Binary gray codes with long bit runs

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Abstract

We show that there exists an n-bit cyclic binary Gray code all of whose bit runs have length at least $n - 3\log_2 n$. That is, there exists a cyclic ordering of $\{0,1\}^n$ such that adjacent words differ in exactly one (coordinate) bit, and such that no bit changes its value twice in any subsequence of $n - 3\log_2 n$ consecutive words. Such Gray codes are 'locally distance preserving' in that Hamming distance equals index separation for nearby words in the sequence.

Keywords: cyclic binary Gray code, Hamming distance preserving, Hamilton cycle, Hamilton circuit, n-cube, gap, spread, threshold, minimum run length.

1 Introduction

We use the language of graph theory, but where a *circuit* is a closed walk with no repeated internal vertices. A *cycle* is an orbit of a permutation acting on a set. The n-cube Q_n is the graph whose vertices are the words of length n on the alphabet $\{0,1\}$; two vertices are adjacent if they differ in exactly one coordinate. The *transition* of an edge vw in Q_n is the index $\delta_{vw} \in \{1,2,\ldots,n\}$ of the coordinate (or *bit*) in which v and w differ.

An n-bit (cyclic, binary) *Gray code* is a Hamilton circuit in Q_n. Frank Gray [3] described an elementary family of 'reflected' Gray codes (RGC) which has seen countless applications. Certain applications in engineering, statistics and computer science require specialized Gray codes with properties not possessed by the RGC. We refer to Savage [6] for more information on such variations. This paper is concerned with Gray codes for which any two edges which have equal transitions are well separated along the circuit.

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Formally, the *minimum run length* of a closed walk $W = w_0w_1 \dots w_{k-1}w_0$ in Q_n is defined as

$$\mathrm{mrl}(W) = \min\{||i-j|| : \delta_{w_iw_{i+1}} = \delta_{w_jw_{j+1}},\ i \neq j\}.$$

Here, $||i-j||$ is the smaller of the least residues of $i-j$ and $j-i$ modulo k. (In case W has no repeated transitions, we may define $\mathrm{mrl}(W) = k$.) In some papers, $\mathrm{mrl}(W)$ is called the *threshold*, the *spread*, or the *gap* of W.

Let $\mathrm{mrl}(n)$ be the maximum possible value of $\mathrm{mrl}(W)$ among all Hamilton circuits W in Q_n. We write $\lg n$ for $\log_2 n$. In Section 4 we prove our main result.

Theorem 1 *For $n \geq 2$ we have* $\mathrm{mrl}(n) \geq \lfloor n - 2.001 \lg n \rfloor$.

It is easy to see that $\mathrm{mrl}(n) < n$ for $n > 2$, so this settles a conjecture appearing in [2]:

$$\lim_{n\to\infty} \mathrm{mrl}(n)/n = 1.$$

Constructions in [2] show only that $\mathrm{mrl}(n)/n \geq 2/3 + o(1)$. An earlier construction of Evdokimov [1] proves that $\mathrm{mrl}(n)/n \geq 1/2$.

Gray codes with large minimum run length are used in electronic devices such as the Codacon [4] spectrograph. The 10-bit code used in the Codacon is presented as a sporadic construction in Section 5. Other applications such as the visualization of neural networks [7] take advantage of the 'local Hamming distance preservation' property of such Gray codes: a closed walk $W = (w_i)$ in Q_n satisfies $\mathrm{mrl}(W) \geq m$ if and only if for $0 \leq i, j < k$,

$$||i-j|| \leq m \ \text{ implies } \ d(w_i, w_j) = ||i-j||. \tag{1}$$

(Here $d(w_i, w_j)$ is the number of bits in which w_i and w_j differ.)

Two variations of our problem have been studied. The first is concerned with finding a *longest* circuit in Q_n having a prespecified minimum run length m, where $\mathrm{mrl}(n) < m < n$. This optimal length $\ell(n,m)$ is known [8] to satisfy $\ell(n,m) \geq 2^{n-\lceil m/2 \rceil}$. This is surely far from optimal; it is conjectured in [2] that $\ell(n, n-1)/2^n \to 1$. The second variation imposes an additional separation requirement which provides the circuit with error-correcting capability. Given $d \leq m < n$, one typically seeks a longest circuit W in Q_n satisfying (1) and additionally

$$||i-j|| \geq d \ \text{ implies } \ d(w_i, w_j) \geq d.$$

Such objects are often called *circuit codes*. For $d = m = 2$ this is the *snake-in-a-box* problem. We refer the interested reader to [5] and the references therein.

2 Definitions

A *step permutation* of $V(Q_n)$ is a permutation of $V(Q_n)$ such that each vertex is mapped to one of its neighbouring vertices. A list $\pi_1, \pi_2, \dots, \pi_k$ of step permutations acting on (successive images of) a vertex v naturally defines a walk of length k in Q_n,

$$W(v; \pi_1, \pi_2, \dots, \pi_k) = vv^{\pi_1}v^{\pi_1\pi_2} \dots v^{\pi_1\pi_2\dots\pi_k}.$$

The *stream induced by* $\pi_1, \pi_2, \ldots, \pi_k$ is the set of 2^n walks

$$S(\pi_1, \pi_2, \ldots, \pi_k) = \{W(v; \pi_1, \pi_2, \ldots, \pi_k) : v \in V(Q_n)\}.$$

We say that the stream $S = S(\pi_1, \pi_2, \ldots, \pi_k)$ *induces the permutation* $\pi(S) = \pi_1\pi_2 \ldots \pi_k$ of $V(Q_n)$. We can obviously concatenate any two streams S, S' to obtain a new stream SS'. We may write S^2 for SS. Let $W(v; S)$ denote the walk in S with initial vertex v and terminal vertex $v^{\pi(S)}$. Let $r(v)$ be the length of the orbit of v under $\pi(S)$. Then $W(v; S^{r(v)})$ is a closed walk in the (concatenated) stream $S^{r(v)}$. The *minimum run length* $\mathrm{mrl}(S)$ of the stream S is defined to be the smallest minimum run length among the closed walks in $\{W(v; S^{r(v)}) : v \in V(Q_n)\}$.

We sometimes identify $V(Q_n)$ with the vector space $\mathrm{GF}(2)^n$, denoting by e_i the ith standard vector. For example, $\delta_{vw} = i$ iff $v + w = e_i$. Two examples of step permutations are: the elementary involutions

$$\tau_i : v \mapsto v + e_i, \quad i = 1, \ldots, n$$

and the following modification of τ_i. If uvw is a path of length two in Q_n and $\delta_{vw} = i$, then $\tau_i^{(uvw)}$ is the step permutation defined by

$$\tau_i^{(uvw)} = (uw)\tau_i. \tag{2}$$

The orbit of u under $\tau_i^{(uvw)}$ induces the 4-circuit $uvwzu$ in Q_n, where $z = u^{\tau_i}$. Thus the cycle structure of $\tau_i^{(uvw)}$ is given by

$$(uvwz) \prod \{(xx^{\tau_i}) : x \in V(Q_n) - \{u, v, w, z\}\}.$$

The *transition sequence* of a walk $W = w_0w_1 \ldots w_k$ in Q_n is defined to be the sequence

$$\delta_{w_0w_1}, \delta_{w_1w_2}, \ldots, \delta_{w_{k-1}w_k}.$$

Thus for closed walks, $\mathrm{mrl}(W)$ equals the smallest index separation (modulo k) between two identical entries in its transition sequence. The following might help illustrate these notions.

Example 2 Let $uvw = 000\ 100\ 110$ be a path of length two in Q_3. Then $\delta_{uv} = 1$ and $\delta_{vw} = 2$. The following stream in Q_3 consists of eight walks of length four.

$$\begin{aligned} S &= S(\tau_2^{(uvw)}, \tau_3, \tau_2, \tau_3) \\ &= \{W_1, W_2, \ldots, W_8\} \end{aligned}$$

where

$$\begin{array}{ll} W_1 = 000, 100, 101, 111, 110 & W_5 = 100, 110, 111, 101, 100 \\ W_2 = 001, 011, 010, 000, 001 & W_6 = 101, 111, 110, 100, 101 \\ W_3 = 010, 000, 001, 011, 010 & W_7 = 110, 010, 011, 001, 000 \\ W_4 = 011, 001, 000, 010, 011 & W_8 = 111, 101, 100, 110, 111. \end{array}$$

In this example we have $\pi(S) = (uw) = (000\ 110)$ (the other 6 vertices are fixed points). Concatenating S with itself gives rise to seven closed walks, W_1W_7, W_2, W_3, W_4, W_5, W_6, and W_8. All seven closed walks have minimum run length 2. Thus $\mathrm{mrl}(S)$, being the minimum of these numbers, equals 2.

3 The Construction

We construct a Hamilton circuit in the cartesian product $Q_a \times Q_b \cong Q_{a+b}$ from a stream in Q_a and a Hamilton circuit in Q_b. Let (X, Y) be the bipartition of $V(Q_a)$ into words of even and odd weight.

Lemma 3 *Let S be a stream of length 2^b in Q_a such that X is one of the orbits of $\pi(S)$. Then* $\mathrm{mrl}(a+b) \geq 2\min\{\mathrm{mrl}(S), \mathrm{mrl}(b)\}$.

Proof: Let S be as in the hypothesis, and let $w_0 \in X$. Since $\pi(S)$ cyclically permutes the vertices in X, the concatenated stream $S^{2^{a-1}}$ contains the closed walk

$$W = W(w_0; S^{2^{a-1}}) = w_0 w_1 \ldots w_{2^{a+b-1}-1} w_0.$$

This walk is a concatenation of the 2^{a-1} walks in S which originate in X. That is, $X = \{w_{k2^b} : 0 \leq k < 2^{a-1}\}$. Step permutations map X into Y bijectively, so for each $j \in \{0, \ldots, 2^b - 1\}$ we have that

$$\{w_{k2^b+j} : 0 \leq k < 2^{a-1}\} = \begin{cases} X & \text{if } j \text{ is even} \\ Y & \text{if } j \text{ is odd.} \end{cases} \tag{3}$$

Let $Z = z_0 z_1 \ldots z_{2^b-1} z_0$ be a Hamilton circuit in Q_b with $\mathrm{mrl}(Z) = \mathrm{mrl}(b)$. We assume that the index sets of the bits of Q_a and Q_b are disjoint. By merging the transition sequences of W and the concatenated walk $Z^{2^{a-1}}$, we obtain the following walk in $Q_a \times Q_b$.

$$\begin{array}{rcl} C & = & (w_0, z_0)(w_1, z_0)(w_1, z_1)(w_2, z_1)(w_2, z_2) \ldots (w_{2^b-1}, z_{2^b-1})(w_{2^b}, z_{2^b-1}) \\ & & (w_{2^b}, z_0)(w_{2^b+1}, z_0)(w_{2^b+1}, z_1) \quad \ldots \quad (w_{2\cdot 2^b-1}, z_{2^b-1})(w_{2\cdot 2^b}, z_{2^b-1}) \\ & & (w_{2\cdot 2^b}, z_0)(w_{2\cdot 2^b+1}, z_0)(w_{2\cdot 2^b+1}, z_1) \ \ldots \ (w_{3\cdot 2^b-1}, z_{2^b-1})(w_{3\cdot 2^b}, z_{2^b-1}) \\ & & (w_{3\cdot 2^b}, z_0) \ldots \\ & & \ldots \\ & & (w_{(2^{a-1}-1)\cdot 2^b}, z_0)(w_{(2^{a-1}-1)\cdot 2^b+1}, z_0) \ldots (w_{2^{a-1}\cdot 2^b-1}, z_{2^b-1})(w_0, z_{2^b-1}) \\ & & (w_0, z_0). \end{array}$$

It is immediate that C is a closed walk of length 2^{a+b}, and that

$$\mathrm{mrl}(C) = 2\min\{\mathrm{mrl}(W), \mathrm{mrl}(Z)\} \geq 2\min\{\mathrm{mrl}(S), \mathrm{mrl}(b)\}.$$

It remains to show that C is a Hamilton circuit. Let $(w, z) \in V(Q_a) \times V(Q_b)$. As Z is a Hamilton circuit, there is a unique $j \in \{0, 1, \ldots, 2^b - 1\}$ such that $z = z_j$. By (3) there is a unique k such that $w \in \{w_{k2^b+j}, w_{k2^b+j+1}\}$. By its construction, both pairs (w_{k2^b+j}, z_j), (w_{k2^b+j+1}, z_j) are vertices of C, so $(w, z) \in V(C)$. Since C has length 2^{a+b}, it is a Hamilton circuit in $Q_a \times Q_b$. ■

Remark 4 The (essentially unique) 5-bit Gray code C with $\mathrm{mrl}(C) = 4$ is a special case of the above construction, with $a = 3$ and $b = 2$. Here S is the stream $S(\pi_1, \pi_2, \pi_3, \pi_4)$ where the permutations are involutions defined by $\pi_i : w \mapsto w + e_{f_i(w)}$ $(1 \leq i \leq 4)$, where

$$f_1(w) = 1, \qquad f_2(w) = f_4(w) = 2 + w^1, \qquad f_3(w) = 3 - w^1,$$

and $w^1 \in \{0,1\}$ denotes the first bit of w. The resulting walk W has transition sequence

$$1,3,2,3,\ 1,2,3,2,\ 1,3,2,3,\ 1,2,3,2.$$

The next two lemmas serve to construct a stream S having large minimum run length while satisfying the two hypotheses of Lemma 3. The first lemma heeds the requirement on the orbits of $\pi(S)$. The second lemma shows how to adjust the length of the stream.

Lemma 5 *There exists a stream S' in Q_a ($a \geq 2$) having length $(a-1)(2^a-2)$ such that X is an orbit of $\pi(S')$ and $\mathrm{mrl}(S') = a-1$.*

Proof: Let $x_0y_0x_1y_1 \ldots x_{2^{a-1}-1}y_{2^{a-1}-1}$ be a Hamilton path in Q_a such that $\delta_{x_jy_j} = 1$, for $0 \leq j < 2^{a-1}$. For example, Gray's RGC has this property. The transition sequence of this path is $1, t_1, 1, t_2, 1, \ldots, t_{2^{a-1}-1}, 1$ where $t_j = \delta_{y_{j-1}x_j} \in \{2,3,\ldots,a\}$. For $1 \leq j < 2^{a-1}$, we define the sequence of permutations

$$P_j = \tau_2, \tau_3, \ldots, \tau_{t_j-1}, \tau_{t_j}^{(uvw)}, \tau_{t_j+1}, \ldots, \tau_a, \tau_2, \tau_3, \ldots, \tau_a \tag{4}$$

where uvw is the image of $x_{j-1}y_{j-1}x_j$ under the permutation $\tau_2\tau_3\ldots\tau_{t_j-1}$. We observe that any triple (uvw) appearing in (4) satisfies

$$\delta_{uv} = \delta_{x_{j-1}y_{j-1}} = 1. \tag{5}$$

For $1 \leq j < 2^{a-1}$, we define the stream $S_j = S(P_j)$. Example 2 provides an example of S_j when $a = 3$ and $t_j = 2$. Let

$$S' = S_1S_2\ldots S_{2^{a-1}-1} = S(\pi_1, \pi_2, \ldots, \pi_{(a-1)(2^a-2)}).$$

Each substream $S(\pi_i)$ has length one and consists of the set of edges $\{ww^{\pi_i} : w \in V(Q_a)\}$. By (4) and (5), the set of transitions of the edges in $S(\pi_i)$ is

$$\{\delta_{ww^{\pi_i}} : w \in V(Q_a)\} = \begin{cases} \{k\} & \text{if } \pi_i = \tau_k \\ \{1,k\} & \text{if } \pi_i = \tau_k^{(uvw)} \text{ for some } uvw. \end{cases} \tag{6}$$

For $2 \leq k \leq a$, any two occurrences of τ_k or $\tau_k^{(uvw)}$ in the sequence $\pi_1, \pi_2, \ldots, \pi_{(a-1)(2^a-2)}$ are separated by at least $a-1$ positions. Furthermore any two permutations having the form $\tau_{t_j}^{(uvw)}$ are separated by at least a positions, since they occur only in the first half of each subsequence P_j. Thus by (6) we conclude that $\mathrm{mrl}(S') = a-1$.

Any two elementary involutions τ_k, τ_ℓ commute. Further, for any path uvw with $\delta_{vw} = k$ we have

$$\tau_k^{(uvw)}\tau_\ell = \tau_\ell\tau_k^{(u'v'w')}$$

where $u'v'w'$ is the image of uvw under τ_ℓ. Thus with uvw as in (4) we have

$$\begin{aligned} \pi(S_j) &= \tau_2\tau_3\ldots\tau_{t_j-1}\tau_{t_j}^{(uvw)}\tau_{t_j+1}\ldots\tau_a\tau_2\tau_3\ldots\tau_a \\ &= \tau_{t_j}^{(x_{j-1}y_{j-1}x_j)}\tau_{t_j}\prod_{k\neq t_j}\tau_k^2 \\ &= (x_{j-1}x_j). \end{aligned}$$

Thus $\pi(S') = \prod_{j=1}^{2^{a-1}-1}(x_{j-1}x_j) = (x_0x_{2^{a-1}-1}x_{2^{a-1}-2}\ldots x_1)$ and X is an orbit of $\pi(S')$. ■

Remark 6 It is interesting to minimize the length of S' in the statement of Lemma 5 while retaining the other two properties. For example, in the above construction, one can reduce the length of S' by omitting the second half of any P_j for which $t_j \le t_{j+1}$, and adjusting slightly the definition of uvw. (The permutations $\tau_2, \ldots, \tau_a$ comprising the second half of P_j serve only as 'padding' to separate the modified permutation in P_j from the the modified permutation in P_{j+1}. If $t_j \le t_{j+1}$, then those two permutations will be separated by at least $a-1$ positions, even after $\tau_2, \ldots, \tau_a$ are deleted.) For example, if we use the RGC to define the sequence $(t_j) = (2, 3, 2, 4, 2, 3, 2, 5, \ldots)$, then the relation $t_j \le t_{j+1}$ holds for half of the values j. Modifying each corresponding P_j as above results in a stream having $3/4$ the length of S'.

Further reduction should be possible by writing $(x_0 x_1 \ldots x_{2^{a-1}-1})$ as a product of fewer than $2^{a-1} - 1$ involutions. For example, $(0\,1\,2\,3\,4\,5\,6\,7) = \pi_1 \pi_2$, where $\pi_1 = (1\,7)(2\,6)(3\,5)$ and $\pi_2 = (1\,0)(2\,7)(3\,6)(4\,5)$. Of course one must then express each involution as a product of just a few step permutations, while maintaining the minimum run length requirement.

Lemma 7 *For any even integer $\ell \ge (a-1)(2^a + 2a - 6)$, there exists a stream S in Q_a $(a \ge 2)$ having length ℓ such that X is an orbit of $\pi(S)$ and* $\mathrm{mrl}(S) = a - 1$.

Proof: Let $\ell = (a-1)(2^a - 2) + 2t$ where $t \ge (a-1)(a-2)$. It is a standard result that there exist nonnegative integers α and β such that

$$t = \alpha a + \beta(a-1).$$

Let S', P_j and S_j be as in Lemma 5, and consider the stream $S = S' T^{2\alpha} R^{2\beta}$ where $T = S(\tau_1, \tau_2, \ldots, \tau_a)$ and $R = S(\tau_2, \tau_3, \ldots, \tau_a)$. It is immediate that S has length ℓ and $\pi(S) = \pi(S')$. It remains to show that appending $T^{2\alpha} R^{2\beta}$ does not decrease the minimum run length of S'. Since $\mathrm{mrl}(T^{2\alpha} R^{2\beta}) = a - 1$, we need only check the 'boundary substreams'. It is possible that α or β equals zero, so we should verify that each of the four streams TS_1, RS_1, $S_{2^{a-1}-1}T$, $S_{2^{a-1}-1}R$, has minimum run length at least $a-1$. By (4) and (6) this amounts to verifying that the permutations τ_1 and $\tau_{t_j}^{(uvw)}$ are separated by at least $a-1$ positions in the permutation sequences $\tau_1, \tau_2, \ldots, \tau_a, P_j$ and $P_j, \tau_1, \tau_2, \ldots, \tau_a$. Thus we prove $\mathrm{mrl}(S) = a - 1$. ∎

Putting this together with Lemma 3 yields our basic recurrence.

Corollary 8 *If $(a-1)(2^a + 2a - 6) \le 2^b$, then*

$$\mathrm{mrl}(a+b) \ge 2 \min\{a-1, \mathrm{mrl}(b)\}. \tag{7}$$

Starting with known lower bounds on $\mathrm{mrl}(n)$ for small n as given in Section 5, it is easy to write a computer program that applies (7) in an optimal manner for $n \le 2000$. The result (Figure 1) suggests that $\mathrm{mrl}(n) \ge n - 2\lg n$ for all n. In the next section we prove a bound which is almost this good.

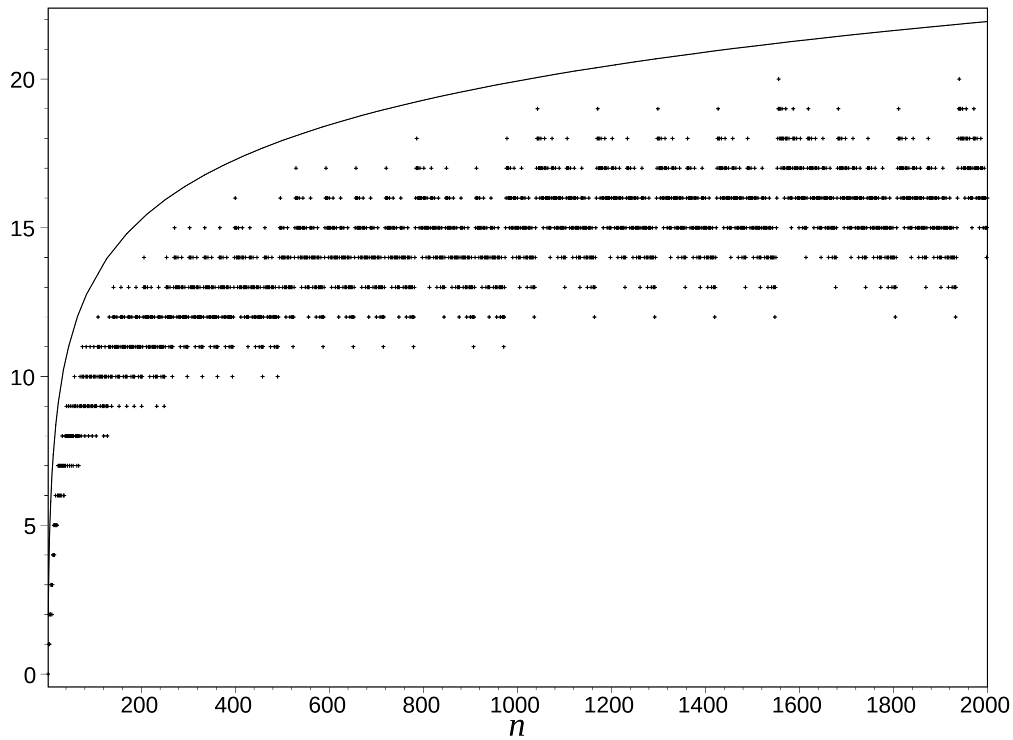

Figure 1: Comparing $2\lg n$ with the upper bound on $n - \mathrm{mrl}(n)$ arising from (7).

4 Proof of Theorem 1

We show that, for any $c > 2$, if the inequality

$$\mathrm{mrl}(n) \geq \lfloor n - c \lg n \rfloor \tag{8}$$

holds for all $n \geq 2$ satisfying

$$\left(2 - \frac{2c \lg n}{n}\right)^c < 4. \tag{9}$$

then (8) holds for all $n \geq 2$. Let $N(c)$ denote the greatest integer n for which (9) holds. For example, $N(3) = 95$. (The function $x \mapsto N(2 + 1/x)$, plotted in Figure 2, is slightly superlinear.) Let $n > N(c)$. There exists an integer $b \geq n/2$ such that $a+b \leq n \leq a+b+1$, where

$$a = b - \lfloor c \lg b \rfloor + 1.$$

We first observe

$$\frac{a+b}{b} \geq 2 - \frac{c \lg b}{b} \geq 2 - \frac{c \lg(n/2)}{n/2} \geq 2 - \frac{2c \lg n}{n} \geq 4^{1/c},$$

and verify that

$$(a-1)(2^a + 2a - 6) \leq a2^a \leq 2a2^b/b^c \leq 2^b.$$

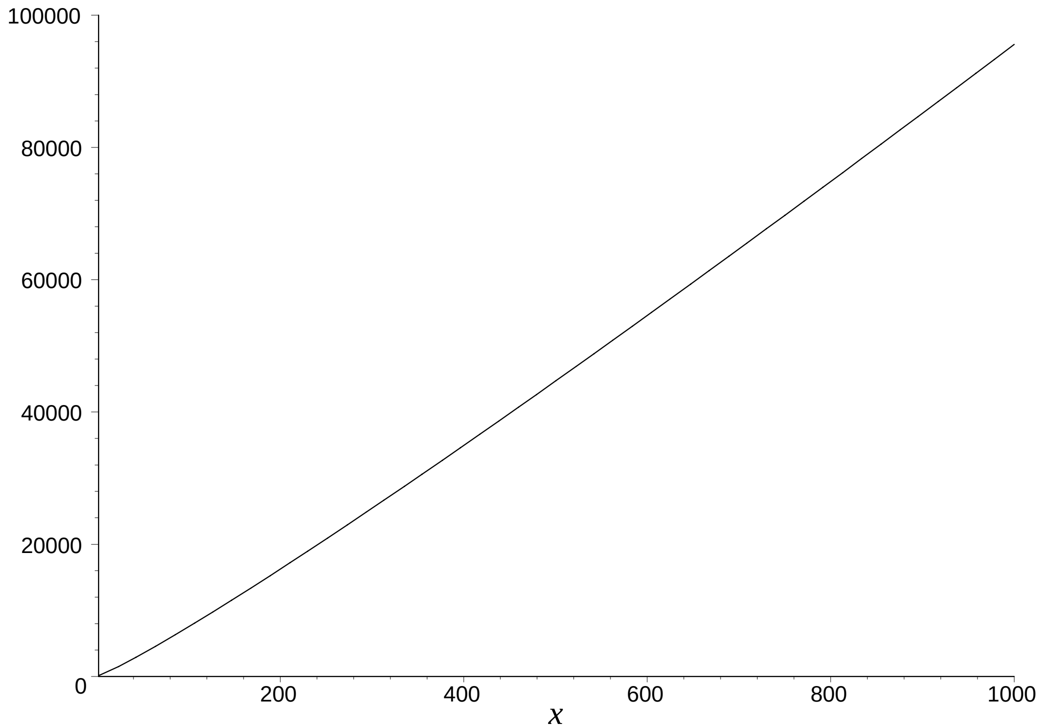

Figure 2: $N(2+1/x)$ versus x.

Using the elementary fact $\mathrm{mrl}(n) \geq \mathrm{mrl}(n-1)$, we apply Corollary 8 inductively as follows.

$$\begin{aligned}
\mathrm{mrl}(n) &\geq \mathrm{mrl}(a+b) \\
&\geq 2\min(a-1, \mathrm{mrl}(b)) \\
&\geq 2\min(b - \lfloor c \lg b \rfloor, b - \lfloor c \lg b \rfloor) \\
&= a + b - \lfloor c \lg b \rfloor - 1 \\
&= a + b + 1 - \left\lfloor c \lg(a+b) + c \lg\left(4^{1/c} b/(a+b)\right)\right\rfloor \\
&\geq n - \lfloor c \lg n \rfloor.
\end{aligned}$$

Using (7) to electronically verify (8) for $n \leq 10^5$, we find that (8) holds for all n when $c = 2.001$. ∎

5 Small n and a Sporadic Construction

The effective application of the construction in Section 3 depends heavily on good lower bounds on $\mathrm{mrl}(n)$ for small values of n. The following table summarizes the best lower bounds $\ell(n)$ that can be achieved by the constructions in [2], and the two sporadic constructions described below. By use of exhaustive computer searches, one can show that the starred values are exact.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\ell(n)$	2^*	2^*	2^*	4^*	4^*	5^*	5	6	8	8	8	9	9	11	11	12	12	14	15	16

We now describe a hitherto unpublished construction which yields a 10-bit Gray code with minimum run length 8. This is the Gray code used in the Codacon device [4]. The construction in Section 3 and those given in [1, 2] can only produce 10-bit codes with minimum run length at most 7.

Let C_4 denote the circuit of length four. We view the 10-cube as a cartesian power

$$Q_{10} \cong G = C_4 \times C_4 \times C_4 \times C_4 \times C_4.$$

We identify each vertex of G with a group element $(a_0, a_1, a_2, a_3, a_4) \in Z_4^5$, where Z_4 is the integers modulo 4. Two vertices u, v are adjacent in G if $u - v = \pm e_i$, where e_i is the ith unit group element, $0 \le i \le 4$. A *G-transition sequence* is any word $T = t_1 t_2 \dots t_k$ from the alphabet $\{0, 1, 2, 3, 4\}$. As usual, if $v \in V(G)$, then T induces a walk $W(v; T) = v_0 v_1 \dots v_k$ in G with $v_0 = v$ and $v_i = v_{i-1} + e_{t_i}$. Let $T^{i \mapsto 4}$ be the set of words that can be obtained from T by replacing *one* copy of the symbol i by the symbol 4. For example, $(0123401)^{1 \mapsto 4} = \{0423401, 0123404\}$.

Let $T_0 = 0123012301230123$. We iteratively (and nondeterministically) define G-transition sequences T_1, T_2 and T_3 by

$$T_{i+1} = T_i' T_i' T_i' T_i', \text{ where } T_i' \in T_i^{i \mapsto 4}.$$

Each $W(v; T_i)$ is a closed walk of length 4^{i+2}. It can be checked by computer, or proved with some effort, that any walk $W(v; T_i)$ $(0 \le i \le 3)$ is a circuit in G. Thus $W(v; T_3)$ represents a 10-bit Gray code C.

The minimum run length of C is at least twice the minimum distance between two occurrences of the same symbol in T_3. For $j \in \{0, 1, 2, 3\}$, two occurrences of symbol j in T_3 are separated by at least four positions. By selecting the sequences T_i' sensibly, the same statement holds for $j = 4$. Thus $\mathrm{mrl}(10) \ge \mathrm{mrl}(C) \ge 8$.

This construction does not appear to easily generalize to higher dimensional cubes. A similar construction using the subgraph $C_4^7 \times C_8^2 \subseteq Q_{20}$ and $T_0 = (01234567)^8$. can be made to produce a noncyclic 20-bit Gray code with minimum run length 16. If T is a word in the alphabet $\{0, 1, \dots, 19\}$ representing its transition sequence, then $T(20)T(20)$ represents a 21-bit cyclic Gray code, whence $\mathrm{mrl}(21) \ge 16$. We omit details here. Other, more complicated schema have produced minor improvements that are not worth mentioning here. We leave as unsolved the problem of generalizing this 10-bit construction in a satisfactory manner.

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