The restricted arc-width of a graph

David Arthur

Department of Mathematics
Duke University, Durham, NC 27708, USA
david.arthur@duke.edu

Submitted: Jun 26, 2003; Accepted: Oct 13, 2003; Published: Oct 23, 2003
MR Subject Classifications: 05C62, 05C83

Abstract

An arc-representation of a graph is a function mapping each vertex in the graph to an arc on the unit circle in such a way that adjacent vertices are mapped to intersecting arcs. The width of such a representation is the maximum number of arcs passing through a single point. The arc-width of a graph is defined to be the minimum width over all of its arc-representations. We extend the work of Barát and Hajnal on this subject and develop a generalization we call restricted arc-width. Our main results revolve around using this to bound arc-width from below and to examine the effect of several graph operations on arc-width. In particular, we completely describe the effect of disjoint unions and wedge sums while providing tight bounds on the effect of cones.

1 Introduction

The notion of a graph's path-width first arose in connection with the Graph Minors project, where Robertson and Seymour [3] introduced it as their first minor-monotone parameter. Since then, applications have arisen in the study of chromatic numbers, circuit layout and natural language processing. More recently, Barát and Hajnal [2] proposed a variant on path-width that leads to the analagous concept of arc-width. Although it has not been as widely studied as path-width, arc-width has similar applications and is an interesting and challenging problem in its own right.

Informally, we define an arc-representation of a graph to be a function mapping each vertex in the graph to an arc on the unit circle in such a way that adjacent vertices are mapped to intersecting arcs. The width of such a representation is the maximum number of arcs passing through a point. The arc-width of a graph is then defined to be the minimum width over all of its arc-representations. We illustrate an optimal arc-representation of C_4, the cycle on 4 vertices, in Figure 1.

Unfortunately, it is very difficult to consider arc-width in isolation. Without other information, even disjoint unions can be incomprehensible in the sense that the arc-width

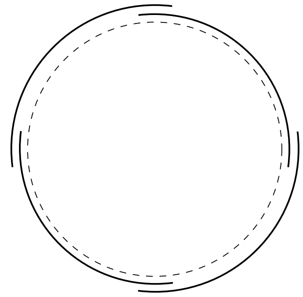

Figure 1: An optimal arc-representation of a cycle on 4 vertices. (The dashed lines represent S^1 and the solid lines represent arcs.)

of the disjoint union of G and H cannot be computed given only the arc-width of G and the arc-width of H. When one looks at more complicated operations, the computations become even more difficult.

To deal with this, we define the *restricted arc-width* of a graph as we define standard arc-width, except that we restrict our attention to arc-representations for which the minimal number of arcs passing through a point is bounded above by some constant. This parameter is a direct generalization of both arc-width and path-width, and it encapsulates information on both.

Using the notion of restricted arc-width, we are able to precisely describe the effect of disjoint unions, wedge sums, and cones on restricted arc-width, and hence on both path-width and arc-width. We also develop a number of results useful for obtaining lower bounds on restricted arc-width, and then show that computing arc-width is $\mathcal{NP}$-complete. Finally, we present a number of directions in which our work could be extended.

2 Preliminaries

Throughout this paper, we will assume that all graphs are finite and simple.

In this section, we review several important definitions and formalize some of the ideas mentioned in the introduction. We begin by defining path-width.

Definition 2.1. *An* interval-representation ϕ *of a graph* G *is a map taking each vertex of* G *to an interval on the real line* $\mathbb{R}$ *in such a way that adjacent vertices are mapped to intersecting intervals. For* $x \in \mathbb{R}$*, we define its width,* $w_\phi(x)$*, to be the number of intervals containing* x*. The* maximum width *of* ϕ*,* $W(\phi)$*, is then given by* $\max_{x\in\mathbb{R}} w_\phi(x)$*. Finally, we define the* path-width *of* G*,* $\mathrm{pw}^*(G)$*, to be the smallest value of* $W(\phi)$ *over all interval-representations* ϕ *of* G*.*

Arc-width is defined similarly.

Definition 2.2. *An* arc-representation ϕ *of a graph* G *is a map taking each vertex of* G *to an arc on the unit circle* S^1 *in such a way that adjacent vertices are mapped to intersecting*

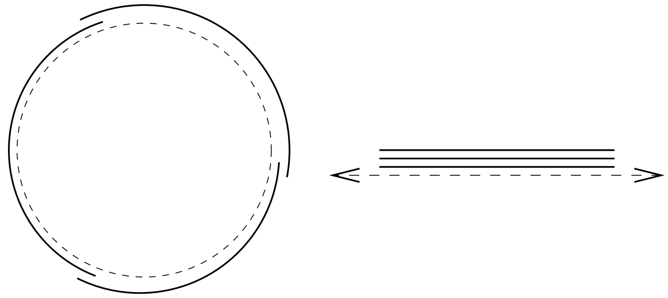

Figure 2: An optimal arc-representation and path-representation of the cycle on 3 vertices. Note that the path-width is larger than the arc-width.

arcs. For $x \in S^1$, we define its width, $w_\phi(x)$, to be the number of arcs containing x. The maximum width *of ϕ, $W(\phi)$, is then given by $\max_{x \in S^1} w_\phi(x)$. Finally, we define the* arc-width *of G, $\mathrm{aw}(G)$, to be the smallest value of $W(\phi)$ over all arc-representations ϕ of G.*

We will show that path-width is no smaller than arc-width, but in general, the two quantities need not be equal (see Figure 2).

Following the lead of Barát and Hajnal [2], we will assume that all arcs are closed, and that they are all proper subsets of S^1. For an arc I, let $l(I)$ denote the counter-clockwise endpoint of I, and let $r(I)$ denote its other endpoint. Also, if ϕ is an arc-representation, we will use $A(\phi)$ to denote the collection of arcs, $\{\phi(v) | v \in V(G)\}$.

Finally, we define restricted arc-width. Informally, we want this to differ from the standard definition of arc-width only in that we restrict ourselves to representations with a certain minimum width. We formalize this as follows.

Definition 2.3. *Let ϕ be an arc-representation of a graph G. We define the* minimum width *of ϕ, $w(\phi)$, to be $\min_{x \in S^1} w_\phi(x)$. We then let $\mathrm{aw}_i(G)$ denote the smallest possible value of $W(\phi)$ over all arc-representations ϕ of G satisfying $w(\phi) \leq i$.*

3 Properties of the Restricted Arc-Width

In this section, we develop some of the most important properties of restricted arc-width. In particular, we show how restricted arc-width is related to path-width and to arc-width, and then we show how $\mathrm{aw}_i(G)$ is related to $\mathrm{aw}_j(G)$ for a fixed graph G.

We begin by showing how restricted arc-width encapsulates information on both arc-width and path-width.

Proposition 3.1. *For any graph G,*

$$\begin{aligned} \mathrm{aw}_\infty(G) &= \mathrm{aw}(G) \qquad \textit{and} \\ \mathrm{aw}_0(G) &= \mathrm{pw}^*(G). \end{aligned}$$

Proof. It follows immediately from the definition of aw_i that $\mathrm{aw}_\infty(G) = \mathrm{aw}(G)$.

To prove the other equality, take an arc-representation ϕ of G with $w(\phi) = 0$ and $W(\phi) = \mathrm{aw}_0(G)$. Let x be a point in S^1 such that $w_\phi(x) = 0$. Then, we can compose ϕ with a projection from x onto the tangent line opposite x to obtain an interval-representation ϕ' of G. It is then easy to check that $W(\phi') = W(\phi) = \mathrm{aw}_0(G)$, so that $\mathrm{pw}^*(G) \leq \mathrm{aw}_0(G)$.

Similarly, given an interval-representation of G, we can compose it with the inverse projection of S^1 onto $\mathbb{R}$ to obtain an arc-representation of G. As above, this implies $\mathrm{aw}_0(G) \leq \mathrm{pw}^*(G)$. The result follows. □

Our next goal is to investigate how $\mathrm{aw}_i(G)$ and $\mathrm{aw}_j(G)$ are related for a fixed graph G. Before we answer this question, however, we must first establish a technical lemma.

Lemma 3.2. *For every graph G and every non-negative integer i, there exists an arc-representation ϕ of G with the following properties:*

1. $w(\phi) \leq i$.

2. $W(\phi) = \mathrm{aw}_i(G)$.

3. *There exists an interval I with positive length that satisfies:*

   i. *I intersects at most i arcs in $A(\phi)$.*

   ii. *Every arc intersecting I contains I.*

Proof. Let ϕ be an arc-representation of G with $w(\phi) \leq i$ and $W(\phi) = \mathrm{aw}_i(G)$. Let x be a point in S^1 with $w(x) \leq i$. Suppose we remove x from S^1 and replace it with an interval I of positive length, extending arcs through I if and only if they contained x. This gives a new arc-representation, and we can easily check that it has the desired properties. □

We are now ready to describe the relationship between aw_i and aw_j.

Proposition 3.3. *Suppose $i > 0$. Then,*

$$\mathrm{aw}_i(G) \leq \mathrm{aw}_{i-1}(G) \leq \mathrm{aw}_i(G) + 1.$$

Proof. It follows immediately from the definition of aw_i that $\mathrm{aw}_i(G) \leq \mathrm{aw}_{i-1}(G)$.

Now, let ϕ be an arc-representation of G with $w(\phi) \leq i$ and $W(\phi) = \mathrm{aw}_i(G)$. Suppose $w(\phi) < i$. Then, we have $\mathrm{aw}_{i-1}(G) \leq W(\phi) < \mathrm{aw}_i(G) + 1$.

On the other hand, suppose $w(\phi) = i$. Let x be a point in S^1 minimizing $w_\phi(x)$. By Lemma 3.2, we can assume without loss of generality that x is contained in an interval I, of positive length, with the property that any arc intersecting I contains I. Since $w(\phi) = i > 0$, there exists some arc J passing through x.

Let ϕ' be the arc-representation of G obtained from ϕ by replacing J with an arc J' containing everything but the interior of I. Since every arc in $A(\phi)$ that intersects I also contains I, we know ϕ' is indeed a valid arc-representation of G. Moreover, $w(\phi') = i - 1$, and

$$\begin{aligned} W(\phi') &\leq W(\phi) + 1 \\ &= \mathrm{aw}_i(\phi) + 1 \end{aligned}$$

It follows that $\mathrm{aw}_{i-1}(G) \leq \mathrm{aw}_i(G) + 1$. □

As a corollary to Proposition 3.3, we know that for a fixed graph G, the sequence $\{\mathrm{aw}_i(G)\}$ is non-increasing and decreases by at most 1 at each step. It is possible that not all such sequences arise in practice, so perhaps these relations could be extended. What we have, however, is already enough to give us an important result due to Barát and Hajnal [2].

Corollary 3.4. *For any graph G,*

$$\left\lceil \frac{\mathrm{pw}^*(G) + 1}{2} \right\rceil \leq \mathrm{aw}(G) \leq \mathrm{pw}^*(G).$$

Proof. Let ϕ be an arc-representation of G with $W(\phi) = \mathrm{aw}(G)$. Since all of the arcs in $A(\phi)$ are closed, $w(\phi) < W(\phi)$. Therefore, if $\mathrm{aw}(G) = n$, we have $\mathrm{aw}_{n-1}(G) = \mathrm{aw}(G)$.

Now, Proposition 3.3 implies

$$\mathrm{aw}_{n-1}(G) \leq \mathrm{aw}_0(G) \leq \mathrm{aw}_{n-1}(G) + n - 1,$$

and so,

$$\mathrm{aw}(G) \leq \mathrm{aw}_0(G) \leq 2\mathrm{aw}(G) - 1.$$

The result now follows from Proposition 3.1. □

4 Lower Bounds on Arc-Width

In general, it is relatively easy to bound the arc-width of a graph from above, since one only requires a single construction to do so. Establishing lower bounds is much more difficult, so in this section, we provide a number of results that can help accomplish this task.

First, recall that H is a *minor* of G if H can be obtained from a subgraph of G by collapsing along zero or more edges. We say aw_i is *minor-monotone* if $\mathrm{aw}_i(H) \leq \mathrm{aw}_i(G)$ whenever H is a minor of G. Extending a known result for path-width and arc-width, we have the following.

Theorem 4.1. aw_i *is minor-monotone for all i.*

Proof. If H is a subgraph of G, then $\mathrm{aw}_i(H) \leq \mathrm{aw}_i(G)$, since any arc-representation of G induces an arc-representation of H by restriction. Therefore, it suffices to show that collapsing along an edge of a graph does not increase its restricted arc-width.

Towards that end, let G be a graph containing adjacent vertices u and v, and let G' be the graph obtained by collapsing along the edge between u and v. Denote the vertex corresponding to u and v in G' by w.

Now, let ϕ be an arc-representation of G. Let ϕ' be the arc-representation of G' defined by setting $\phi'(x) = \phi(x)$ for $x \neq w$ and $\phi'(w) = \phi(u) \cup \phi(v)$. Clearly, this is indeed a valid arc-representation of G', and for all $y \in S^1$, we know $w_{\phi'}(y) \leq w_\phi(y)$. It follows that $\mathrm{aw}_i(G') \leq \mathrm{aw}_i(G)$, as required. □

Although working with minors can be very useful, doing so requires knowing the arc-width of a fairly large class of graphs that could arise as minors. Thus, we also give a more direct result.

Theorem 4.2. *Suppose every vertex in a graph G has degree at least n. Then,*

$$aw_i(G) \geq \max\left\{\left\lceil\frac{n}{2}\right\rceil + 1, n - i + 1\right\}.$$

Proof. Within this proof, we will say I is contained in J to mean $I \subset J$ and $I \neq J$.

Let ϕ be an arc-representation of G with $w(\phi) \leq i$ and $W(\phi) = \mathrm{aw}_i(G)$. Let x be a point on the unit circle with $w_\phi(x) \leq i$. Choose an arc I as follows:

(1) First eliminate from consideration all arcs passing through x. There are at least $n+1$ arcs in $A(\phi)$, so if all of them overlap x, then $\mathrm{aw}_i(G) > n$, and we are done. Therefore, we may assume there exists at least one arc in $A(\phi)$ that does not pass through x, and hence, we have not eliminated every possible arc.

(2) Now, eliminate from consideration all arcs containing other arcs. Let J be any arc not containing x. Then, either J does not contain any other arcs, or J contains some other arc K that does not contain any other arcs. Since no arc contained in J can overlap x, it follows that we still have not eliminated every possible arc.

(3) Finally, choose I among the remaining arcs in such a way as to minimize the clockwise angle from x to $l(I)$.

For the rest of the proof, we will say that for points p and q in S^1, $p < q$ if the clockwise angle from x to p is less than the clockwise angle from x to q.

The vertex corresponding to I in G has degree at least n, so I must intersect at least n other arcs. Since I does not contain any other arc by condition (2) above, at least n arcs intersect at least one of the two endpoints of I. Therefore, at least $\lceil\frac{n}{2}\rceil$ arcs intersect one endpoint of I, and hence, there exists y for which $w_\phi(y) \geq \lceil\frac{n}{2}\rceil + 1$. It follows that $\mathrm{aw}_i(G) \geq \lceil\frac{n}{2}\rceil + 1$.

Now, consider an arc J passing through $l(I)$. Suppose J does not pass through either x or $r(I)$, so that J is entirely contained within the clockwise arc from x to $r(I)$. Let J'

be an arc in J not containing any other arc. Then, J' is also entirely contained inside the clockwise arc from x to $r(I)$. Since $r(J') < r(I)$ and J' cannot be contained within I by (2) above, $l(J') < l(I)$. Now, J' does not contain other arcs and does not intersect x, but it does satisfy $l(J') < l(I)$, which contradicts the choice of I. Thus, any arc passing through $l(I)$ and not $r(I)$ also passes through x. Since $w_\phi(x) \leq i$, at most i arcs pass through $l(I)$ and not $r(I)$.

However, we know at least n arcs other than I pass through either $l(I)$ or $r(I)$, so it then follows that at least $n - i$ of these arcs pass through $r(I)$. Thus, there exists x for which $w_\phi(x) \geq n - i + 1$. It follows that $\mathrm{aw}_i(G) \geq n - i + 1$. □

It is worth mentioning that one can actually relax the conditions needed to bound $\mathrm{aw}(G)$. We do not prove this since it is not directly relevant to our work, but we do give a statement of the result.

Theorem 4.3. *Suppose a graph G contains a vertex v with the property that for every vertex u, the degree of u plus the distance between u and v is at least n. Then,*

$$\mathrm{aw}(G) \geq \left\lceil \frac{n}{2} \right\rceil + 1.$$

One other operation that we consider is the cone of a graph. Specifically, for a graph G, we let $G+v$ denote the graph obtained from G by adding a vertex v and adding edges between it and each vertex in G. The path-width of such a graph is relatively easy to understand.

Proposition 4.4. *For any graph G,*

$$\mathrm{aw}_0(G + v) = \mathrm{aw}_0(G) + 1$$

Proof. We first prove a related statement:

$$\mathrm{aw}_i(G + v) \leq \mathrm{aw}_i(G) + 1. \tag{1}$$

Let ϕ be an arc-representation of G with $w(\phi) \leq i$ and $W(\phi) = \mathrm{aw}_i(G)$. By Lemma 3.2, we can assume there exists an interval I of positive length such that $w_\phi(x) \leq i$ for all $x \in I$, and such that any arc intersecting I contains I. Let ϕ' be the arc-representation of $G+v$ obtained by setting $\phi' = \phi$ on G, and by mapping v to the arc containing everything but the interior of I. Clearly, this is indeed a valid arc-representation of $G+v$. Moreover, $w(\phi') \leq w(\phi)$ and $W(\phi') \leq 1 + W(\phi)$. Thus, (1) follows immediately.

It remains only to show that $\mathrm{aw}_0(G) \leq \mathrm{aw}_0(G + v) - 1$. To do this, we let ϕ' be an arc-representation of $G + v$ with $w(\phi') = 0$ and $W(\phi') = \mathrm{aw}_0(G + v)$. Let $I = \phi'(v)$, and choose x in S^1 such that $w_{\phi'}(x)$ is maximal. Suppose $x \notin I$, and let J be any arc containing x. Recall that I intersects every arc in $A(\phi')$, so in particular, I must intersect J. Thus, we can gradually move x along J until we reach I.

Let K be any arc (J or otherwise) containing x. Suppose moving x to I along J causes x to be no longer contained in K. Then, since K intersects I somewhere, it must span the arc from x to I not spanned by J. In particular, this means that I, J, and K together cover the entire unit circle, which contradicts the fact that $w(\phi') = 0$.

Thus, moving x to I along J keeps x inside all of the arcs originally containing it. However, it also moves x inside I, which did not originally contain it. Thus, we have found a point x' for which $w_{\phi'}(x') > w_{\phi'}(x)$, contradicting our choice of x. Therefore, if $w_{\phi'}(x)$ is maximal, $x \in I$.

Now, ϕ' is an arc-representation of $G + v$, so it induces an arc-representation ϕ of G by restriction. Clearly, $w(\phi) = w(\phi') = 0$, and since $w_{\phi'}(x)$ is maximal only if $x \in I$, we also have $W(\phi) \leq W(\phi') - 1$. Therefore, $\mathrm{aw}_0(G) \leq \mathrm{aw}_0(G + v) - 1$, and the desired result follows. $\square$

Unfortunately, it is impossible to characterize the arc-width of the cone of a graph in a similar fashion. For example, $\mathrm{aw}_i(P_2) = \mathrm{aw}_i(P_3)$ for all i, but one can show $\mathrm{aw}(P_2 + v) \neq \mathrm{aw}(P_3 + v)$. Thus, one needs more information about a graph G than its restricted arc-width to completely determine the arc-width of $G + v$. On the other hand, we can still say a great deal.

Proposition 4.5. *Let G be an arbitrary graph, and let $G' = (G+u)+v$. Then, for $i > 0$,*

$$\mathrm{aw}_i(G) + 2 \geq \mathrm{aw}_i(G') \geq \mathrm{aw}_{i-1}(G) + 1.$$

Proof. It follows immediately from (1) in the proof of Proposition 4.4 that $\mathrm{aw}_i(G) + 2 \geq \mathrm{aw}_i(G')$, so we need only show $\mathrm{aw}_i(G') \geq \mathrm{aw}_{i-1}(G) + 1$.

Let ϕ' be an arc-representation of G' with $w(\phi') \leq i$ and $W(\phi') = \mathrm{aw}_i(G')$. For convenience, we let $n = W(\phi')$. Also, let U and V denote $\phi'(u)$ and $\phi'(v)$. Finally, let ϕ denote the arc-representation that ϕ' induces on G by restriction.

Now, suppose $U \subset V$. Let U' be the arc with endpoints $l(U)$ and $r(V)$ and let V' be the arc with endpoints $l(V)$ and $r(U)$. Note that every arc in $A(\phi')$ must intersect U, so every arc in $A(\phi')$ must also intersect both U' and V'. Thus, if we replace U with U' and V with V', we still have a valid arc-representation of G'. Furthermore, this substitution does not change the number of arcs passing through any given point on the circle, so it also fixes both $W(\phi')$ and $w(\phi')$. Thus, we may assume $U \not\subset V$, and similarly $V \not\subset U$. By switching U and V, we can further assume that $l(U), l(V), r(U)$, and $r(V)$ are arranged clockwise around the circle in that order.

We now consider two cases, based on the value of $W(\phi)$.

Case 1: $W(\phi) < n$.
Suppose the only points in S^1 minimizing w_ϕ are in $U \cup V$. Then $w(\phi) < w(\phi') \leq i$, and since $W(\phi) < n$, we have $\mathrm{aw}_{i-1}(G) \leq \mathrm{aw}_i(G') - 1$, as required.

Otherwise, there exists $a \notin U \cup V$ minimizing w_ϕ. Let $A_1, A_2, \ldots, A_j$ be the arcs containing a. If $V \subset A_k$ for any k, then we replace A_k with V. This will still give a valid arc-representation of G, since V intersects every arc in $A(\phi)$. Moreover, since $V \subset A_k$, making this replacement will decrease $w(\phi)$ without increasing $W(\phi)$. Therefore, $\mathrm{aw}_{i-1}(G) \leq \mathrm{aw}_i(G') - 1$, as required.

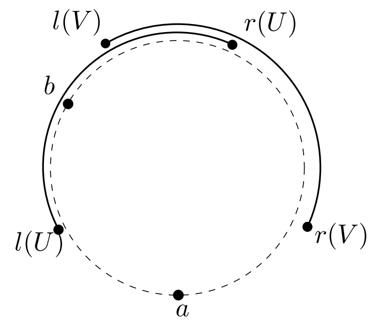

Figure 3: The relative positioning of points and arcs in Case 1.

We now consider the case where $V \not\subset A_k$ for any k. Suppose there is a point b in $U-V$, not contained in A_k for any k, with the property that $w_\phi(b) = n-1$ (see Figure 3). Then, since $a, l(U), b, l(V), r(U)$, and $r(V)$ are arranged clockwise around the circle in that order, any arc containing both b and $r(V)$, but not containing $l(V)$, must also contain a. Since we know this cannot happen, and since every arc in $A(\phi)$ must intersect V somewhere, it follows that all $n-1$ arcs in $A(\phi)$ containing b also contain $l(V)$. However, we know that U and V also contain $l(V)$, which means $w_{\phi'}(l(V)) > n$, contradicting the fact that $W(\phi') = n$. It follows that any b in $U - V$ satisfying $w_\phi(b) = n-1$ is contained in A_k for some k.

Since there are only a finite number of arcs containing a, and none of them contain V, it follows that one of them must contain all $b \in U-V$ with the property that $w_\phi(b) = n-1$. Call this arc A_k. Let $c = \phi^{-1}(A_k)$, and let σ be the arc-representation of G obtained from ϕ by setting $\sigma(c) = U$ and $\sigma(x) = \phi(x)$ for $x \neq c$. Because U intersects every arc in $A(\phi)$, σ is indeed a valid arc-representation of G. Furthermore, since A_k contains a, but U does not, $w(\sigma) \leq w_\sigma(a) < w_\phi(a) \leq i$.

Now, consider $x \in S^1$. If $x \in V$, then since $V \notin A(\sigma)$ and $V \in A(\phi')$, we have $w_\sigma(x) < w_{\phi'}(x) \leq n$. Also, if $x \notin U \cup V$, then $w_\sigma(x) \leq w_\phi(x) < n$. Suppose, on the other hand, that $x \in U - V$. If $w_\phi(x) < n-1$, then $w_\sigma(x) \leq w_\phi(x) + 1 < n$. If $w_\phi(x) \geq n-1$, then $x \in A_k$, so $w_\sigma(x) \leq w_\phi(x) < n$. Therefore, we have $w_\sigma(x) < n$ for all x, and hence $W(\sigma) \leq n-1$. It follows that $\mathrm{aw}_{i-1}(G) \leq n-1$, as desired.

Case 2: $W(\phi) = n$.
Choose $a \notin U \cup V$ so as to minimize $w_\phi(a)$, and let d be the point not in $U \cup V$ closest to $r(V)$ that maximizes $w_\phi(d)$. By possibly reflecting the arc-representation about any diameter of S^1, we can ensure that d is closer to $r(V)$ than a is.

For $x \in U \cup V$, note that $w_\phi(x) < w_{\phi'}(x) \leq n$, so if $w_\phi(x) = n$, then $x \notin U \cup V$. Since $W(\phi) = n$, it follows that $w_\phi(d) = n$. Label the arcs containing d by $D_1, D_2, \ldots, D_n$. These cannot all contain $r(V)$, because if they did, we would have $w_\phi(r(V)) > n$. Since each D_k intersects V somewhere, it follows that there exists some D_k containing $l(V)$ but not containing $r(V)$. Since $d, a, l(U), l(V), r(U)$, and $r(V)$ are arranged clockwise around the circle in that order, it follows that D_k contains both a and $U - V$.

Let $e = \phi^{-1}(D_k)$, and let σ be the arc-representation of G obtained from ϕ by setting $\sigma(e) = U$ and $\sigma(x) = \phi(x)$ for $x \neq e$. Since U intersects every arc in $A(\phi)$, this is indeed a valid arc-representation of G. Also, since V and D_k are both in $A(\phi')$, but not in $A(\sigma)$, it follows that $w_\sigma(x) < w_{\phi'}(x)$ for all $x \in U \cup V \subset D_k \cup V$. Thus, if there exists $x \in U \cup V$ with $w_{\phi'}(x) = i$, then $w(\sigma) < i$. Otherwise, because $w(\phi') \leq i$, we know $w_{\phi'}(a) \leq i$. Since $a \in D_k$ but $a \notin U$, it again follows that $w_\sigma(a) < w_{\phi'}(a) = i$. Therefore, $w(\sigma) < i$ in all cases.

We now show $W(\sigma) < n$. Towards that end, consider $x \in S^1$. As above, if $x \in U \cup V$, then $w_\sigma(x) < w_{\phi'}(x) \leq n$. Suppose, on the other hand, that $x \notin U \cup V$. If $w_\phi(x) = n$, then $x \in D_k$, which implies $w_\sigma(x) < w_\phi(x)$, and if $w_\phi(x) < n$, then $w_\sigma(x) \leq w_\phi(x) < n$. Thus, $w_\sigma(x) < n$ for all x. It follows that $W(\sigma) \leq n-1$, and hence, $\mathrm{aw}_{i-1}(G) \leq n-1$. ☐

Note that since $\mathrm{aw}_i((G+u)+v) \leq \mathrm{aw}_i(G+v)+1$, Proposition 4.5 implies $\mathrm{aw}_i(G+v) \geq \mathrm{aw}_{i-1}(G)$. Summarizing all of these results, we obtain the following.

Theorem 4.6. *For any graph G,*

$$\mathrm{aw}_0(G+v) = \mathrm{aw}_0(G) + 1,$$

and for $i > 0$,

$$\begin{array}{rcll} \mathrm{aw}_i(G)+1 \geq & \mathrm{aw}_i(G+v) & \geq \mathrm{aw}_{i-1}(G) & \textit{and} \\ \mathrm{aw}_i(G)+2 \geq & \mathrm{aw}_i(G+u+v) & \geq \mathrm{aw}_{i-1}(G)+1. & \end{array}$$

We can also modify the proof of Proposition 4.5 to obtain a slightly different result. The details are similar enough that we do not include a full proof, but we still give a statement of the result. First, recall that the *double cone* of a graph G, denoted $G + \overline{K_2}$, is the graph obtained from $(G+u)+v$ by removing the edge between u and v.

Theorem 4.7. *For any graph G, and any integer i,*

$$\mathrm{aw}_i(G) + 2 \geq \mathrm{aw}_i(G + \overline{K_2}) \geq \mathrm{aw}_i(G) + 1.$$

Finally, to demonstrate the power of the techniques that we have developed, we conclude the section by computing $\mathrm{aw}_i(K_n)$.

Corollary 4.8. *Let K_n denote the complete graph on n vertices. Then,*

$$\mathrm{aw}_i(K_n) = \begin{cases} n - i & \text{if } i \leq \lceil \frac{n}{2} \rceil - 1, \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{if } i \geq \lceil \frac{n}{2} \rceil - 1. \end{cases}$$

Proof. We prove this by induction on n. For $n = 1$, the claim is trivial. Now, suppose the result holds for $n = k - 1$.

Then, since $K_k = K_{k-1} + v$,

$$\begin{aligned} \mathrm{aw}_0(K_k) &= \mathrm{aw}_0(K_{k-1}) + 1 \qquad \text{(by Theorem 4.6)} \\ &= k. \end{aligned}$$

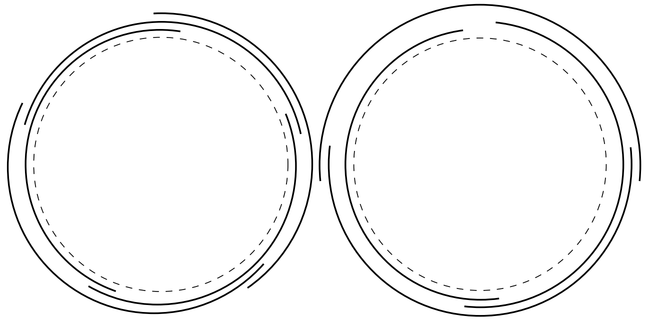

Figure 4: Optimal arc-representations of K_5 (left) and of K_4 (right). Note that we alter the arc-representation of K_{2n} slightly by shortening 2 adjacent arcs to decrease the minimum width to $n-1$.

Similarly, for $0 < i < \lceil \frac{k}{2} \rceil - 1$,

$$\begin{aligned} \mathrm{aw}_i(K_k) &\geq \mathrm{aw}_{i-1}(K_{k-1}) \\ &= k - i, \end{aligned}$$

and

$$\begin{aligned} \mathrm{aw}_i(K_k) &\leq 1 + \mathrm{aw}_i(K_{k-1}) \\ &= k - i, \end{aligned}$$

so $\mathrm{aw}_i(K_k) = k - i$.

By Theorem 4.2, we have for $i \geq \lceil \frac{k}{2} \rceil - 1$,

$$\mathrm{aw}_i(K_k) \geq \left\lceil \frac{k-1}{2} \right\rceil + 1 = \left\lfloor \frac{k}{2} \right\rfloor + 1.$$

Finally, consider the arc-representation ϕ of K_k obtained by mapping each vertex to arcs spaced equally around the circle, of length $\frac{\pi \cdot (k-1)}{k}$ if k is odd, and of length $\pi + \epsilon$ for small ϵ if k is even. It is easy to check that any two such arcs intersect, and hence that this is indeed a valid arc-representation of K_k. Moreover, the maximum number of arcs passing through any point is at most $\lfloor \frac{k}{2} \rfloor + 1$. We can also arrange for there to exist x in S^1 for which $w_\phi(x) \leq \lfloor \frac{k}{2} \rfloor - 1$ (see Figure 4). Thus, $\mathrm{aw}_i(K_k) \leq \lfloor \frac{k}{2} \rfloor + 1$ for $i \geq \lceil \frac{k}{2} \rceil - 1$.

Therefore, the desired result holds for $n = k$, which completes the inductive proof. $\square$

5 Graph Operations

In the previous section, we provided a number of results that give lower bounds on arc-width. These results can often be quite useful, as witnessed by the relatively easy computation of $\mathrm{aw}_i(K_n)$, but they are not always sufficient. In this section, we extend our methods by approaching arc-width in a different fashion. In particular, we focus on the effect of various graph operations, beginning with the disjoint union, denoted $\amalg$.

Theorem 5.1. *Let G_1, G_2, ..., G_n be arbitrary graphs, ordered in such a way that*

$$\mathrm{aw}_0(G_1) \geq \mathrm{aw}_0(G_2) \geq \cdots \geq \mathrm{aw}_0(G_n).$$

For $n > 1$, we have

$$\mathrm{aw}_m(G_1 \amalg G_2 \amalg \cdots \amalg G_n) = \mathrm{aw}_i(G_1),$$

where i is the largest integer satisfying:

1. $i \leq m$,
2. $\mathrm{aw}_i(G_1) \geq \mathrm{aw}_0(G_2) + i$.

Before we prove this theorem, we first need to establish a technical lemma.

Lemma 5.2. *Let G_1, G_2, ..., G_n be arbitrary graphs, ordered in such a way that $\mathrm{aw}_0(G_1) \geq \mathrm{aw}_0(G_2) \geq \cdots \geq \mathrm{aw}_0(G_n)$, and let a_1, a_2, ..., a_n be non-negative integers with $\sum_{j=1}^n a_j \leq m$. Then,*

$$\max_{j\in\{1,2,\ldots,n\}} \Big(\mathrm{aw}_{a_j}(G_j) - a_j\Big) + \sum_{j=1}^n a_j \geq \mathrm{aw}_i(G_1),$$

where i is defined as in Theorem 5.1.

Proof of Lemma 5.2. Let

$$\begin{aligned} b_j &= \mathrm{aw}_{a_j}(G_j) - a_j, \\ b &= \max_{j\in\{1,2,\ldots,n\}} b_j, \qquad \text{and} \\ y &= b + \sum_{j=1}^n a_j. \end{aligned}$$

We wish to show that $y \geq \mathrm{aw}_i(G_1)$ for any choice of $\{a_j\}$. To do this, choose $a_1, a_2, \ldots, a_n$ in such a way that y is minimized, and given the minimal value of y, choose $\{a_j\}$ such that $\sum_{j=1}^n a_j$ is minimized.

Suppose $a_j > 0$ for 2 distinct values of j, and suppose we then decrease both a_j by 1. Note that, by Proposition 3.3, decreasing a_j by 1 increases b_j by at most 2 and fixes b_k for $k \neq j$. Thus, decreasing a_j by 1 for both j will increase b by at most 2, and hence y will not increase. Thus, we have found a choice of $\{a_j\}$ that decreases $\sum_{j=1}^n a_j$ without increasing y, giving a contradiction. Therefore, $a_j > 0$ for at most one value of j.

Now, choose j so that $a_j > 0$, and suppose $b_j < b$. Then, if we decrease a_j by 1, we will increase b_j by at most 2, and hence will increase b by at most 1. On the other hand, $\sum_{j=1}^n a_j$ will decrease by 1, so y will not increase. Thus, we again have found a choice of $\{a_j\}$ that decreases $\sum_{j=1}^n a_j$ without increasing y, giving another contradiction. Therefore, if $a_j > 0$, then $b_j = b$.

Thirdly, suppose $a_j > 0$ for $j \neq 1$. Then, since $b_j = b$, we know $b_j \geq b_1$, and hence,

$$\mathrm{aw}_{a_j}(G_j) - a_j \geq \mathrm{aw}_{a_1}(G_1) - a_1.$$

Since $a_j > 0$, we also know $a_1 = 0$. This implies

$$\mathrm{aw}_{a_j}(G_j) - a_j \quad \geq \quad \mathrm{aw}_0(G_1),$$

and so,

$$\mathrm{aw}_0(G_j) - a_j \quad \geq \quad \mathrm{aw}_0(G_1).$$

However, this is impossible since $\mathrm{aw}_0(G_j) \leq \mathrm{aw}_0(G_1)$ and $a_j > 0$. It follows that $a_2 = a_3 = \cdots = a_n = 0$.

Furthermore, if $a_1 \neq 0$, we have

$$\begin{array}{lll} & \mathrm{aw}_{a_1}(G_1) - a_1 \geq \mathrm{aw}_{a_j}(G_j) - a_j & \text{for all } j > 1 \\ \Rightarrow & \mathrm{aw}_{a_1}(G_1) - a_1 \geq \mathrm{aw}_0(G_j) & \text{for all } j > 1 \quad (\text{since } a_j = 0 \text{ for } j > 1) \\ \Rightarrow & \mathrm{aw}_{a_1}(G_1) - a_1 \geq \mathrm{aw}_0(G_2). & \end{array}$$

On the other hand, if $a_1 = 0$, we also have $\mathrm{aw}_{a_1}(G_1) - a_1 \geq \mathrm{aw}_0(G_2)$. Thus, $\mathrm{aw}_{a_1}(G_1) - a_1 \geq \mathrm{aw}_0(G_2)$ regardless, and since we also know $a_1 \leq m$, it follows that $a_1 \leq i$. Therefore, we have shown that there exists a choice of $\{a_j\}$ minimizing y with the property that $a_2 = a_3 = \cdots = a_n = 0$ and $a_1 \leq i$.

Finally, taking this minimal solution and increasing a_1 as long as $a_1 \leq m$ and $\mathrm{aw}_{a_1}(G_1) - a_1 \geq \mathrm{aw}_0(G_2)$, gives us $a_1 = i$, and $a_2 = a_3 = \cdots = a_n = 0$. Doing this will clearly not increase y. Thus, y achieves its minimum value for $a_1 = i$ and $a_2 = a_3 = \cdots = a_n = 0$. Moreover, for this choice of $\{a_j\}$, it is clear that $y = \mathrm{aw}_i(G_1)$. The result follows. □

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Let $G = G_1 \amalg G_2 \amalg \cdots \amalg G_n$ and let ϕ be an arc-representation of G with $w(\phi) \leq i$. For each j, note that ϕ induces by restriction an arc-representation ϕ_j of G_j.

Now, for any $x \in S^1$, we have,

$$\begin{aligned} w_\phi(x) &= |\phi^{-1}(x)| \\ &= |\phi_1^{-1}(x)| + |\phi_2^{-1}(x)| + \cdots + |\phi_n^{-1}(x)| \\ &= w_{\phi_1}(x) + w_{\phi_2}(x) + \cdots + w_{\phi_n}(x) \\ &\geq w(\phi_1) + w(\phi_2) + \cdots + w(\phi_n) \end{aligned}$$

so it follows that $\sum_{j=1}^n w(\phi_j) \leq m$.

Also, for each j, there exists an x in S^1 such that $|\phi_j^{-1}(x)| = W(\phi_j)$. Thus, there exists an x such that

$$|\phi^{-1}(x)| \geq W(\phi_j) + \sum_{k \neq j} w(\phi_k).$$

It follows that

$$\begin{aligned}
W(\phi) &\geq \max_{j\in\{1,2,\ldots,n\}}(W(\phi_j)+\sum_{k\neq j}w(\phi_k)) \\
&= \max_{j\in\{1,2,\ldots,n\}}(W(\phi_j)-w(\phi_j))+\sum_{j=1}^{n}w(\phi_j) \\
&\geq \max_{j\in\{1,2,\ldots,n\}}(\mathrm{aw}_{w(\phi_j)}(G_j)-w(\phi_j))+\sum_{j=1}^{n}w(\phi_j).
\end{aligned}$$

Thus, by Lemma 5.2, $W(\phi) \geq \mathrm{aw}_i(G_1)$. Since ϕ was chosen arbitrarily, we conclude that $\mathrm{aw}_m(G) \geq \mathrm{aw}_i(G_1)$.

It remains only to show that $\mathrm{aw}_m(G) \leq \mathrm{aw}_i(G_1)$. To do this, we construct an arc-representation ϕ of G, as follows:

1. Let ϕ_1 be an arc-representation of G_1 with $w(\phi_1) \leq i$ and $W(\phi_1) = \mathrm{aw}_i(G_1)$. Let x be a point in S^1 with $w_{\phi_1}(x) \leq i$. By Lemma 3.2, we can ensure that x is contained inside an interval I of positive length which intersects only i arcs. Now, define ϕ to be identical to ϕ_1 on G_1.

2. For $j > 1$, let ϕ_j be an arc-representation of G_j with $w(\phi_j) = 0$ and $W(\phi_j) = \mathrm{aw}_0(G_j)$. By rescaling and rotating each ϕ_j, we can assume $\mathrm{Im}(\phi_j)$ is contained in I for all j, and that $\mathrm{Im}(\phi_j) \cap \mathrm{Im}(\phi_k) = \emptyset$ for $j \neq k$. Define ϕ to be identical to ϕ_j on G_j.

Clearly, ϕ is indeed an arc-representation of G. Also note that for $j > 1$ and $x \in \mathrm{Im}(\phi_j)$, we have $w_\phi(x) \leq i + \mathrm{aw}_0(G_j) \leq \mathrm{aw}_i(G_1)$, and for $x \notin \bigcup_{j=2}^{n}\mathrm{Im}(\phi_j)$, we have $w_\phi(x) \leq W(\phi_1) = \mathrm{aw}_i(G_1)$. Thus, $W(\phi) \leq \mathrm{aw}_i(G_1)$. Moreover, $w(\phi) = i$. Therefore, $\mathrm{aw}_m(G) \leq \mathrm{aw}_i(G_1)$, and the result follows. □

Corollary 5.3. *For any graph G and any integers j, n with $n \geq 2$,*

$$\mathrm{aw}_j(\underbrace{G \amalg G \amalg \cdots \amalg G}_{n}) = \mathrm{pw}^*(G).$$

Proof. Note that for $i > 0$,

$$\mathrm{aw}_i(G) - i \leq \mathrm{aw}_0(G) - i < \mathrm{aw}_0(G).$$

It follows from Theorem 5.1 that $\mathrm{aw}_j(G \amalg G \amalg \cdots \amalg G) = \mathrm{aw}_0(G) = \mathrm{pw}^*(G)$. □

Corollary 5.3 demonstrates that, in a very precise sense, arc-width is a more difficult problem than path-width. Specifically, if we could compute $\mathrm{aw}(G)$ for all G, then we could also compute $\mathrm{aw}(G \amalg G)$, and hence $\mathrm{pw}^*(G)$. It was shown in [1] that computing path-width is $\mathcal{NP}$-complete. Therefore, it follows that computing arc-width is $\mathcal{NP}$-hard. It is then easy to show the following.

Corollary 5.4. *Computing arc-width is $\mathcal{NP}$-complete.*

It is also worth mentioning that Corollary 5.3 implies $\mathrm{aw}(G \amalg G) = \mathrm{pw}^*(G \amalg G)$, thereby achieving the upper bound on arc-width given by Corollary 3.4. Barát and Hajnal [2] ask for a complete description of the set S of graphs G with the property that $\mathrm{aw}(G) = \mathrm{pw}^*(G)$. They show that any tree is in S, and our argument from Corollary 5.3 can be used to show that a very large family of disconnected graphs is in S. Furthermore, our next theorem can be used to show that a very large family of 1-connected graphs is also in S. All of this suggests that S is very large, and hence, that finding a complete description of it would be a difficult task indeed.

We now move on to consider the wedge sum of a number of graphs. Recall that a graph G is *vertex-transitive* if for any vertices u and v in G, there exists a graph automorphism taking u to v.

Theorem 5.5. *Let $G_1, G_2, \ldots, G_n$ be connected, vertex-transitive graphs, each with at least 2 vertices. Order them in such a way that*

$$\mathrm{aw}_0(G_1) \geq \mathrm{aw}_0(G_2) \geq \cdots \geq \mathrm{aw}_0(G_n).$$

Let $G = G_1 \vee G_2 \vee \cdots \vee G_n$ denote the graph obtained from $G_1 \amalg G_2 \amalg \cdots \amalg G_n$ by identifying one vertex v_i in each G_i to a single point, v. Let

$$z = \Big|\{i \mid \mathrm{aw}_0(G_i - v_i) = \mathrm{aw}_0(G_1)\}\Big|.$$

Then, if $z \leq 2$,

$$\mathrm{aw}_m(G) = \mathrm{aw}_m(G_1 \amalg (G_2 - v_2) \amalg (G_3 - v_3) \amalg \cdots \amalg (G_n - v_n)),$$

and if $z \geq 3$,

$$\mathrm{aw}_m(G) = \mathrm{aw}_m(G_1 \amalg (G_2 - v_2) \amalg (G_3 - v_3) \amalg \cdots \amalg (G_n - v_n)) + 1.$$

Proof. We consider 2 cases, based on the value of z.

Case 1: $z \leq 2$
Let $k = \mathrm{aw}_0(G_1)$. We first show that $\mathrm{aw}_m(G) \leq k$.

To do this, we construct an arc-representation ϕ of G. Since $z \leq 2$, we can choose p and q so that $\mathrm{aw}_0(G_i - v_i) < k$ for $i \neq p, q$. We then define ϕ_i as follows:

1. For $i = p$ and for $i = q$, take ϕ_i to be an arc-representation of G_i with $w(\phi_i) = 0$ and $W(\phi_i) = \mathrm{aw}_0(G_i) \leq k$.

2. For $i \neq p, q$, take ϕ_i to be an arc representation of $G_i - v_i$ with $w(\phi_i) = 0$ and $W(\phi_i) = \mathrm{aw}_0(G_i - v_i) < k$.

3. By rotating and rescaling ϕ_i appropriately, we can assume that arcs in $A(\phi_i)$ and arcs in $A(\phi_j)$ are disjoint for $i \neq j$ and that the arcs in $A(\phi_p)$ immediately follow the arcs in $A(\phi_q)$ in a clockwise orientation.

4. Since G_p and G_q are vertex-transitive, we can also assume that $\phi_p(v_p)$ extends the furthest in the clockwise direction of all arcs in $A(\phi_p)$, and that $\phi_q(v_q)$ extends the furthest in the counter-clockwise direction of all arcs in $A(\phi_q)$.

We can now define ϕ as follows:

1. For $x \in G_i - v_i$, define $\phi(x) = \phi_i(x)$.

2. For $x = v$, define $\phi(x)$ to be the arc with counter-clockwise endpoint $l(\phi_p(v_p))$ and clockwise endpoint $r(\phi_q(v_q))$. Call this arc I.

I contains $\phi_p(v_p), \phi_q(v_q)$, and all the arcs in ϕ_i for $i \neq p, q$, so ϕ does indeed give a valid arc-representation of G. Moreover, $w(\phi) = 0$ and

$$W(\phi) \leq \max\Big\{\mathrm{aw}_0(G_p), \mathrm{aw}_0(G_q), \mathrm{aw}_0(G_i - v_i) + 1 \text{ for } i \neq p, q\Big\}.$$

Since we chose p and q such that $\mathrm{aw}_0(G_i - v_i) < k$ for $i \neq p, q$, it follows that $W(\phi) \leq k$. Therefore, we conclude that $\mathrm{aw}_m(G) \leq k$, as desired.

Letting $G' = G_1 \amalg (G_2 - v_2) \amalg (G_3 - v_3) \amalg \cdots \amalg (G_n - v_n)$, we now show that $\mathrm{aw}_m(G) \leq \mathrm{aw}_m(G')$. Towards that end, consider an arc-representation ϕ of G' that minimizes $W(\phi)$. Then, ϕ induces an arc-representation ϕ_1 of G_1 by restriction.

1. Suppose $w(\phi_1) > 0$. By the proof of Theorem 5.1, we can assume ϕ_1 has some interval I contained in some arc J such that

   i. $w_{\phi_1}(x) = w(\phi_1)$ for all $x \in I$; and
   ii. ϕ maps all the vertices in $(G_2 - v_2) \amalg (G_3 - v_3) \amalg \cdots (G_n - v_n)$ to arcs contained in I, and hence, contained in J.

   Since G_1 is vertex-transitive, we can assume $\phi^{-1}(J) = v$. Then, ϕ is already a valid arc-representation of G. Thus, we have $\mathrm{aw}_m(G) \leq \mathrm{aw}_m(G')$, as required.

2. Suppose $w(\phi_1) = 0$. Then $W(\phi) \geq W(\phi_1) = k$. The desired result then follows from the fact that $\mathrm{aw}_m(G) \leq k$.

Thus, in all cases, we have $\mathrm{aw}_m(G) \leq \mathrm{aw}_m(G')$. On the other hand, we can obtain G from G' by inserting extra edges. Since adding edges to a graph clearly does not decrease arc-width, $\mathrm{aw}_m(G) \geq \mathrm{aw}_m(G')$. It follows that

$$\mathrm{aw}_m(G) = \mathrm{aw}_m(G'),$$

as desired.

Case 2: $z \geq 3$
Again, let $k = \mathrm{aw}_0(G_1)$. Since $z \geq 3$, we can choose distinct p, q and r such that $\mathrm{aw}_0(G_p - v_p) = \mathrm{aw}_0(G_q - v_q) = \mathrm{aw}_0(G_r - v_r) = k$. By Theorem 5.1, we also have $\mathrm{aw}_m(G') = k$. It follows that $\mathrm{aw}_m(G) \geq k$.

Now, consider an arc-representation ϕ of G, and suppose $W(\phi) = k$. This induces arc-representations ϕ_i of $G_i - v_i$ for all i. Suppose $w(\phi_p), w(\phi_q)$, and $w(\phi_r)$ are all non-zero. Following the proof of Theorem 5.1, we can decrease $W(\phi)$ by simultaneously decreasing $w(\phi_p), w(\phi_q)$, and $w(\phi_r)$. Therefore, ϕ is sub-optimal, so in particular, $W(\phi) > \mathrm{aw}_m(G) \geq k$, which contradicts the fact that $W(\phi) = k$. Thus, we may assume that $w(\phi_p) = 0$, and hence that $W(\phi_p) = k$. Since $W(\phi) \geq W(\phi_p) + w(\phi_q) + w(\phi_r)$, it follows that $w(\phi_q) = w(\phi_r) = 0$, and hence that $W(\phi_p) = W(\phi_q) = W(\phi_r) = k$. Therefore, there exist x_p, x_q, and x_r in S^1 for which $w_{\phi_p}(x_p) = w_{\phi_q}(x_q) = w_{\phi_r}(x_r) = k$.

Let $I = \phi(v)$. For each i, the graph G_i has at least one edge and is vertex-transitive, so there is an edge between v_i and $G_i - v_i$. Thus, I must intersect at least one arc in each of $A(\phi_p)$, $A(\phi_q)$ and $A(\phi_r)$. Let y_p, y_q, and y_r be points where I intersects an arc in $A(\phi_p)$, $A(\phi_q)$ and $A(\phi_r)$ respectively.

Divide S^1 into 3 disjoint arcs: A_r between x_p and x_q, A_p between x_q and x_r, and A_q between x_r and x_p. Since $W(\phi) = k$, the arc I cannot contain x_p, x_q, or x_r. Thus, we can assume without loss of generality that I is entirely contained within A_r. Since I contains y_r, it follows that y_r is also in A_r. Since G_r is connected, but x_r is not in A_r, there must then exist some arc in $A(\phi_r)$ containing either x_p or x_q. This implies, however, that $W(\phi) > k$, giving a contradiction. Thus, we deduce

$$\mathrm{aw}_m(G) \geq \mathrm{aw}_m(G') + 1.$$

However, G can be obtained by adding a single vertex to G'. One can easily check that this implies

$$\mathrm{aw}_m(G) \leq \mathrm{aw}_m(G') + 1.$$

The result now follows. □

6 Concluding Remarks

Using the notion of restricted arc-width, we were able to determine a number of general bounds on arc-width, and we were also able to precisely describe the effect of disjoint unions, wedge sums, cones, and double-cones on arc-width. Moreover, we used these same techniques to show that computing arc-width is $\mathcal{NP}$-complete.

In doing all of this, several natural questions arose. For example, one might ask:

Question 1. *Are there any other graph operations whose effect on arc-width we can describe?*

While looking at cones, however, we saw that sometimes even restricted arc-width does not encapsulate all the information that we require. Just as we generalized arc-width to obtain extra information, we might have to further generalize restricted arc-width for the same purpose. Thus, we ask:

Question 2. *Is there any generalization of restricted arc-width that will prove better suited to describing how graph operations affect arc-width?*

Recall that Proposition 3.3 guarantees that for a fixed graph G, the sequence $\{aw_i(G)\}$ is non-increasing and decreases by at most 1 each step. We ask whether this result can be strengthened:

Question 3. *What sequences of non-negative integers can arise as the restricted arc-width sequence,* $\{\mathrm{aw}_i(G)\}$*, for a fixed graph* G*?*

Further results in this direction might help to simplify or expand our work on disjoint unions and on cones.

Finally, we know from Theorem 4.1 that restricted arc-width is minor-monotone. Thus, if $X_{i,j}$ denotes the collection of graphs G for which $\mathrm{aw}_i(G) \leq j$, we say $X_{i,j}$ is minor-closed, meaning that if $H \in X_{i,j}$, then every minor of H is in $X_{i,j}$. We can thus describe $X_{i,j}$ by the collection of graphs that cannot arise as minors in $X_{i,j}$. Now, there is a theorem of Robertson and Seymour [4] that states:

Theorem 6.1. *A non-empty minor-closed family of graphs can be characterized by a finite set of excluded minors.*

It is then natural to ask:

Question 4. *What excluded minors characterize* $X_{i,j}$ *for each* i, j*?*

This question is probably very difficult, but even partial results could prove very useful.

For more open problems, we refer the reader to [2].

7 Acknowledgements

The research included in this paper was done at the University of Minnesota Duluth as part of the summer Research Experience for Undergraduates directed by Joseph A. Gallian. I would like to thank Professor Gallian for his support, and I would also like to thank Philip Matchett, Geir Helleloid, and my fellow students for the helpful suggestions they made throughout the summer. Funding for the Duluth REU is provided by NSF grant DMS-0137611 and NSA grant MDA904-02-1-0060.

References

[1] S. Arnborg, D. G. Corneil, and A. Proskurowski, Complexity of finding embeddings in a k-tree, *SIAM J. Alg. Disc. Meth.*, **8** (1987), 277 - 284.

[2] J. Barát and P. Hajnal, The arc-width of a graph, *Electron. J. Combin.*, **8** (2001), #R34.

[3] N. Robertson and P. D. Seymour, Graph Minors I. Excluding a forest, *J. Combin. Theory Ser. B.*, **35** (1983), 39 - 61.

[4] N. Robertson and P. D. Seymour, Graph Minors XX. Wagner's Conjecture, manuscript (1988).