On the functions with values in $[\alpha(G), \overline{\chi}(G)]$

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Abstract

Let

$$\mathcal{B}(G) = \{X : X \in \mathbb{R}^{n\times n}, X = X^T, I \leq X \leq I + A(G)\}$$

and

$$\mathcal{C}(G) = \{X : X \in \mathbb{R}^{n\times n}, X = X^T, I - A(G) \leq X \leq I + A(G)\}$$

be classes of matrices associated with graph G. Here n is the number of vertices in graph G, and $A(G)$ is the adjacency matrix of this graph. Denote $r(G) = \min_{X\in\mathcal{C}(G)} \operatorname{rank}(X)$, $r_+(G) = \min_{X\in\mathcal{B}(G)} \operatorname{rank}(X)$. We have shown previously that for every graph G, $\alpha(G) \leq r_+(G) \leq \overline{\chi}(G)$ holds and $\alpha(G) = r_+(G)$ implies $\alpha(G) = \overline{\chi}(G)$. In this article we show that there is a graph G such that $\alpha(G) = r(G)$ but $\alpha(G) < \overline{\chi}(G)$. In the case when the graph G doesn't contain two chordless cycles C_4 with a common edge, the equality $\alpha(G) = r(G)$ implies $\alpha(G) = \overline{\chi}(G)$. Corollary: the last statement holds for $d(G)$ – the minimal dimension of the orthonormal representation of the graph G.

Let G be a graph with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$. We are interested in studying the functions of the graph G whose values belong to the interval $[\alpha(G), \overline{\chi}(G)]$. Here $\alpha(G)$ is the size of the largest stable set in G and $\overline{\chi}(G)$ is the smallest number of cliques that cover the vertices of G.

It is well known (see, for example, [1]) that for some $\epsilon > 0$ it is impossible to approximate in polynomial time $\alpha(G)$ and $\overline{\chi}(G)$ within a factor of n^ϵ, assuming $P \neq NP$. We suppose that better approximation could be obtained for narrow classes of graphs determined on the basis of a system of functions of graph G "sandwiched" between $\alpha(G)$ and $\overline{\chi}(G)$.

One such function is the well known Lovász function $\theta(G)$ [7], which has many alternative definitions. One of them is based on the notion of the orthonormal labeling of the graph. Orthonormal labeling of dimension k of the graph G is a mapping

$$f : V(G) \rightarrow \mathbb{R}^k,$$

such that $||f(v)||_2 = 1$ for all $v \in V(G)$ and $f(v_i) \cdot f(v_j) = 0$ if $\{v_i, v_j\} \notin E(G)$.

Let $e_i \in \mathbb{R}^k$ be a unit vector which is 0 in all coordinates except the ith coordinate which is equal 1. Then

$$\theta(G) = \min_f \max_{v \in V(G)} \frac{1}{(e_1 \cdot f(v))^2}$$

and

$$\alpha(G) \le \theta(G) \le d(G) \le \overline{\chi}(G),$$

where $d(G)$ is the minimum dimension of the orthonormal labeling of the graph G.

Let

$$\mathcal{B}(G) = \{X : X \in \mathbb{R}^{n \times n}, X = X^T, I \le X \le I + A(G)\}$$

and

$$\mathcal{C}(G) = \{X : X \in \mathbb{R}^{n \times n}, X = X^T, I - A(G) \le X \le I + A(G)\}$$

be classes of matrices associated with graph G. Here n is the number of vertices of graph G, I is identity matrix and $A(G)$ is the adjacency matrix of this graph. Consider two functions of graph G based on these classes:

$$r(G) = \min_{X \in \mathcal{C}(G)} \operatorname{rank}(X),$$

$$r_+(G) = \min_{X \in \mathcal{B}(G)} \operatorname{rank}(X).$$

The function $r_+(G)$ was studied in [2]. It was shown that for every graph G

$$\alpha(G) \le r_+(G) \le \overline{\chi}(G)$$

holds and

$$\alpha(G) = r_+(G) \quad \text{implies} \quad \alpha(G) = \overline{\chi}(G). \tag{1}$$

It is obvious that

$$\alpha(G) \le r(G) \le d(G), r_+(G) \le \overline{\chi}(G).$$

It is was shown in[3] that for $i = 1, 2, 3$ $\;r(G) = i$ iff $d(G) = i$.

Recent results on well known related problem concerning upper bound on $\chi(G)$ in terms of rank of adjacency matrix $A(G)$ are presented in [4, 5, 6].

In this paper we are interested in the following question: can we use the functions $\theta(G), r(G)$ and $d(G)$ in (1) instead of $r_+(G)$? In the common case the answer is negative. The proof is based on the following lemmas.

Lemma 1 *If $\alpha(G) = d(G)$ implies $\alpha(G) = \overline{\chi}(G)$ for every graph G then for every set of unit vectors $S = \{s_1, \ldots, s_n\}$ with $s_i \in \mathbb{R}^k$ there exists an orthogonality-preserving mapping*

$$\varphi : S \to \mathbb{R}^k_+$$

of vectors from S into non-negative unit vectors from $\mathbb{R}^k$ such that $s_i \cdot s_j = 0$ implies $\varphi(s_i) \cdot \varphi(s_j) = 0$.

Proof. Suppose that $\alpha(G) = d(G)$ implies $\alpha(G) = \overline{\chi}(G)$ for every graph G. Then we can construct above mentioned mapping φ for every vector set S.

Let $S = \{s_1, \dots, s_n\}, \quad s_i \in \mathbb{R}^k$ be a given vector set. Then we can construct graph $G = (V(G), E(G))$, where $V(G) = A \cup B,\ A \cap B = \emptyset, A = \{a_1, \dots, a_k\},\ B = \{b_1, \dots, b_n\}$. Assign unit vector $e_i \in \mathbb{R}^k$ to vertex $a_i \in A$ and vector s_j to vertex $b_j \in B$ for $i = 1, \dots, k, \quad j = 1, \dots, n$.

To form edge set $E(G)$:

- join every vertex from A with every vertex from B;
- join b_i and b_j from B iff $s_i \cdot s_j \neq 0$.

It is obvious that A is a maximum stable set of the graph G and $\alpha(G) = d(G) = k$.

Our assumption implies that $\alpha(G) = \overline{\chi}(G)$ and there exists a decomposition

$$B = B_1 \cup \dots \cup B_k,$$

such that every B_i induces a clique in G. So $\varphi : \varphi(s_i) = e_j$ when $b_i \in B_j$, is the required orthogonality-preserving mapping of S into $\mathbb{R}^k_+$. $\square$

Now we'll construct a system of unit vectors from $\mathbb{R}^3$ such that orthogonal-preserving mapping of this set into $\mathbb{R}^3_+$ does not exists.

Lemma 2 *Let*

$$S = \{a, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3\},$$

be a system of vectors from $\mathbb{R}^3$, *where*

$$a = (1, 1, 1)^T,$$

$$b_1 = (-1, 1, 0)^T, b_2 = (1, 0, -1)^T, b_3 = (0, -1, 1)^T,$$
$$c_1 = (1, 1, 0)^T, c_2 = (1, 0, 1)^T, c_3 = (0, 1, 1)^T,$$
$$d_1 = (-1, 1, 1)^T, d_2 = (1, -1, 1)^T, d_3 = (1, 1, -1)^T,$$
$$e_1 = (1, 0, 0)^T, e_2 = (0, 1, 0)^T, e_3 = (0, 0, 1)^T.$$

Then orthogonality-preserving mapping φ *of the set* S *into a set of unit vectors from* $\mathbb{R}^3_+$ *does not exists.*

Proof. Suppose that the above mentioned mapping φ exists. Then φ can be chosen in such a way that every vector from S is mapped into one of the vectors from $\{e_1, e_2, e_3\}$.

Indeed, let φ' be a orthogonality-preserving mapping from S into a set of unit vectors from $\mathbb{R}^3_+$. Then for every $s \in S$ and any i such that $e_i \cdot \varphi'(s) > 0$ let $\varphi(s) = e_i$.

We may suppose without loss of generality that $\varphi(e_i) = e_i,\ i = 1, 2, 3$.

Let's suppose that $\varphi(a) = e_1$. This implies $\varphi(b_1) = e_2, \varphi(b_2) = e_3, \varphi(c_1) = e_1, \varphi(c_2) = e_1, \varphi(d_2) = e_2, \varphi(d_3) = e_3$. But then $\varphi(c_3)$ has to be orthogonal to every vector from $\{e_1, e_2, e_3\}$.

Let's suppose that $\varphi(a) = e_2$. This implies $\varphi(b_1) = e_1, \varphi(b_3) = e_3, \varphi(c_1) = e_2, \varphi(c_3) = e_2, \varphi(d_1) = e_1, \varphi(d_3) = e_3$. But then $\varphi(c_2)$ has to be orthogonal to every vector from $\{e_1, e_2, e_3\}$.

Let's suppose that $\varphi(a) = e_3$. This implies $\varphi(b_2) = e_1, \varphi(b_3) = e_2, \varphi(c_2) = e_3, \varphi(c_3) = e_3, \varphi(d_1) = e_1, \varphi(d_2) = e_2$. But then $\varphi(c_1)$ has to be orthogonal to every vector from $\{e_1, e_2, e_3\}$. $\square$

Lemmas 1 and 2 imply the following theorem.

Theorem 1 *There exists a graph G such that $\alpha(G) = d(G)$ and $\alpha(G) < \overline{\chi}(G)$.*

Corollary 1 *There exists graphs G such that $\alpha(G) = \theta(G) = r(G)$ and $\alpha(G) < \overline{\chi}(G)$.*

The following theorem shows that implication (1) holds for the function $r(G)$ (and, hence, for $d(G)$) in some cases.

Theorem 2 *If the graph G is free of two chordless cycles C_4 with a common edge then $\alpha(G) = r(G)$ implies $\alpha(G) = \overline{\chi}(G)$.*

Proof. Suppose that $\alpha(G) = r(G)$ and $\alpha(G) = \operatorname{rank}(X),\ X \in \mathcal{C}(G)$. Without loss of generality $M = \{1, \ldots, \alpha(G)\}$ is the maximum stable set of the graph G with n vertices. Then

$$X = \begin{pmatrix} I_{\alpha(G)} & Y \\ Y^T & Z \end{pmatrix}$$

and $Z = Y^T Y$.

This means that the following orthonormal labeling f of dimension $\alpha(G)$ of the graph G exists. If vertex $i \in M$ then $f(i) = e_i \in \mathbb{R}^{\alpha(G)}$, if vertex $j \in V \backslash M$ then $f(j)$ is equal to the $(j - \alpha(G))$th column of the matrix Y.

Let's show that for any three vertices l, i, j such that $l \in M,\ i, j \in V \backslash M$, if vertices i and j are non-adjacent and $e_l \cdot f(i) \neq 0$, $e_l \cdot f(j) \neq 0$ (hence, l is adjacent to i and j), then a vertex $m \in M$ $(m \neq l)$ exists such that $e_m \cdot f(i) \neq 0$ and $e_m \cdot f(j) \neq 0$ (hence, m is adjacent to vertices i and j also).

Because i and j are non-adjacent, we have

$$f(i) \cdot f(j) = \sum_{s=1}^{\alpha(G)} (e_s \cdot f(i))(e_s \cdot f(j)) = 0.$$

But the summand $(e_l \cdot f(i))(e_l \cdot f(j))$ isn't equal 0 in the last sum. Hence, at least one more non-zero summand exists. Let it be mth summand

$$(e_m \cdot f(i))(e_m \cdot f(j)) \neq 0.$$

Hence, vertex $m \in M$ is adjacent to vertices i and j.

Let

$$V(G) = V_1 \cup \cdots \cup V_q,\ q \geq \alpha(G)$$

be a decomposition of the vertex set of the graph G into q non-empty subsets such that

- $l \in V_l,\ l = 1, \ldots, \alpha(G)$;
- if $i \in V(G) \backslash M,\ i \in V_l, 1 \leq l \leq \alpha(G)$, then $e_l \cdot f(i) \neq 0$;
- if $i, j \in V_l, 1 \leq l \leq q$, then vertices i and j are adjacent.

It is obvious that such a decomposition exists. For example, $V(G)$ can be decomposed into n non-empty subsets.

Every V_i induces a clique in the G and, hence, $\overline{\chi}(G) \leq q$.

We suppose without loss of generality that no set V_i from $\{V_1, \ldots, V_{\alpha(G)}\}$ can be extended with vertices from $V_{\alpha(G)+1}, \ldots, V_q$.

Let's suppose that $\alpha(G) < \overline{\chi}(G)$. Then $S = V_{\alpha(G)+1} \cup \cdots \cup V_q \neq \oslash$. Let $x \in S$ be an arbitrary vertex from S. Then vertex $l \in M$ exists such that $e_l \cdot f(x) \neq 0$ (because $f(x) \neq 0$). The set V_l can't be extended with vertex x. Hence the vertex $x_l \in V_l$ exists that isn't adjacent to x. Then the vertex $m \in M,\ m \neq l$ should exist that is adjacent to x and x_l and $e_m \cdot f(x) \neq 0$, $e_m \cdot f(x_l) \neq 0$.

A vertex $x_m \in V_m$ exists that is non-adjacent to x because the set V_m can't be extended with x. Then vertex $y \in M$ exists such that $y \neq m$ and y is adjacent to x and x_m. Note, that vertices l and y may coincide.

If $y \neq l$, then there are two chordless cycles C_4 with common edge in $G: (l, x_l, m, x, l)$ and (m, x, y, x_m, m). If $y = l$, then such cycles exist also. They are (l, x_l, m, x, l) and (l, x, m, x_m, l). $\square$

Corollary 2 *Let the graph G be free of two chordless cycles C_4 with a common edge. Then $\alpha(G) = d(G)$ implies $\alpha(G) = \overline{\chi}(G)$.*

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