Tight estimates for eigenvalues of regular graphs

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Abstract

It is shown that if a d-regular graph contains s vertices so that the distance between any pair is at least $4k$, then its adjacency matrix has at least s eigenvalues which are at least $2\sqrt{d-1}\cos(\frac{\pi}{2k})$. A similar result has been proved by Friedman using more sophisticated tools.

1 The main result

Let $G=(V,E)$ be a simple d-regular graph on n vertices. let A be its adjacency matrix, and let $\lambda_1(G)=d \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$ be its eigenvalues. Alon and Boppana ([1], see also [9], [11]) proved that for any fixed d and for any infinite family of of d-regular graphs G_i, $\liminf\ \lambda_2(G_i) \geq 2\sqrt{d-1}$. This bound is sharp (at least when $d-1$ is a prime congruent to 1 modulo 4), as shown by the construction of Lubotzky, Phillips and Sarnak [9], obtained, independently, by Margulis [10]. In fact, in [1] it is conjectured that almost all d-regular graphs G on n vertices satisfy $\lambda_2(G) \leq 2\sqrt{d-1}+o(1)$ (as n tends to infinity). This has recently been proved by Friedman [6].

More generally, Serre has shown (see [3], [4]) that for any fixed r and for any infinite family of d-regular graphs G_i, $\liminf\ \lambda_r(G_i) \geq 2\sqrt{d-1}$. The same result has been proved by Friedman already in [5].

In this note we give an elementary, simple proof of this result, following the method of [11]. Our estimate is a bit better than that of [11] even for the case $r=2$ (also studied by Kahale in [7]), and matches in the first two terms the estimate of Friedman in [5], but the proof is completely elementary.

Theorem 1 *Let G be a d-regular graph containing s vertices so that the distance between any pair of them is at least $4k$. Then $\lambda_s(G) \geq 2\sqrt{d-1}\cos(\frac{\pi}{2k})$.*

2 The proof

Let $G = (V, E)$ be a d-regular graph, where $d \geq 3$. Let $v \in V$ be a vertex, define $V_0 = \{v\}$, and let V_i be the set of all vertices of distance i from v. Put $\alpha = \frac{\pi}{2k}$, $n_i = |V_i|$, and define

$$y_i = \frac{\cos((i-k)\alpha)}{(d-1)^{i/2}}.$$

Therefore, $y_0 = y_{2k} = 0$ and $y_i \geq 0$ for all $0 \leq i \leq 2k$. It is not difficult to check that the sequence y_i is unimodal, that is, there exists a unique i_0 such that

$$y_0 < y_1 < \ldots < y_{i_0} < y_{i_0+1} \geq \ldots \geq y_{2k}.$$

For $d \geq 5$, $i_0 = 0$, whereas for $d = 4$ or 3 it may be 1 or 2.

To see that this is indeed the case note that the ratio y_{i+1}/y_i is precisely

$$\frac{\cos\alpha - \tan((i-k)\alpha)\sin\alpha}{\sqrt{d-1}},$$

which is a decreasing function of i for all admissible values of i, and hence we simply let $i_0 + 1$ be the smallest i for which this ratio is at most 1. (Such an i exists, as for $i = k$ the ratio is $\cos\alpha/\sqrt{d-1} \leq 1$.) If $d \geq 5$, then the above ratio is at most 1 already for $i = 1$, since in this case the ratio $y_2/y_1 = 2\cos\alpha/\sqrt{d-1} \leq 1$. Thus, in this case, $i_0 + 1 = 1$. For $d = 3, 4$ note that the ratio y_{i+1}/y_i is also equal to $\frac{\sin((i+1)\alpha)}{\sqrt{d-1}\sin(i\alpha)}$. The function $f_i(\alpha) = \frac{\sin((i+1)\alpha)}{\sin(i\alpha)}$ is at most $(i+1)/i$ for all admissible values of α, since $g(\alpha) = i\sin(i\alpha)f_i(\alpha) - (i+1)\sin(i\alpha)$ satisfies $g(0) = 0$ and $g'(\alpha) = i(i+1)\cos((i+1)\alpha) - i(i+1)\cos(i\alpha) \leq 0$ whenever $0 \leq i\alpha \leq (i+1)\alpha \leq \pi$. Therefore, $\frac{y_{i+1}}{y_i} \leq \frac{i+1}{i\sqrt{d-1}}$ which is at most 1 for $i = 2$ and $d = 4$, and at most 1 for $i = 3$ and $d = 3$. It follows that for all $d \geq 3$, $i_0 + 1 \leq 3$, whereas for $d \geq 5$, $i_0 + 1 = 1$.

Put $x_i = y_{i+i_0}$ for all i, $0 \leq i \leq 2k - i_0$. Therefore, x_i is monotone non-increasing for $i \geq 1$, and $x_{2k-i_0} = 0$. Let A be the adjacency matrix of G, whose rows and columns are indexed by the vertices of G, and define a vector $x(v)_{v\in V}$, by putting $x(v) = x_i$ for all $v \in V_i$ and $x(v) = 0$ otherwise. The main part of the proof is the following lemma.

Lemma 1 *The following inequality holds*

$$x^t A x \geq [2\sqrt{d-1}\cos\alpha] \cdot x^t x.$$

Proof : For any two subsets X, Y of V, let $e(X, Y) = \{(x, y) : x \in X, y \in Y, xy \in E\}$ denote the number of edges between X and Y (note that if $X = Y$ then each edge with both ends in X is counted twice).

First note that

$$x^t x = \sum_{i=0}^{2k-i_0} n_i x_i^2. \tag{1}$$

Also:

$$x^t Ax = dx_0x_1 + \sum_{i=1}^{2k-i_0-1} [e(V_{i-1}, V_i)x_{i-1} + e(V_i, V_i)x_i + e(V_i, V_{i+1})x_{i+1}]x_i. \qquad (2)$$

Since $0 \leq x_0 < x_1$

$$dx_0x_1 \geq dx_0^2 \geq [2\sqrt{d-1}\cos\alpha]x_0^2 \qquad (3)$$

Consider the term

$$e(V_{i-1}, V_i)x_{i-1} + e(V_i, V_i)x_i + e(V_i, V_{i+1})x_{i+1}. \qquad (4)$$

Note that $e(V_{i-1}, V_i) \geq n_i$, and that $e(V_{i-1}, V_i) + e(V_i, V_i) + e(V_i, V_{i+1}) = dn_i$. Also note that if $i > 1$, then, by our definition of the values $x_i = y_{i+i_0}$, $x_{i-1} \geq x_i \geq x_{i+1}$, and therefore the minimum possible value of the term (4) is obtained when $e(V_{i-1}, V_i) = n_i$, $e(V_i, V_i) = 0$ and $e(V_i, V_{i+1}) = (d-1)n_i$. For $i = 1$, V_0 consists of a single vertex and hence certainly $e(V_0, V_1) = n_1 = d$, and since $x_1 \geq x_2$ the minimum of the term (4) is again obtained when $e(V_1, V_1) = 0$ and $e(V_1, V_2) = (d-1)n_1$. It follows that in any case the term (4) satisfies

$$e(V_{i-1}, V_i)x_{i-1} + e(V_i, V_i)x_i + e(V_i, V_{i+1})x_{i+1} \geq n_ix_{i-1} + (d-1)n_ix_{i+1}$$

$$= n_i \frac{\sqrt{d-1}}{(d-1)^{(i+i_0)/2}}\left(\cos[(i+i_0-k-1)\alpha] + \cos[(i+i_0-k+1)\alpha] \right)$$

$$= n_i \frac{2\sqrt{d-1}}{(d-1)^{(i+i_0)/2}} \cos\alpha \cos[(i+i_0-k)\alpha] = [2\sqrt{d-1}\cos\alpha]n_ix_i.$$

Substituting this and (3) in (2) we conclude that $x^tAx \geq [2\sqrt{d-1}\cos\alpha]\sum_{i=0}^{2k-i_0-1} n_ix_i^2$. This, together with (1) and the fact that $x_{2k-i_0} = 0$ implies the assertion of the lemma. $\square$

To complete the proof of the theorem note that by the variational definition of the eigenvalue $\lambda_s(G)$ it is equal to the maximum, over all subspaces W of dimension s, of the minimum value of z^tAz as z ranges over all unit vectors in W. Given s vertices v_i as in the theorem, we can construct for each of them a vector $x^{(i)}$ as in the lemma, and observe that as the distance between the supports of any two of these vectors exceeds 1, for every vector z in their span

$$z^tAz \geq [2\sqrt{d-1}\cos\alpha]z^tz.$$

This completes the proof of the theorem. $\square$

3 Concluding remarks

- For large k, $\cos(\frac{\pi}{2k})$ is $1 - \frac{\pi^2}{8k^2} + O(\frac{1}{k^4})$. In particular, for any d-regular graph G with diameter r, $\lambda_2(G) \geq 2\sqrt{d-1}[1 - \frac{2\pi^2}{r^2} + O(\frac{1}{r^4})]$, matching the estimate in [5].

- Similar arguments can be used to show that if a d-regular graph G contains s vertices so that the distance between any pair is at least $4k$ and so that there is no odd cycle that lies within distance $2k$ of any of the vertices, then G has at least s eigenvalues which are smaller than $-2\sqrt{d-1}\cos(\frac{\pi}{2k})$. The proof is analogous to that of Theorem 1, one simply defines the numbers x_i with the same absolute values as in this proof, but with alternating signs. This has also been proved by Friedman in [5], see also [8] and [2] for related results.

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