Dynamic One-Pile Blocking Nim

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Abstract

The purpose of this paper is to solve a class of combinatorial games consisting of one-pile counter pickup games for which the number of counters that can be removed on each successive turn changes during the play of the game. Both the minimum and the maximum number of counters that can be removed is dependent upon the move number. Also, on each move, the opposing player can block some of the moving player's options. This number of blocks also depends upon the move number.

There is great interest in generalizations and modifications of simple, deterministic two-player "take-away-games" — for a nice survey, see chapter 4 of [1]. We discuss here a modification where the player-not-to-move may effect the options of the other player. Modifications of this type have been called Muller twists in the literature. See [4]. In [3], we discuss games in which the number of counters that can be removed depends on the number removed in the previous move.

We begin with some notation. The set of integers is denoted by $\mathbb{Z}$, the positive integers by $\mathbb{N}$ and the nonnegative integers by $\mathbb{N}_0$. If $a, b \in \mathbb{Z}$ with $a \leq b$, then $[a, b]$ denotes $\{x \in \mathbb{Z} : a \leq x \leq b\}$.

Rules of the Game:
We are given three sequences $(c_i \in \mathbb{N}_0)_{i\in\mathbb{N}}$, $(m_i \in \mathbb{N})_{i\in\mathbb{N}}$ and $(M_i \in \mathbb{N})_{i\in\mathbb{N}}$ which satisfy the following conditions:

$$\forall i \in \mathbb{N}, \quad c_i \leq c_{i+1} \tag{1}$$

and

$$u_i = M_i - m_i - c_i \text{ for each } i \in \mathbb{N} \text{ and } \forall i \in \mathbb{N}, 0 \leq u_i \leq u_{i+1}. \tag{2}$$

These two conditions imply that $(M_i - m_i \in \mathbb{N}_0)_{i\in\mathbb{N}}$ is a nondecreasing sequence.

There are two players and a pile of counters. These two players alternate removing counters from the single pile according to the following rules: denote by $k \in \mathbb{N}$ the move-counter and by $p_k \in \mathbb{N}_0$ the pile size before the k-th move. Then the player to make the k-th move must remove from the pile any number of counters $x \in [m_k, M_k]$ satisfying $x \leq p_k$. There is also a further restriction set by the other player for the selection of x: before a player makes his k-th move, the opponent can prohibit up to c_k of the current options. Therefore a player cannot move if p_k is less than the smallest available option. Condition (1) and condition (2) can be interpreted as saying that the number of at most blocked options c_i as well as the number of at least available options $u_i + 1$ must be a nondecreasing function of the turn number i. The game ends as soon as one of the two players cannot move, and this player is called the loser.

As an example, look at the third move of such a game. Suppose, $[m_3, M_3] = [5, 10]$, $c_3 = 2$ and $p_3 = 15$. Since $c_3 = 2$, the opponent of the player-to-move, can block at most two of the moving player's six options. Suppose that he denies the removal of 6 counters and the removal of 10 counters from the available interval $[5, 10]$. This means the player-to-move can remove from the 15 counter pile either $5, 7, 8$ or 9 counters. If we modify the example so that $p_3 = 6$ and the opponent prohibits the removal of 5 or 6 counters, the player-to-move can not move at all and loses this game.

Whether the starting player, also called the first player, or the second player will win the game depends therefore only on the pile size at the beginning.

The possible pile sizes, numbers in $\mathbb{N}_0$, which are a lost position for the player-to-move are called safe positions. The complement is called the unsafe positions, these are the winning positions for the player-to-move. These two sets of pile sizes will be characterized in the following as a set of disjoint intervals of maximal length in $\mathbb{N}$.

Theorem 1. *The safe positions of the game are*

$$\bigcup_{k\in\mathbb{N}_0} [A_k, B_k - 1] \quad \textit{with}$$

$$A_k = \sum_{i=1}^{k} M_{2i-1} + \sum_{i=1}^{k} m_{2i} + \sum_{i=1}^{2k} (-1)^i c_i \quad \textit{and}$$

$$B_k = \sum_{i=1}^{k} M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i \quad .$$

Of course, the unsafe positions are the remaining positions and are the set

$$\bigcup_{k \in \mathbb{N}_0} [B_k, A_{k+1} - 1].$$

Of course these formulas also give the safe and unsafe intervals at each turn by indexing upwards. Let us first show that all of the above intervals of integers exist. This means

Lemma 1. $\forall k \in \mathbb{N}_0 : \quad A_k < B_k \quad$ *and* $\quad B_k < A_{k+1} \quad .$

Proof. Let k any fixed nonnegative integer. From (1) we have

$$\forall i \in \mathbb{N} : \quad c_{2i} \leq c_{2i+1} \leq c_{2i+2} \tag{3}$$

and, summing up from 1 to k and increasing the middle sum by c_1 and the right sum by c_2, we get

$$\sum_{i=1}^{k} c_{2i} \leq \sum_{i=1}^{k} c_{2i+1} = \sum_{i=2}^{k+1} c_{2i-1} \leq \sum_{i=1}^{k+1} c_{2i-1} \leq c_2 + \sum_{i=1}^{k} c_{2i+2} = \sum_{i=1}^{k+1} c_{2i}, \tag{4}$$

and from (2) we have

$$\forall i \in \mathbb{N} : \quad u_{2i-1} \leq u_{2i} \leq u_{2i+1} \quad , \tag{5}$$

and here we get by summing up and increasing the right term by u_1

$$\sum_{i=1}^{k} u_{2i-1} \leq \sum_{i=1}^{k} u_{2i} \leq \sum_{i=1}^{k} u_{2i+1} = \sum_{i=2}^{k+1} u_{2i-1} \leq \sum_{i=1}^{k+1} u_{2i-1}. \tag{6}$$

Adding up (4) and (6) and replacing these u_{2i-1} and u_{2i} by their definition we get

$$\begin{aligned} &\sum_{i=1}^{k} (M_{2i-1} - m_{2i-1} - c_{2i-1}) + \sum_{i=1}^{k} c_{2i} \\ \leq \; &\sum_{i=1}^{k} (M_{2i} - m_{2i} - c_{2i}) + \sum_{i=1}^{k+1} c_{2i-1} \\ \leq \; &\sum_{i=1}^{k+1} (M_{2i-1} - m_{2i-1} - c_{2i-1}) + \sum_{i=1}^{k+1} c_{2i} \end{aligned} \tag{7}$$

which can be rearranged to

$$\sum_{i=1}^{k} M_{2i-1} + \sum_{i=1}^{k} m_{2i} + \sum_{i=1}^{2k} (-1)^i c_i$$

$$\leq \sum_{i=1}^{k} M_{2i} + \sum_{i=1}^{k} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i \tag{8}$$

$$\leq \sum_{i=1}^{k+1} M_{2i-1} + \sum_{i=1}^{k} m_{2i} - m_{2k+1} + \sum_{i=1}^{2k+2} (-1)^{i} c_i \ .$$

Finally increasing the middle side by m_{2k+1} and the right side even more by $m_{2k+1}+m_{2k+2}$ we get exactly

$$A_k < B_k < A_{k+1}. \tag{9}$$

□

For any $k \in \mathbb{N}_0$, we will show if the pile size $n \in [A_k, B_k - 1]$, then the player-to-move can either not move at all or can be forced by the blocking move of his opponent to reduce the pile size to a value $n' \in [B_{k-1}, A_k - 1]$, and if the pile size n is in $[B_k, A_{k+1} - 1]$ can independently of the blocking move of his opponent always decrease the pile size to a value $n' \in [A_k, B_k - 1]$.

Proof of Theorem 1. First, by prohibiting the smallest c_1 options to move, the player-to-move is forced to take at least $m_1 + c_1$ counters from the pile. So, if the starting pile size is less than $m_1 + c_1$ the first player will obviously lose because there are not sufficiently many counters in the pile. For all pile sizes greater or equal $m_1 + c_1$ he can move and the game will continue with his opponent to move. Exactly this is described by the first safe interval $[A_0, B_0 - 1] = [0, m_1 + c_1 - 1]$. For this reason the first player will try to generate for his opponent a pile size of the interval $[0, m_2 + c_2 - 1]$ because then the second player cannot move on his 2-nd turn. By the blocking options of his opponent, the first player may be forced to take at least $m_1 + c_1$ or not more than $M_1 - c_1$ counters from the pile. Thus from the interval $[m_1 + c_1, M_1 - c_1 + m_2 + c_2 - 1]$ he will always be able to create a pile size in the first safe interval. But this interval is indeed the first unsafe interval $[B_0, A_1 - 1]$.

Before making the induction step, let us parameterize the A_k and B_k more detailed by the given sequences $(M_j)_j$, $(m_j)_j$ and $(c_j)_j$:

$$A_k((M_{2i-1})_i, (m_{2i})_i, (c_i)_i) = \sum_{i=1}^{k} M_{2i-1} + \sum_{i=1}^{k} m_{2i} + \sum_{i=1}^{2k} (-1)^{i} c_i$$

and

$$B_k((M_{2i})_i, (m_{2i-1})_i, (c_i)_i) = \sum_{i=1}^{k} M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i \quad .$$

Then we can write

$$A_{k+1}((M_{2i-1})_i, (m_{2i})_i, (c_i)_i) = \sum_{i=1}^{k+1} M_{2i-1} + \sum_{i=1}^{k+1} m_{2i} + \sum_{i=1}^{2k+2} (-1)^{i} c_i$$

$$= M_1 - c_1 + \sum_{i=2}^{k+1} M_{2i-1} + \sum_{i=1}^{k+1} m_{2i} + \sum_{i=2}^{2k+2} (-1)^i c_i \tag{10}$$

$$= M_1 - c_1 + \sum_{i=1}^{k} M_{2i+1} + \sum_{i=1}^{k+1} m_{2i} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_{i+1}$$

$$= M_1 - c_1 + B_k((M_{2i+1})_i, (m_{2i})_i, (c_{i+1})_i)$$

and

$$B_k((M_{2i})_i, (m_{2i-1})_i, (c_i)_i) = \sum_{i=1}^{k} M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i$$

$$= m_1 + c_1 + \sum_{i=1}^{k} M_{2i} + \sum_{i=2}^{k+1} m_{2i-1} + \sum_{i=2}^{2k+1} (-1)^{i+1} c_i \tag{11}$$

$$= m_1 + c_1 + \sum_{i=1}^{k} M_{2i} + \sum_{i=1}^{k} m_{2i+1} + \sum_{i=1}^{2k} (-1)^{i+2} c_{i+1}$$

$$= m_1 + c_1 + A_k((M_{2i})_i, (m_{2i+1})_i, (c_{i+1})_i) \ .$$

Now we proceed by induction of the index k of the safe respectively unsafe intervals. The k-th safe interval will be that interval where the first player (the player-to-move) is forced by his opponent to reduced the pile size to a position into the $k-1$ -th unsafe interval. Because he is always able to decrease the pile size by at least m_1+c_1, but at most by M_1-c_1 counters, this must be $[B_{k-1}((M_{2i})_i, (m_{2i-1})_i, (c_i)_i)), A_k((M_{2i-1})_i, (m_{2i})_i, (c_i)_i)]$ — the indices of the M_j-, m_j- and c_j-numbers have to be increased by 1 because the turn counter also proceeds by one — increased at the left end by $M_1 - c_1$ and at the right end by $m_1 + c_1$. But exactly this the relations (10) — replace here k by $k-1$ — and (11) state.

Similarly, the k-th unsafe interval will be that interval where the player-to-move is able to reduce the pile size to a position into the $k-1$ -th safe interval independently of the blocking move of the opponent. Because of the number of at least and at most to-be-remove counters, this must be $[A_k((M_{2i-1})_i, (m_{2i})_i, (c_i)_i), B_k((M_{2i})_i, (m_{2i-1})_i, (c_i)_i))]$ — again the indices of the M_j-, m_j- and c_j-numbers have to be increased by 1 because the turn counter also proceeds by one — increased at the left end by $m_1 + c_1$ and at the right end by $M_1 - c_1$. Indeed, this also the relations (10) and (11) state. □

Summary: Rewriting the interval boundaries A_k and B_k of the safe/unsafe intervals like

$$A_k = \sum_{i=1}^{k} (M_{2i-1} - c_{2i-1}) + \sum_{i=1}^{k} (m_{2i} + c_{2i}) = A_k((M_{2i-1} - c_{2i-1})_i, (m_{2i} + c_{2i})_i)$$

and

$$B_k = \sum_{i=1}^{k} (M_{2i} - c_{2i}) + \sum_{i=1}^{k+1} (m_{2i-1} + c_{2i-1}) = B_k((M_{2i} - c_{2i})_i, (m_{2i-1} + c_{2i-1})_i)$$

suggest that the structure of the Dynamic One-pile Blocking Nim may be not effected by the introduction of blocking-options of the opponent — what is proved in this article. This fact is due to the conditions (1) and (2), which guarantee that, as in the classic game of Nim [2], any move of a player must always switch from a safe/unsafe position to the *next smaller* unsafe/safe interval of positions. Thus the possible route of the pile size is a non-branching path on the safe/unsafe intervals. Therefore the necessary and sufficient conditions in place of (1) and (2) are that $\forall i \in \mathbb{N} : u_i \geq 0$ and Lemma 1 holds. This lemma is equivalent to the condition $\forall k \in \mathbb{N}_0 : \quad 0 < m_{2k+1} + c_{2k+1} + \sum_{i=1}^{k}(M_{2i} - m_{2i} - 2c_{2i} - M_{2i-1} + m_{2i-1} + 2c_{2i-1}) < M_{2k+1} - c_{2k+1} + m_{2k+2} + c_{2k+2}$.

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