if (x,y) and (y,z) are edges of D then so is (x,z).

In [141] and [143] Al-Addasi, AbuGhneim, and Al-Ezeh determined precisely the values of n for which P_n^k $(k \geqslant 2)$ are divisor graphs and proved that for any integer $k \geqslant 2$, C_n^k is a divisor graph if and only if $n \leqslant 2k+2$. In [25] they gave a characterization of the graphs G and H for which $G \times H$ is a divisor graph and a characterization of which block graphs are divisor graphs. (Recall a graph is a *block graph* if every one of its blocks is complete.) They showed that divisor graphs form a proper subclass of perfect graphs and showed that cycle permutation graphs of order at least 8 are divisor graphs if and only if they are perfect. (Recall a graph is *perfect* if every subgraph has chromatic number equal to the order of its maximal clique.) In [142] Al-Addasi, AbuGhneim, and Al-Ezeh proved that the contraction of a divisor graph along a bridge is a divisor graph; if e is an edge of a divisor graph that lies on an induced even cycle of length at least 6, then the contraction along e is not a divisor graph; and they introduced a special type of vertex splitting that yields a divisor graph when applied to a cut vertex of a given divisor graph.

AbuHijleh, AbuGhneim, and Al-Ezeh [25] prove that for any tree T, T^2 is a divisor graph if and only if T is a caterpillar and the diameter of T is less than six. For any caterpillar T and a positive integer k with $\text{diam}(T) < 2k$, they show that T^k is a divisor graph. Moreover, for a caterpillar T and $k \geqslant 3$ with $\text{diam}(T) = 2k$ or $\text{diam}(T) = 2k + 1$, they show that T^k is a divisor graph if and only if the centers of T have degree two. In [26] AbuHijleh, AbuGhneim, and Al-Ezeh prove that the k-th power Q_n^k of Q_n is a divisor graph if and only if $n = 2, 3$ or $n \geqslant 4$ and $k \geqslant n-1$ hold. In the case of the n-dimensional folded-hypercube FQ_n (that is, the graph obtained from Q_n by adding to it a perfect matching that connects opposite pairs of the vertices of Q_n) they show that FQ_n is a divisor graph for odd n, but not for even $n \geqslant 4$. They also prove $(FQ_n)^k$ is not a divisor graph if and only if $2 \leqslant k \leqslant \lceil n/2 \rceil$, where $n \geqslant 5$.

Ganesan and Uthayakumar [1012] proved that $G \odot H$ is a divisor graph if and only if G is a bipartite graph and H is a divisor graph. Frayer [953] proved $K_n \times G$ is a divisor graph for each n if and only if G contains no edges and $\overline{K_n \times K_2}$ $(n \geqslant 3)$ is a divisor graph. Vinh [3402] proved that for any $n > 1$ and $0 \leqslant m \leqslant n(n-1)/2$ there exists a divisor graph of order n and size m. She also gave a simple characterization of divisor graphs due to Chartrand, Muntean, Saenpholphat, and Zhang [685]. Gera, Saenpholphat, and Zhang [1068] established forbidden subgraph characterizations for all divisor graphs that contain at most three triangles. Tsao [3230] investigated the vertex-chromatic number, the clique number, the clique cover number, and the independence number of divisor graphs and their complements. In [2774] Seoud, El Sonbaty, and Mahran discuss here some necessary and sufficient conditions for a graph to be a divisor graph.

In [1660] Kasthuri, Karuppasamy, and Nagarajan introduced the notions of SD-divisor labelings and SD-divisor cordial labelings as follows. Let G be finite, simple graph G with n vertices and f be a bijection from $V(G)$ to $\{1, 2, \ldots, n\}$. Define a function f' induced by f by assigning $f'(uv) = 1$ if $S = f(u) + f(v)$ divides $D = |f(u) - f(v)|$ and $f'(uv) = 0$, otherwise. They say that f' is a SD-*divisor* labeling if $f'(uv) = 1$ for all edges uv. In [2620] Revathi and Mary Jeya Jothi [2620] provided SD-divisor labelings for paths and cycles and certain path and cycle related graphs. They further find SD-divisor labelings of some subdivision graphs, coronas, and splitting graphs, and prove that stars, complete graphs, and wheels do not admit SD-divisor labelings.

A graph G with n vertices has a *modular multiplicative divisor* (MMD) labeling if there exist a bijection f from vertices of G to the set of all natural numbers from 1 to n such that when the edge uv is labeled $f(u)f(v)(\mathrm{mod}\, n)$, then n divides the sum of all edge labels of G. They prove that r-dimensional butterfly networks admit MMD labengs. Revathi, Dharmakkan, and Jeya Jothi [2618] proved that $K_{l,m,n}$ admits a modular multiplicative divisor labeling. Revathi and Rajeswari [2621] proved P_n, $P_{a,b}$ (the graph that connects two vertices by means of b internally disjoint paths of length a each), the shadow graph of a path, and $P_n \times P_1$, (where n is not a multiple of 6) admit modular multiplicative divisor labelings. They also discuss the upper bound for the number of edges in a modular multiplicative divisor graphs. In [2619] Revathi and Rajeswari proved that the split graphs of cycles, helms, and flower graphs admit modular multiplicative divisor labelings. [2621] new [2619] new

A *divisor 3-equitable* labeling of a graph G is a bijection $d : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that induced map d^* defined on the edges of G by, for any edge uv with $d(u) \leqslant d(v)$, $d^*(uv)) = 1$ if $d(v)/d(u) = 1$; $d^*(uv) = 2$ if $d(v)/d(u) = 2$; and $d^*(uv) = 0$, otherwise, has the property that the number of edges labeled with i and the number of edges labeled with j differ by at most 1 for all i and j. A graph that admits a divisor 3-equitable labeling is called a *divisor 3-equitable* graph. The existence and non-existence of divisor 3-equitable labelings have been determined for wheels [1373], complete graphs and stars [1374], ladders, triangular snakes, and lollipop graphs [2706], and the degree splitting graph of ladders [2591]. [1373] new [1374] new [2706] new [2591] new

7.29 Other Kinds of Labelings

Zumkeller labelings

A positive integer n is said to be a *Zumkeller* number if all the positive factors of n can be partitioned into two disjoint parts so that the sum of the two parts are equal. (For example, 6 partitions as $\{1, 2, 3\}$ and $\{6\}$ and 20 partitions as $\{2, 12\}$ and $\{1, 3, 4, 6\}$. An injective function f from the vertices of a graph G to the natural numbers N is said to be a *Zumkeller* labeling of G, if the induced function $f^* : E(G) \to N$ defined by $f^*(xy) = f(x)f(y)$ is a Zumkeller number for all edges xy. A graph that admits a Zumkeller labeling is called a *Zumkeller* graph. Balamurugan, Thirusangu, Thomas, and Murali [403] investigated the existence of Zumkeller labelings of paths, cycles, and ladders. Wilson and Bebincy [3478] proved that the splitting graph of paths, the total graphs of paths, the shadow graphs of paths, and the middle graphs of paths admit Zumkeller labelings. Wilson and Bebincy investigated the existence of Zumkeller labelings for closed helms, double wheels, sunflowers, flower graphs, and the prisms of wheels. Patodia and Saikia [2370] provided algorithms to label complete bipartite graphs, wheels, cycles, and paths with m-Zumkeller numbers. [403] new [3478] new [2370] new

A simple graph $G(V, E)$ is said to be a *k-Zumkeller* graph if there is an injection f from the vertices of G to the natural numbers such that when each edge xy is assigned the label $f(x)f(y)$, the resulting edge labels are k distinct Zumkeller numbers. Balamurugan, Thirusangu, and Thomas [402] proved that twig graphs admit a 4-Zumkeller labelings. Basher [478] showed that the super subdivision of paths, cycles, combs, ladders, crowns, circular ladders, grids, and prism are k-Zumkeller graphs. An injection is called a *t-m-Zumkeller* labeling of a graph G if, for every edge uv, the induced function defined by $f^*(uv) = f(u)f(v)$ is an m-Zumkeller number and $|f^*(E)| = t$, where t denotes the [402] new [478] new number of distinct m-Zumkeller numbers on the edges of G. A graph that admits a $t-m$-Zumkeller labeling is called a *t-m-Zumkeller graph.*. In [2371] Patodia and Saikia gave *t-m*-Zumkeller labelings for paths, cycles, combs graphs, ladders graphs, and twigs. [2371] new

A positive integer is said to be a *half-Zumkeller* number if its proper positive divisors can be partitioned into two disjoint non-empty subsets of equal sum. A *half-Zumkeller* labeling of a graph is an injective mapping f from the vertex set into the set of natural numbers N such that the induced mapping f^* from the edges to N given by $f^*(uv) = f(u)f(v)$ is a half-Zumkeller number. A graph that admits a half-Zumkeller labeling is called a *half-Zumkeller* graph. Zeen El Deen, Elmahdy, Elkholy, and El Sherbiny [3574] gave half-Zumkeller labelings for stacked books $(K_{m,1} \times P_n)$, $P_m \times P_n$, prisms of ladders, grids, gears, and flowers. They also showed that if G is a half-Zumkeller graph and H is a non-totally disconnected subgraph of G, then H is a half-Zumkeller graph. [3574] new

In 2015 Murali, Thirusangu, and Meenakshi [2209] introduced *Zumkeller cordial* labeling of graphs as an injective function f from the vertices to the natural numbers such that the induced function f^* from the edges to $\{0, 1\}$ defined by $f^*(xy) = f(x)f(y)$ is 1 if $f(x)f(y)$ is a Zumkeller number and 0 otherwise with the condition that the number of edges labeled with 1 and the number labeled with 0 differ by at most 1. They proved the existence of Zumkeller cordial labeling for paths, cycles, and stars. Murali, Thirusangu, and Balamurugan [2207] showed the existence of Zumkeller cordial labelings for helms, wheels, flower graphs, and crowns. [2209] new [2207] new

Lucky labelings

A labeling of the verticies of a graph with natural numbers is said to be *lucky* if the sum of labels over all neighbors of vertex v and the sum of labels over all neighbors of vertex u are distinct whenever u and v are adjacent. (An isolated vertex is defined to have sum 0). The smallest nonzero label of a lucky labeling of a graph G is called the *lucky number* of the graph and is denoted by $\eta(G)$. A lucky labeling of a graph that is injective is called a *proper lucky* graph. The proper lucky number of graph G is denoted by $\eta_p(G)$. Sateesh Kumar and Meenakshi [2719] determined the proper lucky numbers for $K_{m,n}$, friendship graphs, certain triangular books, and certain rectangular books. Chiranjilal Kujur, Xavier, and Raja [1770] gave proper lucky labelings for K-identified triangular meshs and K-identified Sierpinski gasket graphs. Ahadi, Dehghan, Kazemi, and Mollaahmadi [75] proved that for a given planar 3-colorable graph G, determining whether $\eta(G) = 2$ is NP-complete, and that for every $k \geqslant 2$, it is NP-complete to decide whether $\eta(G) = k$ for a given graph G. Using algebraic methods Czerwiński, Grytczuk, and Zelazny [770] proved that $\eta(T) \leqslant 2$ for every tree T, and $\eta(G) \leqslant 3$ for every bipartite planar graph G. They obtained a bound for the lucky number in terms of the acyclic chromatic number and conjecture that for every graph G, $\eta(G) \leqslant \chi(G)$. In [1769] Kujur gave the lucky number and proper lucky number for the bloom graph (a particular planar, tripartite, 4-regular graph). Elrokh, Sateesh Kumar, and Meenaksh [820] computed the lucky numbers for jelly fish graphs, cocktail party graphs, and crowns. Sateesh Kumar and Meenakshi [2718] determined the lucky numbers and proper lucky numbers for quadrilateral snakes, double quadrilateral snakes, alternate quadrilateral [2719] new [1770] new [75] new [770] new [1769] new [820] new [2718] new snakes, and double alternate quadrilateral snakes. Xavier and Rathi [3494] determined the proper lucky numbers of hexagonal meshes and honeycomb networks. Indira, Selvam, and Thirusangu [1327] gave proper lucky labelings for extended duplicate graphs of quadrilateral snakes. [3494] new [1327] new

A labeling of the vertices of a graph with natural numbers is said to be a *lucky edge* labeling if for any two adjacent edges the sums of the vertex labels incident to the two edges are distinct. Thesmallest positive integer k for which a graph G has a lucky edge labeling from the set $\{1, 2, \ldots, k\}$ is called the *lucky number* of G and denoted by $\eta(G)$. A graph that admits a lucky edge labeling is called a *lucky edge graph*. Esakkiammal, Thirusangu, and Seethalakshmi [896] showed that the super subdivisions of stars and wheels are lucky edge labeled graphs and determined their lucky numbers. In [2212] Murugan and Chitra [2212] provided lucky edge labelings for P_n, C_n and $P_n \odot C_n$. In [2214] they proved that triangular snakes, book with triangular pages, and $P_n \times C_3$ (triangular prisms) are lucky edge graphs. Indira, Selvam, and Thirusangu [1327] gave lucky edge labelings for extended duplicate graphs of quadrilateral snakes. Chitra and Murugan proved that graphs obtained from two copies P_n in which the ith vertex of one path is joined with the $(i+1)$th vertex of the second copy, fish graphs subindexgraphsfish (the one-point union of a vertex of C_n and a vertex of K_3), butterfly graphs, double triangular snakes, flower graphs, and P_n^2 are lucky edge graphs. Nagarajan and Priyadharsini [2253] provided lucky edge labelings and lucky numbers for lotus graphs, prisms, and $C_{2m}@P_t$. Mohana priya and S Santhiya [2182] gave lucky edge labelings for graphs obtained by joining C_m and C_n with an edge, bistars, wheels, fans, and butterfly graphs. Murugan and Chitra [2213] gave lucky edge labeling of fans, spider, and twig graphs. Aishwarya [128] provided lucky edge labelings and luck numbers for ladders, shell graphs, and books with triangular pages. [896] new [2212] new [2212] new [2214] new [1327] new [2253] new [2182] new [2213] new [128] new

S_3 cordial remainder labelings

In [2025] Lourdusamy, Jenifer Wency, and Patrick introduced the concept of the group S_3 cordial remainder labeling as follows. Let $G(V, E)$ be a graph and let $g : V \to S_3$ be a function. For each edge xy assign the label r where r is the remainder when $o(g(x))$ is divided by $o(g(y))$ or $o(g(y))$ is divided by $o(g(x))$ according as $|g(x)| \geqslant |g(y)|$ or $|g(y)| \geqslant |g(x)|$. The function g is called a *group S_3 cordial remainder* labeling of G if $|v_g(x) - v_g(y)| \leqslant 1$ and $|e_g(1) - e_g(0)| \leqslant 1$, where $v_g(x)$ denotes the number of vertices labeled with x and $e_g(i)$ denotes the number of edges labeled with i $(i = 0, 1)$. A graph G that admits a group S_3 cordial remainder labeling is called a *group S_3 cordial remainder* graph . They prove the following graphs admit a group S_3 cordial remainder labeling: paths, cycles, stars, bistars, complete bipartite graphs, wheels, fans, combs, crowns, the lotus inside a circle, double fans, ladders, slanting ladders, and triangular ladders. In [1377] they proved that shadow graph of cycle and path, splitting graph of cycle, armed crown, umbrella graphs, and dumbbell graphs admit a group S_3 cordial remainder labeling. Also they proved that certain snake related graphs are a group S_3 cordial remainder graphs. In [2026, 2027, 2028], they investigated the behavior of group S_3 cordial remainder labelings of subdivision of stars, subdivision of bistars, subdivision of wheels, subdivision of combs, subdivision of crowns, subdivision of fans, subdivision of ladders, helms, flower graphs, closed helms, gears, sunflowers, triangular snakes, quadrilateral snakes, square of paths, the duplication of a vertex by a new edge in path and cycle graphs, the duplication of an edge by a new vertex in paths and cycles, and total graphs of cycles and paths.

Even sum labelings

Andharia and Kaneria [194] introduced the concept of even sum labeling as follows. A graph $G = (V, E)$ is said to admit *even sum* labeling if there exist an injective function $f : V(G) \rightarrow \{0, \pm 2, \pm 4, \ldots, \pm 2|V(G)|\}$ such that the induced mapping $f^* : E(G) \rightarrow \{2, 4, \ldots, 2|E(G)|\}$ defined by $f^*(uv) = f(u) + f(v), \forall uv \in E(G)$ is bijective. The function f is called an *even sum* labeling of G. The graph that admits even sum labeling is called an *even sum* graph. In [194], [1581], [1582], and [195] Andharia and Kaneria proved the following graphs are even sum graphs: slanting ladders SL_n $(1 < n < 9)$, P_n, C_{4n}, the complete bipartite graphs, $P_m \times P_n$, mirror graphs $M(P_n)$ $(2 \leqslant n \leqslant 6)$, jelly fish graphs, the splitting graph of stars, the degree splitting graph of stars, the splitting graph of $K_{2,n}$, the splitting graph of $K_{1,n,n}$, jewel graphs, triangular books, the triangular book graph with a bookmark, $P_m\,(+)\,\overline{K_n}$, and $(\overline{K_n} \cup P_3) + 2K_1$.

Odd-even sum labelings

Monika and Murugan [2186] proved the graphs obtained by the following duplications admit odd-even sum labelings: the apex vertex of stars, the pendent vertices of stars, the apex vertex of stars by an edge, a vertex of P_n $(n \geqslant 2)$, an edge by a vertex of P_n $(n \geqslant 3)$, a pendant vertex by an edge of P_n $(n \geqslant 2)$, and a pendent edge of P_n $(n \geqslant 2)$ by edge, They also proved that C_3 is not an odd-even sum graph. [2186] new

Prime distance labelings

Eggleton, Erdős, and Skilton, [866] call a graph a *prime distance* graph if its vertices can be labeled with distinct integers in such a way that for any two adjacent vertices, the absolute difference of their labels is a prime number. Laison, Starr, and Walker [1797] prove that trees, cycles, and bipartite graphs are prime distance graphs, and that Dutch windmill graphs and paper mill graphs are prime distance graphs if and only if the Twin Prime Conjecture and the Polignac's Conjecture that for every positive even integer n there are infinitely many cases of two consecutive prime numbers with difference n are true, respectively. Parthiban and David [2349] proved the following graphs admit prime distance labelings: the graph obtained by the duplication of every vertex by an edge in any prime distance graph H if the Twin prime conjecture is true; the graph obtained by fusing any two vertices v_i and v_j, where $d(v_i, v_j) \geqslant 3$, of C_n; the graph obtained by duplicating any vertex of C_n $(n \geqslant 6)$; and the graph obtained from m copies of C_n by joining a vertex in the ith copy of C_n to a vertex in the $(i+1)$th copy of C_n where $1 \leqslant i \leqslant m-1$.

2-odd labelings Laison, Starr, and Walker [1797] say a graph G is a 2-*odd graph* if its vertices can be labeled with distinct integers such that for any two adjacent vertices, the absolute difference of their labels is either an odd integer or 2. They give a characterization of 2-odd graphs in terms of edge colorings and use this characterization to determine which circulant graphs of a particular form are 2-odd and to prove results on circulant prime distance graphs.

Pir and Parathiban [2400] proved that diamonds, double fans, double alternate triangular snakes, *king* graphs, which consist of the $m \times n$ chessboard with edges being the moves that a king chess piece can make, and antiprisms admit 2-odd labelings.

In [16] Abirami, Parthiban, and Srinivasan prove the following graphs admit 2-odd labelings: W_n ($n \geqslant 4$), fans, graphs with $2n$ verticies obtained from an n-cycle and joining every two consecutive edges to a new vertex, and shell graphs $C(n, n-3)$ ($n \geqslant 4$). Abirami, Parthiban, and Srinivasan [15] proved the following graphs admit 2-odd labelings: helms H_n ($n \geqslant 4$), double wheels, umbrellas, graphs obtained by performing the duplication of a vertex by an edge at all the vertices of 2-odd graph if the Twin Prime conjecture is true, $K_a + mK_1$, butterfly graphs $B_{n,m}$ ($n \geqslant 3$) if Goldbach's conjecture is true, and graphs obtained from a cycle by performing duplication of a vertex by a vertex at all the vertices of the cycle if Goldbach's conjecture is true.

Absolute difference labelings

In [2857] Shalini and Dhayaran introduced the notion of *absolute differences of cubic and square difference* labelings of a graph as labelings for which every edge label is the absolute difference of the cubes of the vertices and the difference of the squares of the vertices. They proved that paths, stars, cycles, fans, windmills, and wheels admit differences of cubic and squared labelings. They also observed that the labels of the edges must be even. Shalini, Gowri, and Dhayabaran [2858] proved that barbells, cycle cactuses, coconut trees, shells, and dragons admit absolute differences of cubic and squared labelings. In [3345], [3346], [3347], [3348], [3349], [3350], [3351], [3352], [3143], [3353], and [3354] Mathew Varkey and Sunoj proved that planar grids, web graphs, kayak paddle graphs, snake graphs, armed crowns, fans, friendship graphs, windmill graphs, cycles, wheels, gears, helms, 2-tuple graphs, middle graphs, total graphs, shadow graphs barbells, $K_{2,n}^{(m)}$, and $K_{1,n} \odot 2P_m$ admit absolute difference of cubic and square sum labelings.

Quotient labelings

In [3122] Sumathi and Rathi introduced the notion of quotient labeling as follows. Let $G(V, E)$ be a finite, non-trivial, simple,undirected graph. For a one-one assignment $f : V \to \{1, 2, \ldots, |V|\}$ let $f^*(uv) = \lfloor f(u)/f(v) \rfloor$. Then f^* is a said to be a *quotient labeling* of G if $f^* : E \to \{1, 2, \ldots, |V|\}$ and $f(u) > f(v)$. (The edge labels need not be distinct.) The maximum value of $f^*(E(G))$, denoted $ql(f^*)$, is called the *q-labeling number* of G. The *quotient labeling number* of G, $QL(G)$, is the minimum value among all $ql(f^*)$. They found the quotient labeling number of the following graphs: the graph obtained by duplicating an arbitrary vertex of a cycle by an edge, the graph obtained by duplicating an arbitrary edge of cycle by a vertex, two copies C_n sharing a common edge, the twig graphs, the splitting graph of a path, the composition graph $P_m[P_2]$ and joint sum of two copies of C_n. In [3123] Sumathi and Rathi determine the quotient number of various families of ladder-related graphs. In [3124] Sumathi and Rathi provided the quotient labeling number of quadrilateral snakes, double quadrilateral snakes, alternate triangular snakes, alternate double triangular snakes, and the subdivision of triangular and quadrilateral snakes along the main path.

Oblong labelings

Oblong numbers are the numbers of dots that can be placed in rows and columns in a rectangular array, each row containing one more dot than each column. (The first five oblong numbers are 2, 6, 12, 20, and 30.) In [2518] Prema and Murugan proved the following graphs have oblong labelings: H-graphs, jelly fish, shrubs, $G \odot K_1$ where G is an H-graph that has an oblong labeling, banana trees, and graphs obtained by identifying an endpoint of $K_{1,3}$ with an end vertex of P_n. Muthumanickavel and Murugan [2234] investigated the existence of oblong sum labelings of the union of graphs involving stars and subdivisions of stars and bistars. They also prove that helms do not admit oblong labelings.

Signed cordial labelings

In 2011 two versions of cordial labelings utilizing -1 and 1 as vertex and edge labels were introduced. Devaraj and Delphy [810] defined the notion of signed cordial graphs [810] new as follows. A graph $G(V, E)$ is called *signed cordial* if the edges can be assigned -1 and 1 in such a way that when each vertex v is assigned the product of the labels of the edges incident with v the resulting graph has the property that the number of vertices labeled with i and the number of edges labeled with i differ by at most 1 for $i = -1$ and 1. A graph is called a *signed cordial* graph if it admits a signed cordial labeling. Similarly, Babujee and Loganathan [501] say graph $G(V, E)$ is *signed product cordial* if [501] new the verticies can be assigned -1 and 1 in such a way that when each edge assigned the product of its endpoints, the number of vertices labeled with i and the number of edges labeled with i differ by at most 1 for $i = -1$ and 1. A graph is called *signed product cordial* if it admits a signed product cordial labeling. In [2240] Nada, Elrokh, Elmshtaye, [2240] new and El-hay gave necessary and sufficient conditions for which cones $(\overline{K_n} + C_m)$ and their second powers are signed product cordial. In [501] Babujee and Loganathan proved that [501] new paths, cycles, stars, and bistars are signed product cordial. Devaraj and Delphy [810] [810] new investigated signed-cordiality of complete graphs, books, Jahangir graphs, flower graphs, and Petersen graphs. Cynthia and Padmavathy [768] proved that $C_n \times P_n$ and certain [768] new banana trees admit signed cordial and signed product cordial labelings.

***k*-vertex weighting**

In [2900] Shiu, Laub, and Ng [2900] introduced the following notion. For a simple, finite [2900] new and undirected graph $G(V, E)$ of order n, a k-vertex weighting of a graph G is a mapping [2900] new $w : V(G) \to \{1, \ldots, k\}$. A k-vertex weighting induces an edge labeling $f_w : E(G) \to N$ such that $f_w(uv) = w(u) + w(v)$. Such a labeling is called an *edge-coloring k-vertex weighting* if $f_w(e) \neq f_w(e')$ for any two adjacent edges e and e'. They determined the minimum k that admit an edge-coloring k-vertex weighting for the following graphs: wheels, gears, grids, $P_2 \times C_n$, lollipops, theta graphs, and two cycles joined by a path (*long dumbells*).

SD-harmonious

For a graph G with q edges, an injection $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is said to be SD-*harmonious labeling* if the induced function $f^* : E(G) \to \{0, 2, \ldots, 2q-2\}$ defined by $f(uv) = S + D \pmod{2q}$ is bijective, where $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$, for every edge uv in $E(G)$. In [2023] Lourdusamy, Wency, and Patrick investigated SD-harmonious labelings of paths, trees, star related graphs, and the disjoint union of graphs. [2023] new

Dividing graceful labeling

Zahraa, Nabeel, and Fawzi [3568] say a graph $G(V, E)$ has a *dividing graceful labeling* if there is a one-to-one mapping ϕ from V to $\{1, 2, 3, \ldots, |E|\}$ such that for every edge uv the induced function ϕ^* defined by $\phi^*(uv) = \lceil(\phi(u) + \phi(v))/|E|\rceil$ has the property that $\phi^*(E) = \{1, 2, 3, \ldots, |E|\}$. They showed the following graph have dividing graceful labelings: stars, graphs obtained by joining the centers of two copies of $K_{1,t}$, and spiders with diameter 4. [3568] new

Square product labeling

In [2156] Mirajkar and Sthavarmath introduced the notion a *square product* graph as one for which there exists a bijection f from $V(G)$ to $\{1, 2, 3, \ldots, p\}$ that induces an injective labeling f^* from $E(G)$ to N defined by $f^*(uv) = f(u)^2 f(v)^2$. In [2155] they proved that $C_n \odot K_1$ and certain kinds of cactus graphs are square product graphs. [2156] new [2155] new

***E*-super arithmetic graceful**

In [2738] Sekar and Varatharajaperumal define a (p, q) graph G to be E-*super arithmetic graceful* if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ such that $f(E(G)) = \{1, 2, \ldots, q\}$, $f(V(G) = \{q+1, q+2, \ldots, q+p\}$, and the induced mapping f^* given by $f^*(uv) = f(u) + f(v) - f(uv)$ for $uv \in E(G)$ has the range $\{p+q+1, p+q+2, \ldots, p+2q\}$. They provided E-super arithmetic graceful lablings for four families of cycle related graphs. [2738] new

Numbering of graphs

In [1286] Ichishima, Oshima, and Takahashi define a *numbering* f of a graph G of order n as a labeling that assigns distinct elements of the set $\{1, 2, \ldots, n\}$ to the vertices of G. The strength $\text{str}_f(G)$ of a numbering $f : V(G) \to \{1, 2, \ldots, n\}$ of G is defined by $\text{str}_f(G) = \max\{f(u) + f(v) \mid uv \in E(G)\}$, that is, $\text{str}_f(G)$ is the maximum edge label of G and the strength $\text{str}(G)$ of a graph G itself is $\text{str}(G) = \min\{\text{str}_f(G) \mid f \text{ is a numbering of } (G)\}$. They proved necessary and sufficient conditions for the strength of a graph G of order n to meet $\text{str}(G) = 2n - 2\beta(G) + 1$ and $\text{str}(G) = n + \delta(G) = 2n - 2\beta(G) + 1$, where $\beta(G)$ and $\delta(G)$ denote the independence number and the minimum degree of G, respectively. They answer open problems posed [1286] new by Gao, Lau and Shiu, and the earlier result leads us to determine a formula for the strength of graphs containing a particular class of graphs as a subgraph. They also extend what is known in the literature about k-stable properties. In [1285] Ichishima, López, [1285] new Muntaner-Batle, Takahashi present a sharp lower bound for the strength of a graph in terms of its domination number as well as its (edge) covering and (edge) independence number. They also provide a necessary and sufficient condition for the strength of a graph to attain the earlier bound in terms of their subgraph structure and establish a sharp lower bound for the domination number of a graph under certain conditions.

In [1287] Ichishima, Oshima, and Takahashi define an *edge numbering* f of a graph [1287] new G of size m as a labeling that assigns distinct elements of the set $\{1, 2, \dots, m\}$ to the edges of G. The *edge-strength*, $\text{estr}(G)$, of G is defined as the minimum $\text{estr}_f(G)$ where $\text{estr}_f(G) = \max\{f(e_1) + f(e_2) \mid e_1, e_2$ are adjacent edges of $G\}$ over all edge numberings of G. They present formulas for $\text{estr}(G)$ when G is the forest whose components are stars of order at least three and the complete bipartite graph whose partite sets consist of at least two vertices. The edge-strength of a graph G is the strength of the line graph of G, and thus this work extends what was known about the edge-strength and strength.

Fibonacci range labelings

A simple finite graph G with p vertices and q edges is said to be a *Fibonacci range* graph if there is a bijective $f : V(G) \to \{F_2, F_3, F_4, \dots, F_{p+1}\}$, where F_i is the ith Fibonacci number, such that the induced edge labels given by $f^*(uv) = \lceil (f(u)^2 + f(v)^2)/(f(a) + f(b)) \rceil$ or $f^*(uv) = \lfloor (f(u)^2 + f(v)^2)/(f(a) + f(b)) \rfloor$ can be assigned subscripts $1, 2, \dots, n$ in such a way that the ratios $e - 1/e_2, e_2/e_3, \dots, e_{n-1}/e_n$ approach the golden ratio $\phi = 1.618\dots$. In [2318] Odyuo and Mercy classified k-copies of shell graph $C[n, (n-3)]k$ with a union [2318] new of K_2 for $n = 4$, $2K_2$ for $n = 6$, and $3K_2$ for $n = 8$ having a common end vertex joined to the apex of the shell are Fibonacci range graphs.

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