# Some remarks on the Plotkin bound

Jörn Quistorff
Speckenreye 48
22119 Hamburg, Germany
joern.quistorff@hamburg.de

Submitted: Nov 24, 2001; Accepted: Jun 17, 2003; Published: Jun 27, 2003
MR Subject Classifications: 94B65

**Abstract**

In coding theory, Plotkin's upper bound on the maximal cadinality of a code with minimum distance at least $d$ is well known. He presented it for binary codes where Hamming and Lee metric coincide. After a brief discussion of the generalization to $q$-ary codes preserved with the Hamming metric, the application of the Plotkin bound to $q$-ary codes preserved with the Lee metric due to Wyner and Graham is improved.

## 1 Introduction

Let $K$ be a set of cardinality $q \in \mathbf{N}$ and $d^K : K \times K \to \mathbf{R}$ be a metric. Consider $R := K^n$ with $n \in \mathbf{N}$ and $d^R((v_1, ..., v_n), (w_1, ..., w_n)) := \sum_{i=1}^{n} d^K(v_i, w_i)$. Then $(K, d^K)$ and $(R, d^R)$ are finite metric spaces.

A subset $C \subseteq R$ is called a (block) code of length $n$. If $|C| \geq 2$ then its minimum distance is defined by $d(C) := \min\{d^R(v, w) \in \mathbf{R}^+ | v, w \in C \text{ and } v \neq w\}$. The observation of the metric properties of $(R, d^R)$ and of its subsets is an essential part of coding theory. The value $u(R, d^R, d)$ (or briefly $u(d)$), defined as the maximal cardinality of a code $C \subseteq R$ with minimum distance $d(C) \geq d$, is frequently considered.

The determination of $u(d)$ is a fundamental and often unsolved problem but some lower and upper bounds are well known. This paper deals with the following condition on the parameters of a code which gives Plotkin's upper bound on $u(d)$. Similar formulations are given by Berlekamp [1] and Răduică [8].

Let $d > 0$ and $u \in \mathbf{N} \setminus \{1\}$. Put $J := \{0, ..., u-1\}$. If $u(d) \geq u$ then

$$d\binom{u}{2} \leq n \max\left\{ \sum_{\{j,k\} \subseteq J} d^K(v_1^{(j)}, v_1^{(k)}) | (v_1^{(0)}, ..., v_1^{(u-1)}) \in K^u \right\} =: nP_{(K,d^K)}(u). \qquad (1)$$

This condition is easy to prove by estimating $\sum_{\{v,w\} \subseteq C} d^R(v, w)$.

If instead of $P_{(K,d^K)}(u)$ an upper bound $Q_{(K,d^K)}(u)$ is known then inequality (1) can be replaced by

$$d\binom{u}{2} \leq nQ_{(K,d^K)}(u). \tag{2}$$

The most common finite metric spaces in coding theory are the ($n$-dimensional $q$-ary) Hamming spaces $(R, d_H)$. Here, the Hamming metric can be introduced by

$$d_H((v_1, ..., v_n), (w_1, ..., w_n)) = \sum_{i=1}^{n} d_H(v_i, w_i)$$

and

$$d_H(v_i, w_i) = \begin{cases} 0 & \text{if } v_i = w_i \\ 1 & \text{if } v_i \neq w_i. \end{cases}$$

Furthermore, $A_q(n, d)$ is usually used instead of $u(R, d_H, d)$.

Other common finite metric spaces in coding theory consider $R = K^n$ with $K = \mathbf{Z}/q\mathbf{Z}$ together with the Lee metric $d_L$ which can be introduced by

$$d_L((v_1, ..., v_n), (w_1, ..., w_n)) = \sum_{i=1}^{n} d_L(v_i, w_i)$$

and

$$d_L(v_i, w_i) = \min\{|v_i - w_i|, q - |v_i - w_i|\}. \tag{3}$$

Whenever, like on the right-hand side of equation (3), an order $\leq$ is used in $\mathbf{Z}/p\mathbf{Z}$, their elements have to be represented by elements of $\{0, ..., p-1\} \subseteq \mathbf{Z}$. The spaces $(R, d_L)$ should be called Lee spaces.

In case of $q \leq 3$, the metrics $d_H$ and $d_L$ are identical. Lee [3] noticed that also the case $((\mathbf{Z}/4\mathbf{Z})^n, d_L)$ can be reduced to $((\mathbf{Z}/2\mathbf{Z})^{2n}, d_H)$, using the transformation $0 \mapsto (0,0)$, $1 \mapsto (0,1)$, $2 \mapsto (1,1)$, $3 \mapsto (1,0)$. The pathological case $q = 1$ is usually omitted.

After a brief discussion of the Plotkin bound in Hamming spaces, the paper considers this bound in Lee spaces.

# 2 Hamming Spaces

Plotkin [6] introduced his bound in case of $q = 2$ where Hamming and Lee metric coincide. In terms of condition (1), he used $P_2^H(u) := P_{(\{0,1\},d_H)}(u) = \lfloor \frac{u+1}{2} \rfloor (u - \lfloor \frac{u+1}{2} \rfloor)$ and proved the existence of an $m \in \mathbf{N}$ with

$$A_2(n, d) \leq 2m \leq \frac{2d}{2d - n} \tag{4}$$

if $2d > n$. MacWilliams/Sloane [5] mentioned in this case the equivalent bound

$$A_2(n, d) \leq 2 \left\lfloor \frac{d}{2d - n} \right\rfloor. \tag{5}$$

Berlekamp [1] considered the generalization to $q$-ary Hamming spaces. In terms of $P_q^H := P_{(\mathbf{Z}/q\mathbf{Z},d_H)}$ and $Q_q^H$, he showed $P_q^H(u) \leq Q_q^H(u) = \frac{u^2(q-1)}{2q}$. This result yields the bound

$$A_q(n,d) \leq \frac{dq}{dq - n(q-1)} \quad \text{if} \quad dq > n(q-1).$$

Quistorff [7] determined

$$P_q^H(u) = \binom{u}{2} - b\binom{a+1}{2} - (q-b)\binom{a}{2} \tag{6}$$

if $u = aq + b$ with $a, b \in \mathbf{N}_0$ and $b < q$. An equivalent statement can be found in Bogdanova et al. [2]. The results (1) and (6) imply e.g. the tight upper bound $A_3(9,7) \leq 6$. Vaessens/Aarts/Van Lint [9] formerly mentioned this and similar examples for $q = 3$ as an implication of Plotkin [6] and also solved the case $a = b = 1$ in (6) with arbitrary $q \in \mathbf{N} \setminus \{1\}$. Mackenzie/Seberry's [4] bound on $A_3(n,d)$ with $3d > 2n$ is incorrect. The adequate use of their method leads to

$$A_3(n,d) \leq \max\left\{3\left\lfloor\frac{d}{3d-2n}\right\rfloor, 3\left\lfloor\frac{d}{3d-2n} - \frac{2}{3}\right\rfloor + 1\right\} \quad \text{if} \quad 3d > 2n$$

which is equivalent to the application of (6).

# 3 Lee Spaces

Put $P_q^L(u) := P_{(\mathbf{Z}/q\mathbf{Z},d_L)}(u)$. Wyner/Graham [10] proved

$$P_q^L(u) \leq Q_q^L(u) := \begin{cases} \frac{u^2(q^2-1)}{8q} & \text{if } q \text{ is odd} \\ \frac{u^2}{8}q & \text{if } q \text{ is even} \end{cases}$$

as an application of the Plotkin bound in Lee spaces, cf. also Berlekamp [1]. The stronger inequality

$$P_q^L(u) \leq \left\lfloor Q_q^L(u) \right\rfloor \tag{7}$$

follows by definition. In order to improve formula (7), some preparation is necessary.

**Lemma 1** *Let $q, u \in \mathbf{N} \setminus \{1\}$ and $m \in \{1, ..., u-1\}$. Let $J := \mathbf{Z}/u\mathbf{Z}$ and $v^{(j)} \in \mathbf{Z}/q\mathbf{Z}$ with $j \in J$ and $v^{(j)} \leq v^{(k)}$ for $j < k$. Then*

$$\sum_{j \in J} d_L(v^{(j)}, v^{(j+m)}) \leq mq \tag{8}$$

*and equality holds in estimation* (8) *iff*

$$d_L(v^{(j)}, v^{(j+m)}) = \begin{cases} v^{(j+m)} - v^{(j)} & \text{if } j < u - m \\ q + v^{(j+m)} - v^{(j)} & \text{if } j \geq u - m \end{cases} \tag{9}$$

*is valid.*

Proof:
$$\sum_{j\in J} d_L(v^{(j)}, v^{(j+1)}) \leq q + v^{(0)} - v^{(u-1)} + \sum_{j\in J\setminus\{u-1\}} v^{(j+1)} - v^{(j)} = q$$
and hence
$$\begin{aligned}\sum_{j\in J} d_L(v^{(j)}, v^{(j+m)}) &\leq \sum_{j\in J}\sum_{l=0}^{m-1} d_L(v^{(j+l)}, v^{(j+l+1)}) \\ &\leq \sum_{l=0}^{m-1}\sum_{j\in J} d_L(v^{(j)}, v^{(j+1)}) \\ &\leq mq.\end{aligned}$$
All estimates turn out to be equalities iff condition (9) is valid. □

Put
$$N_q^L(u) := \begin{cases} \frac{u^2-1}{8}q & \text{if } u \text{ is odd} \\ \frac{u(u-2)}{8}q + \frac{u}{2}\left\lfloor\frac{q}{2}\right\rfloor & \text{if } u \text{ is even}\end{cases}$$
with $u \in \mathbf{N}\setminus\{1\}$. Clearly, $\frac{u^2-1}{8} \in \mathbf{N}$ if $u$ is odd and $\frac{u(u-2)}{8} \in \mathbf{N}_0$ if $u$ is even.

**Theorem 2** *Let $q, u \in \mathbf{N}\setminus\{1\}$. Then $P_q^L(u) \leq N_q^L(u)$ holds true.*

Proof: Let $v^{(j)} \in \mathbf{Z}/q\mathbf{Z}$ with $j \in J := \mathbf{Z}/u\mathbf{Z}$. Without loss of generality, let $v^{(j)} \leq v^{(k)}$ for $j < k$.

(i) Let $u$ be odd. Then
$$\begin{aligned}\sum_{\{j,k\}\subseteq J} d_L(v^{(j)}, v^{(k)}) &= \sum_{m=1}^{\frac{u-1}{2}}\sum_{j\in J} d_L(v^{(j)}, v^{(j+m)}) \\ &\leq \sum_{m=1}^{\frac{u-1}{2}} mq \quad = \quad N_q^L(u)\end{aligned}$$
follows by Lemma 1.

(ii) Let $u$ be even. Then
$$\begin{aligned}\sum_{\{j,k\}\subseteq J} d_L(v^{(j)}, v^{(k)}) &= \sum_{m=1}^{\frac{u}{2}-1}\sum_{j\in J} d_L(v^{(j)}, v^{(j+m)}) + \sum_{j\in J; j<\frac{u}{2}} d_L(v^{(j)}, v^{(j+\frac{u}{2})}) \\ &\leq \sum_{m=1}^{\frac{u}{2}-1} mq + \frac{u}{2}\left\lfloor\frac{q}{2}\right\rfloor \quad = \quad N_q^L(u)\end{aligned}$$
follows by Lemma 1.

Hence, in both cases $P_q^L(u) \le N_q^L(u)$ is valid. $\square$

Theorem 2 improves formula (7) in many cases. E.g. $N_8^L(3) = 8 < 9 = \lfloor Q_8^L(3) \rfloor$ and $N_9^L(6) = 39 < 40 = \lfloor Q_9^L(6) \rfloor$ hold true.

The following statements will prove coincidence between $P_q^L(u)$ and $N_q^L(u)$ if $q$ is odd or $u$ is small, relative to $q$. Put $f(u) := 1$ if $u$ is odd and $f(u) := 2$ if $u$ is even.

**Lemma 3** *Let $q, u \in \mathbf{N} \setminus \{1\}$. Let $q$ be even or $f(u)q \ge u - 1$. Let $\lfloor \frac{jq}{u} \rfloor, \lfloor \frac{kq}{u} \rfloor \in \mathbf{Z}/q\mathbf{Z}$ with $j, k := j + m \in \mathbf{Z}/u\mathbf{Z}$ and $1 \le m \le \lfloor \frac{u-1}{2} \rfloor$ as well as $0 \le j, k < u$. Put*

$$\left\lfloor \widetilde{\frac{kq}{u}} \right\rfloor := \begin{cases} \lfloor \frac{kq}{u} \rfloor & \text{if } j < u - m \\ q + \lfloor \frac{kq}{u} \rfloor & \text{if } j \ge u - m. \end{cases}$$

*Then $d_L(\lfloor \frac{jq}{u} \rfloor, \lfloor \frac{kq}{u} \rfloor) = \lfloor \widetilde{\frac{kq}{u}} \rfloor - \lfloor \frac{jq}{u} \rfloor \le \lfloor \frac{q}{2} \rfloor$ is valid.*

Proof: It holds true that $\left\lfloor \widetilde{\frac{kq}{u}} \right\rfloor \le \frac{(j + \lfloor \frac{u-1}{2} \rfloor)q}{u}$ and $\lfloor \frac{jq}{u} \rfloor \ge \frac{jq - (u-1)}{u}$.

(i) Let $u$ be odd. Then $\left\lfloor \widetilde{\frac{kq}{u}} \right\rfloor - \lfloor \frac{jq}{u} \rfloor \le \lfloor \frac{\frac{u-1}{2}q + (u-1)}{u} \rfloor = \lfloor (\frac{q}{2} + 1)(1 - \frac{1}{u}) \rfloor$. If $q$ is even then $\left\lfloor \widetilde{\frac{kq}{u}} \right\rfloor - \lfloor \frac{jq}{u} \rfloor \le \lfloor \frac{q}{2} \rfloor$. If $q \ge u - 1$ then $\left\lfloor \widetilde{\frac{kq}{u}} \right\rfloor - \lfloor \frac{jq}{u} \rfloor \le \lfloor (\frac{q}{2} + 1)\frac{q}{q+1} \rfloor = \lfloor \frac{q+1}{2} - \frac{1}{2(q+1)} \rfloor \le \lfloor \frac{q}{2} \rfloor$.

(ii) Let $u$ be even. Then $\left\lfloor \widetilde{\frac{kq}{u}} \right\rfloor - \lfloor \frac{jq}{u} \rfloor \le \lfloor \frac{(\frac{u}{2} - 1)q + (u-1)}{u} \rfloor = \lfloor (\frac{q}{2} + 1) - \frac{q+1}{u} \rfloor$. If $q$ is even then $\left\lfloor \widetilde{\frac{kq}{u}} \right\rfloor - \lfloor \frac{jq}{u} \rfloor \le \lfloor \frac{q}{2} \rfloor$. If $2q \ge u - 1$ then $\left\lfloor \widetilde{\frac{kq}{u}} \right\rfloor - \lfloor \frac{jq}{u} \rfloor \le \lfloor \frac{q+1}{2} - \frac{2(q+1) - u}{2u} \rfloor \le \lfloor \frac{q}{2} \rfloor$.

Hence, $d_L(\lfloor \frac{jq}{u} \rfloor, \lfloor \frac{kq}{u} \rfloor) = \lfloor \widetilde{\frac{kq}{u}} \rfloor - \lfloor \frac{jq}{u} \rfloor$. $\square$

In case of $q = 3$, $u = 5$, $j = 3$, $m = 2$, Lemma 3 can not be applied. Here, $k = 0$, $\lfloor \frac{jq}{u} \rfloor = 1$, $\lfloor \frac{kq}{u} \rfloor = 0$, $\lfloor \widetilde{\frac{kq}{u}} \rfloor = 3$, $\lfloor \widetilde{\frac{kq}{u}} \rfloor - \lfloor \frac{jq}{u} \rfloor = 2 > 1 = \lfloor \frac{q}{2} \rfloor$ and $d_L(\lfloor \frac{jq}{u} \rfloor, \lfloor \frac{kq}{u} \rfloor) = 1$. A similar example is $q = 3$, $u = 8$, $j = 5$, $m = 3$.

**Lemma 4** *Let $q, u \in \mathbf{N} \setminus \{1\}$ and $u$ be even. Let $\lfloor \frac{jq}{u} \rfloor, \lfloor \frac{kq}{u} \rfloor \in \mathbf{Z}/q\mathbf{Z}$ with $j, k := j + \frac{u}{2} \in \mathbf{Z}/u\mathbf{Z}$ and $0 \le j < \frac{u}{2} \le k < u$. Then $d_L(\lfloor \frac{jq}{u} \rfloor, \lfloor \frac{kq}{u} \rfloor) = \lfloor \frac{q}{2} \rfloor$ is valid.*

Proof: It holds true that $\frac{(j + \frac{u}{2})q - (u-1)}{u} \le \lfloor \frac{kq}{u} \rfloor \le \frac{(j + \frac{u}{2})q}{u}$ and $\frac{jq - (u-1)}{u} \le \lfloor \frac{jq}{u} \rfloor \le \frac{jq}{u}$. Hence, $\lfloor \frac{kq}{u} \rfloor - \lfloor \frac{jq}{u} \rfloor \le \lfloor \frac{q}{2} + \frac{u-1}{u} \rfloor \le \lfloor \frac{q+1}{2} \rfloor$ and $q - \lfloor \frac{kq}{u} \rfloor + \lfloor \frac{jq}{u} \rfloor \le \lfloor \frac{q}{2} + \frac{u-1}{u} \rfloor \le \lfloor \frac{q+1}{2} \rfloor$. This yields $d_L(\lfloor \frac{jq}{u} \rfloor, \lfloor \frac{kq}{u} \rfloor) = \lfloor \frac{q}{2} \rfloor$. $\square$

**Theorem 5** *Let $q, u \in \mathbf{N} \setminus \{1\}$. Let $q$ be even or $f(u)q \ge u - 1$. Then $P_q^L(u) = N_q^L(u)$.*

Proof: Put $v^{(j)} := \lfloor \frac{jq}{u} \rfloor$ for $j \in J := \mathbf{Z}/u\mathbf{Z}$ with $0 \le j < u$.

(i) Let $u$ be odd. Then

$$\begin{aligned}
P_q^L(u) \quad &\geq \quad \sum_{\{j,k\}\subseteq J} d_L(v^{(j)}, v^{(k)}) &=& \sum_{m=1}^{\frac{u-1}{2}} \sum_{j\in J} d_L(v^{(j)}, v^{(j+m)}) \\
&&=& \sum_{m=1}^{\frac{u-1}{2}} mq \\
&&=& N_q^L(u)
\end{aligned}$$

by Lemma 1 and 3.

(ii) Let $u$ be even. Then

$$\begin{aligned}
P_q^L(u) \quad &\geq \quad \sum_{\{j,k\}\subseteq J} d_L(v^{(j)}, v^{(k)}) \\
&= \quad \sum_{m=1}^{\frac{u}{2}-1} \sum_{j\in J} d_L(v^{(j)}, v^{(j+m)}) + \sum_{j\in J; j<\frac{u}{2}} d_L(v^{(j)}, v^{(j+\frac{u}{2})}) \\
&= \quad N_q^L(u)
\end{aligned}$$

by Lemma 1, 3 and 4.

Theorem 2 completes the proof. □

If $u$ is considerable greater than $q$, the Plotkin bound is usually weak and other well known upper bounds, e.g. the Hamming bound, give stronger results. Hence, it seems not to be fatal that $P_q^L(u)$ is not determined in all these cases. The final theorem gives at least a lower bound on $P_q^L(u)$. According to Theorem 5, it is sufficent to consider only odd values of $q$. The following convention is used. Extending inequality (1) by $u \in \{0, 1\}$, one gets $P_{(K,d^K)}(u) = 0$ and hence $P_q^L(0) = P_q^L(1) = 0$.

**Theorem 6** *Let $q, u \in \mathbf{N} \setminus \{1\}$ and $q$ be odd. Let $u = aq + b$ with $a, b \in \mathbf{N}_0$ and $b < q$. Then*

$$P_q^L(u) \geq a(u+b)\frac{q^2-1}{8} + P_q^L(b) \tag{10}$$

Proof: Put $J_s := \{0, ..., q-1\} \times \{s\}$ with $s \in \{0, ..., a-1\}$ as well as $J_a := \{(\lfloor \frac{jq}{b} \rfloor, a) | j \in \{0, ..., b-1\}\}$. Put $v^{(r,s)} := r$ for all $(r, s) \in J := \bigcup_{s=0}^{a} J_s$. Using the proof of Theorem 5, it follows that

$$\sum_{\{j,k\}\subseteq \bigcup_{s=0}^{a-1} J_s} d_L(v^{(j)}, v^{(k)}) = a^2 \sum_{\{j,k\}\subseteq J_0} d_L(v^{(j)}, v^{(k)}) = a^2 P_q^L(q)$$

and

$$\sum_{\{j,k\}\subseteq J_a} d_L(v^{(j)}, v^{(k)}) = P_q^L(b)$$

as well as

$$\sum_{j\in\bigcup_{s=0}^{a-1} J_s;k\in J_a} d_L(v^{(j)},v^{(k)}) = 2ab\sum_{i=0}^{\frac{q-1}{2}} i = ab\frac{q^2-1}{4}.$$

Hence,

$$P_q^L(u) \geq \sum_{\{j,k\}\subseteq J} d_L(v^{(j)},v^{(k)}) = a(u+b)\frac{q^2-1}{8} + P_q^L(b)$$

is valid. □

One might conjecture equality in (10). The combination of the formulas (7) and (10) proves e.g. $P_3^L(5) = \left\lfloor Q_3^L(5)\right\rfloor = 8 < 9 = N_3^L(5)$ and $P_3^L(8) = \left\lfloor Q_3^L(8)\right\rfloor = 21 < 22 = N_3^L(8)$.

For some applications, let $u(d) \geq u \in \mathbf{N}\setminus\{1\}$.

(i) Let $u = 3$. Inequality (2) and Theorem 2 imply the condition $3d \leq qn$. Theorem 5 shows that inequality (1) cannot improve this condition.

(ii) Let $u = 4$ and use (2). If $q$ is even then $3d \leq qn$ follows again. If $q$ is odd then the stronger condition $6d \leq (2q-1)n$ follows. In both cases, an improvement by (1) is impossible.

(iii) Let $u = 5$. Inequality (2) implies $10d \leq 3qn$. Only in case of $q = 3$, an improvement by (1) is possible: $5d \leq 4n$.

(iv) Let $q$ be even and $u$ be odd. Then inequality (1) implies the same condition for $u$ and $u+1$, since $\binom{u}{2}^{-1} P_q^L(u) = \binom{u+1}{2}^{-1} P_q^L(u+1)$.

(v) Let $q$ be even. Then $\binom{u}{2}^{-1} P_q^L(u) > \frac{q}{4}$ and $\lim_{u\to\infty}\binom{u}{2}^{-1} P_q^L(u) = \frac{q}{4}$. Hence, inequality (1) turns out to be a tautology iff $4d \leq qn$.

# References


[1] Berlekamp, E.R.: *Algebraic Coding Theory*, McGraw-Hill, New York, 1968.

[2] Bogdanova, G.T. / Brouwer, A.E. / Kapralov, S.N. / Östergård, P.R.J.: Error-Correcting Codes over an Alphabet of Four Elements, *Des. Codes Cryptogr.*, 23 (2001), 333-342.

[3] Lee, C.Y.: Some Properties of Nonbinary Error-Correcting Codes, *IRE Trans. Inform. Theory*, 4 (1958), 77-82.

[4] Mackenzie, C. / Seberry, J.: Maximal Ternary Codes and Plotkin's Bound, *Ars Comb.*, 17A (1984), 251-270.

[5] MacWilliams, F.J. / Sloane, N.J.A.: *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, New York, Oxford, 1977.

[6] Plotkin, M.: Binary Codes with Specified Minimum Distance, *Univ. Penn. Res. Div. Report* 51-20 (1951); *IRE Trans. Inform. Theory*, 6 (1960), 445-450.

[7] Quistorff, J.: *Simultane Untersuchung mehrfach scharf transitiver Permutationsmengen und MDS-Codes unter Einbeziehung ihrer Substitute*, Habilitationsschrift, Univ. Hamburg, 1999; Shaker Verlag, Aachen, 2000.

[8] Răduică, M.: Marginile Plotkin si Ioshi relativ la coduri arbitrar metrizate, *Bul. Univ. Braşov*, C 22 (1980), 115-120.

[9] Vaessens, R.J.M. / Aarts, E.H.L. / van Lint, J.H.: Genetic Algorithms in Coding Theory - a Table for $A_3(n, d)$, *Discrete Appl. Math.*, 45 (1993), 71-87.

[10] Wyner, A.D. / Graham, R.L.: An Upper Bound on Minimum Distance for a $k$-ary Code, *Inform. Control*, 13 (1968), 46-52.