On a two-sided Turán problem

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Abstract

Given positive integers n, k, t, with $2 \leq k \leq n$, and $t < 2^k$, let $m(n, k, t)$ be the minimum size of a family $\mathcal{F}$ of nonempty subsets of $[n]$ such that every k-set in $[n]$ contains at least t sets from $\mathcal{F}$, and every $(k-1)$-set in $[n]$ contains at most $t-1$ sets from $\mathcal{F}$. Sloan et al. determined $m(n, 3, 2)$ and Füredi et al. studied $m(n, 4, t)$ for $t = 2, 3$. We consider $m(n, 3, t)$ and $m(n, 4, t)$ for all the remaining values of t and obtain their exact values except for $k = 4$ and $t = 6, 7, 11, 12$. For example, we prove that $m(n, 4, 5) = \binom{n}{2} - 17$ for $n \geq 160$. The values of $m(n, 4, t)$ for $t = 7, 11, 12$ are determined in terms of well-known (and open) Turán problems for graphs and hypergraphs. We also obtain bounds of $m(n, 4, 6)$ that differ by absolute constants.

1 Introduction

We consider an extremal problem for set systems. Given integers n, k, t, with $2 \leq k \leq n$, and $t < 2^k$, a family $\mathcal{F} \subset 2^{[n]} \setminus \emptyset$ is a (k, t)-*system* of $[n]$ if every k-set in $[n]$ contains at least t sets from $\mathcal{F}$, and every $(k-1)$-set in $[n]$ contains at most $t-1$ sets from $\mathcal{F}$. Let $m(n, k, t)$ denote the minimum size of a (k, t)-system of $[n]$. This threshold function first arose in problems on computer science [10, 11] (although the notation $m(n, k, t)$ was not used until [6]). It was shown in [11] that $m(n, k, t) = \Theta(n^{k-1})$ for $1 < t < k$ and $m(n, 3, 2) = \binom{n-1}{2} + 1$. In [6], $m(n, 4, 3)$ was determined exactly for large n and it was shown that for fixed k, $m(n, k, 2) = (1 + o(1))T_{k-1}(n, k, 2)$, where $T_r(n, k, t)$ is the generalized Turán number. For fixed k and $t < 2^k$, the order of magnitude of $m(n, k, t)$

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was determined in [9]. A special case of this result is the following proposition, where $\binom{a}{\leq b} = \sum_{i=1}^{b} \binom{a}{i}$.

Proposition 1. **[9]** *$m(n,k,1) = \binom{n}{k}$, $m(n,k,k) = n$, $m(n,k,2^k-2) = \binom{n}{\leq k-1}$ and $m(n,k,2^k-1) = \binom{n}{\leq k}$.*

In this paper we study $m(n,k,t)$ for $k = 3, 4$. The case $k = 3$ is not very difficult: Proposition 1 determines $m(n,3,t)$ for $t \in \{1,3,6,7\}$ and [11] shows that $m(n,3,2) = \binom{n-1}{2} + 1$. The remaining cases $t = 4$ and $t = 5$ are covered below.

Proposition 2.

$$m(n,3,t) = \begin{cases} n + \binom{n}{2} - \lfloor n^2/4 \rfloor & t = 4, \\ n + \binom{n}{2} - \lfloor n/2 \rfloor & t = 5. \end{cases}$$

The main part of this paper is devoted to $m(n,4,t)$, a problem which is substantially more difficult than the case $k = 3$. As mentioned above, both $m(n,4,2)$ and $m(n,4,3)$ were studied in [6]. It was shown in [11] how these two functions apply to frequent sets of Boolean matrices, a concept used in knowledge discovery and data mining. Perhaps the determination of $m(n,4,t)$ for other t will have similar applications.

The cases $t = 1, 4, 14, 15$ are answered by Proposition 1 immediately. In this paper we obtain the exact values of $m(n,4,t)$ for $t = 5, 8, 9, 10, 13$. Our bounds for $m(n,4,6)$ differ only by an absolute constant. For $t = 7, 11, 12$, we determine $m(n,4,t)$ exactly in terms of well-known (and open) Turán problems in extremal graph and hypergraph theory. Perhaps this connection provides additional motivation for investigating $m(n,k,t)$ (the first connection between $m(n,k,t)$ and Turán numbers was shown in [6] via $m(n,k,2) = (1+o(1))T_{k-1}(n,k,2)$).

For a family of r-uniform hypergraphs $\mathcal{H}$, let $\mathrm{ex}(n,\mathcal{H})$ be the maximum number of edges in an n vertex r-uniform hypergraph $\mathcal{G}$ containing no member of $\mathcal{H}$ as a subhypergraph. The (2-uniform) cycle of length l is written C_l. The complete 3-uniform hypergraph on four points is $K_4^{(3)}$, and the 3-uniform hypergraph on four points with three edges is $H(4,3)$. An $(n,3,2)$-packing is a 3-uniform hypergraph on n vertices such that every pair of vertices is contained in at most one edge. The packing number $P(n,3,2)$ is the size of a maximal $(n,3,2)$-packing. Note that the maximal packing is a Steiner system when $n \equiv 1$ or $3 \pmod 6$.

Theorem 3 (Main Theorem).

$$m(n,4,5) = \binom{n}{2} - 17,$$

when $n \geq 160$ and

$$\binom{n}{2} - 190 < m(n,4,6) \leq \binom{n}{2} - 5,$$

when $n \geq 8$. Furthermore,

$$\begin{aligned}
m(n,4,7) &= n + \binom{n}{2} - ex(n,\{C_3,C_4\}),\\
m(n,4,8) &= n + \binom{n}{2} - 2n/3,\\
m(n,4,9) &= n + \binom{n}{2} - 1,\\
m(n,4,10) &= n + \binom{n}{2},\\
m(n,4,11) &= n + \binom{n}{2} + \binom{n}{3} - ex(n,K_4^{(3)}),\\
m(n,4,12) &= n + \binom{n}{2} + \binom{n}{3} - ex(n,H(4,3)),\\
m(n,4,13) &= n + \binom{n}{2} + \binom{n}{3} - P(n,3,2).
\end{aligned}$$

It is worth recalling the known results for the three Turán numbers and the packing number $P(n,3,2)$ in Theorem 3 above.

- It is known that $(\frac{1}{2\sqrt{2}}+o(1))n^{3/2} \leq \mathrm{ex}(n,\{C_3,C_4\}) \leq (\frac{1}{2}+o(1))n^{3/2}$ (Erdős-Rényi [3], Kővari-Sós-Turán [7]). Erdős and Simonovits [4] conjectured that $ex(n,\{C_3,C_4\}) = (\frac{1}{2\sqrt{2}}+o(1))n^{3/2}$.

- It is known that $(5/9)\binom{n}{3} \leq \mathrm{ex}(n,K_4^{(3)}) \leq (0.592+o(1))\binom{n}{3}$ (Turán [14], Chung-Lu [2]). It was conjectured [14] that the lower bound is correct (Erdős offered \$1000 for a proof).

- It is known $(2/7+o(1))\binom{n}{3} \leq \mathrm{ex}(n,H(4,3)) \leq (1/3-10^{-6}+o(1))\binom{n}{3}$ (Frankl-Füredi [5], Mubayi [8]). It was conjectured [8] that $\mathrm{ex}(n,H(4,3)) = (2/7+o(1))\binom{n}{3}$.

- Spencer [12] determine $P(n,3,2)$ exactly:

$$P(n,3,2) = \begin{cases} \lfloor \frac{n}{3}\lfloor\frac{n-1}{2}\rfloor\rfloor - 1 & \text{if } n \equiv 5 \pmod 6,\\ \lfloor \frac{n}{3}\lfloor\frac{n-1}{2}\rfloor\rfloor & \text{otherwise.}\end{cases}$$

This paper is organized as follows. In Section 2 we describe the main idea in the proofs and prove Proposition 2. The Main Theorem (Theorem 3) is proved in Section 3.

Most of our notations are standard: $[n] = \{1,2,\ldots,n\}$. For a set system $\mathcal{F}$, let $\mathcal{F}_t$ denote the family of t-sets in $\mathcal{F}$, let $\mathcal{F}_{\leq t} = \cup_{i\leq t}\mathcal{F}_i$ and $\mathcal{F}_{\geq t} = \cup_{i\geq t}\mathcal{F}_i$. If $a \in \mathcal{F}$ and $b \notin \mathcal{F}$, we simply write $\mathcal{F} - a$ for $\mathcal{F} \setminus \{a\}$ and $\mathcal{F} + b$ for $\mathcal{F} \cup \{b\}$. Given a set X and an integer a, let $2^X = \{S : S \subseteq X\}$, $\binom{X}{a} = \{S \subset X : |S| = a\}$, $\binom{X}{\leq a} = \{S \subset X : 1 \leq |S| \leq a\}$ and $\binom{X}{\geq a} = \{S \subset X : |S| \geq a\}$. We write $\mathcal{F}(X)$ for $\mathcal{F} \cap 2^X$. An r-graph on X is a (hyper)graph whose edges are r-subsets of X. All sets or subsets considered in this paper are nonempty unless specified differently.

2 Ideas in the proofs and $m(n,3,t)$

In this section we make some basic observations on $m(n,k,t)$ and prove Proposition 2.

Recall that a (k,t)-system $\mathcal{F} \subseteq 2^{[n]} \setminus \emptyset$ satisfies the following two conditions:

Property D (DENSE): Every k-set in $[n]$ contains at least t sets from $\mathcal{F}$,
Property S (SPARSE): Every $(k-1)$-set in $[n]$ contains at most $t-1$ sets from $\mathcal{F}$.

The main idea in our proofs is to work with optimal (k,t)-systems which are defined as follows.

Definition 4. *Suppose that $\mathcal{F}$ is a (k,t)-system of $[n]$. We say that $\mathcal{F}$ is optimal if $|\mathcal{F}| = m(n,k,t)$ and $\sum_{S\in\mathcal{F}} |S|$ is minimal among all (k,t)-system of $[n]$ with size $m(n,k,t)$.*

The advantage of considering optimal (k,t)-systems $\mathcal{F}$ is that it allows us to assume certain structure on $\mathcal{F}$: if $\mathcal{F}$ does not have such a structure, we always modify $\mathcal{F}$ to $\mathcal{F}'$ such that $\mathcal{F}'$ is a (k,t)-system with $\sum_{S\in\mathcal{F}'} |S| < \sum_{S\in\mathcal{F}} |S|$, a contradiction to the optimality of $\mathcal{F}$. A typical modification of $\mathcal{F}$ is replacing a set in $\mathcal{F}$ by one of its subsets. Because the new system still satisfies Property **D**, we only need to check Property **S** in this case.

For example, if $\mathcal{F}$ is an optimal (k,t)-system for $t \geq 2^{k-1}$, then we may assume that

$$A \in \mathcal{F} \Rightarrow \left(2^A \setminus \emptyset\right) \subset \mathcal{F}. \qquad (\star)$$

Indeed, if $A \in \mathcal{F}$ has a nonempty subset $B \notin \mathcal{F}$, then $\mathcal{F}' = \mathcal{F} - A + B$ is also a (k,t)-system, because Property **S** holds trivially (any $(k-1)$-set of $[n]$ has at most $2^{k-1} - 1 \leq t-1$ nonempty subsets). Since $\sum_{S\in\mathcal{F}'} |S| < \sum_{S\in\mathcal{F}} |S|$, this contradicts the optimality of $\mathcal{F}$.

Now we consider $m(n,3,t)$ for $3 \leq t \leq 7$. Applying Proposition 1 directly, we have $m(n,3,3) = n$, $m(n,3,6) = \binom{n}{\leq 2}$ and $m(n,3,7) = \binom{n}{\leq 3}$.

Proof of Proposition 2. We determine $m(n,3,t)$ exactly for $t = 4,5$. Recall that $\mathcal{F}(S) = \mathcal{F} \cap 2^S$ for a set system $\mathcal{F}$ and a set S.

Let $\mathcal{F}$ be an optimal $(3,t)$-system with $4 \leq t \leq 5$. Since $t \geq 4 \geq 2^2$, we may assume that $(\star)$ holds in $\mathcal{F}$. First, we claim that $\binom{[n]}{1} \subset \mathcal{F}$. Suppose instead, that there exists some $a \in [n]$ such that $\{a\} \notin \mathcal{F}$. Pick a 3-set $T = \{a,b,c\}$. Since $\{a\} \notin \mathcal{F}$, by $(\star)$, we know that $\mathcal{F}$ does not contain $\{a,b\}, \{a,c\}$ and T as well. Thus $|\mathcal{F}(T)| \leq 3$, a contradiction to Property **D**. Second, we claim that $\mathcal{F} \subset \binom{[n]}{\leq 2}$. Suppose instead, that there exists a set $T \in \mathcal{F}_3$. Then $|\mathcal{F}(T)| = 7$ by $(\star)$ and consequently $\mathcal{F}' = \mathcal{F} - T$ is a $(3,t)$-system of cardinality $|\mathcal{F}| - 1$, contradicting the optimality of $\mathcal{F}$.

When $t = 4$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on n vertices in which every set of 3 vertices has at least one edge, *i.e.*, $\overline{\mathcal{F}_2}$, the complement of $\mathcal{F}_2$ is a K_3-free graph. Thus $|\mathcal{F}_2| \geq \binom{n}{2} - ex(n, K_3) = \binom{n}{2} - \lfloor n^2/4 \rfloor$. Consequently $m(n,3,4) = n + |\mathcal{F}_2| \geq n + \binom{n}{2} - \lfloor n^2/4 \rfloor$. On the other hand, $\binom{[n]}{1} \cup E(G)$ is a $(3,4)$-system, where G is a complete bipartite graph with two color classes of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Consequently $m(n,3,4) = n + \binom{n}{2} - \lfloor n^2/4 \rfloor$.

When $t = 5$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on n vertices in which every 3 vertices have at least two edges. Therefore $\overline{\mathcal{F}_2}$ is a matching M and $|\mathcal{F}_2| = \binom{n}{2} - |M| \geq$

$\binom{n}{2} - \lfloor n/2 \rfloor$. Consequently $m(n,3,5) \geq n + \binom{n}{2} - \lfloor n/2 \rfloor$ and equality holds for the $(3,5)$-system $\mathcal{F} = \binom{[n]}{1} \cup E(G)$, where G is a complete graph except for a matching of size $\lfloor n/2 \rfloor$. $\square$

3 The values of $m(n,4,t)$

Applying Proposition 1, we obtain that $m(n,4,1) = \binom{n}{4}$, $m(n,4,4) = n$, $m(n,4,14) = \binom{n}{\leq 3}$ and $m(n,4,15) = \binom{n}{\leq 4}$. In this section we prove Theorem 3, *i.e.*, determine $m(n,4,t)$ for $5 \leq t \leq 13$. We consider the cases $7 \leq t \leq 13$ in Section 3.1. The more difficult cases $t = 5, 6$ are studied in Section 3.2 and 3.3, respectively.

3.1 The cases $7 \leq t \leq 13$

Our proof is facilitated by the following four lemmas, whose proofs are postponed to the end of this section.

In Lemmas 5 - 8, $\mathcal{F}$ is an optimal $(4,t)$-system.

Lemma 5. *If* $2 \leq t \leq 14$*, then* $\mathcal{F}_4 = \emptyset$*.*

Lemma 6. *If* $7 \leq t \leq 10$*, then* $\mathcal{F}_3 = \emptyset$*.*

Lemma 7. *If* $7 \leq t \leq 14$*, then* $\binom{[n]}{1} \subset \mathcal{F}$*.*

Lemma 8. *If* $11 \leq t \leq 14$*, then* $\binom{[n]}{2} \subset \mathcal{F}$*.*

Proof of Theorem 3 for $7 \leq t \leq 13$:
By Lemmas 5, 6 and 7, we conclude that

$$\binom{[n]}{1} \subset \mathcal{F} \subset \binom{[n]}{\leq 2} \qquad \text{for } 7 \leq t \leq 10.$$

Clearly, when $t = 10$, $\mathcal{F} = \binom{[n]}{\leq 2}$ and consequently $m(n,4,10) = \binom{n}{\leq 2}$.

When $t = 9$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on $[n]$ in which every 4-set has at least 5 edges. Then there is at most one edge absent from $\mathcal{F}_2$, or $|\mathcal{F}_2| \geq \binom{n}{2} - 1$. Consequently $m(n,4,9) \geq n + \binom{n}{2} - 1$ and equality holds when $\mathcal{F} = \binom{[n]}{\leq 2} \setminus e$ for some $e \in \binom{[n]}{2}$.

When $t = 8$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on $[n]$ in which every 4-set has at least 4 edges. Therefore, $\overline{\mathcal{F}_2}$ contains no K_3, S_3 (a star with 3 leaves), or P_3 (a path of length 3). Thus all connected components of $\overline{\mathcal{F}_2}$ have size at most 3 and each component is either an edge or P_2. So $|\overline{\mathcal{F}_2}| \leq \lfloor 2n/3 \rfloor$ and $|\mathcal{F}| \geq n + \binom{n}{2} - \lfloor 2n/3 \rfloor$. Consequently $m(n,4,8) = n + \binom{n}{2} - \lfloor 2n/3 \rfloor$ and the optimal system is $\binom{[n]}{\leq 2} \setminus E(G)$, where G is the union of disjoint copies of P_2 and P_1 covering $[n]$ with maximum copies of P_2.

When $t = 7$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on $[n]$ in which every 4-set has at least 3 edges. Let G be a graph on $[n]$ with $E(G) = \overline{\mathcal{F}_2}$. Then G contains no copies of C_4 or C_3^+ (C_3 plus an edge). If C_3 is also absent in G, then $e(G) \leq ex(n, \{C_3, C_4\})$. Otherwise, assume that G contains $t(\geq 1)$ copies of C_3 on a vertex-set T. Because G is C_3^+-free, the copies of C_3 must be vertex-disjoint and

$$e(G) = 3t + e(G \setminus T) \leq 3t + ex(n - 3t, \{C_3, C_4\}) \leq ex(n, \{C_3, C_4\}),$$

where the last inequality is an easy exercise. Consequently $m(n, 4, 7) \geq n + \binom{n}{2} - ex(n, \{C_3, C_4\})$ and equality holds when $\mathcal{F} = \binom{[n]}{\leq 2} \setminus E(G)$, where G is an extremal graph without C_3 or C_4.

By Lemma 5, 7 and 8, we conclude that

$$\binom{[n]}{\leq 2} \subset \mathcal{F} \subset \binom{[n]}{\leq 3} \quad \text{for } 11 \leq t \leq 13.$$

When $t = 11$, $\mathcal{F}_3 = \mathcal{F} \setminus \binom{[n]}{\leq 2}$ is the edge set of a 3-graph in which every 4-set has at least one hyper-edge. In other words, the 3-graph $([n], \overline{\mathcal{F}_3})$ contains no $K_4^{(3)}$ and therefore $|\overline{\mathcal{F}_3}| \leq ex(n, K_4^{(3)})$. Consequently $m(n, 4, 11) \geq \binom{n}{\leq 3} - ex(n, K_4^{(3)})$ and equality holds when $\mathcal{F} = \binom{[n]}{\leq 3} \setminus \mathcal{H}$, where $\mathcal{H}$ is the edge set of an extremal 3-graph without $K_4^{(3)}$.

By a similar argument, we obtain that $m(n, 4, 12) \geq \binom{n}{\leq 3} - ex(n, H(4, 3))$ and equality holds when $\mathcal{F} = \binom{[n]}{\leq 3} \setminus \mathcal{H}$, where $\mathcal{H}$ is the edge set of an extremal 3-graph without $H(4, 3)$.

Finally, when $t = 13$, $\overline{\mathcal{F}_3}$ is an $(n, 3, 2)$-packing since every 4-set of $[n]$ contains at most one hyper-edge of $\overline{\mathcal{F}_3}$. Since $|\overline{\mathcal{F}_3}| \leq P(n, 3, 2)$, we have $m(n, 4, 13) \geq \binom{n}{\leq 3} - P(n, 3, 2)$ and equality holds when $\overline{\mathcal{F}_3}$ is a maximal $(n, 3, 2)$-packing. $\square$

Before verifying Lemma 5, we start with a technical lemma, which is very useful in the cases $5 \leq t \leq 7$.

Lemma 9. *Suppose that $t \in \{5, 6, 7\}$ and $\mathcal{F}$ is an optimal $(4, t)$-system. Fix a set $P \in \binom{[n]}{\leq 2} \setminus \mathcal{F}$ and let*

$$\mathcal{T} = \{T \in \binom{[n]}{3} : T \supset P, |\mathcal{F}(T)| = t - 1\}. \tag{1}$$

If $\mathcal{T} \subset \mathcal{F}$, then $T \notin \mathcal{F}$ for every 3-set $T \supset P$.

Proof. Suppose instead, that there exists a 3-set $T_0 \supset P$ and $T_0 \in \mathcal{F}$. If $\mathcal{T} = \emptyset$, then let $\mathcal{F}' = \mathcal{F} - T_0 + P$. It is clear that $\mathcal{F}'$ satisfies Property **D**. $\mathcal{F}'$ also satisfies Property **S** because $|\mathcal{F}'(Y)| = |\mathcal{F}(Y)| + 1 \leq t - 2 + 1 = t - 1$ for every 3-set $Y \supset P$. Therefore $\mathcal{F}'$ is a $(4, t)$-system, a contradiction to the optimality of $\mathcal{F}$.

Now assume that $\mathcal{T} \neq \emptyset$. We claim that $\mathcal{F}' = \mathcal{F} - \mathcal{T} + P$ is a $(4, t)$-system, contradicting the optimality of $\mathcal{F}$. To check Property **D**, we only need to consider those 4-sets S which contain two members T_1, T_2 of $\mathcal{T}$ (because $|\mathcal{F}'(Q)| = |\mathcal{F}(Q)|$ for every 4-set Q that contains at most one member of $\mathcal{T}$). Since $|\mathcal{F}(S)| \geq |\mathcal{F}(T_1)| + |\mathcal{F}(T_2)| - |\mathcal{F}(P)| \geq 2(t-1) - 2 = 2t - 4 \geq t + 1$ (using the assumption that $t \geq 5$), we have $|\mathcal{F}'(S)| \geq t + 1 - 2 + 1 = t$. On the other hand, $\mathcal{F}'$ also satisfies Property **S** since for every 3-set $Y \supset P$, $|\mathcal{F}'(Y)| = |\mathcal{F}(Y)| = t - 1$ if $Y \in \mathcal{T}(\subset \mathcal{F})$, otherwise $|\mathcal{F}'(Y)| = |\mathcal{F}(Y)| + 1 \leq t - 2 + 1 = t - 1$. $\square$

Proof of Lemma 5. We are to show that $\mathcal{F}_4 = \emptyset$ for $2 \leq t \leq 14$.

When $8 \leq t \leq 14$, $(\star)$ holds in $\mathcal{F}$ (since $t \geq 2^3$). We may thus assume that $\mathcal{F}$ contains no 4-set, otherwise removing these 4-sets results in a smaller $(4, t)$-system, a contradiction to the optimality of of $\mathcal{F}$.

Let $2 \leq t \leq 7$. Suppose to the contrary, that there exists a set $S \in \mathcal{F}_4$. We may assume that $|\mathcal{F}(S)| = t$, otherwise S could be removed from $\mathcal{F}$. Let $\mathcal{T} = \binom{S}{3} \setminus \mathcal{F}$.

Case 1. $\mathcal{T} \neq \emptyset$.

Suppose that $T_0 \in \mathcal{T}$ has the minimal value of $|\mathcal{F}(T)|$ among all $T \in \mathcal{T}$. We claim that $|\mathcal{F}(T_0)| \leq t - 2$. Suppose instead, that $|\mathcal{F}(T_0)| \geq t - 1$. If $|\mathcal{T}| < 4$, then there exists $T_1 \in \binom{S}{3} \cap \mathcal{F}$. Because $T_1, S \in \mathcal{F}$, we have $|\mathcal{F}(S)| \geq |\mathcal{F}(T_0)| + 2 \geq t - 1 + 2 > t$, a contradiction to the assumption that $|\mathcal{F}(S)| = t$. If $|\mathcal{T}| = 4$, then for every $T \in \binom{S}{3}$, we have $|\mathcal{F}(T)| \geq t - 1$ and $T \notin \mathcal{F}$. Since $|\cup_{T \in \binom{S}{3}} \mathcal{F}(T)| = |\mathcal{F}(S) \setminus S| = t - 1$, we have $\mathcal{F}(T_1) = \mathcal{F}(T_2) \neq \emptyset$ for every $T_1, T_2 \in \binom{S}{3}$. But this is impossible because $\cap_{i=1}^4 T_i = \emptyset$. Now let $\mathcal{F}' = \mathcal{F} - S + T_0$. Trivially $\mathcal{F}'$ satisfies Property **D** and because $|\mathcal{F}'(T_0)| \leq t-1$, $\mathcal{F}'$ satisfies Property **S** as well. Thus $\mathcal{F}'$ is a $(4, t)$-system, a contradiction to the optimality of $\mathcal{F}$.

Case 2. $\mathcal{T} = \emptyset$, *i.e.*, $\binom{S}{3} \subset \mathcal{F}$.

Note that this case does not exist for $t = 2, 3, 4$, because it implies that $|\mathcal{F}(S)| \geq 4 + 1$, a contradiction to the assumption $|\mathcal{F}(S)| = t$.

When $t = 5$, we know that $\mathcal{F}(S) = \{S\} \cup \binom{S}{3}$. Pick any two elements $a, b \in S$ and consider $\mathcal{T} = \{\{a, b, c\} : |\mathcal{F}(\{a, b, c\})| = 4\}$. Since $\mathcal{F}(\{a, b\}) = \emptyset$, it must be the case that $\mathcal{F}(T) = \{\{c\}, \{c, a\}, \{c, b\}, \{c, a, b\}\}$ for every $T = \{a, b, c\} \in \mathcal{T}$. In particular, $\mathcal{T} \subset \mathcal{F}$. We may therefore apply Lemma 9 to conclude that $T \notin \mathcal{F}$ for every 3-set $T \subset \{a, b\}$. This is a contradiction to the assumption that $T \in \mathcal{F}$ for all $T \in \binom{S}{3}$.

When $t = 6, 7$, since $|\mathcal{F}(S)| \leq 7$ and $\binom{S}{\geq 3} \subset \mathcal{F}$, we have $|\mathcal{F} \cap \binom{S}{\leq 2}| \leq 2$. Consequently there exist $a, b \in S$ such that $\mathcal{F}(\{a, b\}) = \emptyset$. Since $\mathcal{T} = \{\{a, b, c\} : |\mathcal{F}(\{a, b, c\})| = t\} = \emptyset$, we may again apply Lemma 9 and derive a contradiction as in the previous paragraph. $\square$

Proof of Lemma 6. We are to show that $\mathcal{F}_3 = \emptyset$ for $7 \leq t \leq 10$. Suppose to the contrary, that there exists a set $T \in \mathcal{F}_3$. We now separate the case $t = 7$ and the cases $t = 8, 9, 10$.

Case 1. $t = 7$.

Since $|\mathcal{F}(T)| < 7$ (by Property **S**), there exists a set $P \in \binom{T}{\leq 2} \setminus \mathcal{F}$. Define $\mathcal{T}$ as in (1), trivially $\mathcal{T} \subset \mathcal{F}$. We may apply Lemma 9 to conclude that $T \notin \mathcal{F}$, a contradiction.

Case 2. $t = 8, 9, 10$.

Since $t \geq 2^3$, we may assume that $(\star)$ holds in $\mathcal{F}$. In particular, if $T \in \mathcal{F}_3$, then $|\mathcal{F}(T)| = 7$. Let $\mathcal{D} = \{S \in \binom{[n]}{4} : S \supset T, |\mathcal{F}(S)| = t\}$. If $\mathcal{D} = \emptyset$, then $\mathcal{F}' = \mathcal{F} - T$ satisfies Property **D** and is thus a $(4,t)$-system of size $|\mathcal{F}| - 1$, a contradiction. Now suppose that $|\mathcal{D}| = 1$ and $\{a\} \cup T$ is the only element of $\mathcal{D}$. Since $t < 11$, at least one of $\{a\}, \{a,b\}, \{a,c\}, \{a,d\}$, say $\{a\}$, is not contained in $\mathcal{F}$. Let $\mathcal{F}' = \mathcal{F} - T + \{a\}$. $\mathcal{F}'$ satisfies Property **S** trivially. Consider a 4-set $S \supset T$ of $[n]$. If $S \neq \{a\} \cup T$, then $|\mathcal{F}(S)| \geq t+1$ and $|\mathcal{F}'(S)| \geq t$. If $S = \{a\} \cup T$, then $|\mathcal{F}'(S)| = |\mathcal{F}(S)| = t$. This means that $\mathcal{F}'$ satisfies Property **D** and consequently $\mathcal{F}'$ is a $(4,t)$-system, a contradiction.

Now we assume that there exist $a_1, a_2 \in [n]$ such that $\{a_i\} \cup T \in \mathcal{D}$ for $i = 1, 2$. We will show that when $8 \leq t \leq 10$, there are two vertices $v_1, v_2 \in T$ such that $|\mathcal{F}(\{a_1, a_2, v_1, v_2\})| < t$, contradicting Property **D**.

Define $\mathcal{F}_{\{a_i\}}(T) = \mathcal{F}(\{a_i\} \cup T) - \mathcal{F}(T)$ for $i = 1, 2$. Since $|\mathcal{F}(T)| = 7$, we have $|\mathcal{F}_{\{a_i\}}(T)| = 1, 2, 3$ for $t = 8, 9, 10$, respectively. Using $(\star)$, we thus know that $\{a_i\} \subseteq \mathcal{F}_{\{a_i\}}(T) \subset \mathcal{F}_{\leq 2}$ for every $t \in \{8, 9, 10\}$.

- When $t = 8$, we have $\mathcal{F}_{\{a_i\}}(T) = \{\{a_i\}\}$ for $i = 1, 2$. Thus $|\mathcal{F}(\{a_1, a_2, b, c\})| \leq 6 < 8$ for any $b \neq c \in T$.
- When $t = 9$, we have $\mathcal{F}_{\{a_1\}}(T) = \{\{a_1\}, \{a_1, c\}\}$ and $\mathcal{F}_{\{a_2\}}(T) = \{\{a_2\}, \{a_2, d\}\}$, for not necessarily distinct $c, d \in T$. Consequently $|\mathcal{F}(\{a_1, a_2, b, c\})| \leq 8 < 9$ for some $b \in T \setminus \{c, d\}$.
- When $t = 10$, we may assume that $\mathcal{F}_{\{a_1\}}(T) = \{\{a_1\}, \{a_1, b\}, \{a_1, d\}\}$ and $\mathcal{F}_{\{a_2\}}(T) = \{\{a_2\}, \{a_2, c\}, \{a_2, d\}\}$, where $c, b \in T$ are not necessarily distinct. If $c \neq b$, then $|\mathcal{F}(\{a_1, a_2, b, c\})| \leq 8 < 10$. Otherwise, $|\mathcal{F}(\{a_1, a_2, b, w\})| \leq 8 < 10$, where $w = T \setminus \{c, d\}$. $\square$

Proof of Lemma 7. Let $7 \leq t \leq 14$. We are to show that $\binom{[n]}{1} \subset \mathcal{F}$. Suppose instead, say, that $\{n\} \notin \mathcal{F}$.

For $t \geq 8$, consider a set $S \in \binom{[n]}{4}$ and $S \ni n$. We know that no set from $\mathcal{F}(S)$ contains n (otherwise $(\star)$ forces $\{n\} \in \mathcal{F}$). Thus $|\mathcal{F}(S)| \leq 7 < t$, a contradiction to Property **D**.

For $t = 7$, consider a set $T \in \binom{[n-1]}{3}$. By Property **S** and Property **D**, we have $|\mathcal{F}(T)| \leq 6$ and $|\mathcal{F}(\{n\} \cup T)| \geq 7$. Then there exists a set $P \in \mathcal{F}(\{n\} \cup T)$ such that $P \supset n$. Let $\mathcal{F}' = \mathcal{F} - P + \{n\}$. For any $Y \in \binom{[n]}{3}$ and $n \in T$, we have $|\mathcal{F}(Y)| \leq 5$ (because $\{n\}, Y \notin \mathcal{F}$). Therefore $\mathcal{F}'$ satisfies Property **S** and is thus a $(4,t)$-system, a contradiction. $\square$

Proof of Lemma 8. We are to show that $\binom{[n]}{2} \subset \mathcal{F}$ for $11 \leq t \leq 13$. Suppose to the contrary, that there exist $a, b \in [n]$ such that $\{a, b\} \notin \mathcal{F}$. Pick any two elements $v_1, v_2 \in [n] \setminus \{a, b\}$ and consider $D = \{a, b, v_1, v_2\}$. Since $(\star)$ holds, we have $\{a, b, v_1\}, \{a, b, v_2\} \notin \mathcal{F}$ (otherwise $\{a, b\} \in \mathcal{F}$). Together with $\{a, b\}$ and D, this gives us four members of $(2^D \setminus \emptyset) \setminus \mathcal{F}$. Consequently $|\mathcal{F}(D)| \leq 11$, which contradicts Property **D** when $t = 12, 13$. Now assume that $t = 11$. Then $|\mathcal{F}(D)| = 11$ and $|\mathcal{F}(\{a, v_1, v_2\})| = |\mathcal{F}(\{b, v_1, v_2\})| = 7$. Let $\mathcal{F}' = \mathcal{F} - \{a, v_1, v_2\} + \{a, b\}$. $\mathcal{F}'$ satisfies Property **S** trivially. To check Property **D**, we consider all the 4-sets S containing $\{a, v_1, v_2\}$. If $S = \{a, b, v_1, v_2\}$, then $|\mathcal{F}'(S)| = |\mathcal{F}(S)| > 11$. Otherwise, $S = \{a, v_1, v_2, v_3\}$ for some $v_3 \in [n] \setminus \{a, b, v_1, v_2\}$. Since $|\mathcal{F}(\{a, v_i, v_j\})| = 7$ for any $i \neq j$, only S and $\{v_1, v_2, v_3\}$ could be absent from $\mathcal{F}(S)$ and consequently $|\mathcal{F}(S)| \geq 13$. We thus have $|\mathcal{F}'(S)| = |\mathcal{F}(S)| - 1 \geq 13 - 1 > 11$. Therefore $\mathcal{F}'$ is a $(4, 11)$-system, a contradiction to the optimality of $\mathcal{F}$. □

3.2 $m(n, 4, 5)$

In this section we prove that $m(n, 4, 5) = \binom{n}{2} - 17$. Before the proof, we introduce the following extensions of the Turán number:

Definition 10. *A family $\mathcal{G} \in \binom{[n]}{i}$ is called a Turán-$_i(n, k, t)$-system if every k-set of $[n]$ contains at least t members of $\mathcal{G}$. The* generalized *Turán number $T_i(n, k, t)$ is defined as the minimum size of a Turán-$_i(n, k, t)$-system.*

Replacing all the instances of i by $\geq i$ in the previous paragraph, we obtain the non-uniform *Turán number $T_{\geq i}(n, k, t)$.*

In the proof we will consider $T_3(k, 4, 1) = \binom{k}{3} - \mathrm{ex}(k, K_4^{(3)})$. Turán [14] conjectured that $T_3(k, 4, 1)$ is achieved by the following 3-graph $\mathcal{H}_k$ (referred to as *Turán's 3-graph*). Partition $[k]$ into $A_1 \cup A_2 \cup A_3$, where $\lfloor k/3 \rfloor \leq |A_i| \leq \lceil k/3 \rceil$. The edges of $\mathcal{H}_k$ are 3-sets which are either contained in some A_i or contain two vertices of A_i and one of $A_{i+1 \pmod 3}$. It is known [13] that Turán's conjecture holds for $k \leq 13$. For larger k, the following lower bound of de Caen [1] suffices for our purpose:

$$T_3(k, 4, 1) \geq \frac{k(k-1)(k-3)}{18}. \tag{2}$$

We also need the following simple lemma on $T_{\geq 1}(n, k, t)$.

Lemma 11. [9] *$T_{\geq 1}(n, k, t) = n - k + t$ for $1 \leq t \leq k$.*

Let $\mathcal{F}$ be an optimal $(4, 5)$-system with $A = \{a : \{a\} \in \mathcal{F}\}$, $B = [n] - A$ and assume $|A| = k$. By Lemma 5, we may assume that $\mathcal{F}$ contains no 4-sets. In order to show that $|\mathcal{F}| \geq \binom{n}{2} - 17$, our proof consists of three stages described in Section 3.2.1 – 3.2.3. The proof leads to a construction achieving this bound, which we present in Section 3.2.3 as well.

3.2.1 Stage 1

We start with Claim 12 which reflects a rough picture of $\mathcal{F}$ and in turn implies a (weak) lower bound (4) for $|\mathcal{F}|$.

Given two disjoint sets $C, D \in [n]$, we write $\mathcal{F}(C, D) = \{S \in \mathcal{F} : S \cap C \neq \emptyset$ and $S \cap D \neq \emptyset\}$.

Claim 12. *1.* $(\mathcal{F}(A))_2$ *is a matching in* A.

2. $\overline{(\mathcal{F}(B))_2}$ *contains no matching of size 2 or star with 3 edges.*

3. $|\mathcal{F}(A,B)| \geq (n-k)(k-2) + |\mathcal{F}_{1,2}(A,B)|$, *where* $\mathcal{F}_{1,2}(A,B) = \{T \in \mathcal{F}_3 : |T \cap A| = 1, |T \cap B| = 2\}$.

4. $|(\mathcal{F}(A))_3| \geq k(k-2)(k-4)/24$.

Proof. Part 1: Property **S** prevents $\mathcal{F}(A)$ from containing two adjacent (graph) edges. Thus $(\mathcal{F}(A))_2$ is a matching.

Part 2: We first claim that

$$\text{If } P \in \binom{B}{2} \setminus \mathcal{F} \text{ and } P \subset T, |T| = 3, \text{ then } T \notin \mathcal{F}. \tag{3}$$

In fact, if Y is 3-set of $[n]$ such that $Y \supset P$ and $|\mathcal{F}(Y)| = 4$, then $Y \in \mathcal{F}$. We may therefore apply Lemma 9 to conclude that $T \notin \mathcal{F}$.

If there are $a, b, c, d \in B$ such that $\{a, b\}, \{c, d\} \notin \mathcal{F}$, then $(\mathcal{F}(\{a, b, c, d\}))_3 = \emptyset$ by (3). Consequently $|\mathcal{F}(\{a, b, c, d\})| \leq 4$, a contradiction to Property **D**. Therefore, $\overline{\mathcal{F}(B)}$ contains no two vertex-disjoint (graph) edges. A similar argument shows that $\overline{\mathcal{F}(B)}$ contains no star with 3 edges.

Part 3: Consider a vertex $b \in B$ and a 3-subset T of A. Since $\{b\} \notin \mathcal{F}$, $|\mathcal{F}(T)| \leq 4$ and $|\mathcal{F}(\{b\} \cup T)| \geq 5$, we have $|\mathcal{F}(\{b\}, T)| \geq 1$. Define $\mathcal{G}_b = \{Y \setminus \{b\} : Y \in \mathcal{F}(\{b\}, A)\}$ for every $b \in B$. Then $\mathcal{G}_b$ is a set system of $\binom{A}{\leq 2}$ such that every 3-set in A contains at least one member of $\mathcal{G}_a$, in other words, $\mathcal{G}_b$ is a Turán-$_{\geq 1}(k, 3, 1)$-system. By Lemma 11, we have $|\mathcal{H}_b| \geq T_{\geq 1}(k, 3, 1) = k - 2$. Repeating this for all $b \in B$, we have

$$|\{S \in \mathcal{F}(A, B) : |S \cap B| = 1\}| = \sum_{b \in B} |\mathcal{G}_b| \geq (n-k)(k-2).$$

Consequently $|\mathcal{F}(A,B)| \geq (n-k)(k-2) + |\mathcal{F}_{1,2}(A,B)|$.

Part 4. Now we give a crude lower bound for $(\mathcal{F}(A))_3$. From Part 1, we know that $(\mathcal{F}(A))_2$ is a matching $M = \{\{x_i, y_i\}\}_{i=1}^m$. Let

$$\mathcal{D} = \{S \in \binom{A}{4} : |S \cap \{x_i, y_i\}| \leq 1 \text{ for every } \{x_i, y_i\} \in M\}.$$

By Property **D**, every 4-set in $\mathcal{D}$ contains at least one member of $(\mathcal{F}(A))_3$. Since $\mathcal{D}$ is minimal when $m = \lfloor k/2 \rfloor$, we may assume that $m = \lfloor k/2 \rfloor$ when estimating $(\mathcal{F}(A))_3$ from below. The usual averaging arguments thus give the following lower bound (for even k, the case when k is odd yields an even larger bound):

$$(\mathcal{F}(A))_3 \geq \frac{|\mathcal{D}|}{k-6} = \frac{k(k-2)(k-4)(k-6)}{4!(k-6)} = \frac{k(k-2)(k-4)}{24}. \qquad \square$$

The consequence of Claim 12 is the following lower bound.

$$\begin{aligned} |\mathcal{F}| &\geq |(\mathcal{F}(A))_1| + |(\mathcal{F}(A))_3| + |\mathcal{F}(A,B)| + |(\mathcal{F}(B))_2| + |(\mathcal{F}(B))_3| \\ &\geq k + \frac{k(k-2)(k-4)}{24} + (n-k)(k-2) + |\mathcal{F}_{1,2}(A,B)| \\ &\quad + \binom{n-k}{2} - 2 + |(\mathcal{F}(B))_3|. \end{aligned} \tag{4}$$

3.2.2 Stage 2

In this stage we first prove Claim 13, $(\mathcal{F}(A))_2 = \emptyset$, which not only implies that $(\mathcal{F}(A))_{\geq 2}$ is a Turán-$_3(k,4,1)$-system, but also makes it possible to find more details about $\mathcal{F}(A,B)$ and $\mathcal{F}(B)$, which are summarized in Claim 14. Claim 13 and 14 together describe the fine structure of $\mathcal{F}$. This leads to an improved lower bound (5) for $|\mathcal{F}|$.

Let us first sketch the idea behind the proof of Claim 13. Suppose that $\{a_1, a_2\} \in (\mathcal{F}(A))_2$. Then at least one of $B_i = \{b \in B : \{a_i, b\} \notin \mathcal{F}\}$, $i = 1, 2$ has size $|B|/2$ and consequently either $|\mathcal{F}_{1,2}(A,B)|$ or $|(\mathcal{F}(B))_3|$ is at least $3(n-k)$. But because of (4), $|\mathcal{F}|$ is larger than the trivial upper bound $\binom{n}{2}$, which is a contradiction.

Claim 13. $(\mathcal{F}(A))_2 = \emptyset$ *provided that* $n \geq 160$.

Proof. Note that (4) and $|\mathcal{F}| \leq \binom{n}{2}$ imply that $k = O(n^{1/3})$ as $n \to \infty$ (in particular, when $n \geq 20$, $k < n/2$).

Suppose instead, that $\{a_1, a_2\} \in (\mathcal{F}(A))_2$. Pick a vertex $b \in B$. By Property **S**, at most one of $\{a_1, b\}$ and $\{a_2, b\}$ is contained in $\mathcal{F}$. Without loss of generality, we may assume that B has a subset B_1 of size $\frac{n-k}{2}$, such that $\{a_1, b\} \notin \mathcal{F}$ for every $b \in B_1$. Consider $\mathcal{T}_{a_1} = \{T \in \mathcal{F}_3 : a_1 \in T, |T \cap B_1| = 2\}$. If $|\mathcal{T}_{a_1}| \geq 3(n-k)$, then (4) implies that (when $n \geq 30$),

$$\begin{aligned} |\mathcal{F}| &\geq k + \frac{k(k-2)(k-4)}{24} + (n-k)(k-2) + \binom{n-k}{2} - 2 + |\mathcal{T}_{a_1}| \\ &\geq \binom{n}{2} + (n-k) + \frac{k(k-2)(k-4)}{24} - \binom{k}{2} + k - 2 \\ &\geq \binom{n}{2} + n - 29 > \binom{n}{2}, \end{aligned}$$

a contradiction to the trivial upper bound that $|\mathcal{F}| \leq \binom{n}{2}$, where the third inequality follows from the fact

$$\min_{k \geq 0} \frac{k(k-2)(k-4)}{24} - \binom{k}{2} = -26.125 \quad \text{(achieved by } k = 11\text{)}.$$

We may therefore assume that $|\mathcal{T}_{a_1}| < 3(n-k)$. Let $\mathcal{P} = \{P \in \binom{B_1}{2} : \{a_1\} \cup P \in \mathcal{T}_{a_1}\}$

and $\mathcal{T} = \{T \in \binom{B_1}{3} : \binom{T}{2} \cap P = \emptyset\}$. Then $|\mathcal{P}| = |\mathcal{T}_{a_1}|$, and therefore

$$\begin{aligned} |\mathcal{T}| &\geq \binom{(n-k)/2}{3} - |\mathcal{P}|(\frac{n-k}{2} - 2) \\ &> \frac{(n-k)(n-k-2)(n-k-4)}{48} - \frac{3}{2}(n-k)^2 \\ &\geq 3(n-k), \quad \text{when } n-k \geq 80 \text{ or } n \geq 160. \end{aligned}$$

On the other hand, we have $\mathcal{T} \in \mathcal{F}$ for every $T \in \mathcal{T}$ because $|\mathcal{F}(\{a_1\} \cup T)| \geq 5$ and $\mathcal{F}(\{a_1\}, T) = \emptyset$. Consequently $|(\mathcal{F}(B))_3| \geq |\mathcal{T}| > 3(n-k)$. Using this lower bound for $|(\mathcal{F}(B))_3|$ in (4), we obtain $|\mathcal{F}| \geq k(k-2)(k-4)/24 + \binom{n}{2} + n - 2 > \binom{n}{2}$, a contradiction. $\square$

Note that we make no effort to optimize the constant 160 in Claim 13.

With the help of Claim 13, we are able to see the fine structure of $\mathcal{F}$ as follows.

Claim 14. *1. $(\mathcal{F}(A, B))_3 = \emptyset$.*

2. $(\mathcal{F}(B))_3 = \emptyset$ and $|\mathcal{F}(B)| = |(\mathcal{F}(B))_2| \geq \binom{|B|}{2} - 1$.

3. For every $a \in A$, we have $|\{b \in B : \{a, b\} \in \mathcal{F}\}| \geq n - k - 2$. Consequently $|(\mathcal{F}(A, B))_2| \geq k(n - k - 2)$.

Proof. Part 1. Let T_0 be a 3-set of $[n]$ with $T_0 \cap A \neq \emptyset$ and $T \cap B \neq \emptyset$.

If $T_0 \cap B = \{b_1, b_2\} \notin \mathcal{F}$, then $T_0 \notin \mathcal{F}$ by (3). If $T_0 \supset \{a, b\} \notin \mathcal{F}$ for some $a \in A$ and $b \in B$, then we consider all the 3-sets $T \supset P$ such that $|\mathcal{F}(T)| = 4$. If $|T \cap B| = 2$, then it must be the case that $T \in \mathcal{F}$. Otherwise, assume that $T \cap A = \{a, a'\}$. Since $\{a, a'\} \notin \mathcal{F}$ by Claim 13, we also have $T \in \mathcal{F}$. We may therefore apply Lemma 9 to conclude that $T_0 \notin \mathcal{F}$.

Finally we assume that $P \in \mathcal{F}$ for every $P \in \binom{T_0}{2}$ and $P \not\subset A$. Either $|T_0 \cap B| = 2$ or $|T_0 \cap A| = 2$, we always have $|(\mathcal{F}(T_0))_{\leq 2}| = 4$. Therefore $T_0 \notin \mathcal{F}$ by Property **S**.

We thus conclude that $(\mathcal{F}(A, B))_3 = \emptyset$.

Part 2. Suppose instead, that there exists $T \in (\mathcal{F}(B))_3$. Then we know that $\binom{T}{2} \subset \mathcal{F}$ by (3). Let $\mathcal{D} = \{S \in \binom{[n]}{4} : T \subset S, |\mathcal{F}(S)| = 5\}$. We may assume that $\mathcal{D} \neq \emptyset$, otherwise $|\mathcal{F}(S)| \geq 6$ for every $S \supset T$, and T could be removed from $\mathcal{F}$ without hurting Property **D**. Consider a set $S = \{a\} \cup T \in \mathcal{D}$. Either $a \in A$ or $a \in B$. If $a \in A$, then $\mathcal{F}(\{a\}, T) = \emptyset$; if $a \in B$, then $|\mathcal{F}(\{a\}, T)| = 1$.

We claim that $|\mathcal{D}| = 1$. Suppose instead, that $\mathcal{D}$ contains two members $\{a_1\} \cup T$ and $\{a_2\} \cup T$. If $a_1, a_2 \in A$, then we consider $S_0 = \{a_1, a_2, b, c\}$ for any two vertices $b, c \in T$. From Part 1 we know that $(\mathcal{F}(S_0))_3 = \emptyset$. We also have $\{a_1, a_2\} \notin \mathcal{F}$ by Claim 13. Consequently $|\mathcal{F}(S_0)| = 3 < 5$, a contradiction to Property **D**. If $a_1, a_2 \in B$, then there are two vertices $b, c \in T$ such that $\{a_1, b\}, \{a_2, c\} \notin \mathcal{F}$. This already contradicts Claim 12 Part 2. Finally, assume that $a_1 \in A$, $a_2 \in B$ and $\{a_2, d\} \in \mathcal{F}$ for some $d \in T$. Consider

$S_0 = \{a_1, a_2, b, c\}$, where $\{b, c\} = T \setminus \{d\}$. We know that $(\mathcal{F}(S_0))_3 = \emptyset$ from Part 1 and (3), $(\mathcal{F}(S_0)_2) \subseteq \{\{a_1, a_2\}, \{b, c\}\}$ from our assumption. Consequently $|\mathcal{F}(S_0)| = 3 < 5$, again a contradiction.

Now assume that $S_0 = \{a\} \cup T$ is the unique element of D. Let $\mathcal{F}' = \mathcal{F} - T + \{a, b, c\}$ for any two vertices $b, c \in T$. $\mathcal{F}'$ satisfies Property **S** since $|\mathcal{F}'(\{a, b, c\})| \leq 3$. $\mathcal{F}'$ also satisfies Property **D** because $|\mathcal{F}'(S)| \geq 6-1 = 5$ for every $S \in \binom{[n]}{4} \setminus S_0$ and $|\mathcal{F}'(S_0)| = |\mathcal{F}(S_0)| = 5$. $\mathcal{F}'$ is thus another optimal $(4, 5)$-system. However, if $a \in A$, then $\mathcal{F}'$ contradicts Part 1. If $a \in B$, then $\mathcal{F}'$ contradicts (3), because $\{a, b, c\} \in \mathcal{F}'$ and at least one of $\{a, b\}$ and $\{a, c\}$ is not in $\mathcal{F}'$.

We thus conclude that $(\mathcal{F}(B))_3 = \emptyset$.

Now it is easy to see why there are no three vertices $a, b, c \in B$ such that $\{a, b\}, \{a, c\} \notin \mathcal{F}$. In such a case, since $(\mathcal{F}(B))_3 = \emptyset$, we have $|\mathcal{F}(\{a, b, c, d\})| \leq 4$ for any $d \in B \setminus \{a, b, c\}$, a contradiction to Property **D**. Together with Claim 12 Part 2, we conclude that $\mathcal{F}(B)$ misses at most one edge on B.

Part 3. Suppose instead, that there exists an $a \in A$ and $b_1, b_2, b_3 \in B$ such that $\{a, b_i\} \notin \mathcal{F}$ for all i. Part 1 and Part 2 together imply that $S = \{a, b_1, b_2, b_3\}$ contains no 3-set from $\mathcal{F}$. Thus, $|\mathcal{F}(S)| \leq 4$, contradicting Property **D**. □

We now refine (4) by applying Claims 13 and 14:

$$\begin{aligned} |\mathcal{F}| &= |(\mathcal{F}(A))_1| + |(\mathcal{F}(A))_3| + |(\mathcal{F}(A, B))_2| + |(\mathcal{F}(B))_2| \\ &\geq k + T_3(k, 4, 1) + k(n - k - 2) + \binom{n-k}{2} - 1 \\ &= g(k) + \binom{n}{2} - 1, \end{aligned} \tag{5}$$

where $g(k) = T_3(k, 4, 1) - k - \binom{k}{2}$.

3.2.3 Stage 3

In this stage, we complete the proof that $m(n, 4, 5) = \binom{n}{2} - 17$ by analyzing (5).

Since $T_3(k, 4, 1)$ is known for $k \leq 13$, we compute $g(k)$ exactly for $0 \leq k \leq 11$ and obtain that $\min_{0 \leq k \leq 11} g(k) = g(7) = g(8) = -16$. For $k \geq 12$, using the inequality (2), we have $g(k) \geq k(k-1)(k-3)/18 - k - \binom{k}{2} \geq -12$. Putting these together, we have

$$\min_{k \geq 0} g(k) = g(7) = g(8) = -16 \tag{6}$$

Applying (6) to (5), we obtain that $|\mathcal{F}| \geq \binom{n}{2} - 17$.

Claims 13, 14 and (6) lead us to the following construction, which gives a $(4, 5)$-system of cardinality $\binom{n}{2} - 17$.

Construction 1: Partition $[n]$ into $A \cup B$, where $|A| = k = 7$ or 8. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, where $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are defined as follows:

- $\mathcal{F}_1 = \{\{a\} : a \in A\}$.
- Let M be a union of k disjoint copies of P_2 (paths of length 2) whose middle vertices are in A and end vertices make up a subset B_0 of B. Let $\mathcal{F}_2 = \binom{B}{2} \setminus \{e\} \cup (A \times B) \setminus M$ for some $e \in B \setminus B_0$.
- $\mathcal{F}_3$ is the edge set of the Turán 3-graph $\mathcal{H}_k$ on A.

We thus conclude that $m(n, 4, 5) = \binom{n}{2} - 17$. $\square$

3.3 $m(n, 4, 6)$

Let $\mathcal{F}$ be an optimal $(4, 6)$-system. Define A, B, $\mathcal{F}(A)$, $\mathcal{F}(B)$, $\mathcal{F}(A, B)$ as in Section 3.2 and assume that $|A| = k$. We first define another threshold function.

Definition 15. *A $\widehat{(4,2)}$-system of $[n]$ is a set system $\mathcal{G} \subseteq \binom{[n]}{2} \cup \binom{[n]}{3}$ such that every 4-set of $[n]$ contains at least two members of $\mathcal{G}$ and every 3-set of $[n]$ contains at most two members of $\mathcal{G}$. Let $\hat{m}(n, 4, 2)$ denote the minimum size of such a set system.*

The lower bound $|\mathcal{F}| \geq \binom{n}{2} - 190$ in Theorem 3 is the consequence of the following claim, whose proof is postponed to the end of this section.

Claim 16. *$\mathcal{F}$ has the following properties.*

1. *There exists no $T \in \mathcal{F}_3$ which contains $a \in A$ and $b \in B$ such that $\{a, b\} \notin \mathcal{F}$.*
2. *$\mathcal{F}(B) = \binom{B}{2}$.*
3. *$|\mathcal{F}(A, B)| \geq k(n - k) - 4$.*
4. *$|\mathcal{F}_{\geq 2}(A)| \geq \hat{m}(k, 4, 2) \geq 2\binom{k}{7} / \binom{k-3}{4} = k(k-1)(k-2)/105$.*

Claim 16 also suggests a general way to construct $(4, 6)$-systems of $[n]$: Partition $[n]$ into $A \cup B$ with $|A| = k$ for any $0 \leq k \leq n$. Let $\mathcal{F} = \mathcal{F}(A) \cup \mathcal{F}(A, B) \cup \mathcal{F}(B)$, with $\mathcal{F}(B) = \binom{B}{2}$, $\mathcal{F}(A, B) = A \times B$ and $\mathcal{F}(A) = (\mathcal{F}(A))_1 \cup (\mathcal{F}(A))_{\geq 2}$, where $(\mathcal{F}(A))_1 = \{\{a\} : a \in A\}$ and $(\mathcal{F}(A))_{\geq 2}$ is a $\widehat{(4,2)}$-system on A.
In particular, the following construction gives a $(4, 6)$-systems of $[n]$ of size $\binom{n}{2} - 5$.

Construction 2: Let $k = 8$ and A, B, $\mathcal{F}(B)$, $\mathcal{F}(A, B)$ and $(\mathcal{F}(A))_1$ are defined as above. We construct $(\mathcal{F}(A))_{\geq 2}$ as follows.

- Suppose that $A = A_1 + A_2$, where $A_1 = \{u_1, u_2, u_3, u_4\}$ and $A_2 = \{v_1, v_2, v_3, v_4\}$. Let $E_0 = \{\{u_1, v_1\}, \{u_1, v_2\}, \{u_2, v_3\}, \{u_3, v_3\}, \{u_4, v_4\}\}$. Let $(\mathcal{F}(A))_2 = (A_1 \times A_2) - E_0$.
- $(\mathcal{F}(A))_3 = \{\{u_1, u_2, u_3\}, \{u_2, u_3, u_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}\}$.

Proof of Theorem 3 for $t = 6$. The upper bound $m(n, 4, 6) \le \binom{n}{2} - 5$ follows from Construction 2. Claim 16 gives the lower bound for $|\mathcal{F}| = m(n, 4, 6)$ as follows.

$$\begin{aligned} |\mathcal{F}| &= |(\mathcal{F}(A))_1| + |(\mathcal{F}(A))_{\ge 2}| + |\mathcal{F}(A, B)| + |\mathcal{F}(B)| \\ &\ge k + \frac{k(k-1)(k-2)}{105} + k(n-k) - 4 + \binom{n-k}{2} \\ &\ge \binom{n}{2} - 190, \end{aligned} \tag{7}$$

where the last inequality follows from $\min_{k\ge 0} k + k(k-1)(k-2)/105 - \binom{k}{2} = -186$ (achieved by $k = 35$). $\square$

Remark: Actually, we almost determine $m(n, 4, 6)$ exactly in terms of $\hat{m}(k, 4, 2)$ (off only by 4). To see this, replace $k(k-1)(k-2)/105$ by $\hat{m}(k, 4, 2)$ in (7) and get an upper bound following the general construction:

$$k + \hat{m}(k, 4, 2) + k(n-k) + \binom{n-k}{2} - 4 \le |\mathcal{F}| \le k + \hat{m}(k, 4, 2) + k(n-k) + \binom{n-k}{2},$$

for some $k \ge 0$. The knowledge of $\hat{m}(k, 4, 2)$ for small values of k may lead to the final settlement of $m(n, 4, 6)$.

Proof of Claim 16.

Part 1: Suppose that there are two vertices $a \in A$ and $b \in B$ such that $\{a, b\} \notin \mathcal{F}$. For a 3-set $T \supset \{a, b\}$, if $|\mathcal{F}(T)| = 5$, then $T \in \mathcal{F}$. We may therefore apply Lemma 9 to conclude that $T \notin \mathcal{F}$ for every 3-set $T \supset \{a, b\}$.

Part 2: We first claim that (3) holds in $\mathcal{F}$. In fact, when $P \in \binom{B}{2}$, we have $\{T \in \binom{[n]}{3} : P \subset T, |\mathcal{F}(T)| = 5\} = \emptyset$. Then we can apply Lemma 9 to obtain (3).

Next, we show that if there exists a set $T \in (F(B))_3$, then we obtain a contradiction. The proof is similar to that of Claim 14 Part 2. First, we claim that $\mathcal{D} = \{S \in \binom{n}{4} : T \subset S, |\mathcal{F}(S)| = 6\}$ has exactly one member. Suppose instead, for example, that $\mathcal{D}$ contains $S_1 = \{a\} \cup T$ and $S_2 = \{b\} \cup T$ for some $a \in A$ and $b \in B$ (other cases are similar). It means that there are exactly two sets from $\mathcal{F}(S_1)$, $\mathcal{F}(S_2)$ which contain a, b, respectively. From Part 1, we know that these two sets in S_1 must be $\{a\}$ and $\{a, b_1\}$ and the two sets in S_2 must be $\{b, b_2\}$ and $\{b, b_3\}$, where $b_1, b_2, b_3 \in T$. Assume that, for example, b_1, b_2, b_3 are all distinct. Consider $S_3 = \{a, b, b_1, b_2\}$. It is easy to see that $|\mathcal{F}(S_3)| \le 5$, a contradiction to Property **D**. Now assume that $\{a\} \cup T$ is the unique member of $\mathcal{D}$, for example, for some $a \in A$. Let $\mathcal{F}' = \mathcal{F} - T + \{a, b_1, b_2\}$. It is easy to see that $\mathcal{F}'$ is an optimal $(4, 6)$-system. However, since $\{a, b_1, b_2\} \in \mathcal{F}'$ and $\{a, b_2\} \notin \mathcal{F}'$, this contradicts Part 1.

Since $(F(B))_3 = \emptyset$, $\binom{B}{2} \subset \mathcal{F}$ follows from Property **D**. Consequently $\mathcal{F}(B) = \binom{B}{2}$.

Part 3: We first show that for every $a \in A$, there is at most one vertex $b \in B$ such that $\{a, b\} \notin \mathcal{F}$. Suppose instead, that $\{a, b_1\}, \{a, b_2\} \notin \mathcal{F}$ for some $a \in A$. Consider $S = \{a, b_1, b_2, c\}$ for any vertex $c \in B \setminus \{b_1, b_2\}$. From Part 1 and 2 we know that

$(\mathcal{F}(S))_3 = \emptyset$. Consequently $|\mathcal{F}(S)| \le 5$, a contradiction to Property **D**. Second, for each $b \in B$, there are at most two vertices $a_1, a_2 \in A$ such that $\{a_i, b\} \notin \mathcal{F}$ for $i = 1, 2$. Suppose instead, that there are three vertices $a_1, a_2, a_3 \in A$ such that $\{a_i, b\} \notin \mathcal{F}$. Since $\mathcal{F}(\{a_1, a_2, a_3\}, \{b\}) = \emptyset$, we have $\mathcal{F}(\{a_1, a_2, a_3\}) = \mathcal{F}(\{b, a_1, a_2, a_3\})$, which either violates Property **D** or Property **S**.

Now consider the bipartite graph G whose edge set is $(A \times B) - (\mathcal{F}(A, B))_2$. By the argument in the previous paragraph, G consists of vertex-disjoint edges or 2-paths whose centers are in B. On the other hand, two independent edges $\{a_i, b_i\}$, $i = 1, 2$ in G (where $a_i \in A$) imply that $\{a_1, a_2\} \in \mathcal{F}$ (by Property **D**). If G contain 3 independent edges $\{a_i, b_i\}$, then $|\mathcal{F}(\{a_1, a_2, a_3\})| = 6$, a contradiction to Property **S**. Therefore G has at most 4 edges which are from two disjoint 2-paths.

Part 4. Clearly $\mathcal{F}_{\ge 2}(A)$ is a $\widehat{(4,2)}$-system of $[k]$. Let us count the number of triples in a $\widehat{(4,2)}$-system $\mathcal{G}$ of $[k]$. The following lemma implies that every 7-set of $[k]$ contains at least two triples from $\mathcal{G}$. We omit its proof because it is an easy case analysis.

Lemma 17. *Every $\widehat{(4,2)}$-system of $[7]$ contains at least two triples.* $\square$

Using Lemma 17 and the averaging argument, we have $\hat{m}(k, 4, 2) \ge 2\binom{k}{7}/\binom{k-3}{4}$. $\square$

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