Propagation of mean degrees

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Abstract

We propose two alternative measures of the local irregularity of a graph in terms of its vertex degrees and relate these measures to the order and the global irregularity of the graph measured by the difference of its maximum and minimum vertex degree.

1 Introduction

All graphs will be simple and finite. Let $G = (V, E)$ be a graph of order $n = |V|$. The degree and the neighbourhood of a vertex $u \in V$ will be denoted by $d(u)$ and $N(u)$. The maximum and minimum degree of G will be denoted by $\Delta(G)$ and $\delta(G)$.

A graph G is usually called *regular* if $\Delta(G) = \delta(G)$ which trivially implies that $d(u) = d(v)$ for all edges $uv \in E$. In view of this convention, we considered in [5] the expressions $\Delta(G) - \delta(G)$ and $\max\{|d(u) - d(v)|, uv \in E\}$ as suitable measures of the *global* and *local irregularity* of G, respectively. The main results of [5] are asymptotically tight lower bounds on the order of a connected graph in terms of its global and local irregularity. The intuition behind these bounds is that the global irregularity of a connected graph with bounded local irregularity can only be large if its order is large.

Following suggestions of M. Kouider and J.-F. Saclé [3] we will consider here two alternative measures of local irregularity. Again, our main results relate the order of the graph, its global irregularity and one of these measures.

A reasonable requirement for a possible measure of local irregularity is that it should be zero for a connected graph if and only if the global irregularity is zero. It is easy to see that $\Delta(G) - \delta(G) = 0$ for a connected graph G if and only if

$$\sum_{v \in N(u)} |d(v) - d(u)| = 0 \text{ for every } u \in V \tag{1}$$

or

$$\frac{1}{d(u)} \sum_{v \in N(u)} |d(v) - d(u)| = 0 \text{ for every } u \in V. \tag{2}$$

The terms in (1) and (2) are the total and the mean *deviation* of the degrees of the neighbours from the degree of some vertex. For further notions of irregularity in graphs cf. e.g. [1] and [2].

2 Results

Throughout this section let $G = (V, E)$ be a connected graph of order $n \geq 2$, maximum degree Δ and minimum degree δ. Let $V_i = \{u \in V | d(u) = i\}$ and $n_i = |V_i|$ for $i \in \mathbf{Z}$. The proof of the next lemma should remind the reader of *Markov's inequality* (cf. e.g. [4])

Lemma 1 *Let $\delta \leq i \leq \Delta$ and let $\delta + 1 \leq j \leq \Delta$. Let $u_i \in V_i$ and $\nu \in \mathbf{N}_0 = \{0, 1, 2, ...\}$.*
(i) If $\sum_{v \in N(u)} |d(v) - d(u)| \leq \alpha$ for every $u \in V$ and for some $\alpha \in \mathbf{N}$, then

$$\max\{n_k | j - \alpha \leq k \leq j - 1\} > 0$$

and

$$\sum_{\mu : |\mu - i| \leq \nu} |V_\mu \cap N(u_i)| \geq i - \frac{\alpha}{\nu + 1}.$$

(ii) If $\frac{1}{d(u)} \sum_{v \in N(u)} |d(v) - d(u)| \leq \alpha$ for every $u \in V$ and for some $\alpha > 0$, then

$$\max\{n_k | \frac{j}{\alpha + 1} \leq k \leq j - 1\} > 0$$

and

$$\sum_{\mu : |\mu - i| \leq \nu} |V_\mu \cap N(u_i)| \geq (1 - \frac{\alpha}{\nu + 1}) i.$$

Proof: We will only prove (ii). The proof of (i) will then be immediate.

If $\frac{j}{\alpha+1} \leq \delta$, then the first statement is trivial. Hence we assume $\delta < \frac{j}{\alpha+1}$ and $\max\{n_k | \frac{j}{\alpha+1} \leq k \leq j - 1\} = 0$. Since G is connected, there is an edge $uv \in E$ such that $d(u) < \frac{j}{\alpha+1}$ and $d(v) > j - 1$. This implies the contradiction $\frac{|d(v) - d(u)|}{d(u)} > \frac{j - \frac{j}{\alpha+1}}{\frac{j}{\alpha+1}} = \alpha$ and the first part of (ii) is proved. Furthermore, we have

$$\begin{aligned} \sum_{\mu : |\mu - i| > \nu} |V_\mu \cap N(u_i)| &\leq \frac{1}{\nu + 1} \sum_{\mu : |\mu - i| > \nu} |\mu - i| |V_\mu \cap N(u_i)| \\ &\leq \frac{1}{\nu + 1} \sum_{\mu} |\mu - i| |V_\mu \cap N(u_i)| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\nu+1} \sum_{v \in N(u_i)} |d(v) - d(u_i)| \\
&= \frac{1}{\nu+1} \frac{i}{d(u_i)} \sum_{v \in N(u_i)} |d(v) - d(u_i)| \\
&\leq \frac{\alpha}{\nu+1} i.
\end{aligned}
$$

In view of $\sum_\mu |V_\mu \cap N(u_i)| = d(u_i) = i$, the desired bound follows and the proof is complete. □

Theorem 1 *Let $G = (V, E)$ be as above and let $\nu \in \mathbf{N}_0$. Let $\alpha \in \mathbf{N}$ be such that for every $u \in V$*

$$
\sum_{v \in N(u)} |d(v) - d(u)| \leq \alpha.
$$

Then

$$
n \geq \frac{1}{2\nu+\alpha} (\Delta - \delta) \left(\frac{\Delta+\delta}{2} - \frac{\alpha}{\nu+1} \right).
$$

Proof: Let $i_0 = \Delta$ and for $l \geq 1$ let $i_l = \max\{k | k \leq i_{l-1} - 2\nu - 1 \text{ and } n_k > 0\}$. By Lemma 1 (i), $i_l \geq i_{l-1} - 2\nu - \alpha$ is well-defined for $0 \leq l \leq l_{\max} := \lfloor \frac{\Delta-\delta}{2\nu+\alpha} \rfloor$. Since $n_{i_l} > 0$ for $0 \leq l \leq l_{\max}$, we can choose $u_{i_l} \in V_{i_l}$. For $0 \leq l \leq l_{\max}$ the sets $\bigcup_{\mu : |\mu - i_l| \leq \nu} V_\mu$ are mutually disjoint and we obtain by Lemma 1 (i)

$$
\begin{aligned}
n &= \sum_{i=\delta}^{\Delta} n_i \\
&\geq \sum_{l=0}^{l_{\max}} \sum_{\mu : |\mu - i_l| \leq \nu} n_\mu \\
&\geq \sum_{l=0}^{l_{\max}} \sum_{\mu : |\mu - i_l| \leq \nu} |V_\mu \cap N(u_{i_l})| \\
&\geq \sum_{l=0}^{l_{\max}} \left(i_l - \frac{\alpha}{\nu+1} \right) \\
&\geq \sum_{l=0}^{l_{\max}} \left(\Delta - (2\nu+\alpha) l - \frac{\alpha}{\nu+1} \right) \\
&= \left(\Delta - \frac{\alpha}{\nu+1} \right) (l_{\max} + 1) - \frac{2\nu+\alpha}{2} l_{\max} (l_{\max} + 1) \\
&= (l_{\max} + 1) \left(\Delta - \frac{\alpha}{\nu+1} - \frac{2\nu+\alpha}{2} l_{\max} \right) \\
&\geq \frac{\Delta - \delta}{2\nu+\alpha} \left(\Delta - \frac{\alpha}{\nu+1} - \frac{2\nu+\alpha}{2} \frac{\Delta-\delta}{2\nu+\alpha} \right) \\
&\geq \frac{1}{2\nu+\alpha} (\Delta - \delta) \left(\frac{\Delta+\delta}{2} - \frac{\alpha}{\nu+1} \right)
\end{aligned}
$$

which implies the desired result. $\Box$

For $\nu = 0$ we obtain the following corollary.

Corollary 1 *Let $G = (V, E)$ be as above. Let $\alpha \in \mathbf{N}$ be such that for every $u \in V$ $\sum\limits_{v \in N(u)} |d(v) - d(u)| \leq \alpha$. Then*

$$n \geq \frac{1}{2\alpha}(\Delta - \delta)(\Delta + \delta - 2\alpha)$$

and

$$\Delta - \delta \leq \sqrt{2\alpha n} + 2\alpha.$$

Remark 1 *For positive integers $\alpha, l_{\max} \in \mathbf{N}$ let the graph G arise from the $l_{\max} + 1$ disjoint cliques $K_1, K_{\alpha+2}, K_{2\alpha+2}, ..., K_{l_{\max}\alpha+2}$ by deleting one edge $u_l v_l$ in the clique $K_{l\alpha+2}$ for $1 \leq l \leq l_{\max}$ and adding an edge between the unique vertex in the clique K_1 and u_1 and new edges $v_l u_{l+1}$ for $1 \leq l \leq l_{\max} - 1$. It is straightforward to verify that G satisfies the assumptions of Corollary 1. Furthermore, $\Delta = \Delta(G) = l_{\max}\alpha + 1$, $\delta = \delta(G) = 1$ and we obtain for the order n of G that*

$$\begin{aligned} n &= 1 + \sum_{l=1}^{l_{\max}} (l\alpha + 2) \\ &= -1 + \sum_{l=0}^{l_{\max}} (l\alpha + 2) \\ &= -1 + 2(l_{\max} + 1) + \frac{\alpha l_{\max}}{2}(l_{\max} + 1) \\ &= (l_{\max} + 1)(\frac{\alpha l_{\max}}{2} + 2) - 1 \\ &= \frac{1}{2\alpha}(\Delta - \delta + \alpha)(\Delta + \delta + 2) - 1. \end{aligned}$$

Hence Corollary 1 is asymptotically best possible in the sense that the fraction of the given bound and of the order of the constructed graph tends to 1 as $\Delta - \delta = \Delta - 1$ tends to ∞.

Theorem 2 *Let $G = (V, E)$ be as above. Let $\alpha > 0$ and $\nu \in \mathbf{N}_0$ be such that $\frac{\alpha}{\nu+1} < 1$ and for every $u \in V$*

$$\frac{1}{d(u)} \sum_{v \in N(u)} |d(v) - d(u)| \leq \alpha.$$

Then

$$n \geq (1 - \frac{\alpha}{\nu + 1}) \left(\Delta \frac{1 - \left(\frac{1}{\alpha+1}\right)^{(l_{\max}+1)}}{1 - \frac{1}{\alpha+1}} - \frac{2\nu}{\alpha}(l_{\max} + 1) \right)$$

for $l_{\max} = \lfloor \frac{\ln(\Delta) - \ln(\delta + \frac{2\nu}{\alpha})}{\ln(\alpha+1)} \rfloor$.

Proof: Let $i_0 = \Delta$ and for $l \geq 1$ let $i_l = \max\{k | k \leq i_{l-1} - 2\nu - 1 \text{ and } n_k > 0\}$. By Lemma 1 (ii), $i_l \geq \frac{i_{l-1}-2\nu}{\alpha+1}$. This implies that for $l \geq 0$

$$i_l \;\geq\; \frac{\Delta}{(\alpha+1)^l} - \sum_{j=1}^{l} \frac{2\nu}{(\alpha+1)^j} \geq \frac{\Delta}{(\alpha+1)^l} - \frac{2\nu}{\alpha}.$$

Hence i_l is well-defined for $0 \leq l \leq l_{\max} := \lfloor \frac{\ln(\Delta)-\ln(\delta+\frac{2\nu}{\alpha})}{\ln(\alpha+1)} \rfloor$. Let $u_{i_l} \in V_{i_l}$ for $0 \leq l \leq l_{\max}$. For $0 \leq l \leq l_{\max}$ the sets $\bigcup_{\mu:|\mu-i_l|\leq\nu} V_\mu$ are disjoint and we obtain with Lemma 1 (ii)

$$\begin{aligned}
n &= \sum_{i=\delta}^{\Delta} n_i \\
&\geq \sum_{l=0}^{l_{\max}} \sum_{\mu:|\mu-i_l|\leq\nu} n_\mu \\
&\geq \sum_{l=0}^{l_{\max}} \sum_{\mu:|\mu-i_l|\leq\nu} |V_\mu \cap N(u_{i_l})| \\
&\geq (1-\frac{\alpha}{\nu+1}) \sum_{l=0}^{l_{\max}} i_l \\
&\geq (1-\frac{\alpha}{\nu+1}) \sum_{l=0}^{l_{\max}} \frac{\Delta}{(\alpha+1)^l} - \frac{2\nu}{\alpha} \\
&= (1-\frac{\alpha}{\nu+1}) \left(\Delta \frac{1-\left(\frac{1}{\alpha+1}\right)^{(l_{\max}+1)}}{1-\frac{1}{\alpha+1}} - \frac{2\nu}{\alpha}(l_{\max}+1) \right)
\end{aligned}$$

which implies the desired result. □

Remark 2 *For positive integers $\alpha, l_{\max} \in \mathbf{N}$ let the graph G arise from the $l_{\max}+1$ disjoint cliques $K_3, K_{3(\alpha+1)}, ..., K_{3(\alpha+1)^{l_{\max}}}$ by deleting one edge $u_l v_l$ in the clique $K_{3(\alpha+1)^l}$ for $0 \leq l \leq l_{\max}$ and adding new edges $v_l u_{l+1}$ for $0 \leq l \leq l_{\max}-1$. It is straightforward to verify that G satisfies the assumptions of Theorem 2. Furthermore, $\Delta = \Delta(G) = 3(\alpha+1)^{l_{\max}} - 1$, $\delta = \delta(G) = 1$ and we obtain for the order n of G that*

$$\begin{aligned}
n &= \sum_{l=0}^{l_{\max}} 3(\alpha+1)^l \\
&= 3\frac{1-(\alpha+1)^{l_{\max}+1}}{1-(\alpha+1)} \\
&= 3(\alpha+1)^{l_{\max}} \left(\frac{1-(\frac{1}{\alpha+1})^{l_{\max}+1}}{1-(\frac{1}{\alpha+1})} \right)
\end{aligned}$$

and

$$l_{\max} = \frac{\ln\left(\frac{\Delta}{\delta}+1\right) - \ln(3)}{\ln(\alpha+1)}.$$

Again, as in Remark 1 the constructed graph implies that Theorem 2 is asymptotically best possible up to the factor $(1-\frac{\alpha}{\nu+1})$.

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