Coloring, List Coloring, and Painting Squares of Graphs (and other related problems)

Daniel W. Cranston*

Submitted: Dec 1, 2021; Accepted: Mar 7, 2023; Published: Apr 21, 2023

Abstract

We survey work on coloring, list coloring, and painting squares of graphs; in particular, we consider strong edge-coloring. We focus primarily on planar graphs and other sparse classes of graphs.

Mathematics Subject Classifications: 05C15, 05C10, 05C76

Contents

*Department of Computer Science, Virginia Commonwealth University, Richmond, VA; dcranston@vcu.edu. This author was partially supported by NSA Grant H98230-15-1-0013.

1 Introduction

square

The *square* G^2 of a graph G is formed from G by adding an edge between each pair of vertices at distance 2 in G. Over the past 25 years, a remarkable amount of work has focused on bounding the chromatic number of squares of graphs. This problem was first studied by Kramer and Kramer [135, 136] and much of its current popularity is due to Wegner's conjecture on coloring squares of planar graphs [183], which we consider in Section 3.1.

The goal of this survey is to present major open questions on coloring squares, as well as many current best results. In addition, we provide history of most problems, often including partial results, as well as proofs that are particularly enlightening or enjoyable. Our aim is not to be exhaustive (or exhausting), but rather to help the reader get the lay of the land. In the spirit of open access, in addition to standard journal citations, whenever possible we also provide links to freely accessible, often preliminary, versions of the papers. Typically, these are on preprint servers such as the arXiv.

For many results in this survey, we will not say much about the proofs. Likewise, we de-emphasize questions of algorithms and complexity. So it is useful to say a bit now.

The most common technique for proving coloring bounds is coloring greedily in some good vertex order. More often than not, this order is constructed using the discharging method. A standard reference for discharging is *An Introduction to the Discharging Method via Graph Coloring* [64], by the author and West. Most existence proofs using the discharging method naturally yield efficient coloring algorithms. More details are given in [61, Section 6].

The second most common technique employed here is the probabilistic method. To learn about this approach, we recommend the excellent monograph *Graph Colouring and the Probabilistic Method* [151], by Molloy and Reed. At the heart of many probabilistic coloring proofs is the Lovász Local Lemma (LLL), which was originally proved nonconstructively. Much work has focused on giving constructive proofs of LLL (see [153] and the references cited there), so now these probabilistic proofs often also yield efficient algorithms. For instance, entropy compression is a way to show that a randomized algorithm runs in expected polynomial time; for example, see [148] and [77].

We assume standard graph theory terminology, as in Diestel [67], West [184], and Bondy & Murty [22]. However, notation for many types of coloring varies among authors, so we include ours below. For completeness, we also include some definitions. Most of our choices are standard, but we have a few exceptions, particularly in the penultimate paragraph of this subsection. The reader should feel free to skip ahead to Section 2 and only return to this section as needed.

proper vertex coloring
k-coloring
chromatic number, χ
k-list assignment

A *proper vertex coloring* of a graph G assigns to each vertex v of G a color, such that any two adjacent vertices get distinct colors. A *k-coloring* is a proper vertex coloring that uses at most k colors. The *chromatic number* of G, denoted $\chi(G)$, is the smallest k such that G has a k-coloring. When our meaning is clear, we often shorten $\chi(G)$ to χ, and similarly for other parameters, such as χ_ℓ. A *k-list assignment* L for a graph G assigns to each vertex v of G a list of allowable colors, $L(v)$, such that $|L(v)| = k$ for all v. A *proper*

L-*coloring* is a proper vertex coloring φ of G such that each vertex gets a color from its list, i.e., $\varphi(v) \in L(v)$ for all v. The *list chromatic number*, $\chi_\ell(G)$, is the smallest k such that G has a proper L-coloring for every k-list assignment L.

proper L-coloring

list chromatic number, χ_ℓ

The k-*painting game* is played between two players, *Lister* and *Painter*. Fix a graph G. On each round, Lister lists some non-empty subset of the unpainted vertices, and Painter colors (or "paints") some independent subset of these. If Lister lists any vertex k times without Painter painting it, then Lister wins; otherwise, Painter wins. The *paint number* of G, denoted $\chi_p(G)$, also called the *online list chromatic number* of G, is the smallest k for which Painter can always win the k-painting game on G. The term "online" derives from an *online algorithm*, which begins outputting a solution (a list coloring) before the entire input (the list assignment) is known.

k-painting game

Lister, Painter

paint number

For an orientation $\vec{D}$ of a graph G, let $EE(\vec{D})$ (resp. $EO(\vec{D})$) denote the set of spanning Eulerian subgraphs of G with an even (resp. odd) number of edges. An *Alon–Tarsi orientation* of G is one for which $|EE(\vec{D})| \neq |EO(\vec{D})|$; for convenience, we agree that EE contains the spanning subgraph with no edges. The *Alon–Tarsi number* of G, denoted $\chi_{\mathrm{AT}}(G)$, is defined by $\chi_{\mathrm{AT}}(G) := 1 + \min_{\vec{D}} \max_{v \in V(G)} d^+_{\vec{D}}(v)$, where the minimum is taken over all Alon–Tarsi orientations of G. The *coloring number* of G, denoted $\chi_{\mathrm{col}}(G)$, is defined by $\chi_{\mathrm{col}}(G) := 1 + \max_{H \subseteq G} \delta(H)$; here $\delta(H)$ is the minimum degree of H. It is useful to note that every graph G satisfies

Alon–Tarsi orientation

Alon–Tarsi number, χ_{AT}

coloring number, χ_{col}

$$\chi(G) \leqslant \chi_\ell(G) \leqslant \chi_p(G) \leqslant \chi_{\mathrm{AT}}(G) \leqslant \chi_{\mathrm{col}}(G) \leqslant \Delta(G) + 1.$$

The first two inequalities follow directly from the definitions, if the lists are identical or they are revealed progressively. The last inequality is trivial. Alon and Tarsi [6] proved that $\chi_\ell(G) \leqslant \chi_{\mathrm{AT}}(G)$ and Schauz [162] strengthened this to $\chi_p(G) \leqslant \chi_{\mathrm{AT}}(G)$. We can see that $\chi_{\mathrm{AT}}(G) \leqslant \chi_{\mathrm{col}}(G)$ as follows. Form the order $v_1, \ldots, v_n$ by starting from G and repeatedly taking v_i to be a vertex of minimum degree in the remaining subgraph, and then deleting v_i. Now form $\vec{D}$ by orienting each edge as $\overrightarrow{v_iv_j}$ where $i < j$. Since this orientation $\vec{D}$ is acyclic, it has $|EE(\vec{D})| = 1$ and $|EO(\vec{D})| = 0$. This proves that $\chi_{\mathrm{AT}}(G) \leqslant \chi_{\mathrm{col}}(G)$. (It has been shown [75, 70] that for the graph $K_{n,n}$, as $n \to \infty$, each of the differences $\chi_\ell(G) - \chi(G)$, $\chi_p(G) - \chi_\ell(G)$, $\chi_{\mathrm{AT}}(G) - \chi_p(G)$, $\chi_{\mathrm{col}}(G) - \chi_{\mathrm{AT}}(G)$, and $\Delta(G) + 1 - \chi_{\mathrm{col}}(G)$ grows arbitrarily large.)

We denote the chromatic number of the square of G by $\chi^2(G)$; more generally, $\chi^d(G)$ denotes the chromatic number of the d^{th} power of G, denoted G^d (which is formed from G by adding an edge between each pair of vertices at distance no more than d). A *strong edge-coloring* of G is a coloring of the square of the line graph of G. The strong edge-chromatic number is denoted $\chi^s(G)$. A *total coloring* of G colors vertices and edges so that elements get distinct colors whenever they are incident or adjacent. The *total chromatic number* is denoted $\chi''(G)$. An $L(p,q)$-labeling assigns labels (positive integers) to vertices of G so that vertices v and w have labels differing by at least p (resp. at least q) whenever v and w are adjacent (resp. distance 2). The *span* of an $L(p,q)$-labeling is the difference between the largest and smallest labels, and the (p,q)-span of a graph G is the minimum span of an $L(p,q)$-labeling of G; this is denoted $\lambda^{p,q}(G)$. For most of these parameters, it makes sense to define analogues for list coloring, paint number, Alon–Tarsi number, and

χ^2, χ^d

G^d

strong edge-coloring

total coloring

total chromatic number

χ''

span

$\lambda^{p,q}$

coloring number. For G^2, we denote these analogues as $\chi^2(G)$, $\chi_\ell^2(G)$, $\chi_p^2(G)$, $\chi_{\mathrm{AT}}^2(G)$, and $\chi_{\mathrm{col}}^2(G)$. Our focus in this survey, which reflects the focus in the literature, is on $\chi^2(G)$ and $\chi_\ell^2(G)$. However, often proofs of bounds for these parameters actually prove the same bound for $\chi_{\mathrm{col}}^2(G)$, which we mention when applicable.

For a graph G, we write $\Delta(G)$, $\omega(G)$, and $g(G)$ for its maximum degree, its clique number, and its girth (length of its shortest cycle). When G is clear from context, we often shorten these to Δ, ω, and g. The *maximum average degree*, $\mathrm{mad}(G)$, of a graph G is $\max_{H\subseteq G,|V(H)|\geqslant 1} \frac{2|E(H)|}{|V(H)|}$. When G is planar, Euler's formula gives $\mathrm{mad}(G) < \frac{2g}{g-2}$; see Lemma 20. As we will see in Section 3.3, many results initially proved for planar graphs with sufficiently large girth actually hold for the larger class of graphs with bounded maximum average degree.

Δ, ω, g

maximum average degree

mad

2 Graphs in General: Wegner's Conjecture

For every graph G, clearly $\chi(G) \geqslant \omega(G)$. In general, this bound can be quite bad. Mycielski constructed[1] triangle-free graphs with arbitrarily high chromatic number, and Erdős proved the existence of a graph with both chromatic number and girth arbitrarily high. However, both constructions have maximum degree much higher than chromatic number. In contrast, for a given maximum degree, the graph with maximum chromatic number is the complete graph. Wegner [183] believed that something similar is true when we consider graphs raised to a fixed power.

Wegner's Conjecture ([183])**.** *For all integers $k \geqslant 1$ and $D \geqslant 3$, let $\chi^k(D)$ and $\omega^k(D)$ denote, respectively, the maximums over all graphs G with $\Delta \leqslant D$ of $\chi(G^k)$ and $\omega(G^k)$. For all k and D we have $\chi^k(D) = \omega^k(D)$.*

This conjecture is remarkably wide-ranging. Wegner writes: "one cannot expect a general answer, but it would be interesting to settle some cases." The restriction to $D \geqslant 3$ is because the case $D = 1$ is trivial, and the case $D = 2$ is quite easy. (At the end of Section 3.2, we determine $\chi^2(C_n)$, $\chi_\ell^2(C_n)$, and $\chi_{\mathrm{AT}}^2(C_n)$ for each cycle C_n.) The case $k = 1$ is also immediate, by considering greedy coloring and the complete graph K_{D+1}. So we start with $k = 2$ and $D \geqslant 3$.

Greedy coloring shows that $\chi(G) \leqslant \Delta + 1$ for all G. Since G^2 has maximum degree at most Δ^2, we immediately have $\chi^2(G) \leqslant \Delta^2 + 1$. Applying Brooks' Theorem to G^2 shows that equality holds for a connected graph G only if $G^2 = K_{\Delta^2+1}$. Hoffman and Singleton [100] famously used linear algebra to show this is possible only if $\Delta \in \{2, 3, 7, 57\}$. This proof is also outlined in [147, Section 3.1]. The unique realizations for $\Delta = 2$ and $\Delta = 3$ are the 5-cycle and the Petersen graph. For $\Delta = 7$, the only realization is the Hoffman–Singleton graph [100]. For $\Delta = 57$, the question of whether any realization

[1] Given a graph G with $V(G) = \{v_1, \ldots, v_n\}$, we define the Mycielskian $M(G)$ as follows. Let $V(M(G)) := \{v_1, \ldots, v_n, w_1, \ldots, w_n, x\}$. Let $E(M(G)) := E(G) \cup \{v_iw_j | v_iv_j \in E(G)\} \cup \{w_ix | i \in \{1, \ldots, n\}\}$. It is easy to see that $\chi(M(G)) = \chi(G) + 1$ and that if G is triangle-free, then so is $M(G)$.

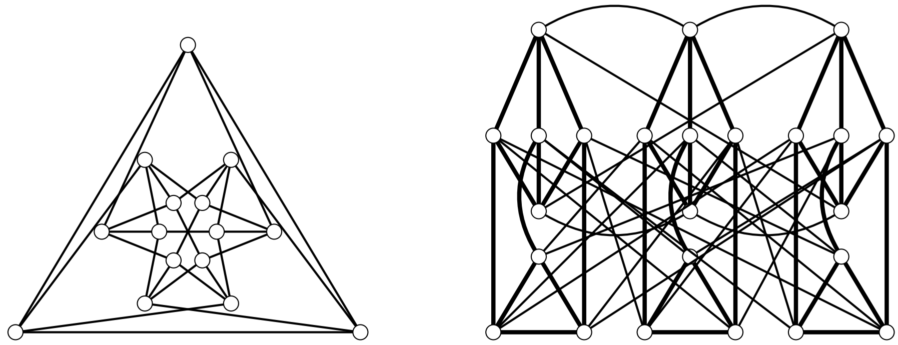


Figure 1: Graphs with $\Delta = 4$ and $\Delta = 5$ that have squares K_{4^2-1} and K_{5^2-1}.

exists remains a major open problem. (Makhnev [145] claimed to disprove the existence of any realization when $\Delta = 57$. However, Faber [78] claimed to refute this proof.)

Erdős, Fajtlowicz, and Hoffman [76] used the same approach to show that when $\Delta \geqslant 3$, no graph G has $G^2 = K_{\Delta^2}$. Elspas [73] constructed graphs G for each $\Delta \in \{4, 5\}$ such that $G^2 = K_{\Delta^2-1}$; see Figure 1. Thus, to prove Wegner's Conjecture for $k = 2$ and $\Delta \in \{4, 5\}$, it suffices to prove that $\chi^2(G) \leqslant \Delta^2 - 1$ whenever $\Delta \in \{4, 5\}$. As a first step, it is useful to prove that $\omega(G^2) \leqslant \Delta^2 - 1$. Fortunately, the above result of Erdős et al. yields the following lemma as an easy corollary; this was first noted in [61].

Lemma 1 ([61]). *If G is connected, $\Delta \geqslant 3$, and $G^2 \neq K_{\Delta^2+1}$, then $\omega(G^2) \leqslant \Delta^2 - 1$.*

Proof. Suppose to the contrary that $G^2 \neq K_{\Delta^2+1}$ and $\omega(G^2) \geqslant \Delta^2$. Let S be a maximum clique in G^2. Erdős et al. [76] showed that S must be a proper subset of $V(G)$. Since G is connected, there exist adjacent vertices v and w with $v \in S$ and $w \notin S$. Note that $d_{G^2}(v) \leqslant \Delta^2$. Since $|S| \geqslant \Delta^2$ and $w \notin S$, the set S contains every neighbor of v in G^2 other than w. In particular, S contains every neighbor of w in G. Now repeating this argument for each neighbor of w in G, we conclude that S contains every vertex at distance at most 2 from w in G, i.e., every neighbor of w in G^2. Thus, $S \cup \{w\}$ is a clique in G^2 of size $|S| + 1$, which contradicts our choice of S as maximum. $\square$

Now we want to show that $G^2 \neq K_{\Delta^2+1}$ implies the stronger result $\chi^2(G) \leqslant \Delta^2 - 1$. The first work in this direction is by Cranston and Kim [61]; they showed that if $\Delta = 3$, G is connected, and G is not the Petersen graph, then $\chi^2_\ell(G) \leqslant \Delta^2 - 1 = 8$. They also conjectured that $\chi^2_\ell(G) \leqslant \Delta^2 - 1$ for every connected graph G such that $\Delta \geqslant 3$ and $G^2 \neq K_{\Delta^2+1}$. Cranston and Rabern [62] proved this bound in the more general setting of Alon–Tarsi number.

Theorem 2 ([62]). *If $\Delta \geqslant 3$ and G is a connected graph other than the Petersen graph, the Hoffman–Singleton graph, or a graph with $\Delta = 57$ and $G^2 = K_{57^2+1}$, then $\chi^2_{AT}(G) \leqslant \Delta^2 - 1$.*

Proof Sketch. The idea of the proof is straightforward; for simplicity we focus on list coloring, although the extension to Alon–Tarsi number follows the same approach. Recall the proof of Brooks' Theorem for list coloring, due to Erdős, Rubin, and Taylor [75]. They choose some connected subgraph H and greedily color the vertices in order of decreasing distance from H. At the time that each vertex v outside of H is colored, it has some uncolored neighbor closer to H, so v can be colored from a list of size Δ. Thus, it suffices to pick H that is degree-choosable, i.e., it can be list colored whenever each vertex v of H has a list of size $d_H(v)$. So the main work in their proof consists in showing that each 2-connected graph (that is neither a complete graph nor an odd cycle) contains such a subgraph H.

The proof of Theorem 2 follows the same outline. Again, we choose some subgraph H of G and color the vertices of G^2 in order of decreasing distance in G from H. Now each vertex v has at least *two* uncolored neighbors when it is colored, so lists of size $\Delta^2 - 1$ suffice. In [75] the choice of H is always an induced even cycle with at most one chord. Here we choose H similarly; when G has sufficiently high girth, H is the square of a shortest cycle in G, possibly with one or two pendant edges. The case of smaller girth is more detailed. In particular, girth 5 requires careful analysis, since it explicitly uses that $G^2 \neq K_{\Delta^2+1}$. □

Now we continue the task of proving Wegner's Conjecture for $k = 2$ and specific values of D. The Petersen graph and the Hoffman–Singleton graph, together with greedy coloring, prove the conjecture for $D \in \{3, 7\}$. For all other D, besides possibly 57, Theorem 2 gives an upper bound. The constructions of Elspas [73] for $D = 4$ and $D = 5$ give matching lower bounds. Thus, for $k = 2$, Wegner's Conjecture is true for all $D \in \{3, 4, 5, 7\}$. The connected graphs achieving the upper upper bound are unique for $D \in \{3, 7\}$. The author also believes this is true for $D \in \{4, 5\}$, and the graphs shown in Figure 1, but he does not know a proof. For easy reference, we present these results in the following theorem.

Theorem 3. *Wegner's Conjecture is trivial for $k = 1$, as shown by the complete graph K_{D+1}. For $k = 2$ it is true for $D \in \{3, 4, 5, 7\}$. (As far as we know, it is open for all other pairs (k, D).)*

We might hope to find similar constructions for larger D. But this seems unlikely. Conde and Gimbert [56] showed that for each D with $6 \leqslant D \leqslant 49$, there exists no graph G with $\Delta = D$ and $G^2 = K_{\Delta^2-1}$. Miller, Nguyen, and Pineda–Villavicencio [146] conjectured this for all $D \geqslant 6$, and proved it for various cases. Miller and Širán [147, p. 13] summarized these results.

Relatively little is known about lower bounds for $\omega^2(D)$. Recall that for each prime power q, there exists a projective plane $\mathcal{P}$ of order q. Brown [40] considered the bipartite incidence graph G of $\mathcal{P}$, which has as its two parts the $q^2 + q + 1$ points and $q^2 + q + 1$ lines of $\mathcal{P}$. Since every pair of lines intersect in a common point, in G^2 the lines form a clique of size $q^2 + q + 1$ (similarly for the points). Since G is $(q + 1)$-regular, $\omega(G^2) = q^2 + q + 1 = \Delta^2 - \Delta + 1$. When $\Delta - 1$ is not a power of a prime, the best bounds known still come from this construction, together with the fact [10] that for Δ sufficiently large there always exists a prime p with $\Delta \geqslant p \geqslant \Delta - \Delta^{0.525}$. This lack of better constructions prompts the following question, which arose from discussion with Goddard.

Question 4. For each integer t, does there exist a constant Δ_t such that all graphs G with $\Delta \geqslant \Delta_t$ satisfy $\omega^2(G) \leqslant \Delta^2 - t$? Can this conclusion be strengthened further to (i) $\chi^2(G) \leqslant \Delta^2 - t$, (ii) $\chi^2_\ell(G) \leqslant \Delta^2 - t$, (iii) $\chi^2_p(G) \leqslant \Delta^2 - t$, or even (iv) $\chi^2_{\mathrm{AT}}(G) \leqslant \Delta^2 - t$?

Before we leave this section, we mention the following related conjecture of Borodin and Kostochka [35]: For every graph G with $\Delta \geqslant 9$, if $\omega(G) < \Delta$, then $\chi(G) \leqslant \Delta - 1$. When restricted to graphs that are squares, this conjecture implies the coloring analogue of Theorem 2. In general the Borodin–Kostochka Conjecture remains open, although Reed [157] used the probabilistic method to prove it for Δ sufficiently large[2]. However, his result is unlikely to help resolve Wegner's conjecture for any further values of D, since quite probably $\omega^2(D) < \Delta^2 - 1$ for all $D \geqslant 6$, with the possible exception of $D = 57$.

3 Planar Graphs and Sparse Graphs

3.1 Wegner's Planar Graph Conjecture

The most well-known conjecture on coloring squares was made by Wegner [183], in 1977.

Wegner's Planar Graph Conjecture ([183]). *If G is planar with maximum degree Δ, then*

$$\chi^2(G) \leqslant \begin{cases} 7 & \text{if } \Delta = 3 \\ \Delta + 5 & \text{if } 4 \leqslant \Delta \leqslant 7 \\ \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1 & \text{if } \Delta \geqslant 8. \end{cases}$$

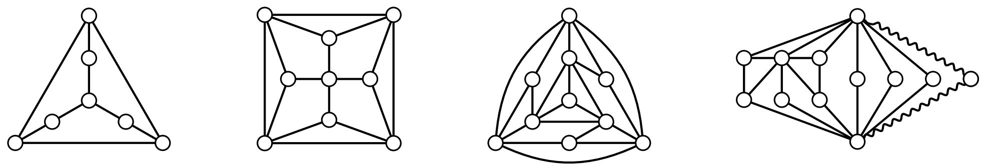


Figure 2: Diameter 2 planar graphs with maximum degrees 3, 4, 5, 7(6) and orders 7, 9, 10, 12(11), respectively. (To form the graph of order 11 from that of order 12, delete the rightmost vertex.) These graphs show that Wegner's Planar Graph Conjecture is best possible for $\Delta \leqslant 7$. The case $\Delta \geqslant 8$ is shown in Figure 3.

For all Δ, Wegner gave constructions showing this conjecture is best possible. Each construction is a diameter 2 graph, that is, a graph G such that G^2 is a clique. For $3 \leqslant \Delta \leqslant 7$, the constructions are shown in Figure 2. (To form the graph with $\Delta = 6$ from that with $\Delta = 7$, delete the rightmost vertex.) For $\Delta \geqslant 8$, Figure 3 below shows the construction when Δ is even; note that all $3s + 1$ vertices of G_s^2 form a clique, so $\chi^2(G_s) = 3s + 1$. When Δ is odd, say $\Delta = 2s + 1$, we add one more vertex v_{s+1}, adjacent to v and w. Hell and Seyffarth [92] proved that, for each $\Delta \geqslant 8$, no planar diameter 2 graph has more vertices than the one constructed by Wegner.

[2] He showed that $\Delta \geqslant 10^{14}$ suffices. However, he commented in the paper that with more detailed analysis, this could be reduced to $\Delta \geqslant 10^6$ and maybe even $\Delta \geqslant 10^3$ (but probably not to $\Delta \geqslant 10^2$).

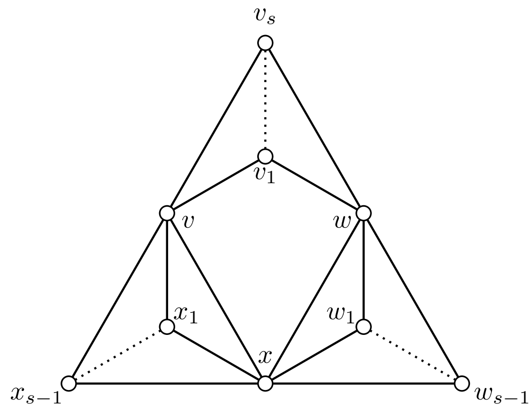


Figure 3: A planar graph G_s with $\Delta(G_s) = 2s$ and $\chi^2(G_s) = \omega^2(G_s) = |V(G_s)| = 3s + 1$.

For planar G, with Δ moderately small, the current best bound on $\chi^2(G)$, is due to Molloy and Salavatipour [152].

Theorem 5 ([152])**.** *If G is a planar graph, then $\chi^2(G) \leqslant \lceil \frac{5}{3}\Delta \rceil + 78$. If also $\Delta \geqslant 241$, then $\chi^2(G) \leqslant \lceil \frac{5}{3}\Delta \rceil + 25$.*

This result does not extend to $\chi^2_{\mathrm{col}}(G)$ or even to $\chi^2_\ell(G)$. However, asymptotically we can do much better. Havet, van den Heuvel, McDiarmid, and Reed [91] proved that Wegner's Planar Graph Conjecture is true in an asymptotic sense, even for list coloring.

Theorem 6 ([91])**.** *For each $\epsilon > 0$ there exists Δ_ϵ such that if G is a planar graph with $\Delta \geqslant \Delta_\epsilon$, then $\chi^2_\ell(G) \leqslant \frac{3}{2}\Delta(1 + \epsilon)$.*

In fact, [91] proved Theorem 6 more generally for every "nice" class of graphs. A class of graphs $\mathcal{G}$ is *nice* if $\mathcal{G}$ is minor-closed and $\mathcal{G}$ does not contain the complete bipartite graph $K_{3,t}$ for some positive integer t. (For example, the class of all planar graphs is nice, since $K_{3,3}$ is non-planar. More generally, for every surface S, the class of graphs embeddable in S is nice.) One consequence of Theorem 6 is that every nice class of graphs satisfies $\omega^2(G) \leqslant \frac{3}{2}\Delta(G)(1 + o(1))$ as $\Delta(G) \to \infty$. For the class of graphs embeddable in each fixed surface S, Amini, Esperet, and van den Heuvel [7], proved the stronger bound $\omega^2(G) \leqslant \frac{3}{2}\Delta(G) + C_S$, for some constant C_S (dependent on S). nice

The proof of Theorem 6 is long and technical. But in a sense, the key idea is that list coloring squares of planar graphs is very similar to list coloring line graphs. This problem was solved asymptotically by Kahn [113], and the proof of the key lemma in [91] follows the approach in [113]. In [91] the authors also posed the following conjecture. It is a special case of the Square List Coloring Conjecture, which we discuss in Section 5.1 and which was previously disproved.

Conjecture 7 ([91])**.** If G is a planar graph, then $\chi^2_\ell(G) = \chi^2(G)$.

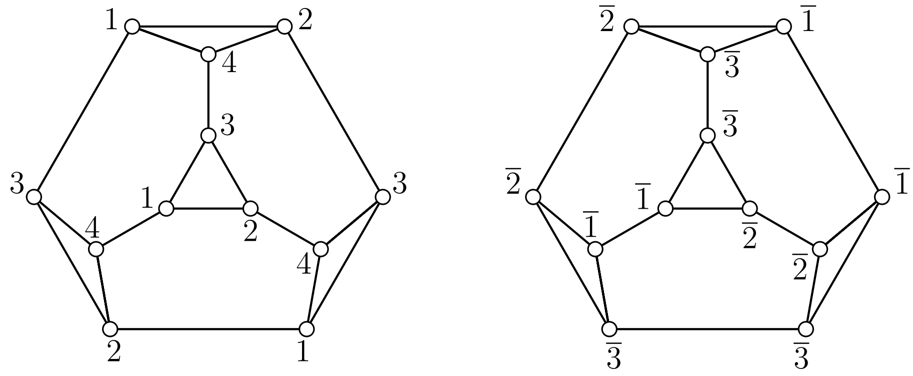


Figure 4: Left: A 4-coloring of G^2. Right: A 4-assignment L such that G^2 admits no L-coloring. This example disproves Conjecture 7; see Theorem 8.

It is difficult to prove general lower bounds on $\chi^2(G)$ that are stronger than $\Delta+1$. So most of the graphs for which Conjecture 7 has been proved are those for which $\chi_\ell^2(G) = \Delta+1$. We discuss these in more detail in Section 3.2. However, in 2022 this conjecture was disproved by Hasanvand [88]. (The conjecture does remain open for graphs with maximum degree at least 4.)

Theorem 8 ([88])**.** *Conjecture 7 is false. Specifically, there exists a cubic claw-free planar graph G of order 12 (see Figure 4) such that $\chi^2(G) = 4 < \chi_\ell^2(G)$. In fact, this graph is the first in an infinite family of (2-connected) cubic claw-free planar graphs satisfying the same inequality.*

The proof is pretty, so we sketch it.

Proof Sketch. Form G from K_4 by subdiving each edge once and taking the line graph; see Figure 4. Clearly $\chi^2(G) \geqslant \omega^2(G) = 4$. And the coloring on the left shows that $\chi^2(G) \leqslant 4$.

To prove that $\chi_\ell^2(G) > 4$, let $\overline{i} := \{1,2,3,4,5\} \setminus \{i\}$ for each $i \in \{1,2,3,4,5\}$. Let L be the 4-assignment shown on the right in Figure 4. It is straightforward to check that the only (four) independent sets of size 3 in G^2 are the color classes in the coloring on the left of Figure 4. Since $|\cup_{v\in V(G)} L(v)| = 5$, if G^2 admits an L-coloring φ, then two of these independent sets of size 3 must each receive a common color.

It is easy to check that in each independent set of size 3 some vertex is missing color i, for each $i \in \{1,2,3\}$. So φ must use color 4 on one independent set of size 3, call it I_1, and must use color 5 on another independent set of size 3, call it I_2. Up to symmetry, we have only two choices for I_1 and I_2: either (a) one of these is color class 4 on the left or (b) both of these are among color classes 1, 2, and 3. In each case, $G^2[V(G) \setminus (I_1 \cup I_2)]$ is a 6-cycle with one chord (forming two 4-cycles). Further, the resulting 2-assignment for this graph is isomorphic to $\{1,2\}, \{1,3\}, \{2,3\}, \{1,2\}, \{1,3\}, \{2,3\}$ (around the 6-cycle) with the ends of the chord having the same list. It is easy to check that for these lists L' the remaining graph admits no L'-coloring.

Now we construct infinitely many such graphs. Form K_4' from the complete graph K_4 by deleting an edge vw and adding pendent edges vv' and ww'. To transform the above graph G to a larger graph G' satisfying the inequality $\chi^2(G') = 4 < \chi^2_\ell(G')$, we delete some edge xy that is not in a 3-cycle and identify x and y, respectively, with vertices v' and w' in K'_4. It is easy to check that a 4-coloring of G^2 extends to a 4-coloring of $(G')^2$ (in exactly two ways). We extend the 4-assignment L for G^2 to a 4-assignment L' for $(G')^2$ by giving each new vertex the list that is already assigned to both x and y. It is straightforward to check that any L'-coloring of $(G')^2$ restricts to an L-coloring of G^2, which cannot exist. Thus, $(G')^2$ has no L'-coloring. Finally, we can apply this construction repeatedly to build the above-mentioned infinite family of counterexamples to Conjecture 7. □

Before returning to Wegner's Planar Graph Conjecture, we mention one other intriguing conjecture from [91].

Conjecture 9 ([91])**.** If $\mathcal{G}$ is a nice class of graphs, then there exists Δ_0 such that if $G \in \mathcal{G}$ and $\Delta(G) \geqslant \Delta_0$, then $\chi^2(G) \leqslant \chi^2_\ell(G) \leqslant \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor + 1$.

This conjecture is best possible, since equality holds for the graphs in Figure 3.

Although Wegner's Planar Graph Conjecture has been solved asymptotically, whenever $\Delta \geqslant 4$ the conjecture remains open. In contrast, the case $\Delta = 3$ has been confirmed by Thomassen [166] and independently by Hartke, Jahanbekam, and Thomas [87].

Theorem 10 ([166, 87])**.** *If G is planar and $\Delta \leqslant 3$, then $\chi^2(G) \leqslant 7$.*

The two proofs of Theorem 10 take strikingly different approaches. The work of Hartke et al. uses a fairly straightforward discharging argument, together with extensive computer case-checking to prove that various configurations are reducible. The proof of Thomassen relies on a detailed structural analysis, and also uses the Four Color Theorem.

Many of the bounds known on $\chi^2(G)$ and $\chi^2_\ell(G)$ for planar graphs actually hold for $\chi^2_{\mathrm{col}}(G)$, although the authors often don't state this fact. To illustrate the idea of such a proof, we show that every planar graph G satisfies $\chi^2_{\mathrm{col}}(G) \leqslant 9\Delta - 3$. Let $v_1, \ldots, v_n$ be an order of the vertices showing that $\chi_{\mathrm{col}}(G) \leqslant 6$. More precisely, let $v_1, \ldots, v_n$ be a vertex order and let $G_i := G \setminus \{v_1, v_2, \ldots, v_{i-1}\}$ such that $d_{G_i}(v_i) \leqslant 5$ for all i. Similarly, let $G^2_i := G^2 \setminus \{v_1, v_2, \ldots, v_{i-1}\}$. We will show, for each i, that $d_{G^2_i}(v_i) \leqslant 9\Delta$.

Recall that v_i has at most five neighbors in G_i; each of these neighbors has at most $\Delta - 1$ other neighbors in G_i, for a total of at most 5Δ neighbors in G^2_i. In addition, v_i may be adjacent in G^2_i to some vertex w such that x is a common neighbor of v_i and w in G, but x precedes v_i in the order. Each such x is followed by at most 4 neighbors in the order, other than v_i. Since v_i is preceded by at most Δ such vertices x, it has at most 4Δ neighbors in G^2_i of this type. Thus, $d_{G^2_i}(v) \leqslant 5\Delta + 4\Delta = 9\Delta$. Hence $\chi^2_{\mathrm{col}}(G) \leqslant 9\Delta + 1$. (More generally [137], if G is k-degenerate, then G^2 is $((2k-1)\Delta)$-degenerate.)

The first result in this direction was due to T. K. Jonas [110]. Work in his dissertation showed, implicitly, that $\chi^2_{\mathrm{col}}(G) \leqslant 8\Delta - 22$ when $\Delta \geqslant 5$. (In fact, Jonas studied $L(2,1)$-labelings, which we consider in Section 5.2.) Wong [185], in his masters thesis, strengthened this to $\chi^2_{\mathrm{col}}(G) \leqslant 3\Delta + 5$. Next, van den Heuvel and McGuinness [168] showed that $\chi^2_{\mathrm{col}}(G) \leqslant 2\Delta + 25$; see also [64, Theorem 6.10] for a simplified proof that $\chi^2_{\mathrm{col}}(G) \leqslant 2\Delta + 34$. About 20 years later, Bousquet, Deschamps, de Meyer, and Pierron [39] proved that if G is planar with $\Delta \geqslant 9$, then $\chi^2(G) \leqslant 2\Delta + 7$. This is the best known bound on $\chi^2(G)$ when $9 \leqslant \Delta \leqslant 31$ (however, the proof at one point requires permuting colors, so it does not extend to list coloring or degeneracy). Earlier the best possible bound on $\chi^2_{\mathrm{col}}(G)$ was proved, for sufficiently large maximum degree, by Agnarsson and Halldórsson [2] and Borodin, Broersma, Glebov, and van den Heuvel [24, 25].

Theorem 11 ([2, 24, 25])**.** *If G is planar with Δ large enough, then $\chi^2_{col}(G) \leqslant \left\lceil \frac{9}{5}\Delta \right\rceil + 1$.*

The first group proved that $\Delta \geqslant 750$ suffices and the second that $\Delta \geqslant 47$ suffices. Each group also gave the same construction (below) showing that this bound is best possible.

Proposition 12 ([2, 25])**.** *For each $\Delta \geqslant 9$, there exists a planar graph G_Δ, with maximum degree Δ, such that G^2_Δ has minimum degree $\lceil \frac{9}{5}\Delta \rceil$.*

Proof Sketch. We just prove the case $\Delta = 5k - 1$, but the other cases are similar. Let H be an icosahedron, embedded in the plane. Note that we can partition the edges of H into five perfect matchings; denote one of these by M. For each $vw \in E(H) \setminus M$, replace vw by k paths of length 2 joining v and w. For each $vw \in M$, keep vw and add $k - 2$ paths of length 2 joining v and w. Call the resulting graph G_Δ.

It is easy to check that each $v \in V(G_\Delta)$ has either (a) $d_{G_\Delta}(v) = 5k - 1$ or (b) $d_{G_\Delta}(v) = 2$. (a) If $v \in V(G_\Delta)$ and $d_{G_\Delta}(v) = 5k - 1$, then v is a vertex of H and $d_{G^2_\Delta}(v) = k-2+8k+5 = 9k+3$. (b) Suppose instead $v \in V(G_\Delta)$ and $d_{G_\Delta}(v) = 2$; so v arose from some edge e of G. If $e \in E(H)-M$, then $d_{G^2_\Delta}(v) = k-1+2+2(3k+k-2+1) = 9k-1$. If instead $e \in M$, then $d_{G^2_\Delta}(v) = k - 3 + 2 + 2(4k) = 9k - 1$. Thus G^2_Δ has minimum degree $9k - 1 = \lceil \frac{9}{5}\Delta \rceil$, as desired. When $\Delta \neq 5k - 1$, we treat more of the five perfect matchings as we treated M. □

Theorems 5 and 6 both circumvent the "barrier" implied by Proposition 12. To accomplish this, they must rely on coloring techniques more sophisticated than simply coloring greedily in the reverse of a vertex order witnessing the degeneracy of the graph.

We will soon consider coloring squares of planar graphs with high girth. To end this section, we offer a brief motivation. It is natural to search for classes of graphs where we can prove an upper bound on χ^2 closer to the trivial lower bound of $\Delta + 1$. For the class of all planar graphs, the best upper bound we can hope for is $\chi^2 \leqslant \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, as witnessed by the graphs in Figure 3. To prove a stronger upper bound, we must exclude these graphs. One natural approach is to forbid cycles of certain lengths. These examples contain cycles only with lengths in $\{3, 4, 5, 6\}$. But the 3-cycles and 5-cycles are not crucial for the construction. If we subdivide edges vx and wx, the resulting graph G' is bipartite with $\chi^2(G') = 3s$. Thus, to significantly improve the bound, we must exclude either 4-cycles or 6-cycles. One obvious way to do the former is to require girth at least 5. This motivates the topic of the next section.

3.2 Planar Graphs of High Girth

Wang and Lih [182] were among the first to study χ^2 for planar graphs with high girth. (In fact, for these graphs they considered $L(2,1)$-labelings, which we discuss briefly in Section 5.2.) If G has girth at least 5, then clearly the construction in Figure 3 is excluded. This absence of constructions led Wang and Lih to ask whether for Δ sufficiently large the trivial lower bound, $\Delta+1$, is also an upper bound.

Conjecture 13 ([182])**.** For every $k \geqslant 5$ there exists Δ_k such that if G is a planar graph with girth at least k and $\Delta \geqslant \Delta_k$ then $\chi^2(G) = \Delta + 1$.

Borodin et al. [26] proved the Wang–Lih Conjecture for $k \geqslant 7$.

Theorem 14 ([26])**.** *If G is planar with girth at least 7 and $\Delta \geqslant 30$, then $\chi^2(G) = \Delta+1$.*

In contrast, for each integer D at least 3, they constructed [26] a planar graph G_D with girth 6 and $\Delta = D$ such that $\chi^2(G_D) = \Delta+2$; this disproved Conjecture 13 for each $k \in \{5,6\}$. Subsequently, Dvořák et al. [71] gave a slightly simpler construction, which we present below.

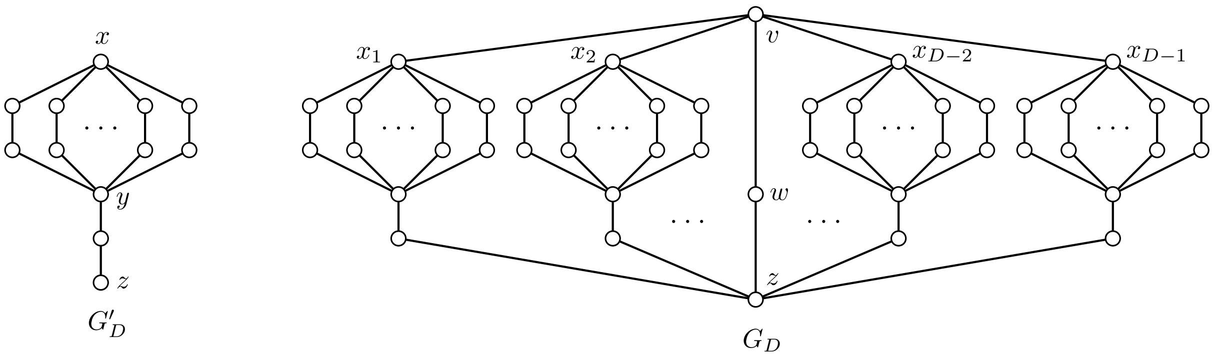


Figure 5: In any $(D+1)$-coloring of the square of G'_D, the $(D-1)$-vertex x and the 1-vertex z must receive distinct colors. Thus, $\chi(G_D^2) \geqslant D+2$.

Proposition 15 ([26, 71])**.** *For every $D \geqslant 3$, there exists a planar graph G_D with girth 6 and $\Delta = D$ such that $\chi^2(G_D) = D+2$; see the right of Figure 5.*

Proof. Consider the graph G'_D on the left in Figure 5, where $d(y) = D$. We show that in any $(D+1)$-coloring of $(G'_D)^2$ vertices x and z must get distinct colors. By symmetry, say that y is colored 1, its common neighbor with z is colored $D+1$, and its other $D-1$ neighbors each get a distinct color from $\{2,\ldots,D\}$. Now x is colored from $\{1, D+1\}$, but z is not. Thus, x and z get distinct colors. To see that $\chi^2(G_D) \geqslant D+2$, let $S = \{x_1,\ldots,x_{D-1},v,w,z\}$ on the right in Figure 5. Note that $S \setminus \{z\}$ induces K_{D+1} in G_D^2. Further, z must get a color distinct from each vertex of $S \setminus \{z\}$. Thus, any proper coloring of G_D^2 uses at least $D+2$ colors. $\square$

These results lead to two natural questions. (i) For each $k \geqslant 7$, what is the smallest D such that every planar G with girth $g \geqslant k$ and $\Delta \geqslant D$ satisfies $\chi^2(G) = \Delta + 1$? Similarly for $\chi^2_\ell(G)$. (ii) For each $g \in \{5, 6\}$, what is the smallest constant C such that every planar G with sufficiently large Δ satisfies $\chi^2(G) \leqslant \Delta + C$? We address these questions in order.

Borodin, Ivanova, and Neustroeva [33] showed that $\chi^2(G) = \Delta + 1$ when G is planar, $\Delta \geqslant D$, and $g \geqslant k$, for each $(D,k) \in \{(3,24),(4,15),(5,13),(6,12),(7,11),(9,10),(16,9)\}$. A few years later, the same authors also extended these results to list coloring. They showed [34] that $\chi^2_\ell(G) = \Delta(G) + 1$ when $(D,k) \in \{(3,24),(4,15),(5,13),(6,12),(7,11),(9,10),(15,8),(30,7)\}$. Shortly thereafter, Ivanova [107] strengthened the bounds for some values of k: If $(D,k) \in \{(5,12),(6,10),(10,8),(16,7)\}$, then $\chi^2_\ell(G) = \Delta(G) + 1$. Later, La and Montassier [139] proved the same upper bound when $(D,k) = (7,9)$.

$\Delta \geqslant$	3	4	5	6	7	10	16	$-$
girth $\geqslant$	24	15	12	10	9	8	7	6

Figure 6: Pairs (D,k), in columns, such that $\chi^2_\ell(G) = \Delta + 1$ if G is a planar graph with maximum degree $\Delta \geqslant D$ and girth $g \geqslant k$. (The entry $(-,6)$ denotes that no such pair $(D,6)$ exists. References are in the preceeding few paragraphs.)

In 2008, Dvořák, Král', Nejedlý, and Škrekovski [71] showed that for $k = 6$, the Wang–Lih Conjecture fails only by 1.

Theorem 16 ([71])**.** *If G is planar with girth at least 6 and $\Delta \geqslant 8821$, then $\chi^2(G) \leqslant \Delta + 2$.*

By using the same core idea as [71] (but a more refined discharging argument), Borodin and Ivanova strengthened this result: in 2009 they showed [27] that $\Delta \geqslant 18$ implies $\chi^2(G) \leqslant \Delta + 2$ and also [28] that $\Delta \geqslant 36$ (and later [29] that $\Delta \geqslant 24$) implies $\chi^2_\ell(G) \leqslant \Delta + 2$. Also, see Theorem 21, below. Furthermore, Dvořák et al. [71] conjectured that a similar result holds for girth 5.

Conjecture 17 ([71])**.** There exists Δ_0 such that if G is a planar graph with girth at least 5 and $\Delta \geqslant \Delta_0$ then $\chi^2(G) = \Delta + 2$.

Conjecture 17 was eventually confirmed by Bonamy, Cranston, and Postle [17].

Theorem 18 ([17])**.** *Conjecture 17 is true.*

Specifically, Bonamy et al. showed that if G is a planar graph with girth at least 5 and $\Delta \geqslant 2{,}689{,}601$, then $\chi^2_{\mathrm{AT}}(G) \leqslant \Delta + 2$. The proof was surprising. While it did rely on reducible configurations, it did not use discharging (only that planar graphs have average degree less than 6). In fact, the key idea was a class of arbitrarily large reducible configurations called "regions" that essentially consist of two high degree vertices, v and w, and all of their low degree neighbors that lie on v,w-paths of length at most 3.

We should mention one other variant of this problem. To avoid the construction in Figure 3 it is essential to forbid 4-cycles, but it is not really necessary to forbid 3-cycles. Zhu, Lu, Wang, and Chen [194] were the first to consider $\chi^2(G)$, when G is a planar graph with no 4-cycle and no 5-cycle. They proved $\chi^2(G) \leqslant \Delta+5$ when $\Delta \geqslant 9$. Cranston and Jaeger [60] strengthened this bound, showing that $\chi^2_{\mathrm{col}}(G) \leqslant \Delta+3$, when $\Delta \geqslant 32$. Zhu, Gu, Sheng, and Lü [192] showed further that $\chi^2_\ell(G) \leqslant \Delta+3$ when $\Delta \geqslant 26$. Finally, Dong and Xu [69] showed that $\chi^2(G) \leqslant \Delta+2$ when $\Delta \geqslant 185,760$.

All the results above forbid both 4-cycles and 5-cycles, but we can also think about just forbidding 4-cycles. (Recall that Figure 3 shows that we must forbid 4-cycles.) Wang and Cai [181] appear to have studied this problem first. They proved that if G is a planar graph with no 4-cycle, then $\chi^2(G) \leqslant \Delta+48$. Choi, Cranston, and Pierron [53] considered the analogous problem for the coloring number and sharpened the bound when Δ is large.

Theorem 19 ([53])**.** *If G is a planar graph with no 4-cycle, then $\chi^2_{col}(G) \leqslant \Delta+73$. If also Δ is sufficiently large*[3]*, then $\chi^2_{AT}(G) \leqslant \Delta+2$.*

The same authors [53] considered the more general problem of coloring the square of a planar graph G with no cycle having length in some finite set S. They showed that there exists a constant C_S such that always $\chi^2(G) \leqslant \Delta + C_S$ if and only if $4 \in S$.

Since Conjectures 13 and 17 are now completely resolved, it is natural to extend this line of research as follows. For every maximum degree Δ and girth g, what is the minimum constant $C_{\Delta,g}$ such that $\chi^2(G) \leqslant \Delta + C_{\Delta,g}$ for each planar graph G with maximum degree Δ and girth at least g? The same question can be asked for χ^2_ℓ. For this problem, the case $\Delta = 3$ has received more attention than any other; see Figure 7.

Let G be a connected graph, possibly non-planar, with $\Delta = 3$. Wegner showed that if G is not the Petersen graph, then $\chi^2(G) \leqslant 8$. Subsequently, this was strengthened [61, 62] to $\chi^2_\ell(G) \leqslant 8$ and eventually $\chi^2_{\mathrm{AT}}(G) \leqslant 8$. For non-planar G, this is sharp, as witnessed by the Wagner graph, which is formed from the 8-cycle by adding an edge joining each pair of vertices at distance 4 on the cycle. Thus, we seek additional hypotheses on G that imply better upper bounds on $\chi^2(G)$ and $\chi^2_\ell(G)$. Recall that Wegner conjectured that $\chi^2(G) \leqslant 7$ for every planar subcubic graph. As noted above, this conjecture was proved by Thomassen [166] and independently by Hartke, Jahanbekam, and Thomas [87]. A natural next step is planar graphs of higher girth.

$\chi^2_\ell \leqslant$	8	7	6	5	4	3
girth $\geqslant$	3	7	9	13	24	$-$

Figure 7: Pairs (k, g_k), in columns, such that $\chi^2_\ell(G) \leqslant k$ if G is a planar graph with maximum degree $\Delta = 3$ and girth at least g_k. (The entry $(3, -)$ denotes that no such pair $(3, g_3)$ exists. References are in the preceeding few paragraphs.)

Let G be a planar graph with $\Delta = 3$ and girth g. Cranston and Kim [61] showed that $\chi^2_\ell(G) \leqslant 7$ if $g \geqslant 7$. Dvořák, Škrekovski, and Tancer [72] showed that (i) $\chi^2_\ell(G) \leqslant 6$ if $g \geqslant 10$, and (ii) $\chi^2_\ell(G) \leqslant 5$ if $g \geqslant 14$, and (iii) $\chi^2_\ell(G) = 4$ if $g \geqslant 24$ (note that (iii) was also proved in [34], mentioned above). Their first result was strengthened by Cranston and Kim [61] and also Havet [89]; both groups showed that if $g \geqslant 9$, then $\chi^2_\ell(G) \leqslant 6$. Their

[3]This proof shows $\Delta \geqslant 6 \cdot 10^{14}$ suffices; but the authors made little effort optimizing the bound on Δ.

second result was strengthened by Havet, who showed that if $g \geqslant 13$, then $\chi^2_\ell(G) \leqslant 5$. Since always $\chi^2(G) \geqslant \Delta + 1$, no restriction on girth implies $\chi^2(G) \leqslant 3$. In particular, $\chi^2(K_{1,3}) = 4$. For result (iii), Borodin and Ivanova slightly weakened the hypothesis for coloring (rather than list coloring) needed to guarantee $\chi^2(G) = 4$. In 2010 they showed [30] that $g \geqslant 23$ suffices; in 2012 they strengthened [31] this to $g \geqslant 22$. In 2021 La and Montassier [138] extended this to $g \geqslant 21$.

To conclude this section, we remark briefly about squares of cycles. Note that $(C_5)^2 = K_5$, so $\chi^2(C_5) = \chi^2_\ell(C_5) = \chi^2_{\mathrm{AT}}(C_5) = 5$. Now consider C_k for some $k \neq 5$. If $3|k$, then $\chi^2_{\mathrm{AT}}(C_k) = 3$; otherwise $\chi^2_{\mathrm{AT}}(C_k) = 4$. Proving the bound for coloring is easy. The bound $\chi^2_{\mathrm{AT}}(C_k) \leqslant 4$ follows from Brooks' Theorem [95]. When $3 \nmid k$, the independence number of C_k^2 is less than $k/3$, so $\chi^2(C_k) = 4$. When $6|k$, Juvan, Mohar, and Škrekovski [111] showed that $\chi^2_\ell(C_k) = 3$. Using ideas from [62, Lemma 16] this result can be strengthened to prove that if $3|k$, then $\chi^2_{\mathrm{AT}}(C_k) = 3$.

3.3 Graph Classes with Bounded Maximum Average Degree

Nearly all results in Sections 3.1 and 3.2 are proved using the discharging method. (One notable exception is Theorem 6, which proves Wegner's conjecture asymptotically; it instead uses the probabilistic method, which we discuss at the end of Section 4.1.) Discussing this technique in depth is outside the scope of this survey. However, for the interested reader, we recommend *An Introduction to the Discharging Method via Graph Coloring* [64].

Many discharging proofs for planar graphs, particularly sparse planar graphs, use planarity in only a very weak sense: to bound the number of edges in any induced subgraph. This observation leads to the notion of *maximum average degree*, denoted $\mathrm{mad}(G)$, and defined as $\mathrm{mad}(G) := \max_{H \subseteq G, |V(H)| \geqslant 1} 2|E(H)|/|V(H)|$. Forests are precisely the class of graphs G with $\mathrm{mad}(G) < 2$. Since every n-vertex planar graph has at most $3n - 6$ edges, it has average degree less than 6. Since planar graphs are hereditary, every planar graph G has $\mathrm{mad}(G) < 6$. More generally, we have the following observation. $\mathrm{mad}(G)$

Lemma 20 (Folklore)**.** *If G is a planar graph with girth at least g, then* $\mathrm{mad}(G) < \frac{2g}{g-2}$.

Proof. We sum the sizes of all $|F(G)|$ faces, which gives the inequality $2|E(G)| \geqslant g|F(G)|$. Substituting this inequality into Euler's formula, and solving for $\mathrm{ad}(G)$, the average degree of G, yields $\mathrm{ad}(G) < \frac{2g}{g-2}$. Each subgraph H is also planar, with girth at least g, so we get the desired result. □

Now $\mathrm{mad}(G)$ gives a natural way to strengthen and refine results about coloring planar graphs, specifically with high girth. One advantage of using mad is that we are no longer restricted to those values of mad that arise from Lemma 20 when g is an integer.

A natural first question, in view of Section 3.2, is whether Conjectures 13 and 17 remain true when the hypotheses are relaxed to $\mathrm{mad}(G) < \frac{2g}{g-2}$. Dolama and Sopena [68] proved that $\chi^2(G) = \Delta + 1$ when $\Delta \geqslant 4$ and $\mathrm{mad}(G) < \frac{16}{7}$, which includes planar graphs with girth at least 16; see also [59]. Cranston and Škrekovski [63] proved that $\chi^2_\ell(G) = \Delta + 1$

Δ	4	5	Δ_ϵ	17	$\Delta_{\epsilon,c}$	8
$\text{mad}(G)$	16/7	$2+\frac{4\Delta-8}{5\Delta+2}$	$3-\epsilon$	3	$4-\frac{4}{c+1}-\epsilon$	4
χ^2_ℓ	$\Delta+1$	$\Delta+1$	$\Delta+1$	$\Delta+2$	$\Delta+c$	$3\Delta+1$

Figure 8: Triples (D,k,C) such that every graph G with $\Delta(G) \geqslant D$ and $\text{mad}(G) < k$ satisfy $\chi^2_\ell(G) \leqslant C$. References appear below (and near the end of the section).

whenever $\Delta \geqslant 5$ and $\text{mad}(G) < 2 + \frac{4\Delta-8}{5\Delta+2}$. In particular, this includes all planar graphs with girth at least $7 + \frac{12}{\Delta-2}$. Hence, it proves the analogue of Conjecture 13 for $g \geqslant 8$. The case $g \geqslant 7$ was subsumed by Bonamy, Lévêque, and Pinlou [18] and strengthened further in [20]. They proved that for each $\epsilon > 0$ there exists Δ_ϵ such that if G is a graph with $\text{mad}(G) < 3-\epsilon$ and $\Delta \geqslant \Delta_\epsilon$, then $\chi^2_\ell(G) = \Delta+1$. So Conjecture 13, with girth at least 7, holds in a much more general context, requiring only mad $< 3-\epsilon$, for any $\epsilon > 0$. Note that this is sharp, since, as we saw in Proposition 15, there exist planar graphs G with girth 6 (and hence mad < 3) such that $\chi^2(G) = \Delta+2$. One of the prettiest results in this area is the following, due to Bonamy, Lévêque, and Pinlou [19].

Theorem 21 ([19]). *If G is a graph with $\text{mad}(G) < 3$ and $\Delta \geqslant 17$, then $\chi^2_\ell(G) \leqslant \Delta+2$.*

This theorem generalizes that of Dvořák et al. for girth 6, and also its strengthenings by Borodin et al., mentioned above. At this point, it would be natural to guess that Theorem 21 extends to the more general case $\text{mad}(G) < \frac{10}{3}$ (the bound for planar graphs with girth 5), at least for Δ sufficiently large. But this generalization is false. Charpentier [49, 20] constructed, for each positive integer C, a family of graphs G with unbounded maximum degree that each has $\text{mad}(G) < 4 - \frac{2}{C+1}$, and $\chi^2(G) = \Delta + C + 1$. Taking $C = 2$ yields a class of counterexamples to our purported generalization.

(Our proof uses the following easy observation. A fractional orientation of a graph G with maximum indegree α, for any $\alpha > 0$, proves that $\text{mad}(G) \leqslant 2\alpha$. To see this, note that the maximum in the definition of $\text{mad}(G)$ can be restricted to induced subgraphs H. For each such H, we have $|E(H)| = \sum_{v\in V(H)} d^+_H(v) \leqslant |V(H)|\alpha$. Thus, $\text{mad}(G) \leqslant 2\alpha$.)

Theorem 22 ([20]). *For all positive integers C and D with $C \leqslant D$, there exists a graph $G_{C,D}$ with $\Delta(G_{C,D}) = D+1$ and $\chi^2(G_{C,D}) \geqslant \Delta(G_{C,D})+C+1$ and $\text{mad}(G_{C,D}) < 4-\frac{2}{C+1}$.*

Proof. To form $G_{C,D}$, begin with a complete bipartite graph $K_{C,D}$ with parts $\{v_1,\ldots,v_C\}$ and $\{w_1,\ldots,w_D\}$; see Figure 9. Add new vertices x and y, where $N(x) := \{v_1,\ldots,v_C,y\}$ and $N(y) := \{w_1,\ldots,w_D,x\}$. Finally, subdivide each edge v_iw_j with a new 2-vertex z_{ij}. It is easy to check that in $G^2_{C,D}$ all of vertices $v_1,\ldots,v_C,w_1,\ldots,w_D,x,y$ form a clique, so $\chi^2(G_{C,D}) \geqslant C+D+2$. Further, $\Delta(G_{C,D}) = D+1$, so $\chi^2(G_{C,D}) \geqslant \Delta(G_{C,D})+C+1$.

Now we must verify that $\text{mad}(G_{C,D}) < 4-\frac{2}{C+1}$. We fractionally orient $E(G_{C,D})$ as follows. Orient each edge v_iz_{ij} towards z_{ij}. Orient each $z_{ij}w_j$ with the fraction $\frac{1}{C+1}$

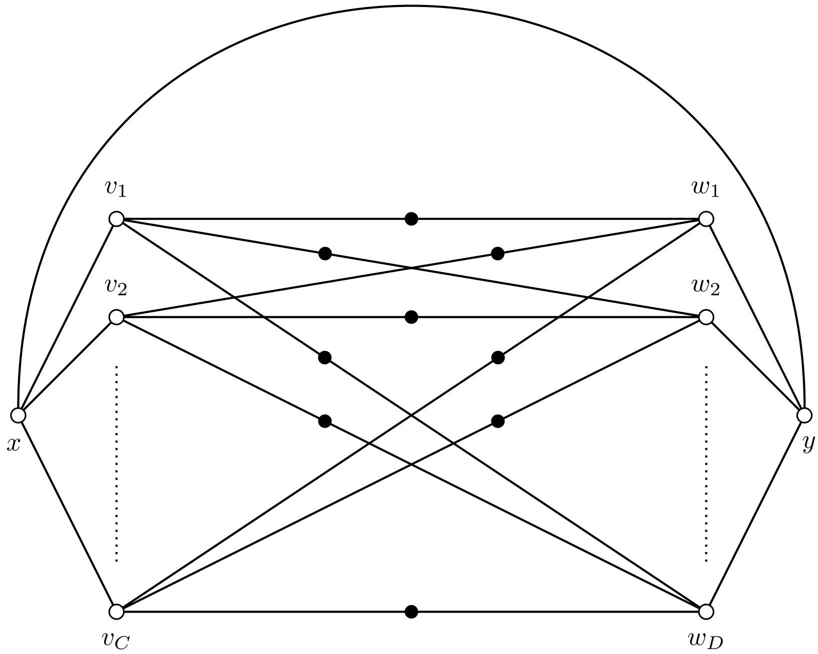


Figure 9: A graph $G_{C,D}$ with $\Delta(G_{C,D}) = D + 1$ and $\chi^2(G_{C,D}) \geqslant \Delta(G_{C,D}) + C + 1$ and $\text{mad}(G_{C,D}) < 4 - \frac{2}{C+1}$, as in Theorem 22.

towards w_j and the rest towards z_{ij}. Orient $v_i x$ towards v_i and orient $w_j y$ towards w_j. Orient xy arbitrarily. This orientation has maximum indegree $2 - \frac{1}{C+1}$, which proves that $\text{mad}(G_{C,D}) \leqslant 2(2 - \frac{1}{C+1})$. Showing the inequality is strict requires a few more details, which we omit. □

Bonamy, Lévêque, and Pinlou [20] suggested that Charpentier's construction is sharp. They asked whether for each $\epsilon > 0$ there exists a constant Δ_ϵ such that if $\text{mad}(G) < 4 - \frac{2}{c+1} - \epsilon$ and $\Delta > \Delta_\epsilon$, then $\chi^2_\ell(G) \leqslant \Delta + c$. (Their result above for mad $< 3 - \epsilon$ proves this for $c = 1$.) In this direction, they showed [18] that if $\text{mad}(G) \leqslant 4 - \frac{40}{c+16}$, then $\chi^2_\ell(G) \leqslant \Delta + c$. For $c \geqslant 3$, Yancey [189] strengthened this bound. He showed that, for each $c \geqslant 3$ and $\epsilon > 0$, if Δ is sufficiently large (in terms of c and ϵ) and $\text{mad}(G) < 4 - \frac{4}{c+1} - \epsilon$, then $\chi^2_\ell(G) \leqslant \Delta + c$.

It is worth noting, in the question above of Bonamy et al., that it is essential to bound $\text{mad}(G)$ strictly away from 4. Charpentier [49] also posed the following conjecture: There exists a constant D such that if G has $\Delta \geqslant D$ and $\text{mad}(G) < 4$, then $\chi^2(G) \leqslant 2\Delta$. This was first disproved by Kim and Park [124]. Using ideas similar to those in the proof of Theorem 22, they constructed graphs G_Δ with $\text{mad}(G_\Delta) < 4$, and maximum degree Δ, but $\chi^2(G_\Delta) = 2\Delta + 2$. The current best known construction, which is due to Hocquard, Kim, and Pierron [96] and presented below, has $\chi^2(G_\Delta) = \frac{5}{2}\Delta$.

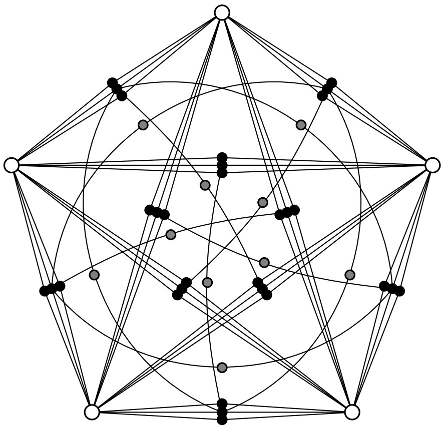


Figure 10: The graph G_D in Theorem 23. In G_D^2, the black vertices form a clique of order $5D/2$.

Theorem 23 ([96]). *For each positive even integer D, there exists a 2-degenerate graph G_D, with* $\text{mad}(G_D) < 4$*, maximum degree D, and $\chi^2(G_D) \geqslant \frac{5D}{2}$. (See Figure 10.)*

Proof. Begin with a copy of K_5 and replace each edge vw with a copy of $K_{2,D/2}$, identifying the vertices in the small part with v and w. Call these $10(D/2) = 5D$ added vertices *new vertices*. Now for each pair of new vertices, x, y, that is at distance 3 or more in the resulting graph, add an additional vertex z_{xy} adjacent to only x and y. Call the resulting graph G_D. Note that in G_D^2 the $5D$ new vertices form a clique. It is easy to check that G_D is 2-degenerate (so has mad < 4) and also that $\Delta(G_D) = 4(D/2) = 2D$. Thus, $\chi(G_D^2) \geqslant \omega(G_D^2) \geqslant 5D = \frac{5}{2}\Delta(G_D)$, as desired. □

To complement Theorem 23, the same authors [96] show that if $\Delta \geqslant 8$ and $\text{mad}(G) < 4$, then $\chi_\ell^2(G) \leqslant 3\Delta + 1$. They also posed the following two questions.

Question 24 ([96]). Is there an integer D such that every graph G with $\Delta(G) \geqslant D$ that is 2-degenerate satisfies $\chi^2(G) \leqslant \frac{5}{2}\Delta(G)$?

Question 25 ([96]). Is there an integer D such that every graph G with $\Delta(G) \geqslant D$ and $\text{mad}(G) < 4$ satisfies $\chi^2(G) \leqslant \frac{5}{2}\Delta(G)$?

4 Strong Edge Coloring

4.1 The Erdős–Nešetřil Conjecture

A *strong edge-coloring* of a graph G colors each edge of G such that edges get distinct colors if either (i) they share a common endpoint or (ii) they each share an endpoint with the same edge. Equivalently, it is a proper coloring of the square of the line graph of G. The *strong edge-chromatic number* of G, denoted $\chi^s(G)$, is the smallest number of colors that allow a strong edge-coloring. At a seminar in Prague in 1985, Erdős and Nešetřil posed the following (see [80]).

strong edge-coloring

$\chi^s(G)$

Conjecture 26. If G is a graph with maximum degree Δ, then

$$\chi^s(G) \leqslant \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even,} \\ \frac{5\Delta^2-2\Delta+1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

The following example shows the conjecture is best possible. When Δ is even, let V_i be an independent set of size $\frac{\Delta}{2}$, for each $i \in \{1, 2, 3, 4, 5\}$. Let $V(G) = \cup_{i=1}^5 V_i$ and for each $v \in V_i$ and $w \in V_j$, add edge vw to $E(G)$ if and only if $(i - j) \equiv 1 \pmod 5$. The resulting graph G is a "5-cycle of independent sets"; since each pair of the $\frac{5}{4}\Delta^2$ edges lies together on a 4-cycle or 5-cycle, all edges must get distinct colors. When Δ is odd, the construction is similar. Now let V_1 and V_2 have size $\frac{\Delta+1}{2}$ and each remaining V_i have size $\frac{\Delta-1}{2}$. In this case the graph has $\frac{5\Delta^2-2\Delta+1}{4}$ edges, and again all edges must get distinct colors. Chung, Gyárfás, Tuza, and Trotter [55] proved that this construction is the unique worst case among graphs where all edges must get distinct colors. Specifically, they showed that the maximum number of edges in a $2K_2$-free graph is exactly given by the upper bound in the Erdős–Nešetřil Conjecture; further, the extremal graphs are unique. In a similar direction, for the line graph $L(G)$ of an arbitrary graph G with maximum degree Δ, Faron and Postle [79] showed that $\omega^2(L(G)) \leqslant \frac{4}{3}\Delta^2$. This also supports the conjecture.

The Erdős–Nešetřil Conjecture is easy for $\Delta \leqslant 2$. For $\Delta = 3$, it was proved by Andersen [9] and also by Horák, He, and Trotter [102]. For $\Delta = 4$, Horák [101] showed that $\chi^s(G) \leqslant 23$, Cranston [57] strengthened this to $\chi^s(G) \leqslant 22$, and Huang, Santana, and Yu [103] strengthened it further to $\chi^s(G) \leqslant 21$. The proofs of [9], [57], and [103] all follow the same approach. For some specified vertex v in G, we color the edges greedily in order of non-increasing distance from v. Thus, at the time each edge is colored, it has at most $2\Delta^2 - 3\Delta$ restrictions on its color, so it uses color at most $2\Delta^2 - 3\Delta + 1$. This is true for all edges except those incident to v. So the hard work consists of showing that we can choose v such that we can finish the coloring. Substituting $\Delta = 3$ gives the desired bound of 10. Substituting $\Delta = 4$ gives 21. To finish the coloring easily, [57] uses a 22^{nd} color, only near v. To save this extra color, [103] considers a wide range of options that work for v, and ultimately shows that every graph contains at least one of them.

For every bipartite graph G, Faudree, Gyárfás, Schelp, and Tuza [80] conjectured that $\chi^s(G) \leqslant \Delta^2$. They proved the weaker statement that if G is such that in every strong edge-coloring each edge must get a distinct color (and G is bipartite), then G has at most Δ^2 edges. Their conjecture was strengthened by Brualdi and Quinn [41], as follows.

Conjecture 27 ([41])**.** If G is bipartite with parts A and B, then $\chi^s(G) \leqslant \Delta_A\Delta_B$, where Δ_A and Δ_B are the maximum degrees, respectively, of vertices in A and B.

This conjecture is sharp, as witnessed by the complete bipartite graph K_{Δ_A,Δ_B}. As a first step, Brualdi and Quinn proved the case when each cycle in G has length divisible by 4. The case $\Delta_A = 1$ is trivial, since the graph is a disjoint union of stars. The case $\Delta_A = 2$ was proved by Nakprasit [154], and the case $\Delta_A = 3$ by Huang, Yu, and Zhou [104].

Mahdian [144] used the probabilistic method to show that if G is C_4-free, then $\chi^s(G) \leqslant (2 + o(1))\frac{\Delta^2}{\ln \Delta}$. (This result was generalized by Vu [174] to a similar upper bound, with a worse multiplicative constant, when any fixed bipartite graph is forbidden as a subgraph.) Mahdian also strengthened the conjecture of Faudree et al. in a different direction.

Conjecture 28 ([144])**.** If G is a graph with no 5-cycle, then $\chi^s(G) \leqslant \Delta^2$.

Greedy coloring (in any order) shows that $\chi^s(G) \leqslant 2\Delta^2 - 2\Delta + 1$. Erdős and Nešetřil specifically asked for $c > 0$ such that $\chi^s(G) \leqslant (2-c)\Delta^2$ for all graphs G. Molloy and Reed [149] provided such a c by using the probabilistic method; see also [151, Chapter 10].

Theorem 29 ([149])**.** *There exists Δ_0 such that whenever $\Delta \geqslant \Delta_0$ we have $\chi^s(G) \leqslant 1.998\Delta^2$.*

Corollary 30 ([149])**.** *There exists $c > 0$ such that $\chi^s(G) \leqslant (2-c)\Delta^2$ for all G.*

Proof. Let $c := \min\{.002, \frac{1}{\Delta_0}\}$. Given a graph G, if $\Delta \geqslant \Delta_0$, then $\chi^s(G) \leqslant 1.998\Delta^2 \leqslant (2-c)\Delta^2$. Otherwise, assume $\Delta < \Delta_0$. Applying Brooks' Theorem to the square of the line graph shows that $\chi^s(G) \leqslant 2\Delta^2 - 2\Delta = (2 - \frac{2}{\Delta})\Delta^2 < (2 - \frac{1}{\Delta_0})\Delta^2 \leqslant (2-c)\Delta^2$. □

The proof of Theorem 29 is quite nice, so we outline it below. We focus on the fact that there exist Δ_0 and $c > 0$ such that $\chi^s(G) \leqslant (2-c)\Delta^2$ for all G with $\Delta \geqslant \Delta_0$, but we don't show that $c = .002$ suffices.

Proof sketch of Theorem 29. For simplicity, assume that G is regular; if not, then we embed it as a subgraph in a regular graph with the same maximum degree Δ. Let H be the square of the line graph of G. We will color H using at most $(2-c)\Delta(G)^2$ colors, for some $c > 0$. Color each vertex of H randomly (and uniformly) from the set $\{1, \dots, \Delta(G)^2\}$ and whenever two adjacent vertices get the same color, uncolor them both.

We can check that, for each vertex v in H, the expected number of colors that are repeated in the neighborhood of v is at least $c'\Delta(G)^2$, for some $c' > 0$ when $\Delta(G)$ is sufficiently large. (This uses that H is the square of a line graph, so $H[N_H(v)]$ has far fewer than $\binom{2\Delta(G)^2}{2}$ edges.) By Talagrand's inequality (see [151, Chapter 10]) we can show that, for each vertex $v \in V(H)$, the number of colors repeated in its neighborhood is close to the expected value with high probability. By the Lovász Local Lemma, there exists some random coloring such that the number of colors repeated in *every* neighborhood is close to its expected value. Given such a partial coloring, we can complete the coloring greedily. This proves the desired result. □

The proof of Theorem 29 breaks the problem of bounding χ^s into two subproblems: (i) proving an upper bound on the density of the subgraph induced by the neighborhood of any vertex in H and (ii) using this density upper bound to prove an upper bound on $\chi(H)$, since $\chi(H) = \chi^s(G)$. Recently, a string of papers has followed this approach, each bounding $\chi^s(G)$ when Δ is sufficiently large. First, Bruhn and Joos [42] showed that $\chi^s(G) \leqslant 1.93\Delta^2$. Bonamy, Perrett, and Postle [21] strengthened this to $\chi^s(G) \leqslant 1.835\Delta^2$. Most recently Hurley, de Joannis de Verclos, and Kang [106] proved that $\chi^s(G) \leqslant 1.772\Delta^2$. Davies, Kang, Pirot, and Sereni [66] also developed a general framework for deriving coloring bounds from bounds on the sparseness of neighborhoods. Their result has several applications, including Johansson's famous bound on the chromatic number of triangle-free graphs, but it does not seem to apply for squares of line graphs.

Recall that Molloy and Reed, in the proof of Theorem 29, constructed their partial coloring of H by coloring each vertex uniformly at random from $\{1, \ldots, \Delta(G)^2\}$ and then uncoloring every vertex that got the same color as some neighbor. This approach is called the Naïve Coloring Procedure. It was introduced by Kahn [112] in his proof that the list-coloring conjecture is true asymptotically, and Molloy and Reed [151] presented many other examples of this technique. The improvements on Theorem 29 given in [42], [21], and [106] all rely crucially on this method. In [42], rather than uncoloring every vertex that got the same color as some neighbor, Bruhn and Joos flipped a coin for each such conflict, and only uncolored the vertices that lost at least one coin flip. They also proved stronger upper bounds on the density of each subgraph $G^2[N(v)]$ and proved stronger bounds on the number of colors repeated in each neighborhood by using the inclusion-exclusion principle.

In [21] and [106], the authors iterated the Naïve Coloring Procedure. That is, they repeatedly used it to color a small fraction of the uncolored vertices. At each iteration, the number of colors available for each uncolored vertex v decreased; but the number of uncolored neighbors of each such v decreased faster. (For each vertex, this property holds with high probability, so the Lovász Local Lemma guarantees that for some partial coloring this property holds for all vertices, simultaneously.) Eventually, each vertex remaining uncolored had fewer uncolored neighbors than available colors, so the partial coloring could be finished greedily.

This iterative approach is often called the semi-random method or the Rödl Nibble (each iteration is a nibble). The Rödl Nibble has been applied successfully to a wide range of problems in combinatorics. Presenting further details is beyond the scope of this survey. However, we direct the interested reader to a recent survey [116] on this topic, as well as to [151, Section V]. Before leaving this topic, we note that variations of this Naïve Coloring Procedure also play central roles in the proofs of Theorems 6 and 38.

4.2 Subcubic Graphs

Within the subject of strong edge-coloring, another line of research has focused on the case $\Delta = 3$. Since $\chi^s(G) \leqslant 10$ for all such graphs, we seek conditions to imply $\chi^s(G) \leqslant 9$ (resp. 8, 7, 6, and 5). Faudree, Schelp, Gyárfás, and Tuza [81] posed the following nice sequence of conjectures for all subcubic graphs G. For easy reference, we present it as a single conjecture with six parts; four of these have been confirmed, one was recently disproved, and the remaining one is still open.

Conjecture 31 ([81])**.** If G is a graph with $\Delta = 3$, then the following six bounds hold.

(a) $\chi^s(G) \leqslant 10$. (Confirmed by Andersen [9] and Horak, He, and Trotter [102].)

(b) $\chi^s(G) \leqslant 9$ if G is bipartite. (Confirmed by Steger and Yu [165].)

(c) $\chi^s(G) \leqslant 9$ if G is planar. (Confirmed by Kostochka, Li, Ruksasakchai, Santana, Wang, and Yu [128].)

(d) $\chi^s(G) \leqslant 6$ if G is bipartite and no 3-vertices are adjacent. (Confirmed by Wu and Lin [187].)

(e) $\chi^s(G) \leqslant 7$ if G is bipartite with no 4-cycle.

(f) $\chi^s(G) \leqslant 5$ if G is bipartite with girth sufficiently large. (Disproved by Lužar, Mačajová, Škoviera, and Soták [141]; also by Cranston [58].)

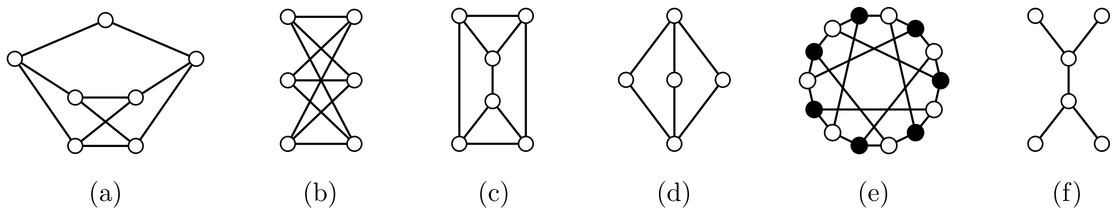


Figure 11: Graphs illustrating that each part of Conjecture 31 is best possible. In each case except (e), the square of the line graph is a clique with order equal to the conjectured upper bound. In contrast, (e) is the Heawood graph H with 21 edges. Now no color can be used on more than 3 edges, so $\chi^s(H) \geqslant 21/3 = 7$.

Wu and Lin [187] proved part (d) in a stronger form. Rather than requiring the graph to be bipartite, they only forbid a single graph H, formed from a 5-cycle by adding a path of length 2 joining two non-adjacent vertices. The disproof of part (f) more generally characterizes [141] the k-regular graphs G with $\chi^s(G) = 2k - 1$; these are graphs which cover the Kneser graph $K(2k-1, k-1)$. So, to disprove part (f), the authors construct cubic bipartite graphs of arbitrarily large girth that have no such cover. An alternative disproof of (f) is given in [58]. We suspect that the bounds in the first 5 parts of Conjecture 31 also hold in the context of list coloring (and possibly even paintability), but we are unaware of any results in this direction.

Many of the parts of Conjecture 31 are now believed to hold in stronger forms.

Conjecture 32. Let G be a subcubic graph.

(a) [97] If G is bridgeless and is neither the Wagner graph nor the graph formed from the complete bipartite graph $K_{3,3}$ by subdividing one edge, then $\chi^s(G) \leqslant 9$.

(b) [141] If G is bridgeless and $|V(G)| \geqslant 13$, then $\chi^s(G) \leqslant 8$.

(c) [141] If G has girth at least 5, then $\chi^s(G) \leqslant 7$.

Note that Conjecture 32(a) strengthens all of Conjecture 31(a,b,c). Conjecture 32(b) even further strengthens Conjecture 32(a). And Conjecture 32(c) strengthens Conjecture 31(e), which still remains open. The authors of [141] also ask whether there exists a girth g_0 such that every cubic graph G with girth at least g_0 satisfies $\chi^s(G) \leqslant 6$.

In the next section, we discuss bounds on χ^s for graphs with bounded maximum average degree. However, when $\Delta \in \{3, 4\}$, the results and proofs tend to be different from the general case. So we include such work here.

Assume that $\Delta = 3$. Hocquard and Valicov [99] showed that if $\mathrm{mad}(G) < \frac{36}{13}$ (resp. $\frac{13}{5}$, $\frac{27}{11}$, $\frac{15}{7}$), then $\chi^s(G) \leqslant 9$ (resp. 8, 7, and 6). A few years later, Hocquard, Montassier, Raspaud, and Valicov [98] weakened the hypotheses on $\mathrm{mad}(G)$. They showed that if $\mathrm{mad}(G) < \frac{20}{7}$ (resp. $\frac{8}{3}$, $\frac{5}{2}$, $\frac{7}{3}$), then $\chi^s(G) \leqslant 9$ (resp. 8, 7, and 6). For all but the bound implying $\chi^s(G) \leqslant 8$, they gave constructions (each with at most eight vertices) showing that the bound on $\mathrm{mad}(G)$ is sharp. This problem has also been studied for list coloring. Ma, Miao, Zhu, Zhang, and Luo [143] showed that if $\mathrm{mad}(G) < \frac{36}{13}$ (resp. $\frac{13}{5}$, $\frac{27}{11}$, $\frac{15}{7}$), then $\chi^s_\ell(G) \leqslant 9$ (resp. 8, 7, 6). These bounds were improved by Zhu and Miao [193], who showed that if $\mathrm{mad}(G) < \frac{14}{5}$ (resp. $\frac{8}{3}$, $\frac{5}{2}$), then $\chi^s_\ell(G) \leqslant 9$ (resp. 8, 7).

Now assume that $\Delta = 4$. Bensmail, Bonamy, and Hocquard [13] showed that if $\mathrm{mad}(G) < \frac{19}{5}$ (resp. $\frac{18}{5}$, $\frac{17}{5}$, $\frac{10}{3}$, $\frac{16}{5}$), then $\chi^s(G) \leqslant 20$ (resp. 19, 18, 17, and 16). Lv, Li, and Yu [142] weakened these hypotheses, showing that if $\mathrm{mad}(G) < \frac{51}{13}$ (resp. $\frac{15}{4}$, $\frac{18}{5}$, $\frac{7}{2}$, $\frac{61}{18}$), then $\chi^s(G) \leqslant 20$ (resp. 19, 18, 17, and 16).

4.3 Planar Graphs and Bounded Maximum Average Degree

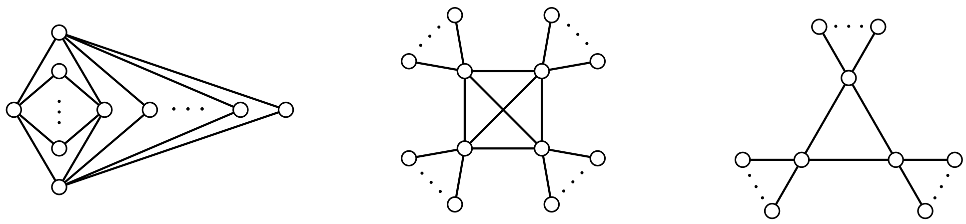
Since planar graphs are sparse, we expect that they should satisfy a stronger upper bound on χ^s. Indeed, Faudree, Schelp, Gyárfás, and Tuza [81] combined Vizing's Theorem and the Four Color Theorem to prove, for every planar graph G, that $\chi^s(G) \leqslant 4\Delta + 4$. They also gave the following construction; see Figure 12(a). Take two copies of $K_{2,\Delta-2}$ and identify the vertices of a 4-cycle in one copy with the vertices of a 4-cycle in the other (so the resulting graph G has maximum degree Δ). Note that G is planar, that $|E(G)| = 4\Delta - 4$, and that a strong edge-coloring of G must give all edges distinct colors. Thus, $\chi^s(G) = 4\Delta - 4$. So, the theorem below is nearly sharp.

Theorem 33 ([81])**.** *If G is a planar graph, then $\chi^s(G) \leqslant 4\Delta + 4$. If also $\Delta \geqslant 7$, then $\chi^s(G) \leqslant 4\Delta$.*

Proof. By Vizing's Theorem, G has an edge-coloring with at most $\Delta + 1$ colors; call these $c_1, \ldots, c_{\Delta+1}$. For each color c_i, form G_i from G by contracting each edge colored c_i. Since G_i is planar, its vertices have a 4-coloring; call its colors $b_{i,1}, b_{i,2}, b_{i,3}, b_{i,4}$. Since the 4-coloring is proper, each pair of edges colored $b_{i,j}$ (for some choice of i and j) must be distance at least two apart in G. Thus, to get a strong edge-coloring of G, we color each edge with its color $b_{i,j}$. This uses at most $4(\Delta + 1)$ colors. If $\Delta \geqslant 7$, then G is class 1, i.e., G has an edge-coloring with only Δ colors [160, 173]; see also [64, Theorem 4.4]. Thus, the proof above now yields $\chi^s(G) \leqslant 4\Delta$. □

When G has girth at least 7, we can use Grötzsch's Theorem to improve the bound [105] in Theorem 33 to $\chi^s(G) \leqslant 3\Delta$. When also $\Delta \geqslant 4$, Ruksasakchai and Wang [159] strengthened this to $\chi^s_\ell(G) \leqslant 3\Delta$.

Hudák, Lužar, Soták, and Škrekovski [105] showed that if G is planar with girth at least 6, then $\chi^s(G) \leqslant 3\Delta+5$. Bensmail, Harutyunyan, Hocquard, and Valicov [14] strengthened this bound to $\chi^s(G) \leqslant 3\Delta + 1$. The current strongest bounds in this direction are due to Choi, Kim, Kostochka and Raspaud [54] and Li, Li, Lv, and Wang [140].



(a) (b) (c)

Figure 12: Figures (a), (b), and (c) illustrate, respectively, that Theorem 33 is nearly sharp, that Theorem 34(i) is sharp, and that Theorem 34(iii) is sharp.

Theorem 34. *(i)[54] If* $\mathrm{mad}(G) < 3$ *and* $\Delta \geqslant 7$*, then* $\chi^s(G) \leqslant 3\Delta$*. (ii)[140] If* $\mathrm{mad}(G) < \frac{26}{9}$ *and* $\Delta \geqslant 7$*, then* $\chi^s(G) \leqslant 3\Delta - 1$*. (iii)[54] If* $\mathrm{mad}(G) < \frac{8}{3}$ *and* $\Delta \geqslant 9$*, then* $\chi^s(G) \leqslant 3\Delta - 3$*.*

All planar graphs of girth at least 6 have mad < 3, so Theorem 34(i) extends the result above of Bensmail et al. when $\Delta \geqslant 7$ and strengthens the bound by 1. The hypothesis $\mathrm{mad}(G) < 3$ is sharp, as follows. Given integers t and Δ, form $G_{t,\Delta}$ from K_t by adding $\Delta - t$ pendant edges at each vertex; see Figure 12(b,c). Note that $\mathrm{mad}(G_{4,\Delta}) = 3$ and $\chi^s(G_{4,\Delta}) = 4\Delta - 6$. In a similar vein, the above bound of $3\Delta - 3$ in (iii) is sharp, since $\chi^s(G_{3,\Delta}) = 3\Delta - 3$ and $\mathrm{mad}(G_{3,\Delta}) = 2$.

Next we return to planar graphs with sufficiently large girth. In a sense, such graphs feel "tree-like", so perhaps a trivial lower bound may hold with equality, as it does for trees. Note that if a graph G has adjacent vertices of degree Δ, then $\chi^s(G) \geqslant 2\Delta - 1$. So we seek sufficient conditions on planar graphs to imply $\chi^s(G) \leqslant 2\Delta - 1$. Borodin and Ivanova [32] showed that $\Delta \geqslant 3$ with $g \geqslant 40\lfloor \Delta/2 \rfloor + 1$ suffices. Chang, Montassier, Pêcher, and Raspaud [47] proved that also $\Delta \geqslant 4$ with $g \geqslant 10\Delta + 46$ suffices, which is a stronger result when $\Delta \geqslant 6$. Most recently, Wang and Zhao [179] weakened the hypotheses further; their result is the best known when $\Delta \geqslant 4$.

Theorem 35 ([179])**.** *If* G *is planar with* $\Delta \geqslant 4$ *and* $g \geqslant 10\Delta - 4$*, then* $\chi^s(G) \leqslant 2\Delta - 1$*.*

Hudák, Lužar, Soták, and Sǩrekovski [105] continued the study of χ^s for planar graphs in terms of both their girth g and their maximum degree Δ. But now they removed the dependence of g on Δ. The following construction provides a lower bound. Given integers g and Δ, form $H_{g,\Delta}$ from the cycle C_g by adding $\Delta - 2$ pendant edges at each cycle vertex. Now $|E(H_{g,\Delta})| = g(\Delta - 2 + 1)$. When g is odd, the maximum size of an induced matching is $(g-1)/2$. So $\chi^s(H_{g,\Delta}) \geqslant \lceil 2g(\Delta - 1)/(g-1) \rceil$. They conjectured that this construction is extremal, up to an additive constant.

Conjecture 36 ([105])**.** There exists a constant C such that if G is any planar graph with girth $g \geqslant 5$ and maximum degree Δ, then

$$\chi^s(G) \leqslant \left\lceil \frac{2g(\Delta - 1)}{g - 1} \right\rceil + C.$$

The best progress on Conjecture 36 is due to Chen, Deng, Yu, and Zhou [51], who proved the following. Let G be connected and planar with girth g, where $g \geqslant 26$. If G is not a subgraph of $H_{g,\Delta}$ (in the previous paragraph), then $\chi^s(G) \leqslant 2\Delta + \lceil 12(\Delta - 2)/g \rceil$.

girth $\geqslant$	3	3	7	$10\Delta - 4$
$\Delta \geqslant$	1	7	1	4
$\chi^s \leqslant$	$4\Delta + 4$	4Δ	3Δ	$2\Delta - 1$

mad $<$	3	26/9	8/3
$\Delta \geqslant$	7	7	9
$\chi^s \leqslant$	3Δ	$3\Delta - 1$	$3\Delta - 3$

Figure 13: Left: Triples $(g, D, f(\Delta))$, in columns, such that every planar graph G with girth at least g and $\Delta(G) \geqslant D$ satisfies $\chi^s(G) \leqslant f(\Delta)$. Right: Triples $(k, D, f(\Delta))$ such that every graph G with $\text{mad}(G) < k$ and $\Delta(G) \geqslant D$ satisfies $\chi^s(G) \leqslant f(\Delta)$. References for both tables appear throughout Section 4.3.

5 Odds and Ends

In this section we briefly touch on a few more related problems.

5.1 Total Coloring and List Coloring Squares

A *total coloring* of a graph G colors its edges and vertices so that every two adjacent or incident elements get distinct colors. The *total chromatic number* $\chi''(G)$, is the minimum number of colors needed for a total coloring. The *total graph* $T(G)$ of a graph G has as its vertices the edges and vertices of G; two vertices of $T(G)$ are adjacent if their corresponding elements are incident or adjacent in G, so $\chi''(G) = \chi(T(G))$. Equivalently, $T(G)$ is the square of its edge-vertex incidence graph; this incidence graph is formed from G by subdividing each edge. Thus, total coloring is a very special case of coloring squares. Undoubtedly, the biggest conjecture in this area is the following, which was posed independently by Bezhad [12] and Vizing [171, 172].

total coloring
χ''
total graph

Conjecture 37 ([12, 171, 172])**.** Every graph G satisfies $\chi''(G) \leqslant \Delta + 2$.

It is easy to check that the conjecture holds for every clique, and that it holds with equality for every even clique, as follows. For the upper bound, let $G := K_{2t+1}$. By Vizing's Theorem, $\chi'(G) \leqslant \Delta + 1 = 2t + 1$. Counting edges implies that each of the $2t+1$ color classes must be a matching of size t; further each vertex has an incident edge in all but one matching. Thus, we can extend the edge-coloring to a total coloring, with no new colors. So $\chi''(K_{2t+1}) = 2t + 1 = \Delta + 1$. Hence, $\chi''(K_{2t}) \leqslant 2t + 1 = \Delta + 2$. Now, we give the matching lower bound for K_{2t}. We must color $\binom{2t}{2} + 2t = 2t^2 + t$ elements. Since each color appears on at most t elements, we need at least $(2t^2 + t)/t = 2t + 1 = \Delta + 2$ colors. Hence, $\chi''(K_{2t}) = \Delta + 2$.

By applying Brooks' Theorem to the total graph, we can see that $\chi''(G) \leqslant 2\Delta$. One approach to improving this bound is to color the vertices first, then the edges. The vertices need at most $\Delta + 1$ colors, and each edge loses colors to only its two endpoints. So this idea shows that $\chi''(G) \leqslant 2 + \chi'_\ell(G)$. Unfortunately, it is non-trivial to bound $\chi'_\ell(G)$ below $2\Delta - 2$ (although this can be done [112, 113]).

The first bound of the form $\chi''(G) \leqslant \Delta + o(\Delta)$ is due to Hind [94], who showed that $\chi''(G) \leqslant \Delta + 2\sqrt{\Delta}$. Chetwynd and Häggkvist [52] improved this to $\chi''(G) \leqslant \Delta + 18\Delta^{1/3}\log(3\Delta)$. And for Δ sufficiently large, Hind, Molloy, and Reed [93] strengthened the bound to $\chi''(G) \leqslant \Delta + \text{poly}(\log \Delta)$. Brualdi [1, p. 437] and Alon [3] asked whether there is some constant C such that $\chi''(G) \leqslant \Delta + C$ for every graph G. The best current bound, due to Molloy and Reed [150] and [151, Chapters 17–18], answers their question affirmatively.

Theorem 38 ([150])**.** *There exists Δ_0 such that if G has $\Delta \geqslant \Delta_0$, then $\chi''(G) \leqslant \Delta + 10^{26}$.*

In the paper, the authors note that by being more careful, they can show that $C = 500$ suffices and probably even $C = 100$ (but most likely not $C = 10$). It seems that no analogue of Theorem 38 is known for list coloring. However, Kahn's results mentioned above for list coloring line graphs [112, 113] do imply that $\chi''_\ell(G) \leqslant \Delta + o(\Delta)$.

Thus far in this survey, we have avoided fractional coloring. Here we make a brief exception. For an introduction to this topic, see [163]. To denote the fractional total chromatic number, we write $\chi''_f(G)$. Kilakos and Reed [120] proved the relaxation of Conjecture 37 for fractional coloring. Specifically, they showed that $\chi''_f(G) \leqslant \Delta + 2$ for every graph G (see also [163, Section 4.6]). Reed conjectured that for every $\epsilon > 0$ and every Δ there exists some girth g such that if G is a graph with girth at least g and maximum degree Δ, then $\chi''_f(G) \leqslant \Delta + 1 + \epsilon$. For $\Delta = 3$ and for Δ even, Kaiser, King, and Král' [115] proved the conjecture in a stronger sense. For each such Δ, they showed there exists g such that if G has maximum degree Δ and girth at least g, then $\chi''_f(G) = \Delta + 1$. Kardoš, Král', and Sereni [119] verified Reed's conjecture for the remaining case, odd Δ (although not in the stronger sense mentioned above).

Now we consider some graph classes for which Conjecture 37 is proved. For bipartite graphs, the result holds trivially, since $\chi''(G) \leqslant \chi(G) + \chi'(G) = 2 + \Delta$. In fact, $\chi''_p(G) \leqslant \Delta + 2$, but the proof is harder. Galvin [83] showed that $\chi'_\ell(G) = \Delta$ for every bipartite graph, and Schauz [161] strengthened this to $\chi'_p(G) = \Delta$. By coloring the vertices first, we get $\chi''_\ell(G) \leqslant 2 + \chi'_\ell(G) = \Delta + 2$; a similar argument shows that also $\chi''_p(G) \leqslant \Delta + 2$. So now suppose G is non-bipartite. For $\Delta = 2$, the problem reduces to computing χ^2 for odd cycles, so $\chi''_{\text{AT}} \leqslant \Delta + 2 = 4$, as noted at the end of Section 3.2. For $\Delta = 3$, the conjecture was proved by Rosenfeld [158] and by Vijayaditya [170]. Kostochka proved it for $\Delta = 4$ [126] and also for $\Delta = 5$, in his dissertation; see [127]. Next, we consider planar graphs.

Theorem 39 ([8])**.** *Every planar graph G with $\Delta \geqslant 7$ satisfies $\chi''(G) \leqslant \Delta + 2$.*

Proof. By the Four Color Theorem, we properly color the vertices with colors $\{1, 2, 3, 4\}$. Since G is planar and $\Delta \geqslant 7$, we can properly color the edges with Δ colors [160, 191]; we use colors $\{3, 4, \ldots, \Delta + 1, \Delta + 2\}$. The only possible conflicts involve edges colored with 3 or 4, so we uncolor all such edges. Now each uncolored edge has available two of the colors $\{1, 2, 3, 4\}$, since only two of these are used on its endpoints. Note that this uncolored subset of edges induces a vertex disjoint union of paths and even cycles (since the edges were properly colored with 3 and 4). Thus, we can color the edges from their lists of available colors, since paths and even cycles have list chromatic number 2. □

The idea of this proof is due to Yap [108, p. 88], but the first presentation with all the details seems to be due to Andersen [8].

For planar graphs with Δ sufficiently large, the bound in Theorem 39 can be improved to $\chi''(G) = \Delta + 1$. Borodin [23] showed that $\Delta \geqslant 14$ suffices. This hypothesis was successively weakened to $\Delta \geqslant 11$ by Borodin, Kostochka, and Woodall [37]; to $\Delta \geqslant 10$ by Wang [180]; and to $\Delta \geqslant 9$ by Kowalik, Sereni, and Škrekovski [132].

When $\Delta \leqslant 8$, it is natural to look for additional hypotheses that imply $\chi''(G) = \Delta+1$. Chen and Wu [50] showed that $\chi''(G) = \Delta + 1$ for planar graphs with $\Delta \geqslant D$ and $g \geqslant k$ whenever $(D, k) \in \{(8, 4), (6, 5), (4, 8)\}$. Borodin, Kostochka, and Woodall [38] strengthened these results by weakening the hypotheses to $(D, k) \in \{(7, 4), (5, 5), (4, 6), (3, 10)\}$. For planar graphs with $\Delta \in \{7, 8\}$, the bound $\chi''(G) = \Delta + 1$ has also been proved when various types of cycles are forbidden; see, for example, [188, 177, 176, 48, 178, 175]. For the case $\Delta = 6$, the best result is $\chi''(G) \leqslant 9$. This was originally proved by Borodin [23], but also follows immediately from the fact that $\chi''(G) \leqslant \Delta + 2$ when $\Delta = 7$.

The most famous conjecture in list coloring states that every line graph G has $\chi_\ell(G) = \chi(G)$. In 2001, Kostochka and Woodall [129] posed an analogous conjecture for squares.

Square List Coloring Conjecture ([129])**.** *For every graph G, we have $\chi^2_\ell(G) = \chi^2(G)$.*

Every graph G clearly satisfies $\chi^2_\ell(G) \geqslant \chi^2(G) \geqslant \Delta + 1$. However, proving a better lower bound on $\chi^2(G)$ or even $\chi^2_\ell(G)$ is typically quite difficult. Thus, the majority of graphs G known to satisfy the Square List Coloring Conjecture are those for which $\chi^2_\ell(G) = \chi^2(G) = \Delta + 1$. This conjecture was proved for many classes of graphs [18, 20, 34, 63, 72], but in general it is false, as shown in 2013 by Kim and Park [123].

Theorem 40 ([123])**.** *The Square List Coloring Conjecture is false. More specifically, for each integer k, there exists a graph G_k such that $\chi_\ell(G^2_k) - \chi(G^2_k) > k$.*

Proof Sketch. A *latin square* is an arrangement of k copies of each of the symbols $1, \dots, k$ in cells of a $k \times k$ grid, so that each symbol appears exactly once in each row and each column. When we overlay one latin square with another, we form in each cell of the grid an ordered pair, (x, y), with x coming from the first square and y from the second. Two latin squares are *mutually orthogonal* if each of the possible ordered pairs appears in exactly one cell of the grid. It is well known that for every prime k, there exists a family of $k - 1$ pairwise mutually orthogonal latin squares.

Kim and Park used these mutually orthogonal latin squares to construct a graph H_k such that $H^2_k = K_{k*(2k-1)}$, the complete multipartite graph with $2k - 1$ parts, each of size k. Clearly, this gives $\chi(H^2_k) = 2k - 1$. Further, it is known [169] that $\chi_\ell(K_{k*r}) > (k-1)\lfloor\frac{2r-1}{k}\rfloor$. In particular, this gives $\chi_\ell(K_{k*(2k-1)}) > 3(k - 1)$. Now taking $G_k = H_{k+2}$ gives $\chi_\ell(G^2_k) - \chi(G^2_k) > k$, as desired. □

Later Kim and Park [122] showed that the graphs G_k in Theorem 40 can also be required to be bipartite. In the direction of the Square List Coloring Conjecture, Zhu asked whether there exists a constant K such that $\chi_\ell^k(G) = \chi^k(G)$ for all $k \geqslant K$ and all graphs G. This question was answered negatively by Kosar, Petrickova, Reiniger, and Yeager [125] and also by Kim, Kwon, and Park [121]. The authors of [91] attempted to partially salvage the Square List Coloring Conjecture, proposing that it holds for all planar graphs. However, this too was disproved. See Conjecture 7 and Theorem 8.

Recall from above that $\chi''(G) = \chi(T(G))$, where the *total graph* $T(G)$ of a graph G has as its vertices the edges and vertices of G; two vertices of $T(G)$ are adjacent if their corresponding elements are incident or adjacent in G. Equivalently, the total graph of G is formed from G by subdividing each edge of G, then taking the square. The disproof of the full Square List Coloring Conjecture (Theorem 40 above) has increased interest in the following special case, which was posed earlier by Borodin, Kostochka, and Woodall [36].

Total List Coloring Conjecture ([36]). *For every graph G, if $T(G)$ is the total graph of G, then $\chi_\ell(T(G)) = \chi(T(G))$.*

In [36] the authors proved the conjecture for every simple planar graph with $\Delta \geqslant 12$. The same paper included a clever averaging argument, which proved the conjecture whenever $\mathrm{mad}(G) \leqslant \sqrt{2\Delta}$. The proof of the latter result was significantly simplified by Woodall [186]; see also [64, Section 4]. A *multicircuit* is a multigraph for which the underlying simple graph is a cycle. Kostochka and Woodall proved the Total List Coloring Conjecture for all multicircuits [130, 131]; however, in general, it remains open.

5.2 L(2,1)-Labeling

An $L(p, q)$-*labeling* of a graph G assigns to each vertex of G a positive integer such that each pair of vertices at distance 1 in G receives integers differing by at least p and each pair of vertices at distance 2 in G receives integers differing by at least q. The *span* of such a labeling is the difference between its largest and smallest integers. The minimum span over all $L(p, q)$-labelings of a graph G is $\lambda^{p,q}(G)$. When $p = q = 1$, the problem is $\lambda^{p,q}$
equivalent to coloring G^2; however, note that $\chi^2(G) = \lambda^{1,1}(G) + 1$. The next most widely studied case is when $p = 2$ and $q = 1$. Griggs and Yeh [86] posed the following intriguing conjecture.

***L*(2, 1)-Labeling Conjecture** ([86]). *Every graph G has $\lambda^{2,1}(G) \leqslant \Delta^2$.*

Consider a greedy $L(2, 1)$-labeling of G in an arbitrary order. For each vertex v, each of at most Δ neighbors of v forbids at most three labels on v. Similarly, each of at most $\Delta(\Delta - 1)$ vertices at distance 2 from v forbids at most one label on v. Thus, a greedy $L(2, 1)$-labeling uses no label larger than $1+3\Delta+\Delta(\Delta-1)$. So $\lambda^{2,1}(G) \leqslant \Delta^2+2\Delta$. Chang and Kuo [46] gave the first major improvement of this bound, showing that $\lambda^{2,1} \leqslant \Delta^2+\Delta$, for every graph. This upper bound was further strengthened to $\Delta^2 + \Delta - 1$ by Král' and Škrekovski [134] and to $\Delta^2 + \Delta - 2$ by Gonçalves [84]. This result of Gonçalves remains the best bound in general, although for Δ sufficiently large Havet, Reed, and Sereni have proved the $L(2,1)$-Labeling Conjecture. In fact, they proved the same upper bound [90] for $L(p,1)$-labeling in general.

Theorem 41 ([90]). *For each $p \geqslant 1$, if G has Δ sufficiently large, then $\lambda^{p,1}(G) \leqslant \Delta^2$.*

Numerous authors have written entire surveys on $L(p,q)$-labelings and their generalizations, such as real number labeling. Thus, we direct the interested reader to these [85, 43, 190]. We close this short section with a sketch of the cute result [86] that the $L(2,1)$-Labeling Conjecture holds for all graphs of diameter 2.

Theorem 42 ([86]). *If G has diameter 2, then $\lambda^{2,1}(G) \leqslant \Delta^2$.*

Proof Sketch. Let $n := |V(G)|$. It is easy to check that $\lambda^{2,1}$ is at most 4 for paths and cycles, so assume $\Delta \geqslant 3$. First, suppose that $n \leqslant 2\Delta + 1$. If $\Delta \geqslant 4$, then label the vertices arbitrarily with distinct elements of $\{0, 2, 4, \ldots, 2(n-1)\}$. This labeling is valid, and it has span at most $2(n-1) \leqslant 2(2\Delta) \leqslant \Delta^2$. If $\Delta = 3$, a slight modification works, labeling at most two vertices with odd labels. So assume instead that $n \geqslant 2\Delta + 2$. Let $d_{\overline{G}}(v)$ denote the degree of v in $\overline{G}$, the complement of G. Since $n \geqslant 2\Delta + 2$, we have $d_{\overline{G}}(v) \geqslant n - (\Delta + 1) \geqslant n/2$ for all v. By Dirac's Theorem, $\overline{G}$ contains a Hamiltonian path; call it $v_1, v_2, \ldots, v_n$. Now label v_i with integer i. Since G is diameter 2, we have $n \leqslant \Delta^2 + 1$, so the span of this labeling is at most Δ^2. Since the labels are distinct, we need only show, for each i, that v_i and v_{i+1} are non-adjacent. This is true because $v_1, v_2, \ldots, v_n$ is a Hamiltonian path in $\overline{G}$. $\square$

This approach was extended by Cole [82], who showed that $\lambda^{2,1}(G) \leqslant \Delta^2$ whenever G has order at most $(\lfloor \Delta/2 \rfloor + 1)(\Delta^2 - \Delta + 1) - 1$.

5.3 Higher Powers

Recall that the k^{th} power, G^k, of a graph G is formed from G by adding an edge between each pair of vertices at distance at most k in G. Let $D_{k,\Delta}$ denote the largest possible degree of a vertex in a k^{th} power of a graph with maximum degree Δ. It is easy to check that $D_{k,\Delta} = \sum_{i=1}^{k} \Delta(\Delta-1)^{i-1} = \Delta((\Delta-1)^k - 1)/(\Delta-2)$. When $k \geqslant 3$ the situation is somewhat simpler than for $k = 2$, since there does not exist any graph $G_{k,\Delta}$ with maximum degree Δ (and $\Delta \geqslant 3$) such that $G_{k,\Delta}^k = K_{D_{k,\Delta}+1}$. This was proved by Damerell [65] and by Bannai and Ito [11]. (Both proofs followed the general approach of Hoffman and Singleton for showing the nonexistence of diameter 2 Moore graphs except when $\Delta \in \{2, 3, 7, 57\}$: studying the eigenvalues of a hypothetical such graph and reaching a contradiction. Recall that this approach was outlined immediately preceeding Lemma 1.) As a result, an analogue of Theorem 2 for $k \geqslant 3$ does not need any exceptional graphs. In fact, Bonamy and Bousquet [16] showed, for each $k \geqslant 3$ and graph G with maximum degree Δ, that $\chi_\ell^k(G) \leqslant D_{k,\Delta} - 1$. And they conjectured something even stronger: For each integer $k \geqslant 3$, except for a finite number of graphs G, every connected graph G satisfies $\chi_\ell^k(G) \leqslant D_{k,\Delta} + 1 - k$.

$D_{k,\Delta}$

The motivation for this conjecture follows the proof sketch we provided of Theorem 2. We can greedily color the vertices in order of non-increasing distance from some subgraph H of diameter at least k. Each vertex v outside H has at least k neighbors in G^k that are closer to H (the first k on a shortest path in G from v to H) and are thus uncolored at the time we color v. Hence, the problem reduces to proving that G always contains a good subgraph H. For coloring (but not list coloring), their conjecture was confirmed by Pierron [156].

Theorem 43. *Fix integers $k \geqslant 3$ and $\Delta \geqslant 3$. For all but finitely many connected graphs G with maximum degree Δ, we have $\chi^k(G) \leqslant D_{k,\Delta}+1-k$. (Here $D_{k,\Delta} = \sum_{i=1}^{k} \Delta(\Delta-1)^{i-1}$.)*

The reasoning above suggests that perhaps also $\chi_p^k(G) \leqslant D_{k,\Delta}+1-k$, but we are unaware of any progress in this direction.

Now we turn to lower bounds. Let $n_{k,\Delta}$ denote the largest order of a graph with maximum degree Δ and diameter k. Bollobás [15] conjectured, for every $\epsilon > 0$, that we have $n_{k,\Delta} > (1-\epsilon)\Delta^k$ for Δ and k both sufficiently large. It seems the best result in this direction is that $n_{k,\Delta} \geqslant \left(\frac{\Delta}{1.6}\right)^k$ for all k and an infinite set of values of Δ. This was proved by Canale and Gómez [45].

It is also natural to consider coloring powers of graphs from some class, such as planar graphs, chordal graphs, or line graphs. In many cases the best known bounds on $\chi^k(G)$ come from bounds on $\chi^k_{\mathrm{col}}(G)$, and most work gives only asymptotic bounds.

Agnarsson and Halldórsson [2] proved that if G is planar, then $\chi^k_{\mathrm{col}}(G) = O(\Delta^{\lfloor k/2 \rfloor})$. This is best possible, as shown by a maximum tree with diameter k and maximum degree Δ. Král' [133] showed that if G is chordal, then $\chi^k_{\mathrm{col}}(G) = O(\sqrt{k}\Delta^{(k+1)/2})$ when k is even and $\chi^k_{\mathrm{col}}(G) = O(\Delta^{(k+1)/2})$ when k is odd. For odd k this is again best possible. Now the construction is similar, but the root of the tree is replaced by a clique on $\Delta/2$ vertices.

For coloring powers of line graphs, greedy coloring trivially gives the bound $\chi^k(L(G)) \leqslant 2\Delta^k$. Kaiser and Kang [114] improved this to $\chi^k(L(G)) \leqslant (2-\epsilon)\Delta^k$ for some $\epsilon > 0$. In a related question, Erdős and Nešetřil [74] asked for the minimum number of edges $h^k(\Delta)$ such that if any graph G has maximum degree at most Δ and at least $h^k(\Delta)$ edges, then its line graph $L(G)$ has diameter at least $k+1$. It is trivial to check that $h_1(\Delta) = \Delta+1$. Chung, Gyárfás, Tuza, and Trotter exactly determined $h_2(\Delta)$; it is $\frac{5}{4}\Delta^2+1$ when Δ is even and slightly smaller when Δ is odd (see the start of Section 4.1). For larger k, Cambie, Cames van Batenburg, de Joannis de Verclos, and Kang [44] showed that $\omega^k(L(G)) \leqslant \frac{3}{2}\Delta^k$. In particular, $h^k(\Delta) \leqslant \frac{3}{2}\Delta^k+1$. This implies that $\chi^k(L(G)) \leqslant 1.941\Delta^k$ for Δ sufficiently large (strengthening the result above of Kaiser and Kang).

Let $f^k(\Delta, g)$ denote the maximum value of $\chi^k(G)$ over all graphs G with girth g and maximum degree Δ. Alon and Mohar [5] determined the asymptotic value of $f^2(\Delta, g)$, when g is fixed and Δ grows. They showed that for $g \leqslant 6$, we have $f^2(\Delta, g) = \Delta^2(1+o(1))$. The upper bound comes trivially from greedy coloring. For $k \geqslant 2$ and $g \geqslant 3k+1$, they showed that there exists a constant C_1 such that $f^k(\Delta, g) \leqslant C_1\Delta^k/\log\Delta$. One way to prove this is using Johansson's result [109] for list coloring triangle-free graphs; also see [151, Chapters 12–13]. He showed that there exists a constant C_2 such that every triangle-free graph G satisfies $\chi_\ell(G) \leqslant C_2\Delta/(\log\Delta)$. Since G^k has girth at least $\lceil g/k \rceil$

and maximum degree $O(\Delta^k)$, the result follows. To prove an asymptotically matching lower bound for $f^k(\Delta, g)$ when $g \geqslant 2k+3$, Kaiser and Kang [114] gave a random construction. (Alon, Krivelevich, and Sudakov [4] extended this result, by weakening the girth hypothesis to a more general sparsity hypothesis.) Kang and Pirot [117] extended this result by proving the same upper bound when excluding cycles of fewer lengths. They further strengthened this result [118] by showing it suffices to forbid a single cycle length.

Theorem 44 ([118])**.** *Let k be a positive integer and ℓ be an even positive integer such that $\ell \geqslant 2k+2$. The supremum of $\chi^k(G)$, over all graphs G with maximum degree Δ and no cycles of length ℓ is $\Theta(\Delta^k/\log k)$, as $\Delta \to \infty$.*

Finally, we remark briefly about coloring exact distance graphs. The *exact distance-k graph* $G^{[\natural k]}$ has as its vertex set $V(G)$. Two vertices are adjacent in $G^{[\natural k]}$ precisely if they are at distance exactly k in G. We will not formally define graph classes with *bounded expansion*, but examples of such classes include all graphs embeddable in any fixed surface and, more generally, all graphs with any fixed graph H forbidden as a minor. Nešetřil and Ossona de Mendez [155, Theorem 11.8] proved the following.

Theorem 45 ([155])**.** *Let $\mathcal{G}$ be a class of graphs with bounded expansion.*

1. *If k is an odd positive integer, then there exists a constant C_1 (as a function of $\mathcal{G}$ and k) such that for every graph $G \in \mathcal{G}$ we have $\chi(G^{[\natural k]}) \leqslant C_1$.*
2. *If k is an even positive integer, then there exists a constant C_2 (as a function of $\mathcal{G}$ and k) such that for every graph $G \in \mathcal{G}$ we have $\chi(G^{[\natural k]}) \leqslant C_2\Delta(G)$.*

The values of C_1 and C_2 arising from the proofs in [155] are very large. These values have been significantly reduced by subsequent work of (among others) Zhu [195], Stavropoulos [164], and van den Heuvel, Kierstead, and Quiroz [167].

6 Open Problems and Conjectures

In this short section, for easy reference we collect some problems and conjectures from throughout this survey that remain open. For consistency, we phrase each as a conjecture, although a few were initially only posed as questions, which we indicate where relevant. In each instance, we provide a brief statement of the problem, possibly a few comments, and a link to the place where the problem or conjecture first appears in this survey.

1. For all integers $k \geqslant 1$ and $D \geqslant 3$, let $\chi^k(D)$ and $\omega^k(D)$ denote, respectively, the maximums over all graphs G with $\Delta \leqslant D$ of $\chi(G^k)$ and $\omega(G^k)$. For all k and D we have $\chi^k(D) = \omega^k(D)$. This is called Wegner's Conjecture. It is trivially true for $k = 1$. For $k = 2$, it is proved for $D \in \{2, 3, 4, 5, 7\}$.
2. For each integer t, there exists a constant Δ_t such that all G with $\Delta \geqslant \Delta_t$ satisfy $\omega^2(G) \leqslant \Delta^2 - t$. This conclusion can be strengthened further to (i) $\chi^2(G) \leqslant \Delta^2 - t$, (ii) $\chi^2_\ell(G) \leqslant \Delta^2 - t$, (iii) $\chi^2_p(G) \leqslant \Delta^2 - t$, and even (iv) $\chi^2_{\mathrm{AT}}(G) \leqslant \Delta^2 - t$. This is Question 4.

3. Every planar graph G with maximum degree $\Delta(G) \geqslant 8$ satisfies $\chi^2(G) \leqslant \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$. This is Wegner's Planar Graph Conjecture (which also conjecture sharp bounds on $\chi^2(G)$ for planar graphs with smaller maximum degree). This conjecture is known to hold asymptotically, even for list coloring (see Theorem 6): For each $\epsilon > 0$ there exists Δ_ϵ such that if G is a planar graph with $\Delta \geqslant \Delta_\epsilon$, then $\chi^2_\ell(G) \leqslant \frac{3}{2}\Delta(1+\epsilon)$.

4. If $\mathcal{G}$ is a minor-closed class of graphs that excludes $K_{3,t}$ for some integer t, then there exists Δ_0 such that if $G \in \mathcal{G}$ and $\Delta(G) \geqslant \Delta_0$, then $\chi^2(G) \leqslant \chi^2_\ell(G) \leqslant \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor + 1$. This is Conjecture 9, due to Havet, van den Heuvel, McDiarmid, and Reed. This conjecture is best possible, since equality holds for the planar graphs in Figure 3.

5. There exists an integer D such that every graph G with $\Delta(G) \geqslant D$ that is 2-degenerate satisfies $\chi^2(G) \leqslant \frac{5}{2}\Delta(G)$. More generally, there exists an integer D such that every graph G with $\Delta(G) \geqslant D$ and $\mathrm{mad}(G) < 4$ satisfies $\chi^2(G) \leqslant \frac{5}{2}\Delta(G)$. These are Questions 24 and 25. They are both best possible due to the construction in Theorem 23.

6. Every graph G with maximum degree Δ satisfies $\chi^s(G) \leqslant 1.25\Delta^2$. This is the Erdős–Nešetřil Conjecture. The best bound for general Δ is $\chi^s(G) \leqslant 1.772\Delta^2$; see the end of Section 4.1.

7. If G is bipartite with parts A and B and maximum degrees (in these parts) Δ_A and Δ_B, then $\chi^s(G) \leqslant \Delta_A\Delta_B$. This is Conjecture 27, due to Brualdi and Quinn.

8. If G has maximum degree Δ and has no 5-cycle, then $\chi^s(G) \leqslant \Delta^2$. This is Conjecture 28, due to Mahdian.

9. If G is subcubic and bipartite and G has no 4-cycle, then $\chi^s(G) \leqslant 7$. This is Conjecture 31(e), due to Faudree, Schelp, Gyárfás, and Tuza.

10. Let G be a subcubic bridgeless graph. If G is neither the Wagner graph nor the graph formed from $K_{3,3}$ by subdiving an edge, then $\chi^s(G) \leqslant 9$. This is Conjecture 32(a), due to Hocquard, Lajou, and Lužar.

11. Let G be a subcubic graph. If G is bridgeless and $|V(G)| \geqslant 13$, then $\chi^s(G) \leqslant 8$. If G has girth at least 5, then $\chi^s(G) \leqslant 7$. These are Conjecture 32(b,c), due to Lužar, Mačajová, Škoviera, and Soták. The latter implies the conjecture above of Faudree, Schelp, Gyárfás, and Tuza.

12. There exists a constant C such that if G is any planar graph with girth $g \geqslant 5$ and maximum degree Δ, then $\chi^s(G) \leqslant \left\lceil \frac{2g(\Delta-1)}{g-1} \right\rceil + C$. This is Conjecture 36, due to Hudák, Lužar, Soták, and Škrekovski.

13. Every graph G satisfies $\chi''(G) \leqslant \Delta + 2$. This is Conjecture 37, posed independently by Bezhad and by Vizing. Asymptotically, the best result on this problem is that there exists a constant C such that $\chi''(G) \leqslant \Delta + C$; see Theorem 38.

14. If graph G has maximum degree Δ, then $\lambda^{2,1}(G) \leqslant \Delta^2$. This is the $L(2,1)$-Labeling Conjecture, due to Griggs and Yeh. It has been proved for Δ sufficiently large; see Theorem 41.

15. Fix integers $k \geqslant 3$ and $\Delta \geqslant 3$. For all but finitely many connected graphs G with maximum degree Δ, we have $\chi_\ell^k(G) \leqslant \sum_{i=1}^{k} \Delta(\Delta-1)^{i-1} + 1 - k$. The same bound holds for $\chi_p^k(G)$. This would be a list coloring, or painting, analogue of Theorem 43. (In fact, both assertions were only posed as questions.)

Acknowledgments

Thanks to Stephen Hartke, Marthe Bonamy, and two anonymous referees for various suggestions that improved the coverage and presentation.

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