Dominance Order and Graphical Partitions

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Abstract

We gave a new criterion for graphical partitions. We derive a new recursion formula, which allows the computation of the number $g(n)$ of graphical partitions of weight n for up to $n > 900$.

1 Introduction

A *partition* λ of *weight* n is a nonincreasing sequence of nonnegative integers $(\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots)$ whose sum is n. The weight is denoted by $|\lambda|$. The number of nonzero elements in the sequence is the *length* of the partition denoted by $l(\lambda)$. The set of all partitions of weight n is denoted by $P(n)$. The number of partitions of weight n is denoted by $p(n)$. There is one partition of weight 0, it is the partition of length 0. A partition is called *graphical* if it is the degree sequence of an undirected simple graph. As each edge is the graph is counted twice, a graphical partition must be of even weight. The partition of weight 0 is graphical as it corresponds to a graph without edges. The set of graphical partitions of weight n is denoted by $G(n)$. The number of graphical partitions is denoted by $g(n)$.

A partition λ is visualized using the *Ferrer's diagram* F_λ, i.e. an array of λ_i left justified boxes in the i-th row of the first quadrant of the plane. The number of boxes on the main diagonal of the Ferrer's diagram F_λ is the *Durfee size* of the partition and denoted by $d(\lambda)$. The subpartition built from the $d(\lambda) \times d(\lambda)$ boxes is called the *Durfee square* of the partition λ. If we count the number of boxes in each column of the Ferrer's diagram F_λ, we get again a partition, which is called the *conjugate* partition and is denoted by λ'. There are several partial orders on the set of all partitions. We are interested in the *dominance order*. A partition λ is dominated by the partition μ, denoted by $\lambda \trianglelefteq \mu$ if $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$ for all $k > 0$. This is a partial order, as there are pairs of partitions which are not comparable. (e.g. $(5,1,1,1,1,1)$ and $(2,2,2,2,2)$).

2 Criterion

It is possible to write a partition λ as a unique tuple of 3 smaller partitions using a decomposition according to the Durfee square. The first partition $L(\lambda)$ is defined to be

$$(\lambda'_1 - d(\lambda), \ldots, \lambda'_{d(\lambda)} - d(\lambda)).$$

The second one $M(\lambda)$ is the Durfee square minus one column. The third one $R(\lambda)$ is defined to be

$$(\lambda_1 - d(\lambda) + 1, \ldots, \lambda_{d(\lambda)} - d(\lambda) + 1).$$

In the following figure these three partitions are marked with R, M and L.

L					
L					
L	L				
M	M	M	R		
M	M	M	R		
M	M	M	R		
M	M	M	R	R	R

The three corresponding partitions are $L((6,4,4,4,2,1)) = (3,1)$, $M((6,4,4,4,2,1)) = (3,3,3,3)$ and $R((6,4,4,4,2,1)) = (3,1,1,1)$. Now it is possible to give a theorem which connects the question of being graphic with the dominance order of partitions.

Theorem 1 *A partition λ is graphical if and only if*

$$L(\lambda) \trianglerighteq R(\lambda).$$

This is a corollary of the criterion of Hässelbarth [Ha] which says that λ is graphical if and only if

$$\sum_{i=1}^{r} (\lambda_i - \lambda'_i + 1) \leq 0$$

for all $1 \leq r \leq d(\lambda)$.

3 Recurrence

Using the above criterion it is possible to compute the number of graphical partitions of weight n. We define

$$G(n) =: G_1(n) \cup \ldots \cup G_d(n)$$

where $G_i(n)$ is the set of graphical partitions of weight n and a Durfee square of size i. To get the partitions in $G_i(n)$ we have to get all pairs of possible $L(\lambda)$ and $R(\lambda)$. The criterion provides a bijection

$$G_i(n) \to \{(\mu, \nu) \text{ with } \mu \trianglerighteq \nu, l(\mu) \leq i, l(\nu) = i, |\nu| + |\mu| = n - (i-1) * i\}.$$

We denote by $g_i(n)$ the number of partitions in $G_i(n)$. The set on the right hand side will be decomposed into smaller subsets well suited for the recursion. We denote by $P(m,k,n,l)$ the set of pairs of partitions (μ, ν) with the following properties

- μ is a partition of weight m with length k
- ν is a partition of weight n with length l
- $\mu \trianglerighteq \nu$

So we get a bijection with $r = n - i * (i-1)$

$$G_i(n) \to \bigcup_{\substack{j=1,\ldots,i \\ s=0,\ldots,r}} P(s, j, r-s, i)$$

Defining $p(m,k,n,l)$ to be the order of $P(m,k,n,l)$ we use this bijection for the computation of $g(n)$. As we are only interested in the number of partitions we get:

$$g_i(n) = \sum_{\substack{j=1,\ldots,i \\ s=0,\ldots,r}} p(s, j, r-s, i)$$

Last step in the algorithm for the computation of the number of graphical partitions is the following bijection:

Lemma 2

$$P(m,k,n,l) \to \bigcup_{\substack{i=0,\ldots,k \\ j=0,\ldots,l}} P(m-k, i, n-l, j)$$

is a bijection given by removal/addition of the first column in the pair of partitions.

Proof:

Take a pair (L, R) of partitions from $P(m, k, n, l)$, so for example $L = (3, 2, 2),\ R = (3, 1, 1, 1)$:

L					
L	L	L			
L	L	L			
M	M	M	R		
M	M	M	R		
M	M	M	R		
M	M	M	R	R	R

We remove in the Ferrer's diagrams the first column of L and R, and get a pair of partitions $\hat{L}, \hat{R}$ of smaller weight:

$\hat{L}$				
$\hat{L}$	$\hat{L}$	$\hat{L}$		
M	M	M		
M	M	M		
M	M	M		
M	M	M	$\hat{R}$	$\hat{R}$

When we check the different cases for the lengths of the partitions L and R we get $\hat{L} \trianglerighteq \hat{R}$, so the pair $\hat{L}, \hat{R}$ is element of $P(m-k, i, n-l, j)$ with $i =$length of the second column of L, and $j =$length of the second column of R. •

Using this recursion on

$$p(m, k, n, l) = \sum_{\substack{i = 0, \ldots, k \\ j = 0, \ldots, l}} p(m-k, i, n-l, j)$$

we computed recursively the number of graphical partitions.

3.1 Properties of $p(m, k, n, l)$

There are several properties of the values $p(m, k, n, l)$ which allow the faster computation of the number of graphical partitions. We have to count the number of pairs of partitions (μ, ν). There is a unique lexicographical minimal partition μ_- with weight m and length k, and a unique lexicographical maximal partition ν^+ with weight n and length l. We have the following lemma

Lemma 3 *For* m, k, n, l *with* $\mu_- \trianglerighteq \nu^+$ *we have* $p(m, k, n, l) = p(m, k, 0, 0)p(n, l, 0, 0)$.

as every partition in the set $P(m, k) := P(m, k, 0, 0)$ dominates any partition from $P(n, l)$.
In the case of $m > n$ we get a telescoping sum (thanks to the referee) which allows a fast computation in this case

Lemma 4 *For* $m > n$ *we have:*

$$\begin{aligned} p(m, k, n, l) &= p(m-1, k-1, n, l) + p(m, k, n-1, l-1) \\ &\quad - p(m-1, k-1, n-1, l-1) + p(m-k, k, n-l, l). \end{aligned}$$

Proof: start with the difference of

$$p(m, k, n, l) = \sum_{\substack{i = 0, \ldots, k \\ j = 0, \ldots, l}} p(m-k, i, n-l, j)$$

and

$$p(m-1, k-1, n, l) = \sum_{\substack{i = 0, \ldots, k-1 \\ j = 0, \ldots, l}} p(m-k, i, n-l, j)$$

giving

$$p(m, k, n, l) - p(m-1, k-1, n, l) = \sum_{j=0,\ldots,l} p(m-k, k, n-l, j)$$

and in the case of $n-1, l-1$:

$$p(m, k, n-1, l-1) - p(m-1, k-1, n-1, l-1) = \sum_{j=0,\ldots,l-1} p(m-k, k, n-l, j).$$

The difference of the last two lines gives the statement of the lemma. •

4 Computation of $g(n)$

Like in the case of Barnes and Savage [BS] it is useful to store the computed values $p(m, k, n, l)$ in a four dimensional table. As k and l are limited by $\lfloor\sqrt{n}\rfloor$ the space requirement of the algorithm is like in their case $O(n^3)$. The telescoping lemma helps to speed up the computation, but it does not reduce the amount of memory necessary to store the intermediate results. The formula for the computation of the number of graphical partitions of weight n computes in the outer loop the number of graphical partitions of n with a fixed Durfee size d. These numbers, which add up to $g(n)$ are listed in the following table.

n/d	1	2	3	4	5	6
2	1					
4	2					
6	3	2				
8	4	5				
10	5	12				
12	6	22	3			
14	7	38	9			
16	8	58	24			
18	9	87	55			
20	10	121	108	5		
22	11	166	195	15		
24	12	218	335	42		
26	13	283	539	98		
28	14	356	832	218		
30	15	445	1247	422	7	
32	16	543	1803	788	23	
34	17	659	2542	1374	65	
36	18	786	3515	2322	158	
38	19	933	4752	3743	356	
40	20	1092	6315	5881	740	
42	21	1274	8278	8931	1441	11
44	22	1469	10683	13318	2653	34
46	23	1689	13619	19339	4699	98
48	24	1924	17182	27612	8016	238
50	25	2186	21435	38656	13257	545
52	26	2464	26490	53412	21267	1143

We computed the number of graphical partitions up to $n > 900$. This table extends the table previously published by Barnes and Savage [BS]. Like in their table we include $p(n)$ the number of partitions and the ratio $g(n)/p(n)$.

n	$g(n)$	$p(n)$	$g(n)/p(n)$	n	$g(n)$	$p(n)$	$g(n)/p(n)$
2	1	2	.5000000	58	264941	715220	.3704328
4	2	5	.4000000	60	357635	966467	.3700436
6	5	11	.4545454	62	480408	1300156	.3695002
8	9	22	.4090909	64	642723	1741630	.3690353
10	17	42	.4047619	66	856398	2323520	.3685778
12	31	77	.4025974	68	1136715	3087735	.3681387
14	54	135	.4000000	70	1503172	4087968	.3677064
16	90	231	.3896103	72	1980785	5392783	.3673029
18	151	385	.3922077	74	2601057	7089500	.3668886
20	244	627	.3891547	76	3404301	9289091	.3664837
22	387	1002	.3862275	78	4441779	12132164	.3661159
24	607	1575	.3853968	80	5777292	15796476	.3657329
26	933	2436	.3830049	82	7492373	20506255	.3653701
28	1420	3718	.3819257	84	9688780	26543660	.3650129
30	2136	5604	.3811563	86	12494653	34262962	.3646693
32	3173	8349	.3800455	88	16069159	44108109	.3643130
34	4657	12310	.3783103	90	20614755	56634173	.3639985
36	6799	17977	.3782054	92	26377657	72533807	.3636601
38	9803	26015	.3768210	94	33671320	92669720	.3633475
40	14048	37338	.3762386	96	42878858	118114304	.3630284
42	19956	53174	.3752961	98	54481054	150198136	.3627278
44	28179	75175	.3748453	100	69065657	190569292	.3624175
46	39467	105558	.3738892	102	87370195	241265379	.3621331
48	54996	147273	.3734289	104	110287904	304801365	.3618353
50	76104	204226	.3726459	106	138937246	384276336	.3615555
52	104802	281589	.3721807	108	174675809	483502844	.3612715
54	143481	386155	.3715632	110	219186741	607163746	.3610010
56	195485	526823	.3710639	112	274512656	761002156	.3607252

To group digits in the larger numbers we included a $'.'$ at every sixth position.

n	$g(n)$	$p(n)$	$g(n)/p(n)$
114	343181668	952050665	.3604657
116	428244215	1188908248	.3601995
118	533464959	1482074143	.3599448
120	663394137	1844349560	.3596900
122	823598382	2291.320912	. 3594426
124	1020807584	2841.940500	. 3591938
126	1263243192	3519.222692	. 3589551
128	1560795436	4351.078600	. 3587146
130	1925513465	5371.315400	. 3584808
132	2371.901882	6620.830889	. 3582483
134	2917.523822	8149.040695	. 3580205
136	3583.515700	10015.581680	. 3577940
138	4395.408234	12292.341831	. 3575728
140	5383.833857	15065.878135	. 3573528
142	6585.699894	18440.293320	. 3571363
144	8045.274746	22540.654445	. 3569228
146	9815.656018	27517.052599	. 3567117
148	11960.467332	33549.419497	. 3565029
150	14555.902348	40853.235313	. 3562974
152	17692.990183	49686.288421	. 3560940
154	21480.510518	60356.673280	. 3558928
156	26048.320019	73232.243759	. 3556946
158	31551.087790	88751.778802	. 3554980
160	38173.235010	107438.159466	. 3553042
162	46134.037871	129913.904637	. 3551123
164	55694.314567	156919.475295	. 3549228
166	67163.674478	189334.822579	. 3547349
168	80909.973315	228204.732751	. 3545499
170	97368.672089	274768.617130	. 3543660
172	117056.456152	330495.499613	. 3541847
174	140584.220188	397125.074750	. 3540048
176	168675.124141	476715.857290	. 3538273
178	202182.888436	571701.605655	. 3536510
180	242116.891036	684957.390936	. 3534773
182	289666.252014	819876.908323	. 3533045
184	346234.896845	980462.880430	. 3531341
186	413474.657328	1.171432.692373	. 3529649
188	493331.835384	1.398341.745571	. 3527977
190	588093.594457	1.667727.404093	. 3526317

n	$g(n)$	$p(n)$	$g(n)/p(n)$
192	700451.190712	1.987276.856363	. 3524678
194	833561.537987	2.366022.741845	. 3523049
196	991134.281267	2.814570.987591	. 3521439
198	1.177516.049387	3.345365.983698	. 3519842
200	1.397805.210533	3.972999.029388	. 3518262
202	1.657968.320899	4.714566.886083	. 3516692
204	1.964994.991232	5.590088.317495	. 3515141
206	2.327052.859551	6.622987.708040	. 3513599
208	2.753697.110356	7.840656.226137	. 3512074
210	3.256081.386335	9.275102.575355	. 3510561
212	3.847232.865612	10.963707.205259	. 3509062
214	4.542341.563460	12.950095.925895	. 3507573
216	5.359127.512113	15.285151.248481	. 3506100
218	6.318223.879596	18.028182.516671	. 3504637
220	7.443670.977177	21.248279.009367	. 3503187
222	8.763432.946593	25.025873.760111	. 3501749
224	10.310044.123494	29.454549.941750	. 3500323
226	12.121309.266199	34.643126.322519	. 3498907
228	14.241160.856051	40.718063.627362	. 3497504
230	16.720586.202163	47.826239.745920	. 3496111
232	19.618767.868192	56.138148.670947	. 3494730
234	23.004324.059046	65.851585.970275	. 3493359
236	26.956798.814985	77.195892.663512	. 3491999
238	31.568326.350604	90.436839.668817	. 3490648
240	36.945596.125431	105.882246.722733	. 3489309
242	43.212042.821600	123.888443.077259	. 3487980
244	50.510448.519684	144.867692.496445	. 3486660
246	59.005849.206367	169.296722.391554	. 3485350
248	68.888924.114697	197.726516.681672	. 3484050
250	80.379859.814364	230.793554.364681	. 3482760
252	93.732799.789716	269.232701.252579	. 3481479
254	109.240907.229098	313.891991.306665	. 3480206
256	127.242219.898679	365.749566.870782	. 3478943
258	148.126317.233645	425.933084.409356	. 3477689
260	172.341932.589627	495.741934.760846	. 3476444
262	200.405745.147874	576.672674.947168	. 3475207
264	232.912328.227060	670.448123.060170	. 3473979

n	$g(n)$	$p(n)$	$g(n)/p(n)$
266	270.545608.772217	779.050629.562167	. 3472760
268	314.091890.030723	904.760108.316360	. 3471548
270	364.454850.689480	1050.197489.931117	. 3470345
272	422.672418.723168	1218.374349.844333	. 3469150
274	489.936412.910522	1412.749565.173450	. 3467963
276	567.614507.770134	1637.293969.337171	. 3466784
278	657.275703.933020	1896.564103.591584	. 3465612
280	760.718950.656347	2195.786311.682516	. 3464448
282	880.006264.424357	2540.952590.045698	. 3463292
284	1017.499729.851133	2938.929793.929555	. 3462143
286	1175.904589.041542	3397.584011.986773	. 3461002
288	1358.317187.000975	3925.922161.489422	. 3459867
290	1568.280617.221370	4534.253126.900886	. 3458740
292	1809.846889.359039	5234.371069.753672	. 3457620
294	2087.648920.451849	6039.763882.095515	. 3456507
296	2406.980630.541347	6965.850144.195831	. 3455401
298	2773.890059.176591	8030.248384.943040	. 3454301
300	3195.282761.990490	9253.082936.723602	. 3453208
302	3679.041523.618584	10657.331232.548839	. 3452122
304	4234.159847.629493	12269.218019.229465	. 3451042
306	4870.896069.907545	14118.662665.280005	. 3449969
308	5600.944751.183391	16239.786535.829663	. 3448902
310	6437.635040.483873	18671.488299.600364	. 3447842
312	7396.150995.787149	21458.096037.352891	. 3446788
314	8493.785631.612121	24650.106150.830490	. 3445739
316	9750.224120.415064	28305.020340.996003	. 3444697
318	11187.869357.515526	32488.293351.466654	. 3443661
320	12832.204376.370829	37274.405776.748077	. 3442631
322	14712.209437.460953	42748.078035.954696	. 3441607
324	16860.826086.379188	49005.643635.237875	. 3440588
326	19315.489319.698561	56156.602112.874289	. 3439575
328	22118.721571.923434	64325.374609.114550	. 3438568
330	25318.812013.118277	73653.287861.850339	. 3437567
332	28970.573903.861784	84300.815636.225119	. 3436571
334	33136.211302.816850	96450.110192.202760	. 3435580
336	37886.285108.888194	110307.860425.292772	. 3434595
338	43300.814969.998852	126108.517833.796355	. 3433615

n	$g(n)$	$p(n)$	$g(n)/p(n)$
340	49470.510879.271020	144117.936527.873832	. 3432640
342	56498.174296.996950	164637.479165.761044	. 3431671
344	64500.263250.927647	188008.647052.292980	. 3430707
346	73608.673091.026153	214618.299743.286299	. 3429748
348	83972.725212.921328	244904.537455.382406	. 3428794
350	95761.423661.105714	279363.328483.702152	. 3427845
352	109165.980705.294681	318555.973788.329084	. 3426901
354	124402.676259.476568	363117.512048.110005	. 3425961
356	141716.056306.961362	413766.180933.342362	. 3425027
358	161382.553841.124116	471314.064268.398780	. 3424097
360	183714.538700.727580	536679.070310.691121	. 3423173
362	209064.891369.055649	610898.403751.884101	. 3422253
364	237832.127202.308431	695143.713458.946040	. 3421337
366	270466.172555.733767	790738.119649.411319	. 3420426
368	307474.832766.566116	899175.348396.088349	. 3419520
370	349431.082262.461473	1.022141.228367.345362	. 3418618
372	396981.225388.693589	1.161537.834849.962850	. 3417721
374	450854.077011.727696	1.319510.599727.473500	. 3416828
376	511871.252448.667437	1.498478.743590.581081	. 3415939
378	580958.724595.080633	1.701169.427975.813525	. 3415055
380	659159.770630.009034	1.930656.072350.465812	. 3414175
382	747649.510890.816168	2.190401.332423.765131	. 3413299
384	847751.191197.762717	2.484305.294265.418180	. 3412427
386	960954.438051.593305	2.816759.503217.942792	. 3411560
388	1.088935.719745.607833	3.192707.518433.532826	. 3410696
390	1.233581.257527.175008	3.617712.763867.604423	. 3409837
392	1.397012.690532.962312	4.098034.535626.594791	. 3408982
394	1.581615.810583.814123	4.640713.124699.623515	. 3408130
396	1.790072.736673.606208	5.253665.124416.975163	. 3407283
398	2.025397.891141.779483	5.945790.114707.874597	. 3406440
400	2.290978.305361.293873	6.727090.051741.041926	. 3405600
402	2.590618.639582.366767	7.608802.843339.879269	. 3404765
404	2.928591.589058.407676	8.603551.759348.655060	. 3403933
406	3.309694.180956.900829	9.725512.513742.021729	. 3403105
408	3.739310.775529.786273	10.990600.063775.926994	. 3402280
410	4.223483.357738.533049	12.416677.403151.190382	. 3401460
412	4.768990.219275.581506	14.023788.883518.847344	. 3400643

n	$g(n)$	$p(n)$	$g(n)/p(n)$
414	5.383433.672948.398244	15.834420.884488.187770	. 3399829
416	6.075338.174381.850575	17.873792.969689.876004	. 3399020
418	6.854259.698107.669457	20.170183.018805.933659	. 3398213
420	7.730908.023798.421590	22.755290.216580.025259	. 3397411
422	8.717282.933540.859360	25.664640.213837.714846	. 3396612
424	9.826826.500401.651249	28.938037.257084.798150	. 3395816
426	11.074592.583508.500810	32.620068.617410.232189	. 3395024
428	12.477436.219213.567055	36.760667.241831.527309	. 3394235
430	14.054224.400690.717799	41.415739.207102.358378	. 3393450
432	15.826071.462575.276026	46.647863.284229.267991	. 3392668
434	17.816600.870069.735539	52.527070.729108.240605	. 3391889
436	20.052237.544857.629067	59.131714.309169.618645	. 3391113
438	22.562532.800060.770285	66.549436.566966.297367	. 3390341
440	25.380526.940366.383372	74.878248.419470.886233	. 3389572
442	28.543152.282052.209031	84.227730.407729.499781	. 3388807
444	32.091682.620530.574852	94.720370.257893.471820	. 3388044
446	36.072232.518846.606193	106.493051.905239.118581	. 3387285
448	40.536314.045019.830519	119.698712.782720.205954	. 3386528
450	45.541454.927569.937980	134.508188.001572.923840	. 3385775
452	51.151887.377895.518187	151.112262.071917.313678	.3385025
454	57.439312.856875.164198	169.723951.046458.040965	. 3384278
456	64.483753.765628.119990	190.581040.442651.931034	. 3383534
458	72.374498.575459.529761	213.948907.032733.069132	. 3382793
460	81.211154.126223.919812	240.123655.613925.192081	. 3382055
462	91.104812.889042.494402	269.435605.212954.994471	. 3381320
464	102.179351.774084.517247	302.253162.872576.636605	. 3380588
466	114.572872.711521.982305	338.987127.249525.432549	. 3379859
468	128.439304.643242.346397	380.095468.763120.598477	. 3379132
470	143.950179.632789.523790	426.088638.015652.413417	. 3378409
472	161.296607.420846.627003	477.535459.708164.115593	. 3377688
474	180.691463.780462.230570	535.069675.351607.262125	. 3376970
476	202.371821.906641.398466	599.397204.782301.852926	. 3376255
478	226.601646.793972.364092	671.304203.896731.807232	. 3375543
480	253.674787.206902.086847	751.666004.194993.125591	. 3374833
482	283.918289.980520.154068	841.457028.742823.649455	. 3374127
484	317.696079.235783.042618	941.761789.114997.698055	. 3373422
486	355.413030.566947.436185	1053.787078.862455.346513	. 3372721
488	397.519491.261433.715764	1178.875491.155735.802646	. 3372022
490	444.516285.148721.676016	1318.520401.612270.233223	. 3371326

n	$g(n)$	$p(n)$	$g(n)/p(n)$
492	496.960262.431221.933377	1474.382572.040363.953132	. 3370633
494	555.470442.345519.505726	1648.308547.066172.438760	. 3369942
496	620.734822.524488.429887	1842.351033.503159.891466	. 3369253
498	693.517913.244644.376036	2058.791472.042884.901563	. 3368568
500	774.669085.011428.835081	2300.165032.574323.995027	. 3367884
502	865.131803.384963.113026	2569.288288.377098.289281	. 3367204
504	965.953855.732975.451076	2869.289850.802400.662045	. 3366525
506	1078.298661.059460.646172	3203.644275.096202.070012	. 3365850
508	1203.457790.569809.977154	3576.209579.998154.653671	. 3365176
510	1342.864809.865594.854721	3991.268758.958164.118300	. 3364506
512	1498.110595.470108.535477	4453.575699.570940.947378	. 3363837
514	1670.960265.197692.006066	4968.405970.488126.319775	. 3363171
516	1863.371903.154222.485007	5541.612982.013113.936133	. 3362508
518	2077.517250.840894.293529	6179.690078.238084.808000	. 3361846
520	2315.804584.308779.792051	6889.839175.409542.385648	. 3361188
522	2580.903985.437582.047712	7680.046623.716094.332553	. 3360531
524	2875.775270.499622.643254	8559.167038.437716.736150	. 3359877
526	3203.698835.512775.321542	9537.015921.990240.021538	. 3359225
528	3568.309730.462613.277337	10624.471981.512075.020731	. 3358576
530	3973.635279.920638.314843	11833.590138.006300.416410	. 3357928
532	4424.136628.837685.204084	13177.726323.474524.612308	. 3357283
534	4924.754598.198148.903956	14671.675272.840783.232475	. 3356640
536	5480.960303.823131.913346	16331.822638.729701.493803	. 3356000
538	6098.811014.380095.758058	18176.312890.390861.435034	. 3355362
540	6785.011787.015484.143370	20225.234604.409151.266221	. 3354725
542	7546.983460.661323.270992	22500.824915.577356.165493	. 3354091
544	8392.937659.823034.254314	25027.695072.821279.146420	. 3353460
546	9331.959509.452339.672600	27833.079238.879849.385687	. 3352830
548	10374.098842.503992.047601	30947.108885.217475.101876	. 3352202
550	11530.470761.851201.437941	34403.115367.205050.943160	. 3351577
552	12813.366486.201931.098406	38237.963520.943177.237554	. 3350954
554	14236.375525.907215.720937	42492.419404.397720.872600	. 3350333
556	15814.520314.233192.845657	47211.555614.160398.040338	. 3349713
558	17564.404554.306849.379544	52445.197947.746313.627407	. 3349096
560	19504.376630.425338.940450	58248.417552.751868.050007	. 3348481
562	21654.709624.095907.037587	64682.073111.542943.380454	. 3347868
564	24037.799541.712545.312555	71813.408056.839596.203570	. 3347257

n	$g(n)$	$p(n)$	$g(n)/p(n)$
566	26678.383616.684151.091417	79716.708303.343130.521599	.3346648
568	29603.780629.611359.695197	88474.026517.495817.981253	. 3346041
570	32844.155485.585087.895823	98175.979536.033971.312388	. 3345437
572	36432.810377.738654.758119	108922.626189.067392.956037	. 3344834
574	40406.505261.273864.035858	120824.433490.320564.237125	. 3344232
576	44805.810415.680682.864354	134003.339931.725153.597473	. 3343633
578	49675.494379.015092.548130	148593.925468.119890.197615	. 3343036
580	55064.950610.484813.034226	164744.698707.340387.584240	. 3342441
582	61028.666818.870688.179003	182619.512839.056823.919887	. 3341848
584	67626.740980.425298.679830	202399.122950.629095.580175	. 3341256
586	74925.448815.380684.522752	224282.898599.046831.034631	. 3340667
588	82997.867525.602669.487564	248490.706844.586261.413858	. 3340079
590	91924.561526.362430.605299	275264.982414.934173.206642	. 3339493
592	101794.335967.391354.819610	304873.003269.975366.031783	. 3338909
594	112705.064903.541372.910902	337609.391590.065169.560935	. 3338327
596	124764.601060.696691.230392	373798.862128.436852.709430	. 3337746
598	138091.775476.065977.362714	413799.241966.727832.978027	. 3337168
600	152817.495306.901705.606360	458004.788008.144308.553622	. 3336591
602	169085.949733.393626.681353	506849.831053.734861.481872	. 3336016
604	187055.933951.850393.240988	560812.778053.476538.349420	. 3335443
606	206902.303112.470251.801908	620420.507127.059714.307352	. 3334872
608	228817.568177.684489.362279	686253.193233.019826.880477	. 3334302
610	253013.647961.452399.720977	758949.605954.969709.105721	. 3333734
612	279723.791649.781906.578355	839212.924798.226411.060795	. 3333168
614	309204.688862.398100.317169	927817.121679.723721.849795	. 3332603
616	341738.784463.948448.266796	1.025613.964982.134990.453294	. 3332041
618	377636.818464.771506.567385	1.133540.704665.979618.906662	. 3331479
620	417240.611615.194905.255263	1.252628.503530.795506.440909	. 3330920
622	460926.121091.974999.638124	1.384011.685831.426958.558879	. 3330362
624	509106.790873.918774.794943	1.528937.881135.168275.063375	. 3329806
626	562237.225936.238472.384947	1.688779.148601.189609.516729	. 3329252
628	620817.219815.967674.710332	1.865044.174831.202682.776536	. 3328699
630	685396.170227.967465.834501	2.059391.647140.527228.529479	. 3328148
632	756577.918078.906296.908312	2.273644.913597.837330.081136	. 3327599
634	835026.051382.395988.454667	2.509808.051552.031608.082535	. 3327051
636	921469.716245.048299.657364	2.770083.477684.418110.395121	. 3326505
638	1.016709.984388.014492.641358	3.056891.244979.232231.862474	. 3325960
640	1.121626.827786.602072.747475	3.372890.185488.482409.685019	. 3325417
642	1.237186.759233.338047.297620	3.721001.072479.541451.508397	. 3324876

n	$g(n)$	$p(n)$	$g(n)/p(n)$
644	1.364451.199264.602452.424663	4.104431.991606.013700.457110	. 3324336
646	1.504585.639672.551644.242118	4.526706.128254.173781.044298	. 3323797
648	1.658869.675649.684241.358424	4.991692.197319.220372.390544	. 3323261
650	1.828707.990132.791601.947044	5.503637.762499.727151.307095	. 3322725
652	2.015642.376610.571304.137220	6.067205.714919.484306.343541	. 3322192
654	2.221364.899627.572482.931246	6.687514.205661.440172.553650	. 3321660
656	2.447732.295924.003825.269655	7.370180.353811.425547.662139	. 3321129
658	2.696781.734491.609424.140355	8.121368.081058.512888.507057	. 3320600
660	2.970748.058134.122963.494038	8.947840.456000.332817.673697	. 3320072
662	3.272082.647087.407057.382802	9.857016.966290.401433.259592	. 3319546
664	3.603474.051224.525351.885322	10.857036.174895.938656.583295	. 3319021
666	3.967870.557560.987133.651938	11.956824.258286.445517.629485	. 3318498
668	4.368504.867675.777863.215047	13.166169.969647.255482.980383	. 3317976
670	4.808921.083435.523461.447446	14.495806.619536.377005.379418	. 3317456
672	5.293004.208720.088866.913117	15.957501.720133.631304.230773	. 3316937
674	5.825012.402614.819454.894906	17.564154.997755.650263.621500	. 3316420
676	6.409612.231875.852908.803564	19.329905.542049.511423.199336	. 3315904
678	7.051917.201670.489551.038777	1.270248.929688.765106.878025	.3315390
680	7.757529.859490.264465.707112	23.402165.235974.892374.954302	.3314876
682	8.532587.803739.101324.785065	25.744258.930034.131533.263392	.3314365
684	9.383813.947348.828453.393455	28.316911.738879.831363.625420	.3313854
688	11.344923.592080.919506.168911	34.245325.433219.728719.773420	.3312838
690	12.471699.488013.896651.711403	37.652317.810725.762600.765183	.3312332
692	13.708565.414096.463673.858784	41.392749.264546.866860.893416	.3311827
694	15.066103.023547.107434.511935	45.498723.689129.703063.649450	.3311324
696	16.555894.604645.352077.748455	50.005385.980149.860746.062163	.3310822
698	18.190616.179789.509008.260961	54.951205.445179.608281.719072	.3310321
700	19.984139.124356.415694.351472	60.378285.202834.474611.028659	.3309822
702	21.951641.078449.121951.690536	66.332699.915362.724119.980694	.3309324
704	24.109726.981512.974424.928791	72.864864.407855.341219.969825	.3308827
706	26.476561.145944.022178.165998	80.029935.953661.656574.123574	.3308332
708	29.072011.353175.813385.104755	87.888253.251761.884175.130183	.3307838
710	31.917806.055993.689642.806479	96.505815.389469.697877.049934	.3307345
712	35.037705.854425.086767.315281	105.954804.374756.131323.439197	.3306853
714	38.457690.525805.807655.349494	116.314155.138696.524440.183805	.3306363
716	42.206162.992149.595203.733066	127.670177.252209.281782.740521	.3305874
718	46.314171.740509.991644.623377	140.117232.974725.477106.760252	.3305387
720	50.815653.333132.453355.417013	153.758476.658245.881594.406593	.3304900
722	55.747696.798594.659665.278707	168.706660.971164.630122.439117	.3304415
724	61.150831.843648.989160.520517	185.085015.885255.746880.625875	.3303932
726	67.069343.000364.790909.509432	203.028206.889569.986197.651315	.3303449
728	73.551612.003580.342261.319587	222.683379.460186.024851.577401	.3302968

n	$g(n)$	$p(n)$	$g(n)/p(n)$
730	80.650490.898600.239564.879402	244.211297.428606.706709.925517	.3302488
732	88.423708.591505.850741.335866	267.787583.558210.323920.375877	.3302009
734	96.934313.793528.032678.451108	293.604071.362025.285843.562670	.3301531
736	106.251157.568762.826790.538184	321.870277.981032.622582.593573	.3301055
738	116.449418.966522.956877.514932	352.815008.795455.957133.215652	.3300580
740	127.611177.530258.752406.622494	386.688105.367749.941220.651375	.3300106
742	139.826036.793992.945433.935250	423.762349.321394.151918.928481	.3299633
744	153.191803.242115.928699.781472	464.335535.850798.483634.138280	.3299161
746	167.815225.581219.295311.353031	508.732731.741838.107613.602755	.3298691
748	183.812799.611964.924619.350744	557.308734.067567.635805.394638	.3298222
750	201.311644.414948.208412.982824	610.450747.117966.916191.771809	.3297754
752	220.450456.090175.710516.586817	668.581296.635294.279311.393900	.3297287
754	241.380545.790304.317092.400955	732.161402.067670.820574.405230	.3296821
756	264.266969.403457.474602.542187	801.694029.333610.862568.750951	.3296356
758	289.289756.827489.405970.746710	877.727848.520950.325159.242658	.3295893
760	316.645249.513934.062456.082785	960.861323.037560.814483.873080	.3295431
762	346.547555.632309.591952.786426	1051.747159.001957.690209.588887	.3294970
764	379.230133.081677.670002.048472	1151.097146.124113.726578.727360	.3294510
766	414.947511.367341.686064.123585	1259.687423.996378.387111.229150	.3294051
768	453.977164.384474.119153.084260	1378.364210.608578.997366.598385	.3293593
770	496.621547.078482.874510.287475	1508.050033.038752.490738.311726	.3293137
780	776.694909.402394.840837.177170	2360.150221.898687.182164.777966	.3290870
790	1211.353075.227951.617473.679900	3683.456542.940343.404363.084600	.3288631
800	1884.120926.878905.266804.491024	5733.052172.321422.504456.911979	.3286418
810	2922.713240.872047.520353.037152	8899.229771.588828.461969.917962	.3284231
820	4521.932323.152782.786266.159359	13777.683783.859651.786576.215682	.3282070
830	6978.188694.133167.705334.437356	21275.399574.724765.449983.360003	.3279933
840	10741.425934.538122.626370.804346	32770.027459.303858.556350.798600	.3277820
850	16493.048943.715545.894087.350086	50349.216918.401212.177548.479675	.3275730
860	25262.619707.922527.216551.496044	77169.232591.877674.590168.543277	.3273664
870	38602.227230.689416.618251.994485	117991.131259.998859.170817.958839	.3271621
880	58846.479520.175142.676222.361808	179980.699075.416049.556058.362840	.3269599
890	89499.347097.873269.012421.364801	273899.386535.208029.575034.561337	.3267599
900	135808.560233.418108.130691.354490	415873.681190.459054.784114.365430	.3265620
910	205616.757446.208107.769772.586031	630018.505076.433611.630379.753807	.3263662

5 Implementation

To implement the recursion we used a hash table to store the results computed in the recursion. This is useful as we do not need to compute all smaller results. Next trick was not to store smaller results, so we decided not to store results which where smaller then 400. But independent of all the tricks the limit was the size of the memory, not the computing time. The following table shows the size of the hash table in MB together with the used CPU time in seconds on a Linux 3GHz/4GB single processor machine.

n	50	100	200	220	300	400	500	600	700	760
MB	0.04	1.3	13	21	59	195	437	863	1332	1889
sec	0	1	29	41	122	380	984	2009	3647	4811

Above table of the number of graphical partitions up to $n = 760$ was computed in a single run, where we filled the hash table with all values necessary to compute all the numbers $g(0), \ldots, g(760)$ of graphical partitions. For the values for $n > 760$ we had to use a different method of computation. We computed the number of pairs $L(\lambda) \trianglerighteq R(\lambda)$ for a given size i of the Durfee square. Afterwards we remove all precomputed values $p(m, k, n, l)$ with $l < i$, and started with the next size of the Durfee square. This method works because as you see in the telescoping lemma, we need for the computation of $p(m, k, n, i)$ other values with parameter i or $i - 1$ only. This reduces the amount of memory necessary for the storage of precomputed values from $O(n^3)$ to $O(n^{2.5})$. (thanks to the referee) This helps in the cases $m! = n$, but in the limiting case $m = n$ we have to recompute new results, but overall this trick reduces the amount of memory necessary to store intermediate results.

So we were able to compute larger values up to $n > 900$. But we didn't have time to compute all the values, which explains the missing values in the above table.

A current version of the table can be fetched from

http://www.mathe2.uni-bayreuth.de/axel/numberofgraphicalpartitions.pdf.

References

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