Even circuits of prescribed clockwise parity

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Abstract

We show that a graph has an orientation under which every circuit of even length is clockwise odd if and only if the graph contains no subgraph which is, after the contraction of at most one circuit of odd length, an even subdivision of $K_{2,3}$. In fact we give a more general characterisation of graphs that have an orientation under which every even circuit has a prescribed clockwise parity. Moreover we show that this characterisation has an equivalent analogue for signed graphs.

We were motivated to study the original problem by our work on Pfaffian graphs, which are the graphs that have an orientation under which every alternating circuit is clockwise odd. Their significance is that they are precisely the graphs to which Kasteleyn's powerful method for enumerating perfect matchings may be applied.

1 Introduction

Consider the three (even) circuits in $K_{2,3}$. Is it possible to find an orientation under which all these circuits are clockwise odd, if the clockwise parity of a circuit of even length is defined as the parity of the number of edges that are directed in agreement with a specified sense? However $K_{2,3}$ is oriented one observes that the total number of clockwise even circuits is odd and therefore it is not possible to find such an orientation. In this paper we present a characterisation, in terms of forbidden subgraphs, of the graphs that have an orientation under which every even circuit is clockwise odd. It will turn out that the non-existence of such an orientation can in a sense always be put down to an even subdivision of $K_{2,3}$. (See Corollary 1.)

We were motivated to study this problem by our work on a characterisation of Pfaffian graphs. A *Pfaffian orientation* of a graph is an orientation under which every alternating circuit is clockwise odd, an alternating circuit being a circuit which is the symmetric difference of two perfect matchings. A *Pfaffian* graph is a graph that admits a Pfaffian orientation. In [3] Kasteleyn introduced a remarkable method for enumerating perfect matchings in Pfaffian graphs, reducing the enumeration to the evaluation of the determinant of the skew adjacency matrix of the Pfaffian directed graph. He has shown that all planar graphs are Pfaffian. However a general characterisation of Pfaffian graphs is still not known. For research in this direction see [4, 5, 6, 1, 7].

Our characterisation of the graphs that admit an orientation under which every even circuit is clockwise odd will be an easy consequence of our main theorem (Theorem 1), which gives a more general characterisation of the graphs that have an orientation under which every even circuit has a prescribed (not necessarily odd) clockwise parity. In Section 2 we will introduce the concept of a signed graph and present an analogue of Theorem 1 for signed graphs (Theorem 2). We will show that each of these theorems implies the other. Finally we prove the theorem for ordinary graphs.

The following definition is fundamental.

Definition 1 *Let G be a graph and J an assignment of clockwise parities to the even circuits of G. An even circuit of G is said to be J-oriented under a given orientation of G if it has the clockwise parity assigned by J. An orientation of G is said to be J-compatible or a J-orientation if every even circuit of G is J-oriented. Otherwise the orientation is J-incompatible. The graph G is said to be J-orientable if G admits a J-orientation, and J-nonorientable otherwise.*

Our main theorem (Theorem 1) characterises J-orientable graphs in terms of forbidden subgraphs. Before we are able to formulate it, we have to introduce two relevant graph operations. To this end we need the following important definition and two lemmas.

Definition 2 *Let G be a graph and J an assignment of clockwise parities to the even circuits of G. A set $\mathcal{S}$ of even circuits in G is said to be J-intractable if the symmetric difference of the circuits in $\mathcal{S}$ is empty and under some orientation of G there are an odd number of circuits in $\mathcal{S}$ that are not J-oriented.*

Observe that the parity of the number of even circuits in a J-intractable set $\mathcal{S}$ that are not J-oriented, with respect to a given orientation, does not depend on the orientation since the reorientation of a single edge changes the clockwise parity of an even number of circuits in $\mathcal{S}$. Therefore under any orientation of G there are an odd number of circuits in $\mathcal{S}$ that are not J-oriented.

Lemma 1 *Let G be a graph and J an assignment of clockwise parities to the even circuits of G. Then G is J-nonorientable if and only if G contains a J-intractable set of even circuits.*

Proof. The existence of a J-intractable set implies that G is J-nonorientable, as can be seen from the remark preceding the formulation of the lemma.

Suppose that G is J-nonorientable and orient G arbitrarily. The existence of a J-orientation of G is equivalent to the solvability of a certain system of linear equations over the field $\mathbb{F}_2$. In these equations the variables correspond to the edges of the even circuits of G. For every even circuit C there is a corresponding equation in which the sum of the variables corresponding to the edges of C is 1 if and only if the clockwise parity of C is not that prescribed by J. A solution of this system is an assignment of zeros and ones to the edges of the even circuits of G. A J-orientation of G can be obtained from the fixed orientation by reorienting precisely those edges to which the solution assigns a 1. The lemma now follows from the solvability criteria for systems of linear equations. □

Let G be a graph and let H be a graph obtained from G by the contraction of the two edges e and f incident on some vertex v in G of degree 2. Thus $EH = EG - \{e, f\}$. We may describe G as an *even vertex splitting* of H. (See Figure 1.) Any even circuit C_H in H is the intersection with EH of a unique even circuit C in G. To any assignment J of clockwise parities to the even circuits in G there corresponds an assignment J_H of clockwise parities to the even circuits in H so that any even circuit C_H in H is assigned the same clockwise parity as C in G. We say that J_H is *induced* by J. If either e or f is incident on a vertex of degree 2 other than v, then it is also true that the intersection with EH of any even circuit C in G yields an even circuit C_H in H. In this case any assignment J_H of clockwise parities to the even circuits in H corresponds to a unique assignment J of clockwise parities to the even circuits in G so that J_H is the assignment induced by J. We then say that J is also *induced* by J_H.

Similarly let H be obtained from G by contracting a circuit A of odd length. Thus $EH = EG - A$. Any even circuit C_H in H is the intersection with EH of a unique even circuit C in G: we have $C \cap EH = C_H$ and if $C \neq C_H$ then $C \cap A$ is the path of even length in A joining the ends of the path C_H in G. To any assignment J of clockwise parities to the even circuits in G there corresponds an assignment J_H of clockwise parities to the even circuits in H so that any even circuit C_H in H is assigned the same clockwise parity as C in G. We say that J_H is *induced* by J.

In the following lemma we summarise some basic facts:

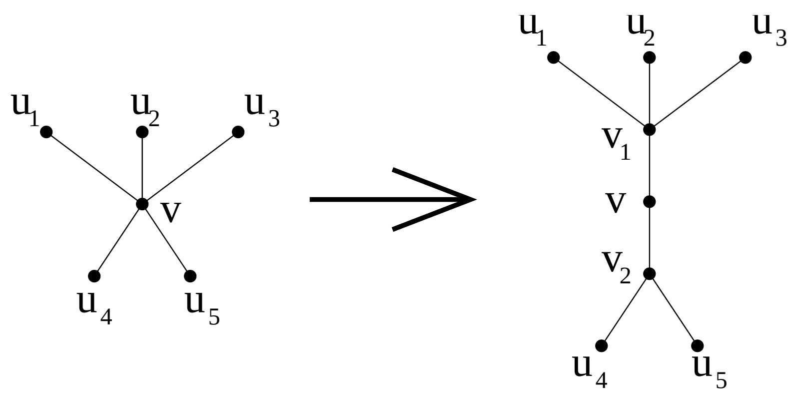

Figure 1: Split vertices to obtain an even vertex splitting.

Lemma 2 *Let G be a graph and J an assignment of clockwise parities to the even circuits of G.*

(1) Let H be a subgraph of G and J_H the restriction of J to the even circuits of H. If G is J-orientable then H is J_H-orientable.

(2) Let H be obtained from G by contracting the two edges incident on a vertex of degree 2. The assignment J induces an assignment J_H of clockwise parities to the even circuits in H. If G is J-orientable then H is J_H-orientable. If either of the two contracted edges is incident on another vertex of degree 2 then G is J-orientable if and only if H is J_H-orientable.

(3) Let H be obtained from G by contracting a circuit of odd length. The assignment J induces an assignment J_H of clockwise parities to the even circuits in H. If G is J-orientable then H is J_H-orientable.

Proof. (2) Every J_H-intractable set of even circuits in H corresponds to a J-intractable set of even circuits in G. If either of the two contracted edges is incident on another vertex of degree two then every J-intractable set of even circuits in G also corresponds to a J_H-intractable set of even circuits in H.

(3) Every J_H-intractable set of even circuits in H corresponds to a J-intractable set of even circuits in G. (Note that if the symmetric difference of some even circuits in H is empty, then the symmetric difference of the corresponding even circuits in G is empty as well, for it is obvious that this symmetric difference is both an even cycle and a subset of $EG - EH$.)

□

In the following three paragraphs we introduce the minimal J-nonorientable graphs which we need in the formulation of our main theorem. We say that an assignment J is *odd* or *even* if it assigns, respectively, an odd or an even clockwise parity to every even circuit.

Let $O_1 = K_{2,3}$ and let O_2 be the graph we obtain from K_4 by subdividing once all edges incident on one fixed vertex. (See Figure 2.) Observe that O_1 and O_2 are J-nonorientable

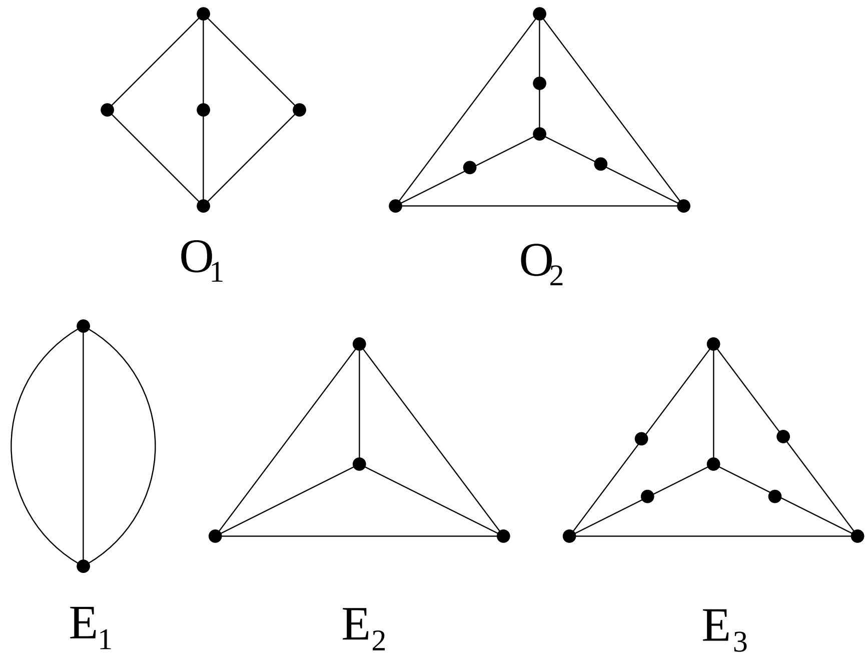

Figure 2:

with respect to the odd assignment J. In fact O_1 and O_2 are J-nonorientable precisely for those assignments J that prescribe an even number of even circuits of these graphs to be of even clockwise parity. For these assignments Lemma 2(3) shows that the J-nonorientability of O_2 can be attributed to the fact that O_1 is J-nonorientable, since the contraction of the triangle in O_2 gives O_1.

Let E_1 be the graph consisting of two vertices and three edges joining them, let $E_2 = K_4$ and let E_3 be the graph we obtain from K_4 by subdividing once each edge in a fixed even circuit. (See Figure 2.) Then E_1, E_2 and E_3 are J-nonorientable with respect to the even assignment J. More generally E_1, E_2 and E_3 are J-nonorientable precisely for those assignments J that prescribe an odd number of even circuits to be of even clockwise parity. Again by Lemma 2(3) the fact that E_2 is J-nonorientable can be put down to the fact that E_1 is J-nonorientable, since the contraction of a triangle in E_2 gives E_1.

A Δ-graph is one of the 10 non-isomorphic graphs that can be obtained from the configuration in Figure 3 by replacing the P_i's independently by paths of length 0, 1 or 2. Each of these graphs has exactly four even circuits and is J-nonorientable if and only if J prescribes an odd number of them to be clockwise even. This observation follows from Lemma 1 because in each of these graphs the set of all even circuits is the only dependent set of even circuits with respect to symmetric difference.

Let H be a graph and let $H_0, H_1, \ldots, H_k$ be graphs such that $H_0 = H$ and, for each $i > 0$, the graph H_i is an even vertex splitting of H_{i-1}. Then H_k is said to be an *even splitting* of H. There is a special case in which, for each $i > 0$, H_i can be obtained from

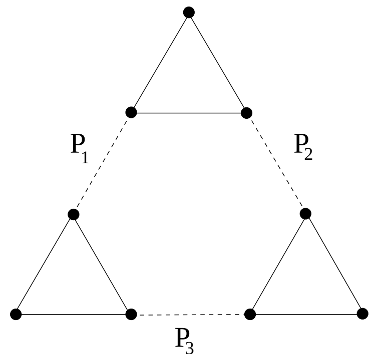

Figure 3: Δ-graphs.

H_{i-1} by subdividing an edge twice. In this case we describe H_k as an *even subdivision* of H. If no vertex of H is of degree greater than 3, then each even splitting of H is also an even subdivision of H. If G is an even splitting of H and J is an assignment of clockwise parities to the even circuits in G, then we may apply the definition of an induced assignment inductively to obtain an assignment J_H of clockwise parities to the even circuits in H. This assignment is also said to be *induced* by J. By applying Lemma 2(2) inductively we find that if G is J-orientable then H is J_H-orientable. Thus if H is J_H-nonorientable then G is J-nonorientable. The converse also holds if G is an even subdivision of H.

Now we formulate our main theorem.

Theorem 1 *Let G be a graph and J an assignment of clockwise parities to the even circuits of G. Then G is J-nonorientable if and only if G contains a J_H-nonorientable even subdivision H of one of O_1, O_2, E_1, E_2, E_3 or of a Δ-graph.*

The "if" direction in the theorem is obvious by Lemma 1 and Lemma 2.

Remark 1 *Note that the fact that H is J_H-nonorientable in the assertion of the theorem is equivalent to the following: if H is an even subdivision of O_i for some i then J prescribes an even number of clockwise even parities to the three even circuits of H, if H is an even subdivision of E_i for some i then J prescribes an odd number of clockwise even parities to the three even circuits of H and if H is an even subdivision of a Δ-graph then J prescribes an odd number of clockwise even parities to the four even circuits of H.*

We obtain the following immediate corollaries.

Corollary 1 *A necessary and sufficient condition for a graph to admit an orientation under which every even circuit is clockwise odd is for it not to contain a subgraph which is, after the contraction of at most one odd circuit, an even subdivision of $K_{2,3}$.*

Proof. For each even subdivision of E_1, E_2, E_3 or a Δ-graph the odd assignment prescribes an even number of clockwise even parities to its set of even circuits and therefore these subdivisions are J-orientable with respect to the odd assignment J.

$\square$

Corollary 2 *A necessary and sufficient condition for a graph to admit an orientation under which every even circuit is clockwise even is for it not to contain a subgraph which is, after the contraction of at most one odd circuit, an even subdivision of E_1 or E_3.*

Proof. For each even subdivision of O_1 or O_2 the even assignment prescribes an odd number of clockwise even parities to its set of even circuits, for both graphs have three even circuits. Moreover for each even subdivision of a Δ-graph the even assignment prescribes an even number of clockwise even parities to its set of even circuits, for these graphs each have four even circuits. Therefore these subdivisions are J-orientable with respect to the even assignment J.

$\square$

2 Signed graphs

A *signed graph* is an ordinary (undirected) graph G together with an assignment of "even" or "odd" to the edges. A circuit of a signed graph is said to be *even* if it has an even number of odd edges. Let J be an assignment of parities to the even circuits of a signed graph. A signed graph is said to be *J-compatible* if there is an assignment of zeros and ones to the edges so that the number of ones in every even circuit has the parity prescribed by J; otherwise it is *J-incompatible*. The notion of a *J-intractable set of even circuits* is defined in a manner analogous to the ordinary case, as in Definition 2. In fact it is easy to see that the analogue of Lemma 1 also holds in the signed case.

The following lemma will lead to the notion of a signed minor.

Lemma 3 *Let G be a signed graph and J an assignment of parities to the even circuits of G.*

(1) Let H be a subgraph of G and J_H the restriction of J to the even circuits of H. If G is J-compatible then H is J_H-compatible.

(2) Let H be obtained from G by contracting an even edge. The assignment J induces an assignment J_H of parities to the even circuits of H. If G is J-compatible then H is J_H-compatible. If the contracted even edge is incident on a vertex of degree 2 in G then the converse is also true.

(3) Let H be obtained from G by resigning the edges on a cut of G. A circuit of H is even if and only if the corresponding circuit in G is even as well and therefore J induces an assignment J_H of parities to the even circuits of H. Then G is J-compatible if and only if H is J_H-compatible.

$\square$

We say that H is a *signed minor* of a signed graph G if G contains a subgraph which can be reduced to H by a sequence of the two *signed minor operations*: contracting even edges and resigning on cuts. Given an assignment J of parities to the even circuits in G, this assignment clearly induces an assignment J_H of parities to H. The previous lemma shows that the J_H-incompatibility of H implies the J-incompatibility of G. Note also that if e and f are the edges incident on a vertex of degree 2, then the signed minor operations permit the contraction of at least one of them. If either e or f is even this is obvious. Otherwise we resign the cut $\{e, f\}$ and afterwards contract either e or f.

Let G be an ordinary unsigned graph. Denote by G^o the signed graph obtained from G by assigning "odd" to every edge. Moreover let Δ^* denote the signed graph obtained from K_3 by replacing every edge with a pair of edges of opposite sign. The following characterisation of J-compatible signed graphs is the analogue of Theorem 1. The advantage of signed graphs is that the number of forbidden minors is smaller than in Theorem 1. Clearly this comes from the fact that the signed minor operations are more general than the even subdivision operation we have used for ordinary graphs.

Theorem 2 *Let G be a signed graph and J an assignment of parities to the even circuits of G. Then G is J-incompatible if and only if G contains a J_H-incompatible minor H isomorphic to E_1^o, K_4^o or Δ^*.*

Theorem 1 and Theorem 2 imply each other. In this section we show that Theorem 1 implies Theorem 2 and indicate briefly how to verify the converse implication.

Let us begin with an overview of the proof that Theorem 1 implies Theorem 2, together with an example to illustrate the construction. We begin with a signed graph G and an assignment J of parities to its even circuits. For example, G could be the first graph in Figure 4, where the even edges are dashed. Thus edges d, e, f are even and the other edges are odd. The even circuits in this example are therefore $C_1 = \{b, c, d, e\}$, $C_2 = \{a, b, e, f\}$ and $C_3 = \{a, c, d, f\}$. Suppose that they are all given the odd parity by a parity assignment J. Now let G^* be the signed graph obtained from G by subdividing each even edge once and giving every edge the odd parity. In our example G^* is the second graph given in Figure 4. Its even circuits are necessarily those of even length: $C_1^* = \{b, c, d', d'', e', e''\}$, $C_2^* = \{a, b, e', e'', f', f''\}$ and $C_3^* = \{a, c, d', d'', f', f''\}$. Each even circuit in G^* is assigned the same parity as the corresponding circuit in G by a parity assignment J^*. Thus in our example each C_i^* is assigned the odd parity by J^*. Now let G' be the unsigned version of G^*, and give it an arbitrary orientation. Figure 4 gives an example of an orientation under which C_1^* is clockwise even but C_2^* and C_3^* are clockwise odd. We construct an assignment J' of clockwise parities to the even circuits of G' by taking the parity assigned by J^* for each clockwise even circuit but the opposite parity for each circuit that is clockwise odd. Thus in our example C_1^* is assigned the odd clockwise parity by J' but C_2^* and C_3^* are assigned the even clockwise parity. Note that the resulting clockwise parities assigned to C_1^*, C_2^* and C_3^* are not all equal even though equal parities were assigned to C_1, C_2 and C_3 by J in our example. On the other hand, if J had assigned the odd parity to C_1 but the even parity to C_2 and C_3 then C_1^*, C_2^* and C_3^* would all have been assigned the odd clockwise parity by J'. Thus if either J or J' assigns equal

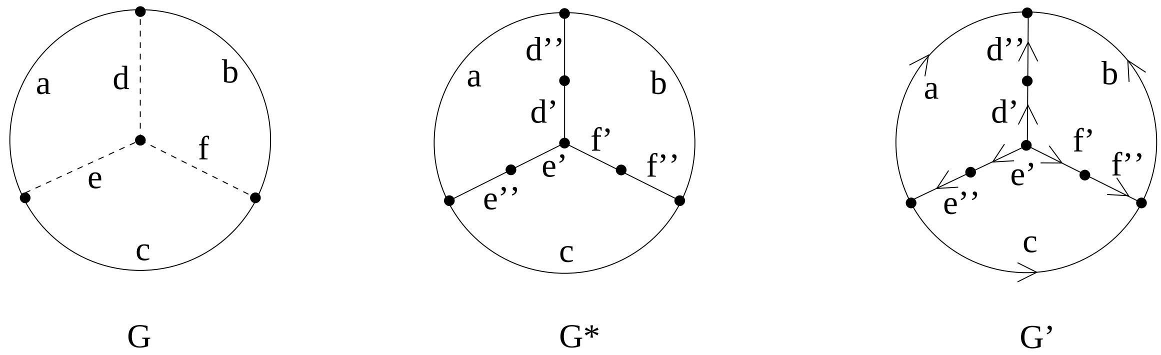

Figure 4:

parities to all even circuits it is not necessarily the case that the other also does. The argument to demonstrate that Theorem 1 implies Theorem 2 continues by showing that G is J-compatible if and only if G' is J'-orientable. Certainly a graph is J-incompatible if it contains one of the J_H-compatible minors H mentioned in Theorem 2. Suppose therefore that G is J-incompatible. Then G' is J'-nonorientable, and therefore contains a J'-nonorientable subgraph H' which is an even subdivision of one of O_1, O_2, E_1, E_2, E_3 or a Δ-graph. The proof concludes by applying the signed minor operations to the corresponding subgraph of G.

Theorem 1 implies Theorem 2. Let G be a signed graph and let J be an assignment of parities to the even circuits of G.

Suppose first that every edge of G is odd. Then the even circuits in G are those of even length. Let G' be the unsigned version of G, fix an arbitrary orientation of G' and let K be the assignment of the consequent clockwise parity to each even circuit in G'. Let $J' = J + K$. Thus J' is the assignment of clockwise parities to the even circuits of G' under which a given even circuit is assigned the same clockwise parity as under J if and only if it is assigned the even clockwise parity under K. If G is J-compatible then there is an assignment of zeros and ones to the edges of G so that the number of ones in any even circuit has the parity prescribed by J. Reversal of the orientation of every edge of G' assigned a 1 in G changes the clockwise parity of a given circuit if and only if the circuit is prescribed the odd parity under J. The resulting orientation of G' is therefore J'-compatible, so that G' is J'-orientable. Conversely if G' is J'-orientable then reversal of this argument shows that G is J-compatible.

On the other hand, suppose some edges of G are even. We may construct a new signed graph G^* by subdividing each even edge once and giving each edge of the new graph the odd parity. Then there is a bijection from the set of even circuits of G onto the set of even circuits of G^* under which each even edge of an even circuit in G is replaced by the pair of odd edges into which it is subdivided in G^*. Therefore G^* inherits from G an assignment J^* of parities to its even circuits so that each even circuit in G is assigned the same parity by J as its image in G^* is assigned by J^*. It is now clear that G is J-compatible if and only if G^* is J^*-compatible. The conclusions in the paragraph above can be applied to

G^*. With G' defined as the unsigned version of G^* and J' defined as in the previous paragraph, we deduce once again that G is J-compatible if and only if G' is J'-orientable.

We have already noted that the existence of a J_H-incompatible minor H of G would imply the J-incompatibility of G. We may therefore suppose that G is J-incompatible. Then G' is J'-nonorientable. Thus, by Theorem 1, G' contains a J'-nonorientable subgraph H' which is an even subdivision of one of O_1, O_2, E_1, E_2, E_3 or a Δ-graph. There is a unique J-incompatible signed subgraph H of G such that EH' consists of the odd edges in H and the union of the paths of length 2 in G' that replace the even edges in H. We shall show that the signed minor operations can be used to reduce H to one of E_1^o, K_4^o, Δ^*.

By sequentially contracting edges incident on vertices of degree 2 we may reduce H to a J-incompatible signed graph H^+ without vertices of degree 2. Note that the parity of the number of odd edges in a path P whose inner vertices are all of degree 2 does not change in this procedure, which is effected by a sequence of resignings of pairs of consecutive odd edges of P and contractions of even edges.

Case 1: Suppose first that H' is an even subdivision of E_1 or O_1. The graph underlying H^+ must be E_1 and all edges have the same parity. If this parity is even, then resign the edges to make them odd. The result is E_1^o.

Case 2: Suppose next that H' is an even subdivision of O_2, E_2 or E_3. The graph underlying H^+ is K_4. If H' is an even subdivision of O_2, then H^+ has three even edges and they are all incident on the same vertex v. Resigning the edges in the vertex cut defined by v gives K_4^o. If H' is an even subdivision of E_2, then H^+ is equal to K_4^o. If H' is an even subdivision of E_3, then H^+ has four even edges. These four edges form a circuit C, and if we resign the edges in a cut defined by two vertices in H^+ which are non-adjacent in C then we obtain K_4^o.

Case 3: Suppose finally that H' is an even subdivision of a Δ-graph. The paths in H^+ corresponding to P_1, P_2 and P_3 in the definition of a Δ-graph are of lengths 0 or 1. Contract every such path that consists of an even edge. The result is isomorphic to one of the graphs in Figure 5, where the even edges are dashed. In each case resign on any cut containing all the edges not in a digon, then contract those edges to obtain Δ^*.

□

Theorem 2 implies Theorem 1. The argument in this direction is a bit more complicated because the signed minor operations are more general than the even subdivision operation we have used for ordinary graphs. We sketch the argument very briefly.

Let G be a graph and J an assignment of clockwise parities to the even circuits of G, and suppose that G is J-nonorientable. Fix an orientation of G. Let G' be the signed graph G^o and let J' be the assignment of parities to the even circuits of G' under which an even circuit is prescribed the parity "even" if and only if, under the fixed orientation, the corresponding even circuit in G has the clockwise parity prescribed by J. Then G' is J'-incompatible. Thus, by Theorem 2, G' contains a $J'_{H'}$-incompatible signed subgraph H' which can be reduced to E_1^o, K_4^o or Δ^* by the signed minor operations. The argument is then completed by investigating the structure of a J_H-nonorientable subgraph H of G

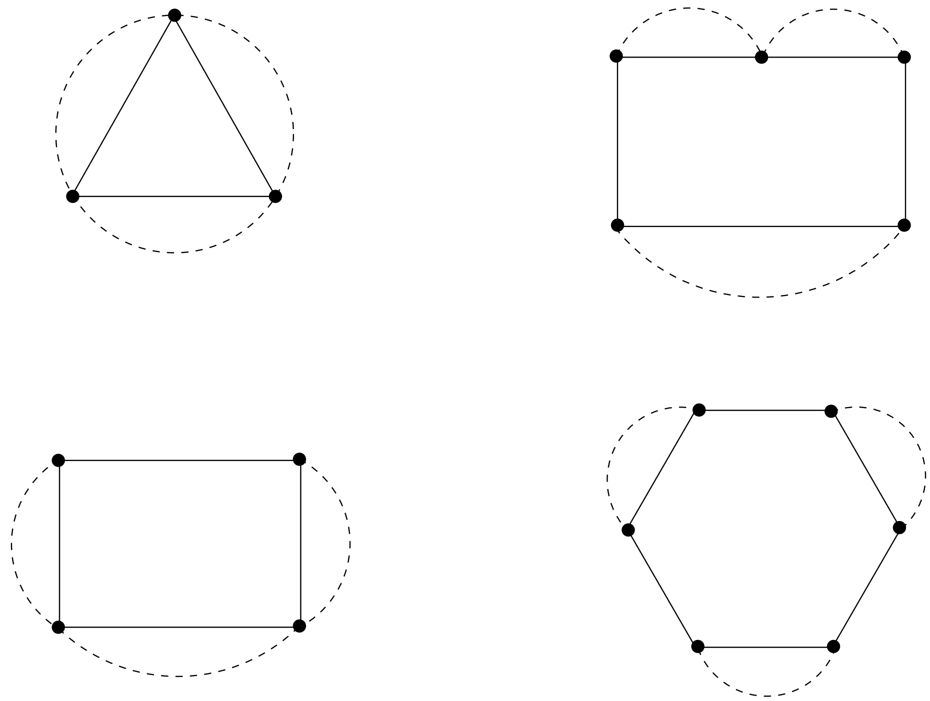

Figure 5:

for which H^o is isomorphic to H'.

3 An arc decomposition theorem

Circuits, non-empty paths and, more generally, subgraphs without isolated vertices are determined by their edge sets and are therefore identified with them in this paper. If u and v are vertices of a path P, then $P[u, v]$ denotes the subpath of P that joins u and v. Given subsets S and T of VG, a path joining a vertex of S to a vertex of T will be called an *(S, T)-path*. If G is a graph and V' is a subset of the vertex set VG of G then $G[V']$ denotes the subgraph of G spanned by V'. Similarly if E' is a subset of the edge set EG of G then $G[E']$ denotes the subgraph of G spanned by E'.

Let H_1 and H_2 be two sets of edges in G. An *$H_1\overline{H_2}$-arc* (or an *$\overline{H_2}H_1$-arc*) is a path in H_1 which joins two distinct vertices in $VG[H_2]$ but does not have an inner vertex in $VG[H_2]$. A $G\overline{H_2}$-arc is also called an *$\overline{H_2}$-arc*.

Definition 3 *A graph G without isolated vertices is said to be even-circuit-connected if for every bipartition $\{H_1, H_2\}$ of EG there exists an even circuit C which meets H_1 and H_2.*

Every even-circuit-connected graph is 2-connected. Indeed, suppose there exists a vertex v such that $G - \{v\}$ is disconnected. Let $H_1, H_2, \ldots, H_k$ be the components of

$G - \{v\}$ and let $H_i' = G[VH_i \cup \{v\}]$ for each i. Then for every circuit C of G there exists an i, $1 \leq i \leq k$, with $C \subseteq EH_i'$, a contradiction.

First we prove a decomposition theorem on even-circuit-connected graphs. Note that in our characterisation of J-nonorientable graphs in terms of forbidden subgraphs, even-circuit-connected graphs are the only graphs of interest since every J-nonorientable graph that is minimal with respect to edge deletion is clearly even-circuit-connected.

Let H be a subgraph of G and C an even circuit in G which includes $EG - EH$ and meets EH. If there are n $C\overline{H}$-arcs, then G is said to be obtained from H by an *n-arc adjunction.* An *arc decomposition* of an even-circuit-connected graph G is a sequence $G_0, G_1, \ldots, G_k$ of even-circuit-connected subgraphs of G such that EG_0 is an even circuit, $G_k = G$ and, for every $i > 0$, G_i is obtained from G_{i-1} by an n-arc adjunction with $n = 1$ or $n = 2$. Moreover we assume that, for each i, every even circuit in G_i which meets $EG_i - EG_{i-1}$ contains $EG_i - EG_{i-1}$. We shall show that every even-circuit-connected graph has an arc decomposition. For this purpose we need the following version of Menger's theorem.

Theorem 3 *[2] Let S and T be sets of at least n vertices in an n-connected graph G. Then there are n vertex disjoint (S,T)-paths such that no inner vertex of these paths is in $VS \cup VT$.*

Lemma 4 *Let H be a non-empty proper even-circuit-connected subgraph of an even-circuit-connected graph G. Then G has an even circuit C that meets EH, admits just one or two $C\overline{H}$-arcs and has the property that $G[H \cup C]$ is even-circuit-connected. Moreover, if G is bipartite or H is not, then C may be chosen to admit just one $C\overline{H}$-arc.*

Proof. Suppose first that G is bipartite. By hypothesis there is an edge e in $EG - EH$. By the 2-connectedness of G and Theorem 3 there are vertex disjoint paths P and Q in $EG - EH$ joining the ends of e to two distinct vertices u and v, respectively, in VH such that neither P nor Q has an inner vertex in VH. Since H is even-circuit-connected and therefore connected, a path R in H joins u to v. Thus $P \cup \{e\} \cup Q \cup R$ is a circuit C in G meeting EH (since $u \neq v$) and having $P \cup \{e\} \cup Q$ as its unique $C\overline{H}$-arc. Moreover C is even since G is bipartite.

Suppose therefore that G is not bipartite. Again we may construct the circuit C as in the previous case, and the proof is complete if C is even. Suppose therefore that C cannot be chosen to be even. Since G is even-circuit-connected there exists an even circuit D which meets EH and $EG - EH$. Let S and T be two distinct $D\overline{H}$-arcs, joining w to x and y to z, respectively. The fact that D meets EH implies $w \neq x$, $y \neq z$ and $\{w, x\} \neq \{y, z\}$. Let U be a path in H joining w to x. Since H is 2-connected there exist two vertex disjoint paths V and W in H joining y and z, respectively, to distinct vertices of U and such that neither has an inner vertex in VU. Let s and t be the ends of V and W, respectively, in VU. By assumption $S \cup U$ is an odd circuit and therefore it includes a path X, joining s to t, such that

$$|X| \equiv |T| + |V| + |W| \pmod{2}.$$

Then $T \cup V \cup W \cup X$ is a circuit C of even length, and the only $C\overline{H}$-arcs are T and possibly S.

Finally suppose that H is not bipartite. Therefore H has an odd circuit O. Since H is even-circuit-connected and therefore 2-connected, there are vertex disjoint paths M and N in H joining w and x, respectively, to distinct vertices p and q in VO but having no inner vertex in VO. Since O is odd, it includes a path Y joining p and q such that

$$|Y| \equiv |M| + |N| + |S| \pmod 2.$$

Then $M \cup N \cup S \cup Y$ is an even circuit C in G, and S is the unique $C\overline{H}$-arc.

It remains to show that $G[H \cup C] =: H'$ is even-circuit-connected. Suppose that $\{K_1, K_2\}$ is a bipartition of EH'. If $K_1 \supseteq EH$ and $K_2 \subseteq EH' - EH$ then C is an even circuit which meets K_1 and K_2. Thus we may assume that $K_l \cap EH =: K'_l \neq \emptyset$ for $l = 1, 2$. By the assumption that H is even-circuit-connected there exists an even circuit in H which meets K'_1 and K'_2, and therefore K_1 and K_2.

□

Lemma 4 shows that every even-circuit-connected graph G has an arc decomposition $G_0, G_1, \ldots, G_n$ with at most one 2-arc adjunction. The single 2-arc adjunction is necessary if and only if G is not bipartite. In this case the arc decomposition can be chosen so that G_1 is obtained from G_0 by a 2-arc adjunction as we see in the following theorem.

Theorem 4 *An even-circuit-connected graph G has an arc decomposition $G_0, G_1, \ldots, G_k$ such that G_i is obtained from G_{i-1} by a single arc adjunction for all $i > 1$.*

Proof. By Lemma 4 let $H_0, H_1, \ldots, H_n$ be an arc decomposition of G such that H_i is obtained from H_{i-1} by a 2-arc adjunction for some $i > 1$ and H_j is obtained from H_{j-1} by a single arc adjunction for all $j \neq i$. If H_i is bipartite, then the theorem holds since H_i may be constructed from G_0 by single arc adjunctions. Therefore we may assume that H_i is non-bipartite. Let $EH_i - EH_{i-1} = P \cup Q$, where P and Q are the two $H_i\overline{H_{i-1}}$-arcs. Let P join w to x and Q join y to z. We distinguish cases according to whether w, x, y, z are distinct.

Case 1. Suppose that w, x, y, z are distinct. Since H_{i-1} is 2-connected, we may assume by Theorem 3, the symmetry of w and x and the symmetry of y and z that H_{i-1} has vertex disjoint paths R joining w to y and S joining x to z. Similarly there are two vertex disjoint (VR, VS)-paths T and U in H_{i-1} having no internal vertex in $VR \cup VS$. Let T join vertex a in VR to vertex b in VS and let U join vertex c in VR to vertex d in VS. With no less generality we may assume that $c \in VR[a, y]$. Then $R[c, a] \cup T \cup S[b, d] \cup U$ is an even circuit C, since H_{i-1} is bipartite. Note also that $P \cup R[w, a] \cup T \cup S[b, x]$ and $Q \cup R[y, c] \cup U \cup S[d, z]$ are odd circuits A and B, respectively, for neither $G[H_{i-1} \cup P]$ nor $G[H_{i-1} \cup Q]$ is an even-circuit-connected graph because H_i is non-bipartite.

Let $G_0 := G[C]$ and $D = P \cup R \cup Q \cup S$. Then D is an even circuit since $D = A + B + C$. Furthermore observe that $G_1 := G[C \cup D]$ is a non-bipartite even-circuit-connected graph obtained from G_0 by a 2-arc adjunction. Thus the assertion follows from Lemma 4.

Case 2. In the remaining case observe that $w \neq x$, $y \neq z$ and $\{w, x\} \neq \{y, z\}$ for there exists an even circuit which includes $P \cup Q$ and meets EH_{i-1}. Thus we may assume that $x = y$ and $|\{w, x, z\}| = 3$ without loss of generality. If there are edges e and f in EH_{i-1} joining x to w and z respectively, then set $R = \{e\}$ and $S = \{f\}$. Otherwise, since H_{i-1} is 2-connected, x is joined in H_{i-1} by an edge g to a vertex v not in $\{w, x, z\}$. Without loss of generality we assume that there are vertex disjoint paths R' and S joining v to w and x to z, respectively. Set $R = R' \cup \{g\}$. By the 2-connectedness of H_{i-1} there exists a path T in $H_{i-1} - \{x\}$ joining a vertex a in VR to a vertex b in VS but having no inner vertex in $VR \cup VS$. Then $C := R[a, x] \cup S[x, b] \cup T$ is an even circuit for H_{i-1} is bipartite. Set

$$D = P \cup R[w, a] \cup T \cup S[b, z] \cup Q.$$

Observe that D is an even circuit since $D = C + P + R + Q + S$ and $P + R$ and $Q + S$ are odd circuits. Finally set $G_0 = G[C]$ and $G_1 = G[C \cup D]$. Then G_1 is a non-bipartite even-circuit-connected graph which can be obtained from G_0 by a 2-arc adjunction. Again the assertion follows from Lemma 4.

□

Remark 2 *In the previous proof G_1 is an even subdivision of E_2, E_3, O_2, V_1, V_2, V_3, V_4 or V_5, where V_1, V_2, V_3, V_4 and V_5 are the graphs depicted in Figure 6. Note that the graphs in Figure 6 are J-orientable with respect to any assignment J of clockwise parities.*

4 Proof of Theorem 1

For the rest of the paper let G be a graph and J an assignment of clockwise parities to the even circuits of G. Assume that G is minimally J-nonorientable with respect to the deletion of an edge. Let $G_0, G_1, \ldots, G_k$ be an arc decomposition of G, where G_i is obtained from G_{i-1} by a single arc adjunction for $i > 1$ and G_1 is isomorphic to an even subdivision of O_1, O_2, E_1, E_2, E_3, V_1, V_2, V_3, V_4 or V_5. Since all possibilities for G_1 are either in the list of graphs in Theorem 1 or J'-orientable with respect to any assignment J', we may assume that $k > 1$. Let $H := G_{k-1}$ and let P be the unique $\overline{H}$-arc. Fix a J-orientation of H and extend it to an orientation of G arbitrarily. Since G is J-nonorientable there exist two even circuits A and B including P such that A does not have the clockwise parity prescribed by J but B does. The following lemma shows that the even circuits A and B can be chosen with these properties so that $G[A \cup B]$ is fairly simple.

Lemma 5 *The even circuits A and B can be chosen so that $G[A \cup B]$ is isomorphic to an even subdivision of O_1, O_2, E_1, E_2, E_3, V_1, V_2, V_3, V_4 or V_5.*

Proof. We assume that A and B have been chosen with the properties above so that $A \cup B$ is minimal.

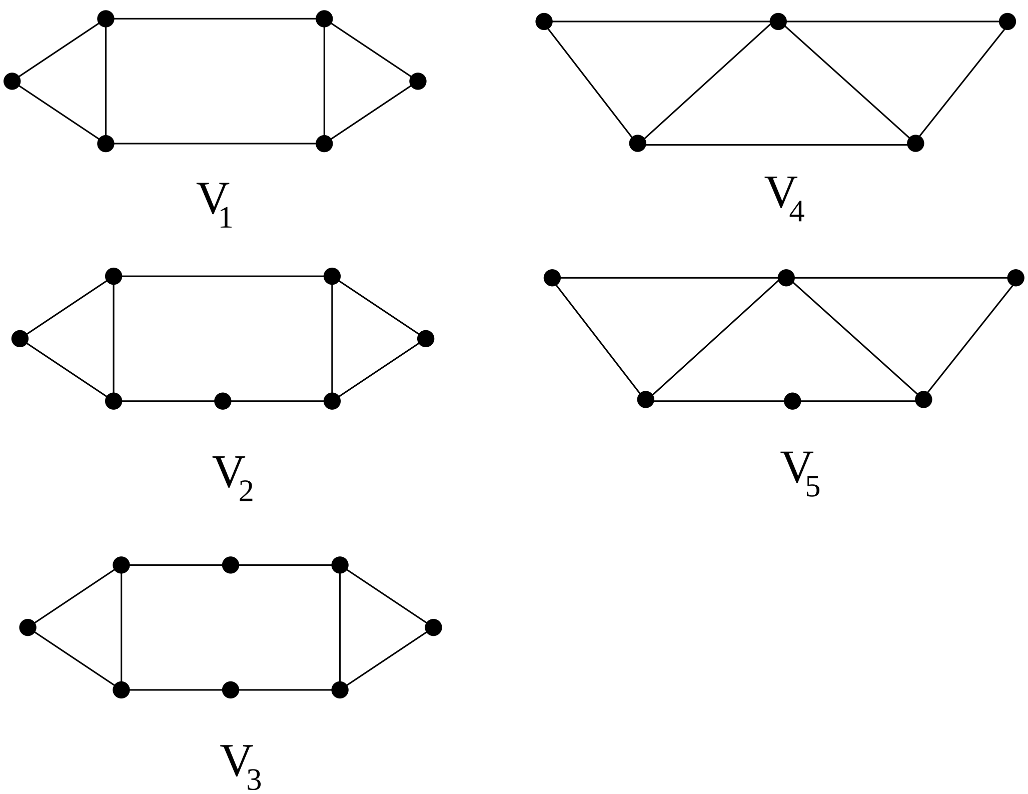

Figure 6:

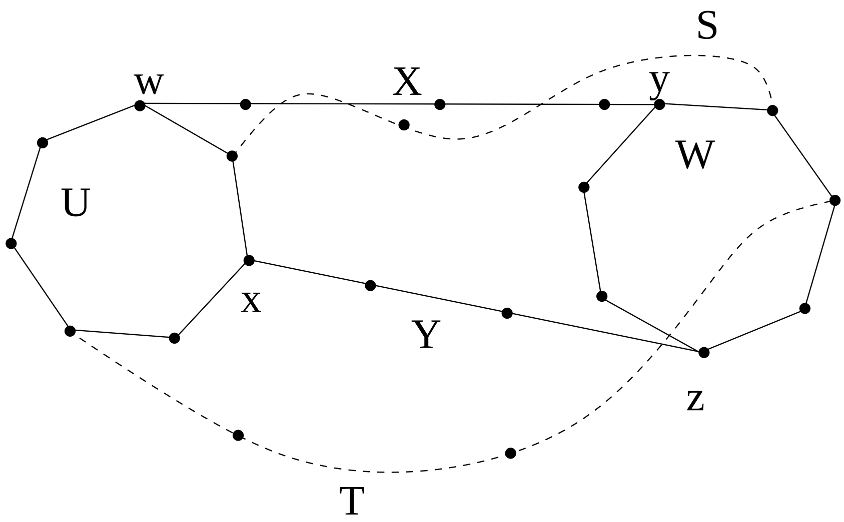

Figure 7: G is generated by the even circuits A and B and the two paths S and T. The paths S and T are dotted because they may intersect X and Y.

Let Q be the first $\overline{A}B$-arc we reach when traversing B in a particular direction starting at P and let R be the first $\overline{A}B$-arc we reach when traversing B in the opposite direction, again starting at P. If there exists an even circuit in $A \cup Q$ which includes $P \cup Q$ or an even circuit in $A \cup R$ which includes $P \cup R$ then let B' be this even circuit. Otherwise there exists an even circuit B' in $A \cup Q \cup R$ which includes $P \cup Q \cup R$.

First we show that B' has the clockwise parity prescribed by J. Suppose the contrary. The minimality of $A \cup B$ implies $A \cup B = B' \cup B$ and therefore $A + B' \subseteq B$. Furthermore $A + B'$ is non-empty and the union of circuits. Therefore $A + B' = B$, a contradiction to $P \subseteq B$.

If there is a unique $\overline{A}B'$-arc then $G[A \cup B']$ is isomorphic to an even subdivision of either O_1 or E_1. Otherwise $G[A \cup B']$ is isomorphic to an even subdivision of O_2, E_2, E_3, V_1, V_2, V_3, V_4 or V_5.

□

If $G[A \cup B]$ is an even subdivision of O_1, O_2, E_1, E_2 or E_3 then we have proved Theorem 1, for in these cases $A+B$ is an even circuit with the clockwise parity prescribed by J since $A+B \subseteq H$. Thus we may assume that for all choices of A and B the symmetric difference $A + B$ is the union of two edge-disjoint odd circuits U and W in H.

Since H is 2-connected there exist vertex disjoint (VU, VW)-paths S and T in H having no internal vertex in $VU \cup VW$. Note that $|VU \cap VW| \leq 1$ and if $|VU \cap VW| = 1$ we may choose S and T so that $VS = VU \cap VW$ and $VT \cap VU \cap VW = \emptyset$. Observe also that $G[S \cup T \cup U \cup W]$ contains exactly two even circuits C and D, and that $C+D = U+W$. In the following lemma we show that G is spanned by the even circuits A and B and the paths S and T. (See Figure 7.)

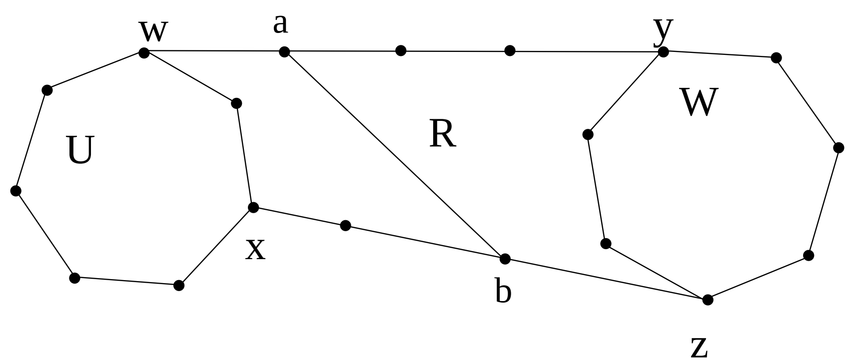

Figure 8: Situation in Lemma 7.

Lemma 6 $G = G[A \cup B \cup S \cup T]$.

Proof. The set $\{A, B, C, D\}$ of even circuits is J-intractable, for C and D both have the clockwise parity prescribed by J since $C \cup D \subseteq EH$. The assertion follows by the minimality of G.

□

The set $(A \cup B) - (U \cup W)$ is the union of two vertex disjoint paths X and Y if $VU \cap VW = \emptyset$. In this case let X join vertex w in U to vertex y in W and let Y join vertex x in U to vertex z in W. If $VU \cap VW \neq \emptyset$ then let X be the path $(A \cup B) - (U \cup W)$ and $VY = VU \cap VW$. Again we let X join vertex w in U to vertex y in W, but we also write $VU \cap VW = \{x\}$ and $z = x$ in this case. (See Figure 7.)

We may assume that A and B are chosen to minimise $A \cap B$. We complete the proof by showing, in the next series of lemmas, that there is no $\overline{A \cup B}$-arc joining a vertex of VX to a vertex of VY, no $\overline{A \cup B}$-arc joining a vertex in $VU - \{w, x\}$ to a vertex in $VW - \{y, z\}$ and no $\overline{A \cup B}$-arc joining a vertex in $(VU \cup VW) - \{w, x, y, z\}$ to a vertex in $VX \cup VY$, and that in the remaining case G is an even subdivision of a Δ-graph and J prescribes the even clockwise parity to an odd number of the even circuits in G.

Lemma 7 *There is no $\overline{A \cup B}$-arc joining a vertex of VX to a vertex of VY.*

Proof. Suppose such an arc R were to exist and that it connects vertex a in VX to vertex b in VY. (See Figure 8.) Being a subpath of S or T by Lemma 6, it cannot join w to x or y to z. Let C denote the unique even circuit in $A \cup B \cup R$ that includes $P \cup R$.

Suppose first that C is J-oriented. Then $\{A, C\}$ is a pair of even circuits, both including P, such that A is not J-oriented but C is. Therefore, by assumption, $A + C$ is the union of two odd circuits. This is, however, a contradiction to the minimality of $A \cap B$ since $A \cap C$ is either $X[w, a] \cup Y[x, b]$ or $X[a, y] \cup Y[b, z]$.

If C is not J-oriented then the same argument works if we replace A by B in the argument above.

□

Either S or T must have an $\overline{A \cup B}$-arc. Without loss of generality suppose that S has an $\overline{A \cup B}$-arc and, throughout the rest of the paper, let R be the first such arc encountered as S is traversed from its end in U to its end in W. From Lemma 7 we conclude that R cannot join a vertex of VX to a vertex of VY. However the next lemma shows that one end of R is in $VX \cup VY$.

Lemma 8 *There is no $\overline{A \cup B}$-arc that joins a vertex in $VU - \{w, x\}$ to a vertex in $VW - \{y, z\}$.*

Proof. First consider the case that $VU \cap VW = \emptyset$. Let Q be an $\overline{A \cup B}$-arc that joins a vertex in $VU - \{w, x\}$ to a vertex in $VW - \{y, z\}$.

Let E and F be the two even circuits in $U \cup W \cup Q \cup Y \cup X[a, y]$, where we assume without loss of generality that $P \subseteq X$. Since $E \cup F \subseteq H$, E and F are of the prescribed clockwise parity and $\{A, B, E, F\}$ is a J-intractable set of even circuits. Therefore $G = G[A \cup B \cup Q]$. Observe that if $G[A \cup E] = G$ or $G[B \cup F] = G$ then $G[A \cup F] \neq G$ and $G[B \cup E] \neq G$. By the symmetry of E and F we therefore assume that $G[A \cup E] \neq G$ and $G[B \cup F] \neq G$.

The symmetric difference $A + E$ is an even circuit in $U \cup W \cup Q \cup X$. Depending on the clockwise parity of $A + E$ either $\{A, E, A + E\}$ or $\{A + E, B, F\}$ is a J-intractable set of even circuits and therefore either $G = G[A \cup E]$ or $G = G[B \cup F]$, a contradiction.

Now we consider the case that $VU \cap VW \neq \emptyset$ and, therefore, $x = z$ and $P \subseteq X$. Let E and F be the two even circuits in $U \cup W \cup Q$. Since $E \cup F \subseteq H$, E and F have the clockwise parity prescribed by J and thus $\{A, B, E, F\}$ is a J-intractable set of circuits. There exists at least one even circuit M that includes Q and X. Moreover either $A + E = M$ or $A + F = M$. Without loss of generality let $A + E = M$. Then either $\{A, E, M\}$ or $\{B, F, M\}$ is a J-intractable set of circuits, which is a contradiction to the minimality of G for both $G[A \cup E]$ and $G[B \cup F]$ are graphs with maximal degree 3, whereas x is of degree at least 4 in G.

□

In the following lemma we show that both ends of R are either in VX or in VY.

Lemma 9 *There is no $\overline{A \cup B}$-arc that joins a vertex in $(VU \cup VW) - \{w, x, y, z\}$ to a vertex in $VX \cup VY$.*

Proof. Without loss of generality we assume that Q is an $\overline{(A \cup B)}$-arc that joins a vertex in $VU - \{w, x\}$ to a vertex a in $VX - \{w\}$. (See Figure 9.)

Let E and F be the two even circuits in $U \cup W \cup Q \cup Y$. If it is not possible to orient P so that E and F have the clockwise parity prescribed by J then $\{C, D, E, F\}$ would be a J-intractable set of circuits. This conclusion would be a contradiction to the minimality of G: $X[w, a]$ is not contained in $C \cup D \cup E \cup F$ since it cannot be contained in $S \cup T$ because S and T are vertex disjoint.

Therefore we may assume that E and F have the prescribed clockwise parity and $\{A, B, E, F\}$ is a J-intractable set of even circuits. Either $A + E$ or $A + F$ is equal to the

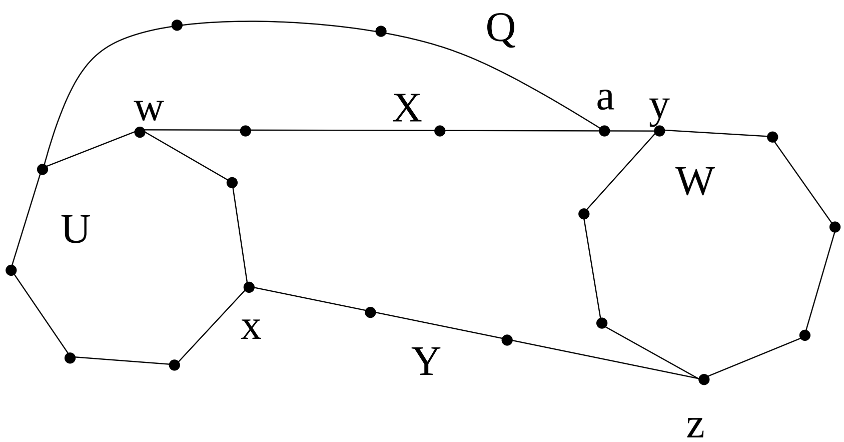

Figure 9: Situation in Lemma 9.

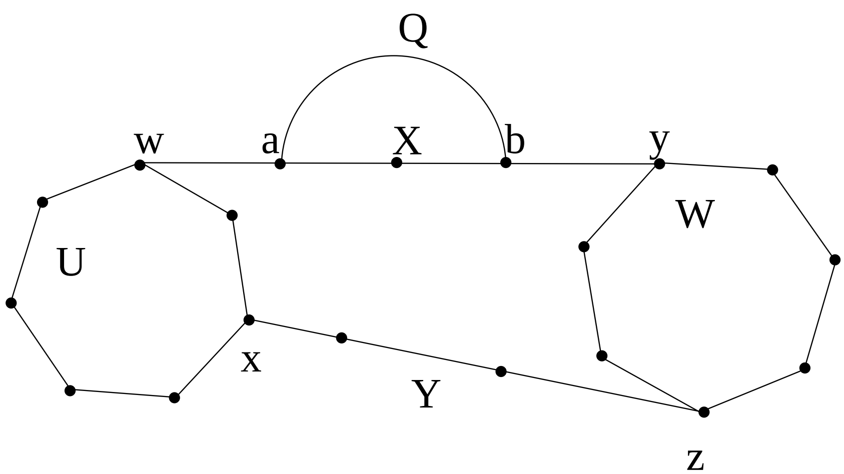

Figure 10: Situation in Lemma 10

unique even circuit in $U \cup Q \cup X$. Without loss of generality we assume that $A + E$ is equal to this even circuit. Then either $\{A, E, A + E\}$ or $\{A + E, B, F\}$ is a J-intractable set of even circuits, which is a contradiction to the minimality of G for neither $W \subseteq A \cup E$ nor $W \subseteq B \cup F$.

□

Thus R joins either two vertices in VX or two vertices in VY, as in Figure 10. In the following lemma we show that G is generated by the even circuits A and B and by the arc R and that G is an even subdivision of a Δ-graph.

Lemma 10 *If there is an $\overline{A \cup B}$-arc that joins two vertices in VX or two vertices in VY, then G is an even subdivision of a Δ-graph and J prescribes the even clockwise parity to an odd number of the even circuits of G.*

Proof. Let Q be an $\overline{A \cup B}$-arc which joins two vertices a and b in VX. We assume that $a \in VX[w, b]$. (See Figure 10.)

First we show that $G = G[A \cup B \cup Q]$. Let E and F be the two even circuits in $U \cup W \cup X[w, a] \cup Q \cup X[b, y] \cup Y$. As in the proofs of the previous lemmas we may orient P so that E and F have the prescribed clockwise parity. Otherwise $G = G[C \cup D \cup E \cup F]$ and we reach a contradiction: the fact that S and T are vertex disjoint implies that $Q \cup X[a, b] \nsubseteq S \cup T$, so that $X[a, b] \nsubseteq C \cup D \cup E \cup F$. We conclude that $\{A, B, E, F\}$ is a J-intractable set of even circuits.

Suppose $Q \cup X[a, b]$ is an even circuit M. Then either $A + E = M$ or $A + F = M$, and without loss of generality we assume $A + E = M$. Consequently, either $\{A, E, M\}$ or $\{B, F, M\}$ is a J-intractable set of even circuits. We now have a contradiction since neither $U \cup W \subseteq A \cup E$ nor $U \cup W \subseteq B \cup F$.

Thus $Q \cup X[a, b]$ is odd and G is an even subdivision of a Δ-graph.

□

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