Rectilinear spanning trees versus bounding boxes

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Abstract

For a set $P \subseteq \mathbb{R}^2$ with $2 \leq n = |P| < \infty$ we prove that $\frac{\text{mst}(P)}{\text{bb}(P)} \leq \frac{1}{\sqrt{2}}\sqrt{n} + \frac{3}{2}$ where $\text{mst}(P)$ is the minimum total length of a rectilinear spanning tree for P and $\text{bb}(P)$ is half the perimeter of the bounding box of P. Since the constant $\frac{1}{\sqrt{2}}$ in the above bound is best-possible, this result settles a problem that was mentioned by Brenner and Vygen (*Networks* **38** (2001), 126-139).

1 Introduction

We consider finite sets of point in the plane $\mathbb{R}^2$ where the distance of two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in $\mathbb{R}^2$ is defined as $\text{dist}(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|$, i.e. $\text{dist}(p, q)$ is the so-called Manhattan or l_1 distance.

For a finite set $P \subseteq \mathbb{R}^2$ let $\text{mst}(P)$ be the minimum total length of a (*rectilinear*) *spanning tree* for the set P, i.e. $\text{mst}(P)$ is the minimum length of a spanning tree in the complete graph whose vertex set is P and in which the edge pq for $p, q \in P$ with $p \neq q$ has length $\text{dist}(p, q)$. Let $\text{steiner}(P)$ be the minimum total length of a (*rectilinear*) *Steiner tree* for the set P, i.e. $\text{steiner}(P) = \min\{\text{mst}(P') \mid P' \subseteq \mathbb{R}^2 \text{ and } P \subseteq P'\}$. Furthermore, let $\text{bb}(P) = \left(\max_{(x_1,y_1)\in P} x_1 - \min_{(x_2,y_2)\in P} x_2\right) + \left(\max_{(x_3,y_3)\in P} y_3 - \min_{(x_4,y_4)\in P} y_4\right)$, i.e. $\text{bb}(P)$ is half the perimeter of the smallest set of the form $[a_1, a_2] \times [b_1, b_2]$ that contains P. This unique set is called the *bounding box* of P.

The three parameters $\text{mst}(P)$, $\text{steiner}(P)$ and $\text{bb}(P)$ are examples of so-called *net models* which are of interest in VLSI design. Clearly, $\text{mst}(P) \geq \text{steiner}(P) \geq \text{bb}(P)$ and it is an obvious problem to study upper bounds on $\text{mst}(P)$ or $\text{steiner}(P)$ in terms of $\text{bb}(P)$.

In [1] Brenner and Vygen prove that (provided $|P| \geq 2$)

$$\frac{\text{mst}(P)}{\text{bb}(P)} \leq \frac{3}{4}\left\lceil\sqrt{|P| - 2}\,\right\rceil + \frac{9}{8} = \frac{3}{4}\sqrt{|P|} + O(1). \tag{1}$$

This result follows from the well-known relation $\text{mst}(P) \leq \frac{3}{2}\text{steiner}(P)$ due to Hwang [4] and the bound $\frac{\text{steiner}(P)}{\text{bb}(P)} \leq \frac{1}{2}\left\lceil \sqrt{|P|-2} \right\rceil + \frac{3}{4}$ due to Brenner and Vygen [1] (cf. also [2]). An example in [1] shows that the smallest-possible constant c in an estimate of the form $\frac{\text{mst}(P)}{\text{bb}(P)} \leq c\sqrt{|P|} + O(1)$ is $c = \frac{1}{\sqrt{2}}$ which is smaller than the factor $\frac{3}{4}$ in (1). With our following main result we close this gap.

Theorem 1 *If $P \subseteq \mathbb{R}^2$ is such that $2 \leq n = |P| < \infty$, then*

$$\frac{\text{mst}(P)}{\text{bb}(P)} \quad \leq \quad \frac{1}{\sqrt{2}}\sqrt{n} + \frac{3}{2}. \tag{2}$$

In the next section we prove Theorem 1.

2 Proof of Theorem 1

The main tool for the proof of Theorem 1 is the following lemma. The construction used in the proof of this lemma is a variation of a construction that goes back to Few [3] and has also been used in [1, 2].

Lemma 1 *Let $P \subseteq [0, a] \times [0, b]$ be such that $a \geq b$ and $3 \leq n = |P| < \infty$. If $t \in \mathbb{N}$ is such that $t \leq n$, then $\text{mst}(P) \leq \frac{1}{2}(a + 4b + at + 2\frac{nb}{t})$.*

Proof: Let P, a, b, n and t be as in the statement. Since $\text{mst}(P)$ is a continuous functions of the coordinates of the points in P, we may assume without loss of generality that $x_1 \neq x_2$ and $y_1 \neq y_2$ for different elements (x_1, y_1) and (x_2, y_2) of P. This implies the existence of real numbers

$$0 = h_0 < h_1 < h_2 < ... < h_t = b$$

such that if we define $L_i = [0, a] \times \{h_i\}$ for $0 \leq i \leq t$ and $S_i = [0, a] \times [h_{i-1}, h_i]$ for $1 \leq i \leq t$, then we have $P \cap L_i = \emptyset$ for $1 \leq i \leq t-1$ and $1 \leq \lfloor \frac{n}{t} \rfloor \leq |S_i \cap P| \leq \lfloor \frac{n}{t} \rfloor + 1$ for $1 \leq i \leq t$ (see Figure 1). Note that this also implies $S_i \cap S_j \cap P = \emptyset$ for $1 \leq i < j \leq t$.

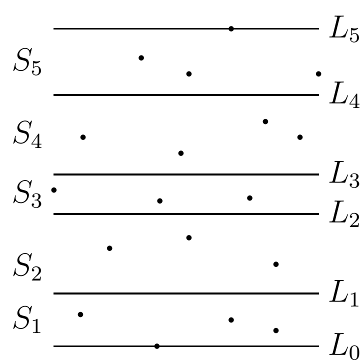

Figure 1

We are now going to assign line segments to each of the L_i's. Let $0 \leq i \leq t$ and let

$$P \cap (S_i \cup S_{i+1}) = \{(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)\}$$

be such that $x_1 < x_2 < ... < x_k$ where $S_0 = S_{t+1} = \emptyset$. For $1 \leq j \leq k-1$ we assign to L_i the three line segments from (x_j, y_j) to (x_j, h_i), from (x_j, h_i) to (x_{j+1}, h_i) and from (x_{j+1}, h_i) to (x_{j+1}, y_{j+1}) (see the left part of Figure 2).

Figure 2

Furthermore, if $i \leq t-2$, then we will assign more line segments to L_i as follows.

If $i \equiv 0 \bmod 4$ or $i \equiv 1 \bmod 4$ let (x', y') be the element of $P \cap (S_{i+2} \cup S_{i+3})$ with the smallest first coordinate.

If $x_1 \leq x'$, then assign to L_i the four line segments from (x_1, y_1) to (x_1, h_i), from (x_1, h_i) to (x_1, h_{i+2}), from (x_1, h_{i+2}) to (x', h_{i+2}) and from (x', h_{i+2}) to (x', y').

If $x_1 > x'$, then assign to L_i the four line segments from (x_1, y_1) to (x_1, h_i), from (x_1, h_i) to (x', h_i), from (x', h_i) to (x', h_{i+2}) and from (x', h_{i+2}) to (x', y').

Among the above line segments the one from (x_1, h_i) to (x_1, h_{i+2}) or from (x', h_i) to (x', h_{i+2}) will be called a *vertically connecting line segment*. Note that if $y_1 \geq h_i$ or $y' \leq h_{i+2}$, the above four segments could be replaced by two or three line segments of smaller total length in an obvious way.

If $i \equiv 2 \bmod 4$ or $i \equiv 3 \bmod 4$, then proceed analogously with x_k and the element of $P \cap (S_{i+2} \cup S_{i+3})$ with the largest first coordinate (see the right part of Figure 2).

Now, the union of all line segments assigned to L_0, L_2, ..., $L_{2\lfloor \frac{t}{2} \rfloor}$ lead to a first spanning tree T_{even} for P and the union of all line segments assigned to L_1, L_3, ..., $L_{2\lfloor \frac{t-1}{2} \rfloor + 1}$ lead to a second spanning tree T_{odd} for P (see Figure 3).

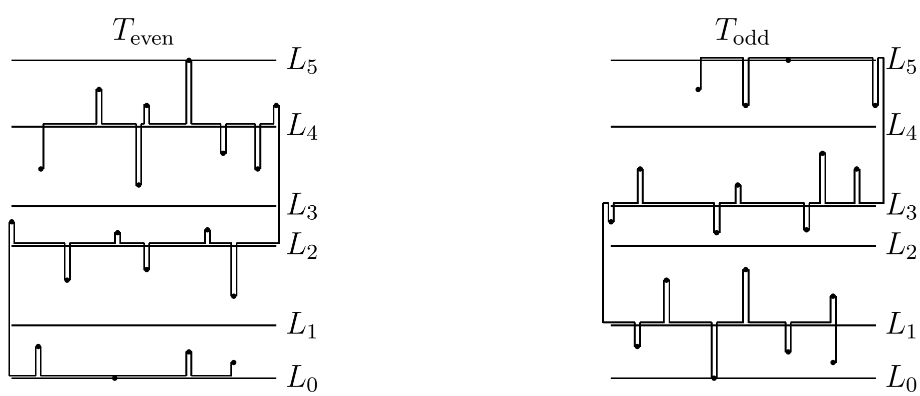

Figure 3

We will now estimate the total length of all line segments assigned to all L_i's.

The total length of all vertical line segments apart from the vertically connecting line segments is at most

$$\sum_{i=1}^{t} 2(h_i - h_{i-1})|S_i \cap P| \leq \sum_{i=1}^{t} 2(h_i - h_{i-1}) \left(\frac{n}{t} + 1\right) = 2b\left(\frac{n}{t} + 1\right).$$

The total length of all vertically connecting line segments is

$$\sum_{i=1}^{2\lfloor \frac{t}{2} \rfloor}(h_{2i} - h_{2(i-1)}) + \sum_{i=1}^{2\lfloor \frac{t-1}{2} \rfloor}(h_{2i+1} - h_{2(i-1)+1}) = (h_{2\lfloor \frac{t}{2} \rfloor} - h_0) + (h_{2\lfloor \frac{t-1}{2} \rfloor + 1} - h_1) \leq 2b.$$

The total length of all horizontal line segments is at most $a(t+1)$.

Altogether, we obtain a total length of $a + 4b + at + 2b\frac{n}{t}$. Since no line segment is used by T_{even} and T_{odd} simultaneously, one of these two trees has a total length of at most $\frac{1}{2}(a + 4b + at + 2b\frac{n}{t})$ which implies the desired result. $\square$

Now we proceed to the proof of Theorem 1. Let $P \subseteq \mathbb{R}^2$ be such that $2 \leq n = |P| < \infty$.

If $n = 2$, then clearly $\frac{\text{mst}(P)}{\text{bb}(P)} = 1 < \frac{1}{\sqrt{2}}\sqrt{n} + \frac{3}{2}$. Hence let $n \geq 3$. We may assume that $[0, a] \times [0, b]$ with $a \geq b$ is the bounding box of P, i.e. $\text{bb}(P) = a + b$. Now Lemma 1 for $t = \left\lceil \sqrt{\frac{b}{a}}\sqrt{2n} \right\rceil \leq n$ and an easy calculation yields $2\text{mst}(P) \leq a + 4b + at + 2\frac{nb}{t} \leq \left(3 + \sqrt{2n}\right) \text{bb}(P)$. Thus $\frac{\text{mst}(P)}{\text{bb}(P)} \leq \frac{1}{\sqrt{2}}\sqrt{n} + \frac{3}{2}$ as desired and the proof is complete. $\square$

References

[1] U. Brenner and J. Vygen, Worst-case ratios of networks in the rectilinear plane, *Networks* **38** (2001), 126-139.

[2] F.R.K. Chung and R.L. Graham, On Steiner trees for bounded point sets, *Geometriae Dedicata* **11** (1981), 353-361.

[3] L. Few, The shortest path and shortest road through n points, *Mathematika* **2** (1955), 141-144.

[4] F.K. Hwang, On Steiner minimal trees with rectilinear distance, *SIAM J. Appl. Math.* **30**(1976), 104-114.