A p, q-analogue of a Formula of Frobenius

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Abstract

Garsia and Remmel (JCT. A 41 (1986), 246-275) used rook configurations to give a combinatorial interpretation to the q-analogue of a formula of Frobenius relating the Stirling numbers of the second kind to the Eulerian polynomials. Later, Remmel and Wachs defined generalized p, q-Stirling numbers of the first and second kind in terms of rook placements. Additionally, they extended their definition to give a p, q-analogue of rook numbers for arbitrary Ferrers boards. In this paper, we use Remmel and Wach's definition and an extension of Garsia and Remmel's proof to give a combinatorial interpretation to a p, q-analogue of a formula of Frobenius relating the p, q-Stirling numbers of the second kind to the trivariate distribution of the descent number, major index, and comajor index over S_n. We further define a p, q-analogue of the hit numbers, and show analytically that for Ferrers boards, the p, q-hit numbers are polynomials in (p, q) with nonnegative coefficients.

1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. Let $[a, b] = \{n \in \mathbb{N} : a \leq n \leq b\}$ where $a, b \in \mathbb{N}$ and let $[n]$ denote the set $[1, n]$. We say that $B_n = [n] \times [n]$ is an n by n array of squares where the columns and rows are labelled from left to right and bottom to top respectively. Each square in the n by n grid will be called a *cell* and we denote the cell in the column i and row j by (i, j). A *board* will be a subset of cells in B_n.

Let $F(b_1, b_2, \ldots, b_n) \subseteq B_n$ denote the board whose column heights from left to right are b_1, b_2, …, b_n. We say that $F(b_1, b_2, \ldots, b_n)$ is a *Ferrers board* if $b_1 \leq b_2 \leq \cdots \leq b_n$.

Given a board $B \subseteq B_n$, we let $R_{k,n}(B)$ denote the set of all k element subsets $\mathbb{P}$ of B such that no two elements lie in the same row or column for nonnegative integers k. Such a subset $\mathbb{P}$ will be called a placement of nonattacking rooks in B. The cells in $\mathbb{P}$ are considered to contain rooks, so that we call $r_{k,n}(B) = |R_{k,n}(B)|$ the kth *rook number* of B. We note that for any board $B \subseteq B_n$, $r_{0,n}(B) = 1$, $r_{1,n}(B) = |B|$, and if $k > n$, then $r_{k,n}(B) = 0$.

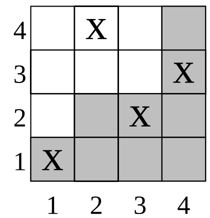

Figure 1: $\mathbb{P}_\sigma$ for $\sigma = 1\ \ 4\ \ 2\ \ 3 \in S_4$.

Given any permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ in the symmetric group S_n, we shall identify σ with the placement $\mathbb{P}_\sigma = \{(1,\sigma_1),(2,\sigma_2),\ldots,(n,\sigma_n)\}$. Let $H_{k,n}(B)$ denote the set of all placements $\mathbb{P}_\sigma$ such that $|\mathbb{P}_\sigma \bigcap B| = k$. Then $h_{k,n}(B) = |H_{k,n}(B)|$ is called the *kth hit number* of B. For example, suppose that $B = F(1,2,2,4) \in B_4$ and $\sigma = 1\ \ 4\ \ 2\ \ 3 \in S_4$. Then $\mathbb{P}_\sigma \in H_{3,4}(B)$ is pictured in Figure 1. The hit numbers and rook numbers are fundamentally related by the following formula of Riordan and Kaplansky [11], called the *hit polynomial*,

$$\sum_{k=0}^{n} h_{k,n}(B)x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k. \tag{1}$$

We define the q-analogues of n, $n!$, and $\binom{n}{k}$ respectively by $[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$, and

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-Stirling numbers of the second kind, denoted $S_{n,k}(q)$ can be defined as the solutions of the recursion

$$S_{n,k}(q) = q^{k-1}S_{n-1,k-1}(q) + [k]_q S_{n-1,k}(q) \tag{2}$$

with initial conditions $S_{0,0}(q) = 1$ and $S_{n,k}(q) = 0$ for $k < 0$ and $k > n$. These q-Stirling numbers of the second kind, introduced by Gould [6], have been given various combinatorial interpretations in terms of partitions, or equivalently, in terms of restricted growth functions ([10], [13], [14], and [15]), $0,1$-tableax ([8] and [9]), and rook placements ([4]).

In [4], Garsia and Remmel gave a combinatorial interpretation for $S_{n,k}(q)$ by q-counting the configurations of $n-k$ nonattacking rooks in the staircase board $S_n = F(0,1,\ldots,n-1)$. More generally, they defined for any Ferrers board $B \subseteq B_n$, the kth q-rook number by

$$r_{k,n}(B,q) = \sum_{\mathbb{P} \in R_{k,n}(B)} q^{u_B(\mathbb{P})},$$

where each rook in $\mathbb{P}$ cancels the cell it occupies plus all of the cells to the right and below it and where $u_B(\mathbb{P})$ is the number of uncancelled cells. In particular, when $B = F(0,1,\ldots,n-1)$, we have

$$S_{n,k}(q) = r_{n-k,n}(S_n, q),$$

since $r_{n-k,n}(S_n, q)$ satifies the recursion given in (2) with the same initial conditions. This can be seen by considering whether or not a placement in $R_{k,n}(S_n)$ contains a rook in the last column of the staircase board S_n.

For each $\sigma \in S_n$, we define the following permutation statistics,

$$Des(\sigma) = \{i \in [n-1] : \sigma(i) > \sigma(i+1)\},$$

$$des(\sigma) = |Des(\sigma)|,$$

$$maj(\sigma) = \textstyle\sum_{i \in Des(\sigma)} i, \quad \text{and}$$

$$comaj(\sigma) = \textstyle\sum_{i \in Des(\sigma)} (n-i).$$

In [4], Garsia and Remmel gave a combinatorial proof of the following q-analogue of a formula of Frobenius [3] relating the Stiring numbers of the second kind to the Eulerian polynomials,

$$\sum_{k=0}^{n} \frac{S_{n,k}(q)[k]!x^k}{(1-x)(1-xq)\cdots(1-xq^k)} = \frac{\sum_{\sigma \in S_n} x^{des(\sigma)+1} q^{maj(\sigma)}}{\prod_{i=0}^{n}(1-xq^i)}. \tag{3}$$

They further defined a q-analogue of the hit numbers for a given board B using the following q-analogue of (1),

$$\sum_{k=0}^{n} h_{k,n}(B,q)x^{n-k} = \sum_{k=0}^{n} r_{n-k,n}(B,q)x^k[k]_q!(1-xq^{k+1})\cdots(1-xq^n). \tag{4}$$

Using three recursions, Garsia and Remmel showed that for Ferrers boards, this polynomial has nonnegative coefficients. That is, they defined three operations on boards from which each Ferrers board could be obtained recursively from an empty board. The first operation, FLIP, replaces a board B by its conjugate board B^*. The second operation, ADD, adds a column of length zero, and the third operation, RAISE, increases each column length by one. With B^*, B^+, and $B\uparrow$ denoting FLIP(B), ADD(B), and RAISE(B) respectively, they obtained the following recursions on the q-hit numbers, $h_{k,n}(B,q)$ as defined by (4),

$$h_{k,n}(B^*,q) = h_{k,n}(B,q),$$
$$h_{k,n+1}(B^+,q) = [n-k+1]_q h_{k,n}(B,q) + q^{n-k}[k+1]h_{k+1,n}(B,q), \quad \text{and}$$
$$h_{k,n}(B\uparrow,q) = h_{k-1,n}(B,q).$$

Garsia and Remmel further showed that their q-analogue of the hit numbers could be realized by q-counting placements of n nonattacking rooks on $[n] \times [n]$ by a certain statistic S. Later, Dworkin [2] and Haglund [7] independently gave explicit combinatorial interpretations of such a statistic.

In this paper we give a p,q-analogue of the formula of Frobenius relating the trivariate distribution of $(des, maj, comaj)$ and the p,q-Stirling numbers of the second kind. In

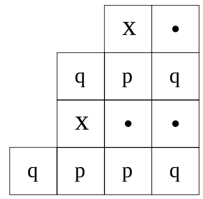

Figure 2: $\mathbb{P} \in R_{2,4}(B)$

addition, we define p, q-hit numbers using a p, q-analogue of (1). We show that for Ferrers boards, the p, q-hit numbers are polynomials in (p, q) with nonnegative coefficients by analytically proving three recursions which are similar to ones proved by Garsia and Remmel for the q-hit numbers.

2 A p, q-analogue of the rook numbers

The p, q-analogues of x and $x!$ are defined by $[x]_{p,q} := p^{x-1} + p^{x-2}q + \cdots + pq^{x-2} + q^{x-1} = (p^x - q^x)/(p - q)$ and $[x]_{p,q}! := [x]_{p,q}[x-1]_{p,q} \cdots [1]_{p,q}$ respectively.

Suppose that $B = F(b_1, b_2, \ldots, b_n) \subseteq B_n$ is a Ferrers board and let $\mathbb{P} \in R_{k,n}(B)$. If the rook $r \in \mathbb{P}$ is in the cell (i, j), then we let r *rook-cancel* those cells in the set $\{(a, j) : i \leq a \leq n\}$. That is, we let each rook cancel the square in which it resides plus all the squares directly to its right. As in [12], we set

$$r_{k,n}(B, p, q) := \sum_{\mathbb{P} \in R_{k,n}(B)} q^{\alpha_B(\mathbb{P}) + \varepsilon_B(\mathbb{P})} p^{\beta_B(\mathbb{P}) - (c_1 + \cdots + c_k)}$$

where $c_1, \ldots, c_k$ are the column labels of the k columns containing the rooks of $\mathbb{P}$ and where

$\alpha_B(\mathbb{P})$ = the number of cells of B which lie above a rook in $\mathbb{P}$ but are not rook-cancelled by any rook in $\mathbb{P}$,

$\beta_B(\mathbb{P})$ = the number of cells of B which lie below a rook in $\mathbb{P}$ but are not rook-cancelled by any rook in $\mathbb{P}$,

$\varepsilon_B(\mathbb{P})$ = the number of cells of B which lie a column with no rook in $\mathbb{P}$ and are not rook-cancelled by any rook in $\mathbb{P}$.

For example, if $B = F(1, 3, 4, 4) \subseteq B_4$ and $\mathbb{P} \in R_{2,4}(B)$ is the placement given in Figure 2, then $\alpha_B(\mathbb{P}) = 1$, $\beta_B(\mathbb{P}) = 3$, $\varepsilon_B(\mathbb{P}) = 3$, $c_1 = 2$, and $c_2 = 3$. So the p, q-contribution of $\mathbb{P}$ to $r_{2,4}(B, p, q)$ is $q^4 p^{-2}$.

As in the $p = 1$ case, $r_{n-k,n}(S_n, p, q)$ gives a p, q-analogue of the Stirling numbers of the second kind. That is, $r_{n-k,n}(S_n, p, q) = S_{n,k}(p, q)$ where $S_{n,k}(p, q)$ is defined by the following recursion.

$$S_{n+1,k}(p,q) = q^{k-1}S_{n,k-1}(p,q) + p^{-(n+1)}[k]_{p,q}S_{n,k}(p,q) \tag{5}$$

with intial conditions $S_{0,0}(p,q) = 1$ and $S_{n,k}(p,q) = 0$ if $k > n$ or $k < 0$.

The following theorem is a special case of the factorization theorem Remmel and Wachs proved in [12] with $i = 0$ and $j = 1$. The reader will recognize their proof as a generalization of that given in [5]. Furthermore, this proof justifies the need for the factor $p^{-(c_1+\cdots+c_k)}$ in the definition of the p,q-rook numbers.

Theorem 1 *Let $B = F(b_1, b_2, \ldots, b_n) \subseteq B_n$ be a Ferrers board. Then*

$$\sum_{k=0}^{n} r_{k,n}(B,p,q)p^{xk+\binom{k+1}{2}}[x]_{p,q}[x-1]_{p,q}\cdots[x-(n-k)+1]_{p,q} = \prod_{i=1}^{n}[x+b_i-(i-1)]_{p,q}. \tag{6}$$

Proof: For the given Ferrers board $B = F(b_1, b_2, \ldots, b_n) \subseteq B_n$, let B_x denote the board obtained from B by adjoining x rows of length n below B. The line dividing B with the x rows of lengths n will be called the *bar*. The x rows below the bar will be labelled from top to bottom by 1 through x.

Assume that $x \geq n$. The factorization is obtained by computing in two different ways the following sum,

$$\sum_{\mathbb{P}\in R_{n,n}(B_x)} q^{\alpha_{B_x}(\mathbb{P})}p^{\beta_{B_x}(\mathbb{P})}. \tag{7}$$

The first way is to consider the contribution to (7) of each column proceding from left to right. By placing a rook in each of the $b_1 + x$ cells in the first column from top to bottom we find that the contributions to (7) are respectively p^{b_1+x-1}, $p^{b_1+x-2}q$, $p^{b_1+x-3}q^2$, …, pq^{b_1+x-2}, q^{b_1+x-1}. So the total contribution from the first column to (7) is $[b_1 + x]_{p,q}$. Regardless of its placement, the rook in the first column will rook-cancel all of the cells to its right. That is, it will cancel exactly one cell in each of the $n-1$ columns to its right. Applying the same argument to the second column, we see that there are $b_2 + x - 1$ cells in which the rook in the second column can be placed, and so the contribution to (7) is $[b_2 + x - 1]_{p,q}$. Again, the rook placed in the second column will rook-cancel exactly one cell in each of the $n-2$ columns to its right. Thus the contribution to (7) from the third column will be $[b_3 + x - 2]_{p,q}$. Continuing in this way, we find that (7) equals

$$\prod_{i=1}^{n}[x+b_i-(i-1)]_{p,q}.$$

To compute (7) in a different way, fix a placement $\mathbb{P}$ of k rooks in B. We wish to compute to sum

$$\sum_{\substack{\mathbb{Q}\in R_{n,n}(B_x)\\ \mathbb{Q}\bigcap B=\mathbb{P}}} q^{\alpha_{B_x}(\mathbb{Q})}p^{\beta_{B_x}(\mathbb{Q})}.$$

For any $\mathbb{Q} \in R_{n,n}(B_x)$ such that $\mathbb{Q} \bigcap B = \mathbb{P}$, it is clear that the contribution of the weight of the cells above the bar to $q^{\alpha_{B_x}(\mathbb{P})} p^{\beta_{B_x}(\mathbb{P})}$ is $q^{\alpha_B(\mathbb{P})+\epsilon_B(\mathbb{P})} p^{\beta_B(\mathbb{P})}$. Suppose that the $n-k$ rooks of $\mathbb{P}$ are in columns $1 \leq c_1 \leq \cdots \leq c_k \leq n$. The cells below the bar in these columns which are not rook-cancelled will all be weighted by a p. The total number of such cells is

$$(x-(c_1-1)) + ((x-(c_2-2)) + \cdots + ((x-(c_k-k)) \\ = kx + \binom{k+1}{2} - (c_1 + \cdots + c_k).$$

That is, for each j, there are $(c_j - 1) - (j-1)$ rooks below the bar which lie to the left of column c_j. So there are $x - (c_j - j)$ uncancelled cells in column c_j that lie in column c_j below the bar. Thus the contribution to $q^{\alpha_{B_x}(\mathbb{P})} p^{\beta_{B_x}(\mathbb{P})}$ of the cells below the bar in columns $c_1, \ldots, c_k$ is $p^{kx+\binom{k+1}{2}-(c_1+\cdots+c_k)}$.

Finally, we must consider the contribution of the cells below the bar in the remaining $n-k$ columns. In the leftmost such column, there are x cells in which we could place a rook. Using the same analysis as above, we find that by placing the rook in the top cell of this column and proceeding downwards, we obtain the following respective p,q-weights: p^{x-1}, $p^{x-2}q$, $\ldots$, pq^{x-2}, q^{x-1}. Thus the contribution of this column to $q^{\alpha_{B_x}(\mathbb{P})} p^{\beta_{B_x}(\mathbb{P})}$ is $[x]_{p,q}$. Regardless of the placement of the rook in the leftmost column below the bar, it will rook-cancel exactly one cell in each of the columns to its right. Thus in the second column from the left, there will be $x-1$ cells in which the rook can be placed. Using the same argument, we find that the second column contributes a factor of $[x-1]_{p,q}$ to $q^{\alpha_{B_x}(\mathbb{P})} p^{\beta_{B_x}(\mathbb{P})}$. Continuing in this way, we find that the contribution of the remaining k columns is $[x]_{p,q}[x-1]_{p,q} \cdots [x-(n-k)+1]_{p,q}$. So,

$$\sum_{\substack{\mathbb{Q} \in R_{n,n}(B_x) \\ \mathbb{Q} \bigcap B = \mathbb{P}}} q^{\alpha_{B_x}(\mathbb{Q})} p^{\beta_{B_x}(\mathbb{Q})} \\ = q^{\alpha_B(\mathbb{P})+\epsilon_B(\mathbb{P})} p^{\beta_B(\mathbb{P})-(c_1+\cdots+c_k)} p^{kx+\binom{k+1}{2}} [x]_{p,q}[x-1]_{p,q} \cdots [x-(n-k)+1]_{p,q}.$$

Thus

$$\sum_{\mathbb{P} \in R_{n,n}(B_x)} q^{\alpha_{B_x}(\mathbb{P})} p^{\beta_{B_x}(\mathbb{P})} \\ = \sum_{k=0}^{n} \sum_{\mathbb{P} \in R_{k,n}(B)} q^{\alpha_B(\mathbb{P})+\epsilon_B(\mathbb{P})} p^{\beta_B(\mathbb{P})-(c_1+\cdots+c_k)} p^{kx+\binom{k+1}{2}} [x]_{p,q}[x-1]_{p,q} \cdots [x-(n-k)+1]_{p,q} \\ = \sum_{k=0}^{n} r_{k,n}(B,p,q) p^{kx+\binom{k+1}{2}} [x]_{p,q}[x-1]_{p,q} \cdots [x-(n-k)+1]_{p,q}.$$

Thus the equality (6) follows.

□

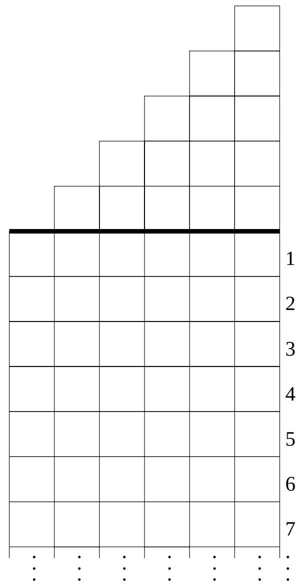

Figure 3: B_∞

3 A p,q-analogue of a formula of Frobenius

In this section we consider a p,q-analogue of (3). Before proving this, let us first consider the p,q-count of all placements of n-nonattacking rooks in the fullboard.

Lemma 2 *For each $n \in \mathbb{N}$,*

$$\sum_{\sigma \in S_n} q^{\alpha_{B_n}(\mathbb{P}_\sigma)} p^{\beta_{B_n}(\mathbb{P}_\sigma)} = [n]_{p,q}!. \tag{8}$$

Proof: This is easily proved by considering the contribution to the lefthand side of (8) of each column of B_n proceding from left to right. Based on the arguments used in the proof of Theorem 1, we find that the total contribution of the ith column from the left is exactly $[n-i+1]_{p,q}$, completing the proof.

□

The idea of our proof the p,q-Frobenius formula will be similar to that of Theorem 1. Suppose $B \subseteq B_n$ is a Ferrers board. Let B_∞ be the board obtained from B by adjoining infinitely many rows of length n below B as pictured in Figure 3. We call the dividing line between B and the added rows the *bar* and we label the added rows from top to bottom by $1, 2, \ldots$. We then have the following.

Theorem 3

$$\frac{1}{1-xp^n}\sum_{\mathbb{P}\in R_{n,n}(B_\infty)} q^{\alpha_{B_\infty}(\mathbb{P})}p^{\beta_{B_\infty}(\mathbb{P})}x^{\max(\mathbb{P})}$$
$$= \sum_{k=0}^{n} \frac{r_{n-k,n}(B,p,q)[k]_{p,q}!p^{\binom{n-k+1}{2}+k(n-k)}x^k}{\prod_{i=0}^{k}(1-xq^ip^{n-i})}. \qquad (9)$$

where

$\max(\mathbb{P})$ = *the level below the bar containing the bottom most rook of* $\mathbb{P}$,

$\alpha_{B_\infty}(\mathbb{P})$ = *the number of uncancelled cells above a rook in* B_∞,

$\beta_{B_\infty}(\mathbb{P})$ = *the number of uncancelled cells below a rook in* B_∞ *but weakly above the row labelled by* $\max(\mathbb{P})$.

Proof: Let's consider the contribution to

$$\sum_{\mathbb{P}\in R_{n,n}(B_\infty)} q^{\alpha_{B_\infty}(\mathbb{P})}p^{\beta_{B_\infty}(\mathbb{P})}x^{\max(\mathbb{P})}$$

from placements with exactly $n-k$ rooks above the bar for each $k=0,1,\dots,n$. As in [4], we can construct a placement making the following three choices:

1. A placement $\mathbb{Q}\in R_{n-k,n}(B)$,
2. k nonnegative integers giving the numbers of rows between the rooks below the bar, labelled p_1, …, p_k from bottom to top, and
3. A placement σ of k nonattacking rooks in the $k\times k$ board that results by considering those cells which lie in a row that contains a rook below the bar but is not contained in a column of a rook that lies above the bar. Note that σ can be considered as an element of S_k.

For example, Figure 4 shows a placement that would be obtained by choosing $\{(3,2)\}\in R_{1,4}(S_4)$, $p_1=2$, $p_2=1$, $p_3=0$, and $\sigma = 2\ 1\ 3$. This given, it is easy to see that the contribution to $q^{\alpha_{B_\infty}(\mathbb{P})}p^{\beta_{B_\infty}(\mathbb{P})}x^{\max(\mathbb{P})}$ can be separated in three parts. Let c_1, …, c_{n-k} be the column numbers of the rooks above the bar.

1. The contribution from the cells above the bar plus the cells below the bar that lie in the columns which contain rooks above the bar is

$$q^{\alpha_B(\mathbb{Q})+\varepsilon_B(\mathbb{Q})}p^{\beta_B(\mathbb{Q})}p^{\max(\mathbb{P})-(c_1-1))+(\max(\mathbb{P})-(c_2-2))+\cdots+(\max(\mathbb{P})-(c_{n-k}-(n-k))}.$$

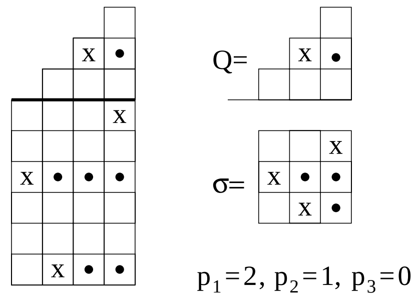

Figure 4: $\mathbb{P}_f$

2. The contribution from those cells below the bar which do not lie in a row with a rook below the bar and which are not counted in 1 is

$$(qp^{k-1})^{p_1}(q^2p^{k-2})^{p_2}\cdots(q^kp^0)^{p_k}.$$

3. The contribution from the cells which lie in either a row or column of a rook below the bar is

$$q^{\alpha_{B_k}(\mathbb{P}_\sigma)}p^{\beta_{B_k}(\mathbb{P}_\sigma)}.$$

It follows that for fixed k, we have

$$\sum_{\substack{\mathbb{P}\in R_{n,n}(B_\infty)\\ |\mathbb{P}\bigcap B|=n-k}} q^{\alpha_{B_\infty}(\mathbb{P})}p^{\beta_{B_\infty}(\mathbb{P})}x^{\max(\mathbb{P})} =$$
$$\sum_{\mathbb{Q}\in R_{n-k,n}(B)} q^{\alpha_B(\mathbb{Q})+\varepsilon_B(\mathbb{Q})}p^{\beta_B(\mathbb{Q})}\sum_{\sigma\in S_k} q^{\alpha_{B_k}(\mathbb{P}_\sigma)}p^{\beta_{B_k}(\mathbb{P}_\sigma)}$$
$$\sum_{p_1\geq 0}\sum_{p_2\geq 0}\cdots\sum_{p_k\geq 0} q^{p_1+2p_2+\cdots+kp_k}p^{(k-1)p_1+(k-2)p_2+p_{k-1}+\sum_{j=1}^{n-k}p_1+\cdots+p_k+k-(c_j-j))}x^{p_1+\cdots+p_k+k} \quad (10)$$

where $c_1, \ldots, c_{n-k}$ are the labels of those columns containing the $n-k$ rooks in $\mathbb{Q}$. Using Lemma 2 and simplifying, we find that (10) equals,

$$r_{n-k,n}(B,p,q)[k]_{p,q}!p^{\binom{n-k+1}{2}+k(n-k)}x^k\prod_{i=1}^{k}\sum_{p_i\geq 0}(xq^ip^{n-i})^{p_i}$$
$$= \frac{r_{n-k,n}(B,p,q)[k]_{p,q}!p^{\binom{n-k+1}{2}+k(n-k)}x^k}{\prod_{i=1}^{k}(1-xq^ip^{n-i})}.$$

Summing (10) over all k and dividing by $\frac{1}{1-xp^n}$ yields

$$\frac{1}{1-xp^n}\sum_{\mathbb{P}\in R_{n,n}(B_\infty)} q^{\alpha_{B_\infty}(\mathbb{P})}p^{\beta_{B_\infty}(\mathbb{P})}x^{\max(\mathbb{P})}$$

$$= \sum_{k=0}^{n} \frac{r_{n-k,n}(B,p,q)[k]_{p,q}!p^{\binom{n-k+1}{2}+k(n-k)}x^k}{\prod_{i=0}^{k}(1-xq^ip^{n-i})}. \quad (11)$$

□

Theorem 4 *For each natural number n,*

$$\sum_{k=0}^{n} \frac{S_{n,k}(p,q)[k]_{p,q}!p^{\binom{n-k+1}{2}+k(n-k)}x^k}{\prod_{i=0}^{k}(1-xq^ip^{n-i})} = \frac{\sum_{\sigma\in S_n} q^{maj(\sigma)}p^{comaj(\sigma)}x^{des(\sigma)+1}}{\prod_{i=0}^{n}(1-xq^ip^{n-i})}. \quad (12)$$

Proof: Let $\mathcal{F} = \{f : \{1,\ldots,n\} \to \mathbb{N} = \{0,1,2,\ldots\}\}$. Then for each $f \in \mathcal{F}$, set

$$|f| := \sum_{i=1}^{n} f(i) \qquad \text{and}$$

$$\max(f) := \max_{i=1,\ldots,n}\{f(i)\}.$$

We will prove (12) by computing in two different ways the sum

$$\frac{x}{1-xp^n}\sum_{f\in\mathcal{F}} x^{\max(f)}q^{|f|}p^{n\cdot\max(f)-|f|}. \quad (13)$$

For a given function $f \in \mathcal{F}$, we order its range values in decreasing order, $k_1 > k_2 > \cdots > k_t$ and for each $i = 1,\ldots,t$, we define

$$A_{k_i} = \{b : f(b) = k_i\}.$$

This given, we associate to f a permutation $\sigma(f) = \sigma = A_{k_1}\!\uparrow A_{k_2}\!\uparrow \cdots A_{k_t}\!\uparrow \in S_n$ where $A_{k_i}\!\uparrow$ is the set of values in A_{k_i} arranged in increasing order. We then define the following values,

$$p_i = \begin{cases} f(\sigma_i) - f(\sigma_{i+1}) & \text{if } 1 \le i \le n-1 \\ f(\sigma_n) & \text{if } i = n. \end{cases}$$

Given these values, it then follows that

$$\max(f) = f(\sigma_1) = p_1 + p_2 + \cdots + p_n \quad \text{and}$$
$$|f| = p_1 + 2p_2 + \cdots + np_n.$$

Now let's consider the possible values for p_i. Note that by the definition of $\sigma(f) = \sigma$ from the function $f \in \mathcal{F}$, if $\sigma_i < \sigma_{i+1}$, then either we switch from set A_{k_j} to $A_{k_{j+1}}$ for some j or $f(\sigma_i) = f(\sigma_{i+1})$. On the other hand, if $\sigma_i > \sigma_{i+1}$, then it must be the case that we are switching from set A_{k_j} to $A_{k_{j+1}}$ for some j. It then follows that

$$\begin{aligned}
&\frac{x}{1-xp^n}\sum_{f\in\mathcal{F}} x^{\max(f)} q^{|f|} p^{n\cdot\max(f)-|f|} \\
&= \frac{x}{1-xp^n}\sum_{\sigma\in S_n}\sum_{p_1\geq\chi(1\in Des(\sigma))}\cdots\sum_{p_n\geq\chi(n\in Des(\sigma))} x^{p_1+\cdots+p_n} q^{p_1+\cdots+np_n} p^{n(p_1+\cdots+p_n)-(p_1+\cdots+np_n)} \\
&= \frac{x}{1-xp^n}\sum_{\sigma\in S_n}\sum_{p_1\geq\chi(1\in Des(\sigma))}(xqp^{n-1})^{p_1}\cdots\sum_{p_n\geq\chi(n\in Des(\sigma))}(xq^n)^{p_n}.
\end{aligned}$$

Now if $i \notin Des(\sigma)$, then

$$\sum_{p_i\geq\chi(i\in Des(\sigma))}(xq^ip^{n-i})^{p_i} = \sum_{p_i\geq 0}(xq^ip^{n-i})^{p_i} = \frac{1}{1-xq^ip^{n-i}}.$$

On the other hand, if $i \in Des(\sigma)$, then

$$\sum_{p_i\geq\chi(i\in Des(\sigma))}(xq^ip^{n-i})^{p_i} = \sum_{p_i\geq 1}(xq^ip^{n-i})^{p_i} = \frac{xq^ip^{n-i}}{1-xq^ip^{n-i}}.$$

So it follows by the definition of des, maj, and $comaj$, that

$$\frac{x}{1-xp^n}\sum_{f\in\mathcal{F}} x^{\max(f)} q^{|f|} p^{n\cdot\max(f)-|f|} = \frac{\sum_{\sigma\in S_n} x^{des(\sigma)+1} q^{maj(\sigma)} p^{comaj(\sigma)}}{\prod_{i=0}^{n}(1-xq^ip^{n-i})}.$$

Now to compute the sum in (13) in another way, note that we can associate to each $f \in \mathcal{F}$ a rook placement $\mathbb{P}_f \in R_{n,n}((S_n)_\infty)$ where $f(i)$ is the number of uncancelled cells above the rook in the ith column. For example, if f is the function with $f(1) = 2$, $f(2) = 5$, $f(3) = 0$, and $f(4) = 2$, then the corresponding rook placement is given in Figure 4.

It is easy to see that $|f| = \alpha_{(S_n)_\infty}(\mathbb{P}_f)$, $n\cdot\max(f) - |f| = \beta_{(S_n)_\infty}(\mathbb{P}_f)$. We claim that $\max(f) + 1 = \max(\mathbb{P}_f)$. To see this, note that in any column the height of the staircase board is the same as the number of cells cancelled by rooks from the left. Therefore, we see that from the definition of f, $\max(\mathbb{P}_f)$ is obtained in the column in which f is maximum. Furthermore, in the column where f is maximum, the rook must be placed in the row $\max(f) + 1$ below the bar and hence $max(f) + 1 = \max(\mathbb{P}_f)$ as claimed.

Thus we have

$$\frac{x}{1-xp^n}\sum_{f\in\mathcal{F}} x^{\max(f)} q^{|f|} p^{n\cdot\max(f)-|f|} = \frac{1}{1-xp^n}\sum_{\mathbb{P}\in R_{n,n}((S_n)_\infty)} q^{\alpha_{(S_n)_\infty}(\mathbb{P})} p^{\beta_{(S_n)_\infty}(\mathbb{P})} x^{\max(\mathbb{P})}. \quad (14)$$

By our comments preceding the theorem, we know that the right hand side of (14) is just

$$\sum_{k=0}^{n} \frac{S_n(p,q)[k]_{p,q}!\ p^{\binom{n-k+1}{2}+k(n-k)} x^k}{\prod_{i=0}^{k}(1-xq^i p^{n-i})}.$$

□

Before proceeding, we pause to observe an interesting corollary that follows from Theorem 4.

Corollary 5 *Let n be a natural number. Then for each integer k,*

$$S_{n,k}(\frac{1}{q},q) = q^{\binom{n+1}{2}+n-k(n+1)} S_{n,k}(q^2). \tag{15}$$

Proof: This is an immediate consequence of Theorem 4. To prove it, we first set $p = 1/q$.

$$\sum_{k=0}^{n} \frac{S_{n,k}(\frac{1}{q},q)[k]_{\frac{1}{q},q}! q^{-\binom{n-k+1}{2}-k(n-k)} x^k}{\prod_{i=0}^{k}(1-xq^i(\frac{1}{q})^{n-i})}$$
$$= \sum_{\sigma\in S_n} q^{maj(\sigma)} (\frac{1}{q})^{comaj(\sigma)} x^{des(\sigma)+1} \prod_{i=0}^{n}(1-xq^i(\frac{1}{q})^{n-i}).$$

Replacing x by xq^n and noting that

$$[k]_{\frac{1}{q},q}! = q^{-\binom{k}{2}}[k]_{q^2}! \text{ and } (\frac{1}{q})^{comaj(\sigma)}(xq^n)^{des(\sigma)+1} = q^{maj(\sigma)+n} x^{des(\sigma)+1}$$

yields

$$\sum_{k=0}^{n} \frac{S_{n,k}(\frac{1}{q},q)[k]_{q^2}! q^{-\binom{k}{2}-\binom{n-k+1}{2}-k(n-k)+nk} x^k}{\prod_{i=0}^{k}(1-xq^{2i})} = \frac{q^n \sum_{\sigma\in S_n} q^{2maj(\sigma)} x^{des(\sigma)+1}}{\prod_{i=0}^{n}(1-xq^{2i})}.$$

Using the fact that $\binom{k}{2}+\binom{n-k+1}{2}+k(n-k) = \binom{n+1}{2}-k$ and simplfying, we get

$$\begin{aligned} \sum_{k=0}^{n} \frac{S_{n,k}(\frac{1}{q},q)[k]_{q^2}! q^{k(n+1)} x^k}{\prod_{i=0}^{k}(1-xq^{2i})} &= \frac{q^{\binom{n+1}{2}+n} \sum_{\sigma\in S_n} q^{2maj(\sigma)} x^{des(\sigma)+1}}{\prod_{i=0}^{n}(1-xq^{2i})} \\ &= q^{\binom{n+1}{2}+n} \sum_{k=0}^{n} \frac{S_{n,k}(q^2)[k]_{q^2}! x^k}{\prod_{i=0}^{k}(1-xq^{2i})}. \end{aligned} \tag{16}$$

Here the last equality follows from (3). It is then easy to see (16) implies our result.

□

We note that one could also prove Corollary 5 directly from the recursions given in (2) and (5).

4 A p, q-analogue of the hit numbers

Let B be a board in B_n and define the *p, q-hit polynomial* of B, denoted $H_B(x, p, q)$, as the following.

$$\sum_{k=0}^{n} h_{k,n}(B, p, q)x^k = \sum_{k=0}^{n} r_{k,n}(B, p, q)[n-k]_{p,q}!p^{\binom{k+1}{2}+k(n-k)} \prod_{l=n-k+1}^{n}(x - q^l p^{n-l}) \qquad (17)$$

We will call the kth coefficient of $H_B(x, p, q)$ the *kth p, q-hit number* of B. We wish to show that $H_B(x, p, q)$ has positive coefficients when $B = F(b_1, \ldots, b_n)$ is a Ferrers board. It is not difficult to see that $H_B(x, p, q) = x^n Q_B(x^{-1}, p, q)$ where

$$Q_B(x, p, q) = \sum_{k=0}^{n} r_{k,n}(B, p, q)x^{n-k}[n-k]_{p,q}!p^{\binom{k+1}{2}+k(n-k)} \prod_{i=n-k+1}^{n}(1 - xp^{n-i}q^i). \qquad (18)$$

It follows from Theorem 3 that if we define $\Phi(x; b_1, \ldots, b_n)$ by

$$\Phi(x; b_1, \ldots, b_n) = \sum_{\mathbb{P} \in R_{n,n}(B_\infty)} q^{\alpha_{B_\infty}(\mathbb{P})} p^{\beta_{B_\infty}(\mathbb{P})} x^{\max(\mathbb{P})}, \qquad (19)$$

then

$$\Phi(x; b_1, \ldots, b_n) = \frac{Q_B(x, p, q)}{\prod_{i=1}^{n}(1 - xq^i p^{n-i})} = \frac{\sum_{k=0}^{n} h_{n-k}(B, p, q)x^k}{\prod_{i=1}^{n}(1 - xq^i p^{n-i})}. \qquad (20)$$

In order to show that $H_B(x, p, q)$ has nonnegative coefficients, we will show that the coefficients in the numerator of $\Phi(x; b_1, \ldots, b_n)$ are nonnegative by using recursions which are similar to ones used by Garsia and Remmel in [4] to prove that the q-hit numbers of Ferrers boards are polynomials in q with nonnegative integer coefficients.

We begin by showing that the hit polynomial of the empty board, $E_n = F(0^n) \in B_n$, has positive coefficients. This follows immediately from equation (11) by noting that all n rooks must be placed below the bar. Namely, we have

Corollary 6 *Let n be a natural number. Then,*

$$\Phi(x; 0^n) = \frac{[n]_{p,q}!x^n}{(1 - xp^{n-1}q)(1 - xp^{n-2}q^2) \cdots (1 - xq^n)}.$$

We now define three geometric operations, each of which preserves the positivity of the p, q-hit polynomial. We call these operations SHIFT, RAISE, and ADD.

The first operation, SHIFT, can only be applied to boards whose first column is empty. In particular, SHIFT replaces the Ferrers board $B = F(0, b_2, \ldots, b_n) \in B_n$ by the Ferrers board $\overleftarrow{B} = F(b_2, \ldots, b_{n-1}, n) \in B_n$. That is, $\overleftarrow{B}$ is the board in B_n obtained from B by shifting all of the columns of B to the left and adding a column of height n to the righthand side of B. It turns out that $\Phi(x; 0, b_2, \ldots, b_n)$ and $\Phi(x; b_2, \ldots, b_n, n)$ have a nice relationship given by the following lemma.

Lemma 7 *Let $B = F(0, b_2, \dots, b_n) \in B_n$. Then*

$$\Phi(x; b_2, \dots, b_n, n) = \frac{1}{x}\Phi(x; 0, b_2, \dots, b_n). \tag{21}$$

Proof: We first note that

$$r_{k,n}(\overleftarrow{B}, p, q) = p^k q^{n-k} r_{k,n}(B, p, q) + [n-k+1]_{p,q} p^{-n+k-1} r_{k-1,n}(B, p, q). \tag{22}$$

The first term on the right hand side of (22) accounts for the placements in $R_{k,n}(\overleftarrow{B})$ with no rook in column n. All such placements can be obtained from placements of k nonattacking rooks in the board B by shifting the board B and the rooks to the left one cell and adjoining a column of length n to the right hand side. Note that in shifting all of the rooks to the left, each of the k column labels decreases by one, resulting in the factor of p^k. Additionally, each of the $n-k$ uncancelled cells in the last column of $\overleftarrow{B}$ will be weighted with a q, accounting for the factor of q^{n-k} in the first term.

The second term accounts for those placements in $R_{k,n}(\overleftarrow{B})$ with a rook in column n. Each of these placements can be obtained from a placement of $k-1$ nonattacking rooks in the board B by shifting the board B and the rooks to left one cell, adjoining a column of length n to the right, and placing the rook in the one of the uncancelled cells in the last column. Again, in shifting, the column labels decrease by one, so we gain a factor of p^{k-1}. Since there is a rook in the last column, there will also be a factor of p^{-n}. Finally, by the usual argument, we find that in placing the rook in the last column a factor of $[n-k+1]_{p,q}$ is gained.

We use this recursion on the p, q-rook numbers of B to write $\Phi(x; b_2, \dots, b_n, n)$ in terms of $\Phi(x; 0, b_2, \dots, b_n)$.

$$\begin{aligned} &\Phi(x; b_2, \dots, b_n, n) \qquad (23)\\ = \quad & \frac{\sum_{k=0}^{n} p^k q^{n-k} r_{k,n}(B, p, q) x^{n-k} [n-k]_{p,q}! p^{\binom{k+1}{2}+k(n-k)} \prod_{i=n-k+1}^{n}(1 - xq^i p^{n-i})}{\prod_{i=1}^{n}(1 - xq^i p^{n-i})} \\ + \quad & \frac{\sum_{k=0}^{n} r_{k-1,n}(B, p, q) x^{n-k} [n-k+1]_{p,q}! p^{\binom{k+1}{2}+k(n-k)-n+k-1} \prod_{i=n-k+1}^{n}(1-xq^i p^{n-i})}{\prod_{i=1}^{n}(1 - xq^i p^{n-i})}. \end{aligned}$$

Note that $r_{-1,n}(B, p, q) = 0$ for any board B and since $B = F(0, b_2, \dots, b_n)$, $r_{n,n}(B, p, q) = 0$. Reindexing the second summation in (23) gives

$$\frac{\sum_{k=0}^{n-1} p^k q^{n-k} r_{k,n}(B,p,q) x^{n-k} [n-k]_{p,q}! p^{\binom{k+1}{2}+k(n-k)} \prod_{i=n-k+1}^{n}(1-xq^ip^{n-i})}{\prod_{i=1}^{n}(1-xq^ip^{n-i})} \tag{24}$$

$$+\frac{\sum_{k=0}^{n-1} r_{k,n}(B,p,q) x^{n-k-1} [n-k]_{p,q}! p^{\binom{k+2}{2}+(k+1)(n-k-1)-n+k} \prod_{i=n-k}^{n}(1-xq^ip^{n-i})}{\prod_{i=1}^{n}(1-xq^ip^{n-i})}$$

$$=\frac{\sum_{k=0}^{n-1} p^k q^{n-k} r_{k,n}(B,p,q) x^{n-k} [n-k]_{p,q}! p^{\binom{k+1}{2}+k(n-k)} \prod_{i=n-k+1}^{n}(1-xq^ip^{n-i})}{\prod_{i=1}^{n}(1-xq^ip^{n-i})}$$

$$+\frac{\frac{1}{x}\sum_{k=0}^{n-1} r_{k,n}(B,p,q) x^{n-k} [n-k]_{p,q}! p^{\binom{k+1}{2}+k(n-k)} (1-xq^{n-k}p^k) \prod_{i=n-k+1}^{n}(1-xq^ip^{n-i})}{\prod_{i=1}^{n}(1-xq^ip^{n-i})}.$$

Cancelling we find that (24) is exactly

$$\frac{1}{x}\Phi(x;0,b_2,\ldots,b_n).$$

□

Corollary 8 *Let $B = F(0,b_2,\ldots,b_n) \in B_n$ be a Ferrers board. Then for each integer k,*

$$h_{k,n}(\overleftarrow{B},p,q) = h_{k-1,n}(B,p,q). \tag{25}$$

Proof: From equation (17) we know that

$$\Phi(x;b_2,\ldots,b_n,n) = \frac{\sum_{k=0}^{n} h_{k,n}(\overleftarrow{B},p,q)x^{n-k}}{(1-xqp^{n-1})(1-xq^2p^{n-2})\ldots(1-xq^n)},$$

and likewise

$$\Phi(x;0,b_2,\ldots,b_n) = \frac{\sum_{k=0}^{n} h_{k,n}(B,p,q)x^{n-k}}{(1-xqp^{n-1})(1-xq^2p^{n-2})\ldots(1-xq^n)}.$$

So it easily follows that

$$\sum_{k=0}^{n} h_{k,n}(\overleftarrow{B},p,q)x^{n-k} = \sum_{k=0}^{n} h_{k,n}(B,p,q)x^{n-k-1}.$$

Taking the coefficient of x^{n-k} on both sides yields the desired result.

□

The second operation, RAISE, replaces the Ferrers board $B = F(b_1, \ldots, b_n)$ with the board $B\uparrow = F(b_1 + 1, \ldots, b_n + 1)$. With this defined, we have the following lemma.

Lemma 9 *If $B = F(b_1, \ldots, b_n)$ is a Ferrers board with $b_n \leq n - 1$, then*

$$\Phi(x; b_1 + 1, \ldots, b_n + 1) = \frac{1}{x}\Phi(x; b_1, \ldots, b_n). \tag{26}$$

Proof: We begin by noting that since $b_n \leq n - 1$, $\max(\mathbb{P}) \geq 1$ for any $\mathbb{P} \in R_{n,n}(B_\infty)$. Now suppose we define the map $\gamma : R_{n,n}(B_\infty) \to R_{n,n}(B\uparrow_\infty)$ so that the rooks in $\gamma(\mathbb{P})$ are placed in the cells directly above the cells containing the rooks of $\mathbb{P}$. Then clearly we see that γ is a p, q-weight-preserving bijection between $R_{n,n}(B_\infty)$ and $R_{n,n}(B\uparrow_\infty)$. We also see that $\max(\gamma(\mathbb{P})) = \max(\mathbb{P}) - 1 \geq 0$. The result now follows by using the definition in (19).

□

Corollary 10 *If $B = F(b_1, \ldots, b_n)$ is a Ferrers board with $b_n \leq n - 1$, then for each integer k,*

$$h_{k,n}(B\uparrow, p, q) = h_{k-1,n}(B, p, q). \tag{27}$$

Proof: This is proved in the same way that Corollary 8 was proved for the SHIFT operation.

□

The third operation, ADD, simply adjoins a column of height zero to the left of the given board. That is, if we apply the ADD operation to the Ferrers board $B = F(b_1, \ldots, b_n)$, the resulting board is $B^+ = F(0, b_1, \ldots, b_n)$ and is contained in B_{n+1}. Before we show how the ADD operation effects $\Phi(x; b_1, \ldots, b_n)$, we must define the p, q-derivative operator, denoted $\delta_{p,q}$. For any formal power series $F(x)$, let

$$\delta_{p,q}F(x) = \frac{F(px) - F(qx)}{x(p - q)}.$$

One can easily check that

$$\delta_{p,q}x^n = [n]_{p,q}x^{n-1}.$$

This is used to prove the following lemma.

Lemma 11 *If $B = F(b_1, \ldots, b_n)$ is a Ferrers board, then*

$$\Phi(x; 0, b_1, \ldots, b_n) = \frac{x\Phi(qx; b_1, \ldots, b_n)}{(1 - xqp^n)} + px^2\delta_{p,q}\Phi(x; b_1, \ldots, b_n). \tag{28}$$

Proof: We begin by noting that

$$\begin{aligned}\Phi(x; b_1, \ldots, b_n)& \\ = &\sum_{\mathbb{P}\in R_{n,n}(B)} q^{\alpha_B(\mathbb{P})+\varepsilon_B(\mathbb{P})}p^{\beta_B(\mathbb{P})} + \sum_{k\geq 1} x^k \sum_{j=1}^{n} \left(p^{j-1}q^{k+b_j-j}[k + b_1 - 1]_{p,q} \times \cdots\right. \\ & \left.\times\ [k + b_{j-1} - j + 1]_{p,q}[k + b_{j+1} - j]_{p,q} \cdots [k + b_n - n + 1]_{p,q}\right). \end{aligned} \tag{29}$$

The first term on the right hand side of (29) accounts for those placements in $R_{n,n}(B_\infty)$ whose bottommost rook is in B, while the second term accounts for those placements in $R_{n,n}(B_\infty)$ whose bottommost is below the bar. The second term further breaks up according to the column in which the bottommost rook is contained. Suppose that the bottommost rook in $\mathbb{P}$ is contained in the kth row below the bar and the jth column for $1 \leq k$ and $1 \leq j \leq n$. Then we can compute

$$q^{\alpha_{B_\infty}(\mathbb{P})}p^{\beta_{B_\infty}(\mathbb{P})}x^{\max(\mathbb{P})} \tag{30}$$

column by column. Clearly, there will be $j-1$ uncancelled cells to the left of the bottommost rook all of which will be weighted with a p. In addition, in the jth column there will be exactly $k + b_j - 1 - (j - 1) = k + b_j - j$ uncancelled cells above the bottommost rook each of which will be weighted with a q. Now in columns $1, \ldots, j-1$, there will be respectively $k + b_1 - 1, \ldots, k + b_{j-1} - j + 1$ uncancelled cells that we have not already considered above. In columns $j+1, \ldots, n$ there will be respectively $k + b_{j+1} - j, \ldots,$ $k + b_n - n + 1$ uncancelled cells. So, by the usual argument we find that the columns $1, \ldots, j-1, j+1, \ldots, n$ respectively contribute a total p,q-count of $[k + b_1 - 1]_{p,q}, \ldots,$ $[k + b_{j-1} - j + 1]_{p,q}$, $[k + b_{j+1} - j]_{p,q}, \ldots, [k - b_n - n + 1]_{p,q}$ to (30).

Now let's consider the board $B^+ = F(0, b_1, \ldots, b_n)$. By definition,

$$\Phi(x; 0, b_1, \ldots, b_n) = \sum_{\mathbb{P}\in R_{n+1,n+1}(B^+)} q^{\alpha_{B^+_\infty}(\mathbb{P})}p^{\beta_{B^+_\infty}(\mathbb{P})}x^{\max(\mathbb{P})}.$$

If $\max(\mathbb{P}) = 0$ for some $\mathbb{P} \in R_{n+1,n+1}(B_\infty)$, then by definition, the bottommost rook of a placement is above the bar. The set of all such placements is empty since there is no way to place $n+1$ nonattacking rooks in at most n nonempty columns. Similarly, we see that the only placements $\mathbb{P} \in R_{n+1,n+1}(B_\infty)$ with $\max(\mathbb{P}) = 1$ are those whose bottommost rook is in column 1. Thus, it follows that,

$$\Phi(x; 0, b_1, \ldots, b_n) = x \sum_{\substack{\mathbb{P}\in R_{n+1,n+1}(B^+) \\ \text{lowest rook in column 1}}} q^{\alpha_{B^+_\infty}(\mathbb{P})}p^{\beta_{B^+_\infty}(\mathbb{P})} + \sum_{k\geq 2} x^k \sum_{\substack{\mathbb{P}\in R_{n+1,n+1}(B^+) \\ max(\mathbb{P})=k}} q^{\alpha_{B^+_\infty}(\mathbb{P})}p^{\beta_{B^+_\infty}(\mathbb{P})}. \tag{31}$$

Now

$$\sum_{k\geq 2} x^k \sum_{\substack{\mathbb{P}\in R_{n+1,n+1}(B^+)\\ max(\mathbb{P})=k}} q^{\alpha_{B^+_\infty}(\mathbb{P})} p^{\beta_{B^+_\infty}(\mathbb{P})} \tag{32}$$

$$= \sum_{k\geq 2} x^k \sum_{\substack{\mathbb{P}\in R_{n+1,n+1}(B^+)\\ max(\mathbb{P})=k\\ \text{lowest rook in column 1}}} q^{\alpha_{B^+_\infty}(\mathbb{P})} p^{\beta_{B^+_\infty}(\mathbb{P})} + \sum_{k\geq 2} x^k \sum_{j=2}^{n+1} \big(p^{j-1} q^{k+b_{j-1}-j}[k-1]_{p,q}$$
$$\times [k+b_1-2]_{p,q}\cdots[k+b_{j-2}-j+1]_{p,q}[k+b_j-j]_{p,q}\cdots[k+b_n-n]_{p,q}\big).$$

After factoring out $[k-1]_{p,q}$ and changing the index of summation, we find that the second term in the right hand side of (32) is

$$px^2\sum_{k\geq 2}[k-1]_{p,q}x^{k-2}\sum_{j=1}^{n} p^{j-1}q^{k+b_j-j-1}[k+b_1-2]_{p,q}\times\cdots \tag{33}$$
$$\times[k+b_{j-1}-j]_{p,q}[k+b_{j+1}-j-1]_{p,q}\cdots[k-b_n-n]_{p,q}.$$

Reindexing with respect to k, we see (33) is equal to

$$px^2\delta_{p,q}\sum_{k\geq 1}x^k\sum_{j=1}^{n}\big(p^{j-1}q^{k+b_j-j-1}[k+b_1-1]_{p,q}\cdots[k+b_{j-1}-j+1]_{p,q}$$
$$\times[k+b_{j+1}-j]_{p,q}\cdots[k-b_n-n+1]_{p,q}\big).$$

Comparing this to (29), we see that the second term of (32) equals

$$px^2\delta_{p,q}\left(\Phi(x;b_1,\ldots,b_n) - \sum_{\mathbb{P}\in R_{n,n}(B)} q^{\alpha_B(\mathbb{P})+\varepsilon_B(\mathbb{P})}p^{\beta_B(\mathbb{P})}\right).$$

Since $\delta_{p,q}\sum_{\mathbb{P}\in R_{n,n}(B)} q^{\alpha_B(\mathbb{P})+\varepsilon_B(\mathbb{P})}p^{\beta_B(\mathbb{P})}=0$, it follows that

$$\Phi(x;0,b_1,\ldots,b_n) = \sum_{k\geq 1} x^k \sum_{\substack{\mathbb{P}\in R_{n+1,n+1}(B^+)\\ max(\mathbb{P})=k\\ \text{lowest rook in column 1}}} q^{\alpha_{B^+_\infty}(\mathbb{P})} p^{\beta_{B^+_\infty}(\mathbb{P})} + px^2\delta_{p,q}\Phi(x;b_1,\ldots,b_n). \tag{34}$$

In order to prove the theorem, it remains to show that the first term in the sum of the right hand side of (34) is just the first term in the sum of (28). We begin by considering the contribution to

$$\sum_{\substack{\mathbb{P}\in R_{n+1,n+1}(B^+)\\ \text{lowest rook in column 1}}} q^{\alpha_{B^+_\infty}(\mathbb{P})} p^{\beta_{B^+_\infty}(\mathbb{P})} x^{max(\mathbb{P})}$$

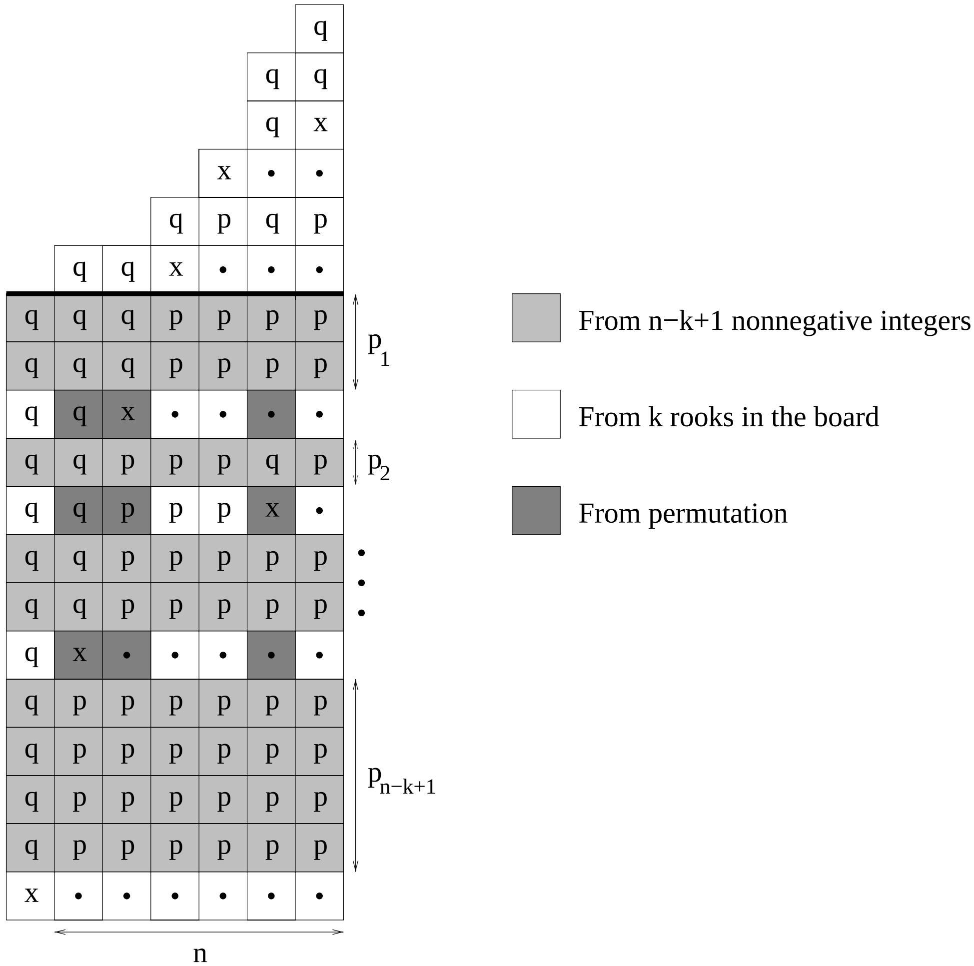

Figure 5: Placement $\mathbb{P} \in R_{n+1,n+1}(B^+)$ with bottommost rook in column 1

coming from the placements with exactly k rooks in B^+ (hence a placement of k rooks in B). Our proof will look similar to our discussion preceding Theorem 4. Namely, an arbitrary placement $\mathbb{P} \in R_{n+1,n+1}(B^+)$ can be obtained by choosing a placement $\mathbb{Q} \in R_{k,n}(B^+) = R_{k,n}(B)$, $n-k+1$ nonnegative integers giving the number of rows between the rooks below the bar, and a permutation $\sigma \in S_{n-k}$ whose corresponding placement $\mathbb{P}_\sigma$ gives the relative positions of the $n-k$ rooks in columns $2,\ldots,n+1$ below the bar as seen in Figure 5.

Again, we consider the p-count, q-count, and x-count coming from the rows above the bar, the rows below the bar with a rook, and the rows below the bar without a rook. As before, the placement $\mathbb{Q}$ yields the following p,q-count for the rows above the bar,

$$q^{\alpha_B(\mathbb{Q})+\varepsilon_B(\mathbb{Q})}p^{\beta_B(\mathbb{Q})}.$$

Next we see that the nonnegative integers $p_1, \ldots, p_{n-k+1}$ give us the following p,q,x-count for the rows below the bar without a rook,

$$\frac{1}{(1-xp^nq)(1-xp^{n-1}q^2)\cdots(1-xp^kq^{n-k+1})}.$$

Finally, the columns containing the k rooks in B along with the choice of $\sigma \in S_{n-k}$ yields the following p,q,x-count for the rows below the bar containing a rook,

$$(qx)^{n-k}[n-k]_{p,q}!p^{(n-k-(c_1-1))+(n-k-(c_2-2))+\cdots+(n-k-(c_k-k))}.$$

We pick up an additional factor of x due to the rook in column 1 below the bar so that the total contribution from the rows which contain a rook below the bar is

$$x(qx)^{n-k}[n-k]_{p,q}!p^{k(n-k)-((c_1-1)+(c_2-2)+\cdots+(c_k-k))}.$$

Thus it follows that

$$\sum_{\substack{\mathbb{P}\in R_{n+1,n+1}(B_\infty^+)\\ |\mathbb{P}\bigcap B^+|=k}} q^{\alpha_{B_\infty^+}(\mathbb{P})}p^{\beta_{B_\infty^+}(\mathbb{P})}x^{\max(\mathbb{P})} = x\ \frac{r_{k,n}(B,p,q)(qx)^{n-k}[n-k]_{p,q}!p^{\binom{k+1}{2}+k(n-k)}}{(1-xp^nq)(1-xp^{n-1}q^2)\cdots(1-xp^kq^{n-k+1})}.$$

Summing over $k=0,\ldots,n$ yields

$$x\ \sum_{k=0}^{n}\frac{r_{k,n}(B,p,q)(qx)^{n-k}[n-k]_{p,q}!p^{\binom{k+1}{2}+k(n-k)}}{\prod_{i=0}^{n-k}(1-(qx)q^ip^{n-i})}. \tag{35}$$

It then follows from Theorem 3 that (35) equals

$$\frac{x\Phi(qx;b_1,\ldots,b_n)}{(1-xqp^n)}.$$

Thus the theorem is proved.

□

The following lemma shows the effect of the $\delta_{p,q}$ operator on an arbitrary function of the form $\frac{Q(x,p,q)}{\prod_{i=1}^{n}(1-xq^ip^{n-i})}$ where $Q(x,p,q)$ is an arbitrary polynomial of x degree n whose coeficients are functions in p and q.

Lemma 12 *Suppose* $Q(x,p,q)=\sum_{k=0}^{n}a_k(p,q)x^k$, *and*

$$\Phi(x;b_1,\ldots,b_n)=\frac{Q_B(x,p,q)}{\prod_{i=1}^{n}(1-xq^ip^{n-i})}.$$

Then

$$\delta_{p,q}\Phi(x;b_1,\ldots,b_n)=\frac{\sum_{k=1}^{n}(a_k(p,q)[k]+a_{k-1}(p,q)p^{k-1}q^k[n-k+1])x^{k-1}}{(1-xp^nq)(1-xp^{n-1}q^2)\cdots(1-xq^{n+1})}.$$

Proof:

$\delta_{p,q}\Phi(x;b_1,\ldots,b_n)$

$$= \frac{1}{x(p-q)}\left(\frac{\sum_{k=0}^{n} a_k(p,q)p^k x^k}{(1-xp^nq)\cdots(1-xpq^n)} - \frac{\sum_{k=0}^{n} a_k(p,q)q^k x^k}{(1-xp^{n-1}q^2)\cdots(1-xq^{n+1})}\right)$$

$$= \frac{\left(\sum_{k=0}^{n} a_k(p,q)p^k x^k\right)(1-xq^{n+1}) - \left(\sum_{k=0}^{n} a_k(p,q)q^k x^k\right)(1-xp^nq)}{x(p-q)(1-xp^nq)(1-xp^{n-1}q^2)\cdots(1-xpq^n)(1-xq^{n+1})}$$

$$= \frac{\left(\sum_{k=0}^{n} a_k(p,q)x^k(p^k-q^k)\right) + \left(\sum_{k=0}^{n} a_k(p,q)x^{k+1}(p^nq^{k+1}-p^kq^{n+1}\right)}{x(p-q)(1-xp^nq)(1-xp^{n-1}q^2)\cdots(1-xpq^n)(1-xq^{n+1})}.$$

Note for our last expression, the lowest term in the first sum of numerator and highest term in the second sum of the numerator are 0 so that we have the following.

$\delta_{p,q}\Phi(x;b_1,\ldots,b_n)$

$$= \frac{\sum_{k=1}^{n}\left(a_k(p,q)(p^k-q^k) + a_{k-1}(p,q)(p^nq^k - p^{k-1}q^{n+1})\right)x^{k-1}}{(p-q)(1-xp^nq)(1-xp^{n-1}q^2)\cdots(1-xpq^n)(1-xq^{n+1})}$$

$$= \frac{\sum_{k=1}^{n}\left(a_k(p,q)[k] + a_{k-1}(p,q)p^{k-1}q^k[n-k+1]\right)x^{k-1}}{(1-xp^nq)(1-xp^{n-1}q^2)\cdots(1-xq^{n+1})}.$$

□

Corollary 13 *Let $B = F(b_1,\ldots,b_n) \in B_n$ be a Ferrers board. Then*

$$h_{k,n+1}(B^+,p,q) = \begin{cases} h_{k,n}(B,p,q)[n-k+1]_{p,q} + h_{k+1,n}(B,p,q)(pq)^{n-k}[k+1]_{p,q} & \text{for } k=0,\ldots,n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Again we use the fact that

$$\Phi(x;b_1,\ldots,b_n) = \frac{\sum_{k=0}^{n} h_{n-k}(B,p,q)x^k}{(1-xp^{n-1}q)(1-xp^{n-2}q^2)\cdots(1-xq^n)}.$$

It immediately follows from this and Lemma 12 that $\delta_{p,q}\Phi(x;b_1,\ldots,b_n)$ equals

$$\frac{\sum_{k=1}^{n}\left(h_{n-k}(B,p,q)[k]_{p,q}+h_{n-k+1}(B,p,q)p^{k-1}q^{k}[n-k+1]_{p,q}\right)x^{k-1}}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}. \tag{36}$$

From Lemma 11 it follows that

$$\Phi(x;0,b_1,\ldots,b_n)=\frac{x\Phi(qx;b_1,\ldots,b_n)}{(1-xqp^{n})}+px^{2}\delta_{p,q}\Phi(x;b_1,\ldots,b_n)$$

$$=\frac{\sum_{k=0}^{n}h_{n-k}(B,p,q)q^{k}x^{k+1}}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}$$

$$+\frac{\sum_{k=1}^{n}p\left(h_{n-k}(B,p,q)[k]_{p,q}+h_{n-k+1}(B,p,q)p^{k-1}q^{k}[n-k+1]_{p,q}\right)x^{k+1}}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}$$

$$=\frac{h_n(B,p,q)x}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}$$

$$+\frac{\sum_{k=1}^{n}\left(h_{n-k}(B,p,q)(q^{k}+p[k]_{p,q})+h_{n-k+1}(B,p,q)(pq)^{k}[n-k+1]_{p,q}\right)x^{k+1}}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}$$

$$=\frac{h_n(B,p,q)x}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}$$

$$+\frac{\sum_{k=1}^{n}\left(h_{n-k}(B,p,q)([k+1]_{p,q})+h_{n-k+1}(B,p,q)(pq)^{k}[n-k+1]_{p,q}\right)x^{k+1}}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}$$

$$=\frac{\sum_{k=2}^{n+1}\left(h_{n-k+1}(B,p,q)[k]_{p,q}+h_{n-k+2}(B,p,q)(pq)^{k-1}[n-k+2]_{p,q}\right)x^{k}}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}$$

$$+\frac{h_n(B,p,q)x}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}. \tag{37}$$

But, by definition

$$\Phi(x;0,b_1,\ldots,b_n)=\frac{\sum_{k=0}^{n+1}h_{n-k+1}(B^{+},p,q)x^{k}}{(1-xp^{n}q)(1-xp^{n-1}q^{2})\cdots(1-xq^{n+1})}. \tag{38}$$

Thus if we multiply $\Phi(x; 0, b_1, \ldots, b_n)$ by $(1 - xp^n q)(1 - xp^{n-1}q^2) \cdots (1 - xq^{n+1})$ and then take the coefficient of x^k in each of (37) and (38) for each integer k yields the desired recursion.

□

We now show that each of the three operations preserves positivity of the coefficients in $Q_B(x, p, q)$. Division by x, as in Lemmas 7 and 9, trivially preserves the positivity. Lemma 12 shows that the same is true for the $\delta_{p,q}$ operator on Φ. This lemma, together with the fact that multiplication by px^2, replacement of x with qx in $\Phi(x, b_1, \ldots, b_n)$, and division by $(1 - xp^n q)$ all preserve positivity of coefficients, proves that positivity is preserved by the ADD operation. Thus all three operations analytically preserve the positivity of the coefficients in $Q_B(x, p, q)$.

Our remarks at the beginning of this section and the next theorem proves that the p, q-hit polynomial does in fact have nonnegative integer coefficients.

Theorem 14 *If $B = F(b_1, \ldots, b_n) \in B_n$ is any Ferrers board, then $Q_B(x, p, q)$ has nonnegative integer coefficients.*

Proof: Our argument is similar to the one given in the proof of Theorem 2.1 in [4]. Namely, we argue by induction on the size of the board. Corollary 6 provides the proof for empty boards (of size zero) with any number of columns. Assume that the theorem is true for boards of size less than N with any number of columns. Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board in B_n of size $N \geq 1$ with m columns of nonzero height.

1. Suppose that $m = n$, that is, $b_i \geq 1$ for each $i = 1, \ldots, n$. Then B could be obtained from the Ferrers board $F(b_1 - 1, \ldots, b_n - 1)$ of size $N - n < N$ by the RAISE operation.

2. Suppose that $m < n$. Here we must consider two subcases.

   (a) Suppose that $b_n = n$. Then by the SHIFT operator, B can be obtained from the Ferrers board $F(0, b_1, \ldots, b_{n-1}) \in B_n$ of size $N - n < N$.

   (b) Suppose that $b_n < n$. Then B can be obtained from a Ferrers board of size N to which case 1 or case 2a applies in $n - \max\{m, b_n\}$ ADD operations.

In any case, we see that B can be obtained from a Ferrers board of size smaller than N in at most $n - \max\{m, b_n\} + 1$ applications of the three operations. Thus the theorem is proved by induction.

□

We will now give an example to illustrate the procedure outlined in the proof of Theorem 14. Figure 6 shows a sequence of operations from which the Ferrers board $F(1, 2, 2, 4)$ can be obtained. The following set of computations shows how to compute the

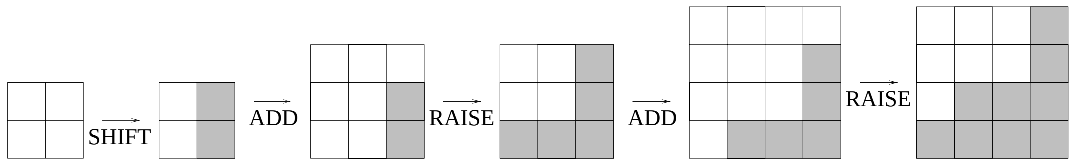

Figure 6: Operations to obtain $F(1,2,2,4)$

function $\Phi(x;1,2,2,4)$. One can easily check the result using the definitions in equations (18) and (19). We begin by applying Corollary 6 with $n=2$ to get

$$\Phi(x;0,0)=\frac{[2]_{p,q}x^2}{(1-xpq)(1-xq^2)}. \tag{39}$$

Lemma 7 together with equation (39) gives the following.

$$\Phi(x;0,2)=\frac{[2]_{p,q}x}{(1-xpq)(1-xq^2)} \tag{40}$$

Applying the ADD operation to $F(0,2)$ results in

$$\begin{aligned}\Phi(x;0,0,2) &= \frac{x}{1-xp^2q}\left(\frac{[2]_{p,q}qx}{(1-xpq^2)(1-xq^3)}\right)\\ &\quad+px^2\left(\frac{[2]_{p,q}(1+xpq^2)}{(1-xp^2q)(1-xpq^2)(1-xq^3)}\right)\\ &= \frac{[2]^2_{p,q}x^2+[2]_{p,q}p^2q^2x^3}{(1-xp^2q)(1-xpq^2)(1-xq^3)}. \end{aligned} \tag{41}$$

by Lemma 11 and equation (40). By application of Lemma 9, we divide (41) through by x to get

$$\Phi(x;1,1,3)=\frac{[2]^2_{p,q}x+[2]_{p,q}p^2q^2x^2}{(1-xp^2q)(1-xpq^2)(1-xq^3)}.$$

Applying the ADD operation once more results in

$$\Phi(x;0,1,1,3) = \frac{x}{1-xp^3q}\left(\frac{[2]_{p,q}^2 qx + [2]_{p,q}p^2q^4x^2}{(1-xp^2q^2)(1-xpq^3)(1-xq^4)}\right)$$

$$+ px^2\left(\frac{[2]_{p,q}^2 + ([2]_{p,q}^2p^2q^2 + [2]_{p,q}^3pq^2)x + [2]_{p,q}p^4q^5x^2}{(1-xp^2q^2)(1-xpq^3)(1-xq^4)}\right)$$

$$= \frac{[2]_{p,q}^3x^2 + (2p^5q^2 + 5p^4p^3 + 5p^3q^4 + 2p^2q^5)x^3 + [2]_{p,q}p^5q^5x^4}{(1-xp^3q)(1-xp^2q^2)(1-xpq^3)(1-xq^4)}.$$

Finally, raising the board $F(0,1,1,3)$ to obtain $F(1,2,2,4)$ results in

$$\Phi(x;1,2,2,4) = \frac{[2]_{p,q}^3x + (2p^5q^2 + 5p^4p^3 + 5p^3q^4 + 2p^2q^5)x^2 + [2]_{p,q}p^5q^5x^3}{(1-xp^3q)(1-xp^2q^2)(1-xpq^3)(1-xq^4)}.$$

In conclusion, we have shown that if B is a Ferrers board, then $h_k(B,p,q)$ is a polynomial in p and q with nonnegative integer coefficients. It is natural to ask whether there is a pair of statistics $s_{1,B}$ and $s_{2,B}$ such that

$$h_k(B,p,q) = \sum_{\substack{\sigma\in S_n \\ |\sigma\cap B|=k}} p^{s_{1,B}(\sigma)}q^{s_{2,B}(\sigma)}. \tag{42}$$

The first author has found such a pair of statistics such that $s_{2,B}$ coincides with Dworkin's statistic for the q-hit numbers. In [1], the first author proves that the combinatorial definition of the $h_k(B,p,q)$'s via (42) is the same as the definition of the $h_k(B,p,q)$'s given by (17) by giving direct combinatorial proofs that the combinatorial definition of the $h_k(B,p,q)$'s satisfies the three recursions (25), (27), and (36). In fact, such direct combinatorial proofs of the three recursions are new even for the q-hit numbers. We should note, however, that there is a significant difference between the q-hit numbers and the p,q-hit numbers. Namely, Dworkin [2] showed that the q-hit numbers defined by equation (4) are polynomials in q with nonnegative coefficients for any skyline board $F(b_1,\ldots,b_n)$. However it is not the case that p,q-hit numbers defined by equation (17) are polynomials in p and q with nonnegative coefficients for all skyline boards $F(b_1,\ldots,b_n)$. We refer the reader to [1] for more details.

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