The plethysm $s_\lambda[s_\mu]$ at hook and near-hook shapes

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Abstract

We completely characterize the appearance of Schur functions corresponding to partitions of the form $\nu = (1^a, b)$ (hook shapes) in the Schur function expansion of the plethysm of two Schur functions,

$$s_\lambda[s_\mu] = \sum_\nu a_{\lambda,\mu,\nu} s_\nu.$$

Specifically, we show that no Schur functions corresponding to hook shapes occur unless λ and μ are both hook shapes and give a new proof of a result of Carbonara, Remmel and Yang that a single hook shape occurs in the expansion of the plethysm $s_{(1^c,d)}[s_{(1^a,b)}]$. We also consider the problem of adding a row or column so that ν is of the form $(1^a, b, c)$ or $(1^a, 2^b, c)$. This proves considerably more difficult than the hook case and we discuss these difficulties while deriving explicit formulas for a special case.

1 Introduction

One of the fundamental problems in the theory of symmetric functions is to expand the plethysm of two Schur functions, $s_\lambda[s_\mu]$, as a sum of Schur functions. That is, we want to find the coefficients $a_{\lambda,\mu,\nu}$ where

$$s_\lambda[s_\mu] = \sum_\nu a_{\lambda,\mu,\nu} s_\nu.$$

In general, the problem of expanding different products of Schur functions as a sum of Schur functions arises in the representation theory of the symmetric group S_n. Specifically, let C_λ be the conjugacy class of S_n associated with a partition λ. Define a function $1_\lambda : S_n \to \mathbf{C}$ by setting $1_\lambda(\sigma) = \chi(\sigma \in C_\lambda)$ for all $\sigma \in S_n$, where for a statement A,

$$\chi(A) = \begin{cases} 0 & \text{if } A \text{ is true} \\ 1 & \text{if } A \text{ is false} \end{cases}$$

Let $\lambda \vdash n$ denote that λ is a partition of the positive integer n. Then the set $\{1_\lambda\}_{\lambda \vdash n}$ forms a basis for $C(S_n)$, the center of the group algebra of S_n. There is a fundamental isometry between $C(S_n)$ and Λ_n, the vector space of homogeneous symmetric polynomials of degree n. This is defined by setting

$$F(1_\lambda) = \frac{1}{z_\lambda} p_\lambda$$

for all $\lambda \vdash n$, where p_λ is the power-sum symmetric function indexed by λ and z_λ is a constant defined below. This map, called the *Frobenius characteristic*, has the remarkable property that irreducible representations of S_n are mapped to Schur functions. That is, if χ^λ is the character of the irreducible representation of S_n associated with the partition λ, then $F(\chi^\lambda) = s_\lambda$. So for any character χ^A of a representation A of S_n, the coefficients a_ν in the expansion

$$F(\chi^A) = \sum_{\nu \vdash n} a_\nu s_\nu$$

give the multiplicities of the irreducible representations in A.

For the plethysm of two Schur functions, the representation that arises is the following (see [9]). For $\lambda \vdash n$, let U_λ denote the irreducible S_n-module corresponding to λ. Also let $\mu \vdash m$ and $U_\mu^{\otimes n}$ denote the n-fold tensor product of U_μ. Then the wreath product of S_n with S_m acts naturally on $U_\lambda \otimes U_\mu^{\otimes n}$. Let χ be the character of the $S_{n \cdot m}$-module which results by inducing the action of the wreath product of S_n with S_m on $U_\lambda \otimes U_\mu^{\otimes n}$ to a representation of $S_{n \cdot m}$. Then $F(\chi) = s_\lambda[s_\mu]$ so that in the expansion

$$s_\lambda[s_\mu] = \sum a_{\lambda,\mu,\nu} s_\nu$$

$a_{\lambda,\mu,\nu}$ is the multiplicity of the irreducible representation indexed by ν in the representation associated with χ.

The notion of plethysm goes back to Littlewood. The problem of computing the $a_{\lambda,\mu,\nu}$ has proven to be difficult and explicit formulas are known only for a few special cases. For example, Littlewood [8] explicitly evaluated $s_{1^2}[s_n]$, $s_2[s_n]$, $s_n[s_2]$, and $s_n[s_{1^2}]$ for all n using generating functions. Thrall [11] has derived the expansion for $s_3[s_n]$. Chen, Garsia, Remmel [4] have given a combinatorial algorithm for computing $p_k[s_\lambda]$. This algorithm can be used to find $s_\lambda[s_\mu]$ by expanding s_λ in the power basis and multiplying Schur functions. Chen, Garsia, Remmel use this algorithm to give formulas for $s_\lambda[s_n]$ when λ is a partition of 3. Foulkes [5] and Howe [6] have shown how to compute $s_\lambda[s_n]$ when λ is a partition of 4. Finally, Carré and Leclerc [3] have found combinatorial interpretations for the coefficients in the expansions of $s_2[s_\lambda]$ and $s_{1^2}[s_\lambda]$, Carbonara, Remmel, Yang [1] have given explicit formulas for $s_2[s_{(1^a,b)}]$ and $s_{1^2}[s_{(1^a,b)}]$, and Carini and Remmel [2] have found explicit formulas for $s_2[s_{(a,b)}]$, $s_{1^2}[s_{(a,b)}]$, and $s_2[s_{k^n}]$.

In this work we obtain explicit formulas when $\nu = (1^a, b)$ (a hook shape), $\nu = (1^a, b, c)$ (a hook plus a row), or $\nu = (1^a, 2^b, c)$ (a hook plus a column). For example, the well-known formula

$$s_\lambda[X - Y] = \sum_{\mu \subseteq \lambda} s_\mu[X](-1)^{|\lambda/\mu|} s_{(\lambda/\mu)'}[Y]$$

shows that $s_\lambda[1 - x] = 0$ unless λ is a hook. This allows us to prove the somewhat surprising fact that there are no hook shapes in the expansion of $s_\lambda[s_\mu]$ unless both λ and μ are hooks, and also gives a new proof of the following result of Carbonara, Remmel, Yang [1]:

$$s_{(1^c,d)}[s_{(1^a,b)}]\big|_{\text{hooks}} = \begin{cases} s_{(1^{a(c+d)+c},b(c+d)-c)} & \text{if } a \text{ is even} \\ s_{(1^{a(c+d)+d-1},b(c+d)-d+1)} & \text{if } a \text{ is odd} \end{cases}$$

Similarly, to study shapes that are hooks plus a row, we examine $s_\lambda[1+x-y]$, employing Sergeev's formula to simplify calculations. This proves considerably more difficult than the hook case and we are only able to derive an explicit formula for a special case. The conjugation rule for plethysm (see below) gives a corresponding formula for shapes of the form a hook plus a column.

We remark that the approach of using expressions like $s_\lambda[1-x]$ and Sergeev's formula was used to find coefficients in the Kronecker product of Schur functions in [10].

We start with the necessary definitions.

2 Notation and Definitions

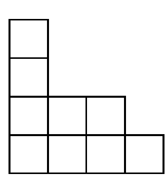
2.1 Partitions and Symmetric Functions

A *partition* λ of a positive integer n, denoted $\lambda \vdash n$, is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. We will often write a partition in the following way:

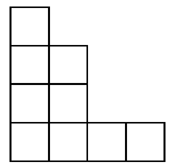
$$(1, 1, 1, 2, 3, 3, 5) = (1^3, 2, 3^2, 5)$$

with the exponent on an entry denoting the number of times that entry appears in the partition. Each integer in a partition λ is called a *part* of λ and the number of parts is the *length* of λ, denoted $l(\lambda)$. So $l(1, 1, 1, 2, 3, 3, 5) = 7$. If $\lambda \vdash n$, we will also write $|\lambda| = n$.

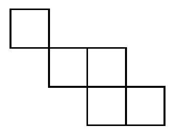
A partition λ can be represented as a *Ferrers diagram* which is a partial array of squares such that the i^{th} row from the top contains λ_i squares. For example, the Ferrers diagram corresponding to the partition $(1, 1, 3, 4)$ is



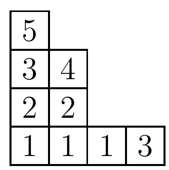
The *conjugate partition*, λ', is the partition whose Ferrers diagram is the transpose of the Ferrers diagram of λ, that is, the Ferrers diagram of λ reflected about the diagonal that extends northeast from the lower left corner. The conjugate of $(1, 1, 3, 4)$ is therefore $(1, 2, 2, 4)$ with Ferrers diagram



If μ and λ are partitions, then $\mu \subseteq \lambda$ if the Ferrers diagram of μ is contained in the Ferrers diagram of λ. For example $(1, 2) \subseteq (1, 3, 4)$. If $\mu \subseteq \lambda$, the Ferrers diagram of the *skew shape* λ/μ is the diagram obtained by removing the Ferrers diagram of μ from the Ferrers diagram of λ. For example $(1, 3, 4)/(1, 2)$ has Ferrers diagram



A *tableau* of shape λ is a filling of a Ferrers diagram with positive integers. A tableau is *column-strict* if the entries are strictly increasing from bottom to top in each column and weakly increasing from left to right in each row. An example of a column-strict tableau of shape $(1, 2, 2, 4)$ is



Let S_N be the symmetric group on N symbols. A polynomial $P(x_1, x_2, \ldots, x_N)$ is *symmetric* if and only if $P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N}) = P(x_1, x_2, \ldots, x_N)$ for all $\sigma = \sigma_1\sigma_2 \cdots \sigma_N \in S_N$.

Let Λ_n be the vector space of all symmetric polynomials that are homogeneous of degree n. The *Schur functions* are a basis of this space, defined combinatorially as follows. For a tableau T, let $T_{i,j}$ be the entry in the cell (i, j) where $(1, 1)$ is the bottom left cell. We assign a monomial to T by defining the weight of T, $w(T)$, to be

$$w(T) = \prod_{(i,j)} x_{T_{i,j}}$$

The Schur functions, $\{s_\lambda\}_{\lambda \vdash n}$, are defined by

$$s_\lambda(x_1, x_2, \ldots, x_N) = \sum_{T \in CS(\lambda)} w(T)$$

where $CS(\lambda)$ is the set of all column-strict tableau of shape λ with entries in the set $\{1, 2, \ldots, N\}$. We note that the Schur function indexed by a partition with one part, $\lambda = (n)$, is the corresponding *homogeneous symmetric function* h_n, and that the Schur function indexed by the partition (1^n) is the *elementary symmetric function* e_n.

We can also extend the definition of Schur functions to *skew Schur functions* by summing over column-strict fillings of a skew diagram.

2.2 Plethysm

We now define *plethysm* as follows. Let R be the ring of formal power series in some set of variables with integer coefficients. Any element $r \in R$ can be written uniquely as $r = \sum_v c_v v$ where v ranges over all monomials in the x_i's and each c_v is an integer. For $k \geq 1$, let $p_k = \sum_{i\geq 1} x_i^k$, the usual power-sum symmetric function. Then define the plethysm of p_k and r by

$$p_k\left[\sum_v c_v v\right] = \sum_v c_v v^k.$$

For any $r \in R$ and any symmetric function f, we then define the plethysm $f[r]$ by the requirement that the map $f \to f[r]$ is a homomorphism from the ring of symmetric functions to R.

In particular, for Schur functions, we use the well-known expansion in terms of the power basis to obtain

$$s_\lambda[X] = \sum_{\mu\vdash n} \frac{\chi_\mu^\lambda}{z_\mu} p_\mu[X]$$

where $X = \sum_{i\geq 1} x_i$, χ_μ^λ is the irreducible S_n character indexed by λ evaluated at the conjugacy class indexed by μ, and

$$z_\mu = 1^{m_1(\mu)} 2^{m_2(\mu)} \cdots n^{m_n(\mu)} m_1(\mu)! m_2(\mu)! \cdots m_n(\mu)!$$

where $m_i(\mu)$ denotes the number of parts of size i in μ.

We will need the following well-known properties (see [9]).

Theorem 2.1 *Let $X = \sum_{i\geq 1} x_i$ and $Y = \sum_{i\geq 1} y_i$. Then*

1. $s_\lambda[X+Y] = \sum_{\mu\subseteq\lambda} s_\mu[X] s_{\lambda/\mu}[Y]$.

2. $s_{\lambda/\mu}[-X] = (-1)^{|\lambda/\mu|} s_{(\lambda/\mu)'}[X]$.

3. $s_\lambda[X-Y] = \sum_{\mu\subseteq\lambda} s_\mu[X](-1)^{|\lambda/\mu|} s_{(\lambda/\mu)'}[Y]$.

We now turn to the problem of finding the coefficients $a_{\lambda,\mu,\nu}$ in the expansion

$$s_\lambda[s_\mu] = \sum_\nu a_{\lambda,\mu,\nu} s_\nu$$

when ν is a hook or a hook plus a row or column. Since a hook plus a row is the conjugate shape of a hook plus a column, we will need the following conjugation formula:

$$s_\lambda[s_\mu]' = \begin{cases} s_\lambda[s_{\mu'}] & \text{if } |\mu| \text{ is even} \\ s_{\lambda'}[s_{\mu'}] & \text{if } |\mu| \text{ is odd} \end{cases} \quad (1)$$

where for any sum $\sum c_\nu s_\nu$, $(\sum c_\nu s_\nu)'$ denotes the sum $\sum c_\nu s_{\nu'}$.

3 The Plethysm $s_\lambda[s_\mu]$ at Hook Shapes

If $s_\lambda[s_\mu] = \sum_\nu a_\nu s_\nu$, define $s_\lambda[s_\mu]\big|_{\text{hooks}} = \sum_{\nu \text{ a hook}} a_\nu s_\nu$. Then we have the following theorem.

Theorem 3.1

1. $s_\lambda[s_\mu]\big|_{\text{hooks}} = 0$ *unless both* λ *and* μ *are hooks.*

2. *If* $\lambda = (1^c, d)$ *and* $\mu = (1^a, b)$,

$$s_{(1^c,d)}[s_{(1^a,b)}]\big|_{\text{hooks}} = \begin{cases} s_{(1^{a(c+d)+c},b(c+d)-c)} & \text{if } a \text{ is even} \\ s_{(1^{a(c+d)+d-1},b(c+d)-d+1)} & \text{if } a \text{ is odd} \end{cases}$$

Again, we note that statement 2 is due to Carbonara, Remmel, Yang [1] but we will give a new, simplified proof here.

Proof. We procede by considering $s_\lambda[s_\mu][X - Y]$ with the substitution $X = 1$ and $Y = x$. If $s_\lambda[s_\mu] = \sum_\nu a_\nu s_\nu$, then

$$s_\lambda[s_\mu][1 - x] = \sum_\nu a_\nu s_\nu[1 - x].$$

Setting $X = 1$ and $Y = x$ in statement 3 of Theorem 2.1 yields

$$s_\nu[1 - x] = \sum_{\rho \subseteq \nu} s_\rho[1](-1)^{|\nu/\rho|} s_{(\nu/\rho)'}[x]$$

Now, a Schur function with one parameter can only be nonzero if the Schur function is indexed by a shape with no columns of height two or more. This follows from the definition in terms of column-strict tableaux. If a Ferrers diagram has a column of height two, a column-strict filling must contain at least two different entries, giving rise to a monomial in at least two variables. So since each Schur function in the sum has one parameter, the terms in the sum are nonzero only if ρ is a row and $(\nu/\rho)'$ is a skew-row, that is, it has no columns of height two or more. This can only happen if $\nu = (1^a, b)$ and $\rho = (b)$ or $(b - 1)$ (see Figure 1). So ν must be a hook. Therefore we have

$$\begin{aligned} s_\lambda[s_\mu][1 - x] &= \sum_\nu a_\nu s_\nu[1 - x] \\ &= \sum_{\nu \text{ a hook}} a_\nu s_\nu[1 - x] \end{aligned}$$

Now, when $\nu = (1^a, b)$, again referring to Figure 1, we have

$$\begin{aligned} s_\nu[1 - x] &= s_{(1^a,b)}[1 - x] \\ &= \sum_{\rho \subseteq (1^a,b)} s_\rho[1](-1)^{|(1^a,b)/\rho|} s_{((1^a,b)/\rho)'}[x] \\ &= s_{b-1}[1](-1)^{a+1}(s_1 s_a)[x] + s_b[1](-1)^a s_a[x] \\ &= (-1)^{a+1}x^{a+1} + (-1)^a x^a \qquad (2) \end{aligned}$$

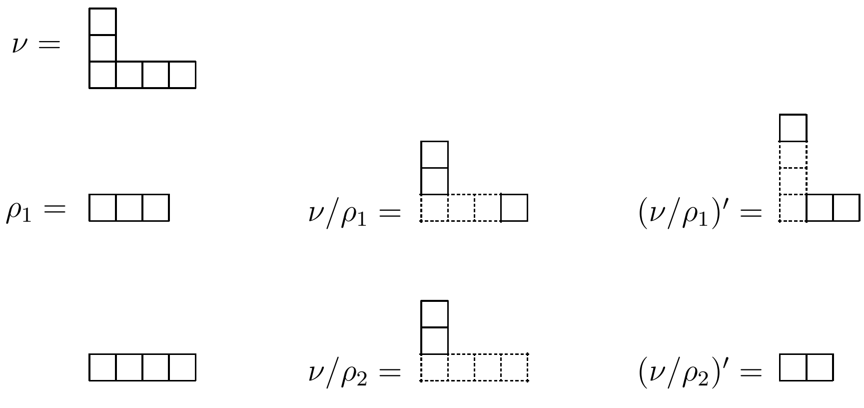


Figure 1: The only ways for $(\nu/\rho)'$ to be a skew row when ρ is a row. In particular ν must be a hook.

So we want to look for sums of this form in the expansion of $s_\lambda[s_\mu][1-x]$.

Since $s_\lambda[s_\mu][1-x] = s_\lambda[s_\mu[1-x]]$, we have $s_\lambda[s_\mu][1-x] = 0$ unless μ is a hook. If $\mu = (1^a, b)$,

$$\begin{aligned} s_\lambda[s_{(1^a,b)}][1-x] &= s_\lambda[s_{(1^a,b)}[1-x]] \\ &= s_\lambda[(-1)^{a+1}x^{a+1} + (-1)^a x^a] \end{aligned}$$

Since the expression in s_λ has one positive and one negative term, the same argument used above for $s_\nu[1-x]$ shows that $s_\lambda[s_{(1^a,b)}][1-x] = 0$ unless λ is a hook. So we have shown that $s_\lambda[s_\mu]|_{\text{hooks}} = 0$ unless both λ and μ are hooks, proving statement 1.

If we now let $\lambda = (1^c, d)$, and for the moment say that a is odd, we have

$$\begin{aligned} s_{(1^c,d)}[s_{(1^a,b)}][1-x] &= s_{(1^c,d)}[(-1)^{a+1}x^{a+1} + (-1)^a x^a] \\ &= s_{(1^c,d)}[x^{a+1} - x^a] \\ &= s_{d-1}[x^{a+1}](-1)^{c+1}(s_1 s_c)[x^a] + s_d[x^{a+1}](-1)^c s_c[x^a] \\ &= (x^{(a+1)})^{(d-1)}(-1)^{c+1}(x^a)^{c+1} + (x^{a+1})^d(-1)^c(x^a)^c \\ &= x^{ad+d-a-1+ac+a}(-1)^{c+1} + x^{ad+d+ac}(-1)^c \\ &= x^{a(c+d)+d-1}(-1)^{c+1} + x^{a(c+d)+d}(-1)^c \end{aligned}$$

Now, this is almost of the form (2). We just need to verify that $a(c+d)+d$ and c have the same parity. Since $a(c+d)+d = ac + d(a+1)$ and a is odd, $a(c+d)+d$ has the same parity as ac, which has the same parity as c. So we have

$$s_{(1^c,d)}[s_{(1^a,b)}][1-x] = x^{a(c+d)+d-1}(-1)^{a(c+d)+d-1} + x^{a(c+d)+d}(-1)^{a(c+d)+d}$$

Again referring to (2), we see that

$$s_{(1^c,d)}[s_{(1^a,b)}][1-x] = s_{(1^{a(c+d)+d-1},l)}[1-x]$$

for some l. It follows from the definition of plethysm that the Schur functions in the expansion

$$s_\lambda[s_\mu] = \sum_\nu a_\nu s_\nu$$

correspond to partitions of size $|\nu| = |\lambda||\mu|$, so we need

$$(c+d)(a+b) = a(c+d) + d - 1 + l.$$

So

$$\begin{aligned} l &= ac + bc + ad + bd - ac - ad - d + 1 \\ &= bc + bd - d + 1 \\ &= b(c+d) - d + 1 \end{aligned}$$

and therefore $s_{(1^c,d)}[s_{(1^a,b)}]\big|_{\text{hooks}} = s_{(1^{a(c+d)+d-1},b(c+d)-d+1)}$ when a is odd.

Similarly, when a is even we have

$$\begin{aligned} s_{(1^c,d)}[s_{(1^a,b)}][1-x] &= s_{(1^c,d)}[(-1)^{a+1}x^{a+1} + (-1)^a x^a] \\ &= s_{(1^c,d)}[x^a - x^{a+1}] \\ &= s_{d-1}[x^a](-1)^{c+1}(s_1 s_c)[x^{a+1}] + s_d[x^a](-1)^c s_c[x^{a+1}] \\ &= (x^a)^{(d-1)}(-1)^{c+1}(x^{a+1})^{c+1} + (x^a)^d(-1)^c(x^{a+1})^c \\ &= x^{ad-a+ac+a+c+1}(-1)^{c+1} + x^{a(c+d)+c}(-1)^c \\ &= x^{a(c+d)+c+1}(-1)^{c+1} + x^{a(c+d)+c}(-1)^c \\ &= x^{a(c+d)+c+1}(-1)^{a(c+d)+c+1} + x^{a(c+d)+c}(-1)^{a(c+d)+c} \\ &= s_{(1^{a(c+d)+c},b(c+d)-c)}[1-x] \end{aligned}$$

which completes the proof. ■

4 The Plethysm $s_\lambda[s_\mu]$ at Near-Hook Shapes

We now consider the problem of finding shapes of the form $(1^a, b, c)$ or $(1^a, 2^b, c)$ in the expansion of $s_\lambda[s_\mu]$. This is considerably more difficult than the hook case and we will only be able to determine an explicit formula for a special case.

To extract shapes that are a hook plus a row, we need to examine $s_\nu[1 + x - y]$. In particular, we show below that $s_\nu[1 + x - y] = 0$ unless ν is contained in a hook plus a row. We will translate our results about hooks plus a row to shapes that are hooks plus a column by using the conjugation rule (1) (we could also compute these directly using $s_\nu[x - 1 - y]$). To simplify our calculations we will use a result known as Sergeev's formula. We note that the calculations can also be performed using techniques similar to those in the previous section. Specifically, we can set $X = 1 + x$ and $Y = y$ in statement 3 of Theorem 2.1 to obtain

$$s_\nu[1 + x - y] = \sum_{\rho \subseteq \nu} s_\rho[1 + x](-1)^{|\nu/\rho|} s_{(\nu/\rho)'}[y]$$

and perform an analysis similar to that in Section 3.

Sergeev's formula also allows us to state a general result about when certain shapes occur in the expansion $s_\lambda[s_\mu] = \sum_\nu a_\nu s_\nu$ based on a restriction on μ.

Before introducing Sergeev's formula, we need a few definitions. First, let $X_m = x_1 + x_2 + \cdots + x_m$ be a finite alphabet and let

$$\delta_m = (m - 1, m - 2, \ldots, 1, 0).$$

Then define

$$X_m^{\delta_m} = x_1^{m-1} x_2^{m-2} \cdots x_{m-1}$$

Next, for a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$, we say that an ordered pair (i, j) is an *inversion* of σ if $i < j$ and $\sigma_i > \sigma_j$. Let $\text{inv}(\sigma)$ denote the number of inversions in σ. Then for a polynomial $P(x_1, \ldots, x_n)$, define the alternant A_n^x by

$$A_n^x P = \sum_{\sigma \in S_n} (-1)^{\text{inv}(\sigma)} P(x_{\sigma_1}, \ldots, x_{\sigma_n})$$

Finally, let Δ be the operation of taking the Vandermonde determinant of an alphabet. Specifically,

$$\Delta(X_m) = \det(x_i^{m-j})_{i,j=1}^m$$

Then we have the following result (see [9]).

Theorem 4.1 (Sergeev's Formula) *Let $X_m = x_1 + \cdots + x_m$ and $Y_n = y_1 + \cdots + y_n$ be two alphabets. Then*

$$s_\lambda[X_m - Y_n] = \frac{1}{\Delta(X_m)\Delta(Y_n)} A_m^x A_n^y X_m^{\delta_m} Y_n^{\delta_n} \prod_{(i,j) \in \lambda} (x_j - y_i)$$

where $(i, j) \in \lambda$ means that the cell (i, j) is in the Ferrers diagram of λ and $(1, 1)$ denotes the bottom left cell. We also set $x_j = 0$ for $j > m$ and $y_i = 0$ for $i > n$.

We need a few more definitions for our first result. Define an *n-hook* to be a partition of the form $(1^{k_1}, 2^{k_2}, \ldots, n^{k_n}, l_1, l_2, \ldots, l_n)$ where $k_i \geq 1$ for $1 \leq i \leq n$ and $l_1 > n$. Similarly define an *n-hook plus a row* to be a partition of the form $(1^{k_1}, 2^{k_2}, \ldots, n^{k_n}, l_1, l_2, \ldots, l_n, l_{n+1})$ where $k_i \geq 1$ for $1 \leq i \leq n$ and $l_1 > n$ and an *n-hook plus a column* to be a partition of the form $(1^{k_1}, 2^{k_2}, \ldots, n^{k_n}, (n+1)^{k_{n+1}}, l_1, l_2, \ldots, l_n)$ where $k_i \geq 1$ for $1 \leq i \leq n+1$ and $l_1 > n$ (see Figure 2). Note that every partition is an n-hook, an n-hook plus a row, or an n-hook plus a column for some n.

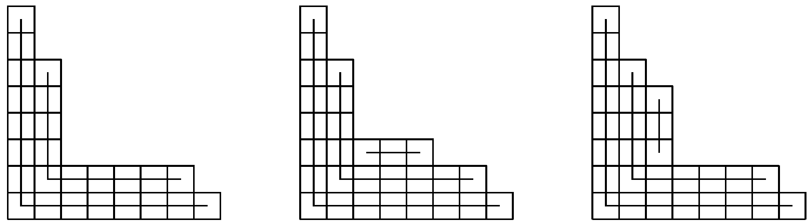


Figure 2: A 2-hook, a 2-hook plus a row, and a 2-hook plus a column.

Also , if $s_\lambda[s_\mu] = \sum_\nu a_\nu s_\nu$, set

$$s_\lambda[s_\mu]|_{\subseteq(n\text{-hook})} = \sum_{\nu \text{ contained in an } n\text{-hook}} a_\nu s_\nu$$

and similarly for $s_\lambda[s_\mu]|_{\subseteq(n\text{-hook+row})}$ and $s_\lambda[s_\mu]|_{\subseteq(n\text{-hook+col})}$.

Our first application of Sergeev's formula is the following:

Theorem 4.2

1. $s_\lambda[s_\mu]|_{\subseteq(n\text{-hook})} = 0$ *if* μ *is not contained in an* n*-hook.*

2. $s_\lambda[s_\mu]|_{\subseteq(n\text{-hook+row})} = 0$ *if* μ *is not contained in an* n*-hook plus a row.*

3. $s_\lambda[s_\mu]|_{\subseteq(n\text{-hook+col})} = 0$ *if* μ *is not contained in an* n*-hook plus a column.*

Proof. For statement 1, we consider $s_\nu[X_n - Y_n]$. If ν is not contained in an n-hook, then the Ferrers diagram of ν contains the cell $(n+1, n+1)$. So the product

$$\prod_{(i,j)\in\nu} (x_j - y_i)$$

in Sergeev's formula for $s_\nu[X_n - Y_n]$ is zero since the factor $x_{n+1} - y_{n+1}$ is zero. Therefore $s_\nu[X_n - Y_n] = 0$ unless ν is contained in an n-hook. So if $s_\lambda[s_\mu] = \sum_\nu a_\nu s_\nu$,

$$\begin{aligned} s_\lambda[s_\mu][X_n - Y_n] &= \sum_\nu a_\nu s_\nu[X_n - Y_n] \\ &= \sum_{\nu\subseteq(n\text{-hook})} a_\nu s_\nu[X_n - Y_n] \end{aligned}$$

But

$$s_\lambda[s_\mu][X_n - Y_n] = s_\lambda[s_\mu[X_n - Y_n]] = 0$$

unless μ is contained in an n-hook. So $s_\lambda[s_\mu]|_{\subseteq(n-\text{hook})} = 0$ if μ is not contained in an n-hook.

For statement 2, we just need to look at $s_\nu[X_{n+1} - Y_n]$. If ν is not contained in an n-hook plus a row, then the Ferrers diagram of ν contains the cell $(n+1, n+2)$. So the product

$$\prod_{(i,j)\in\nu} (x_j - y_i)$$

in Sergeev's formula for $s_\nu[X_{n+1} - Y_n]$ is zero since the factor $x_{n+2} - y_{n+1}$ is zero. Therefore $s_\nu[X_{n+1} - Y_n] = 0$ unless ν is contained in an n-hook plus a row and the result follows as with statement 1.

An analogous argument considering $s_\nu[X_n - Y_{n+1}]$ proves statement 3. ∎

We now turn to the special case of a hook plus a row or column. As a special case of Theorem 4.2, we can start with the following result.

Theorem 4.3

1. $s_\lambda[s_\mu]|_{\subseteq\text{hook+row}} = 0$ *unless* μ *is contained in a hook plus a row.*

2. $s_\lambda[s_\mu]|_{\subseteq\text{hook+col}} = 0$ *unless* μ *is contained in a hook plus a column.*

As in the proof of Theorem 4.2, statement 1 follows from Sergeev's formula for $s_\nu[x_1 + x_2 - y_1]$. For our next theorem we will need this formula evaluated at $x_1 = 1$, $x_2 = x$, and $y_1 = y$. We state this result as a lemma:

Lemma 4.4

1. $s_\nu[1 + x - y] = 0$ *unless* ν *is contained in a partition of the form* $(1^a, b, c)$.

2. *For* $b \geq 1$, $c \geq 2$,

$$s_{(1^a,b,c)}[1 + x - y] = x^{b-1}(1 + x + x^2 + \cdots + x^{c-b})(-y)^a(1 - y)(x - y).$$

3. $x^i(-y)^j s_{(1^a,b,c)}[1 + x - y] = s_{(1^{a+j},b+i,c+i)}[1 + x - y]$.

Proof. We apply Sergeev's formula with $X_2 = x_1 + x_2$, $Y_1 = y_1$ and then substitute $x_1 = 1, x_2 = x$, and $y_1 = y$. The proof of statement 1 is a special case of the proof of Theorem 4.2. For statement 2, we have $\Delta(X_2) = x_1 - x_2$, $\Delta(Y_1) = 1$, $X_2^{\delta_2} = x_1$, $Y_1^{\delta_1} = 1$, $A_2^x P(x_1, x_2) = P(x_1, x_2) - P(x_2, x_1)$, and $A_1^y P(y_1) = P(y_1)$. Also, for $\lambda = (1^a, b, c)$,

$$\prod_{(i,j)\in\lambda} (x_j - y_i) \quad = \quad (x_1 - y_1)x_1^{c-1}(x_2 - y_1)x_2^{b-1}(-y_1)^a$$

So

$$\begin{aligned}
s_{(1^a,b,c)}[x_1+x_2-y_1] &= \frac{1}{x_1-x_2}A_2^x(x_1^c x_2^{b-1}(-y_1)^a(x_1-y_1)(x_2-y_1)) \\
&= \frac{1}{x_1-x_2}(x_1^c x_2^{b-1} - x_1^{b-1}x_2^c)(-y_1)^a(x_1-y_1)(x_2-y_1) \\
&= \frac{x_1^{c-b+1}-x_2^{c-b+1}}{x_1-x_2}(x_1x_2)^{b-1}(-y_1)^a(x_1-y_1)(x_2-y_1) \\
&= (x_1^{c-b}+x_1^{c-b-1}x_2+\cdots+x_1x_2^{c-b-1}+x_2^{c-b}) \\
&\quad \times(x_1x_2)^{b-1}(-y_1)^a(x_1-y_1)(x_2-y_1)
\end{aligned}$$

Substituting $x_1 = 1$, $x_2 = x$, and $y_1 = y$ gives the result.

Statement 3 follows immediately from statement 2. ■

Note that in particular this lemma says that if $s_\lambda[s_\mu] = \sum_\nu a_\nu s_\nu$ then

$$\begin{aligned}
s_\lambda[s_\mu][1+x-y] &= \sum_\nu a_\nu s_\nu[1+x-y] \\
&= \sum_{\nu \subseteq \text{ hook plus a row}} a_\nu s_\nu[1+x-y]
\end{aligned}$$

So we need to look for expressions like those in statement 2 of Lemma 4.4 in the expansion of $s_\lambda[s_\mu][1+x-y]$. This is considerably more difficult than the hook case in the previous section where we were looking for expressions of the form $(-1)^{a+1}x^{a+1}+(-1)^a x^a$ since the factor $1+x+x^2+\cdots+x^{c-b}$ in statement 2 of Lemma 4.4 becomes difficult to deal with when $c \neq b$. As such, we will only derive an explicit formula for the case $b = c$, as follows.

Theorem 4.5

1. $s_\lambda[s_{(1^a,b,b)}]\big|_{\subseteq(\text{hook+row})} = 0$ *unless* λ *is contained in a 2-hook.*

2. *For* $\lambda = (1^n)$,

$$s_{(1^n)}[s_{(1^a,b,b)}]\big|_{\subseteq(\text{hook+row})} = \begin{cases} s_{(1^{na+n-1},n(b-1)+1,n(b-1)+n)} & \text{if } a \text{ is even} \\ \sum_{i=1}^n s_{(1^{na+2n-2i},n(b-1)+i,n(b-1)+i)} & \text{if } a \text{ is odd} \end{cases}$$

3. *For* $\lambda = (n)$,

$$s_{(n)}[s_{(1^a,b,b)}]\big|_{\subseteq(\text{hook+row})} = \begin{cases} \sum_{i=1}^n s_{(1^{na+2n-2i},n(b-1)+i,n(b-1)+i)} & \text{if } a \text{ is even} \\ s_{(1^{na+n-1},n(b-1)+1,n(b-1)+n)} & \text{if } a \text{ is odd} \end{cases}$$

4. *For* $\lambda = (1^k, n-k)$*, with* $k \geq 1$*,* $n-k > 1$*, and* a *even,*

$$\begin{aligned} s_{(1^k,n-k)}\big[s_{(1^a,b,b)}\big]\big|_{\subseteq(\text{hook+row})} &= \sum_{i=1}^{n-k} s_{(1^{na+2n-k-2i},n(b-1)+i,n(b-1)+k+i)} \\ &+ \sum_{i=1}^{n-k-1} s_{(1^{na+2n-k-1-2i},n(b-1)+1+i,n(b-1)+k+i)} \end{aligned}$$

5. *For* $\lambda = (1^k, n-k)$*, with* $k \geq 1$*,* $n-k > 1$*, and* a *odd,*

$$\begin{aligned} s_{(1^k,n-k)}\big[s_{(1^a,b,b)}\big]\big|_{\subseteq(\text{hook+row})} &= \sum_{i=1}^{k+1} s_{(1^{na+n+k+1-2i},n(b-1)+i,n(b-1)+n-k-1+i)} \\ &+ \sum_{i=1}^{k} s_{(1^{na+n+k-2i},n(b-1)+1+i,n(b-1)+n-k-1+i)} \end{aligned}$$

6. *For* $\lambda = (1^k, 2^l, r, s) \vdash n$*, with* $k \geq 0$*,* $l \geq 0$*,* $r \geq 2$*,* $s \geq 2$*, and* a *even,*

$$\begin{aligned} s_{(1^k,2^l,r,s)}\big[s_{(1^a,b,b)}\big]\big|_{\subseteq(\text{hook+row})} &= s_{(1^{na+k+2l+2s},n(b-1)+l+r,n(b-1)+k+l+r)} \\ &+2\sum_{j=0}^{s-r-1} s_{(1^{na+k+2l+2r+2j},n(b-1)+l+s-j,n(b-1)+k+l+s-j)} \\ &+s_{(1^{na+k+2l+2r-2},n(b-1)+l+s+1,n(b-1)+k+l+s+1)} \\ &+\sum_{i=0}^{s-r} \big(s_{(1^{na+k+2l+2r-1+2i},n(b-1)+l+s-i,n(b-1)+k+l+s+1-i)} \\ &\qquad + s_{(1^{na+k+2l+2r-1+2i},n(b-1)+l+s+1-i,n(b-1)+k+l+s-i)}\big) \end{aligned}$$

where the first summation is taken to be empty if $r = s$ *and the second term in the second summation only occurs if* $k \neq 0$*.*

7. *For* $\lambda = (1^k, 2^l, r, s) \vdash n$*, with* $k \geq 0$*,* $l \geq 0$*,* $r \geq 2$*,* $s \geq 2$*, and* a *odd,*

$$\begin{aligned} s_{(1^k,2^l,r,s)}\big[s_{(1^a,b,b)}\big]\big|_{\subseteq(\text{hook+row})} &= s_{(1^{na+2k+2l+r+s},n(b-1)+l+r,n(b-1)+l+s)} \\ &+2\sum_{j=0}^{k-1} s_{(1^{na+2l+r+s+2j},n(b-1)+k+l+r-j,n(b-1)+k+l+s-j)} \\ &+s_{(1^{na+2l+r+s-2},n(b-1)+k+l+r+1,n(b-1)+k+l+s+1)} \\ &+\sum_{i=0}^{k} \big(s_{(1^{na+2l+r+s-1+2i},n(b-1)+k+l+r-i,n(b-1)+k+l+s+1-i)} \\ &\qquad + s_{(1^{na+2l+r+s-1+2i},n(b-1)+k+l+r+1-i,n(b-1)+k+l+s-i)}\big) \end{aligned}$$

where the first summation is taken to be empty if $k = 0$ and the second term in the second summation only occurs if $r < s$.

Before we give a proof, we note that each Schur function indexed by a hook plus a row appears in at most one summation in each of the above formulas. So we can state the following corollary.

Corollary 4.6 *Let $s_\lambda[s_{(1^a,b,b)}] = \sum_\nu a_\nu s_\nu$. Then if ν is a hook plus a row, we have*

1. *$a_\nu = 0$ if λ is not contained in a 2-hook.*
2. *$a_\nu = 0$ or 1 if λ is contained in a hook.*
3. *$a_\nu = 0$, 1, or 2 if λ is contained in a 2-hook and the Ferrers diagram of λ contains the cell $(2,2)$.*

We note that this nice bound on the coefficients does not hold in the general case $s_\lambda[s_{(1^a,b,c)}]$. Indeed, the coefficients grow without bound as $c - b$ becomes large. The first author examines this phenomenon in the special case of two-row shapes in [7].

We now turn to the proof of Theorem 4.5.

Proof of Theorem 4.5. We start by applying Lemma 4.4 to $s_\lambda[s_{(1^a,b,b)}][1+x-y]$:

$$\begin{aligned} s_\lambda[s_{(1^a,b,b)}][1+x-y] &= s_\lambda[s_{(1^a,b,b)}[1+x-y]] \\ &= s_\lambda[x^{b-1}(-y)^a(x+y^2-y-xy)] \\ &= (x^{b-1}y^a)^{|\lambda|} s_\lambda[(-1)^a(x+y^2-y-xy)] \end{aligned}$$

This breaks into cases depending on the parity of a:

$$s_\lambda[s_{(1^a,b,b)}][1+x-y] = \begin{cases} x^{|\lambda|(b-1)}y^{|\lambda|a} s_\lambda[x+y^2-y-xy] & \text{if } a \text{ is even} \\ x^{|\lambda|(b-1)}y^{|\lambda|a}(-1)^{|\lambda|} s_{\lambda'}[x+y^2-y-xy] & \text{if } a \text{ is odd} \end{cases}$$

where the odd case follows from statement 2 of Theorem 2.1. So we need to examine $s_\lambda[x+y^2-y-xy]$. To that end we have the following lemma.

Lemma 4.7

1. *$s_\lambda[x+y^2-xy-y] = 0$ unless λ is contained in a 2-hook.*
2. *For $\lambda = (1^n)$,*
$$s_{(1^n)}[x+y^2-xy-y] = s_{(1^n,n)}[1+x-y]$$
3. *For $\lambda = (n)$,*
$$s_{(n)}[x+y^2-xy-y] = \sum_{i=1}^{n} s_{(1^{2(n-i)},i,i)}[1+x-y]$$

4. For $\lambda = (1^k, n-k)$*, with* $k \geq 1$*,* $n-k > 1$*,*

$$\begin{aligned} s_{(1^k,n-k)}[x+y^2-xy-y] &= \sum_{i=1}^{n-k} s_{(1^{2n-k-2i},i,k+i)}[1+x-y] \\ &\quad + \sum_{i=1}^{n-k-1} s_{(1^{2n-k-1-2i},1+i,k+i)}[1+x-y] \end{aligned}$$

5. $\lambda = (1^k, 2^l, r, s)$*, with* $k \geq 0$*,* $l \geq 0$*,* $r \geq 2$*,* $s \geq 2$*,*

$$\begin{aligned} s_{(1^k,2^l,r,s)}[x+y^2-xy-y] &= s_{(1^{k+2l+2s},l+r,k+l+r)}[1+x-y] \\ &\quad +2\sum_{j=0}^{s-r-1} s_{(1^{k+2l+2r+2j},l+s-j,k+l+s-j)}[1+x-y] \\ &\quad +s_{(1^{k+2l+2r-2},l+s+1,k+l+s+1)}[1+x-y] \\ &\quad +\sum_{i=0}^{s-r}\big(s_{(1^{k+2l+2r-1+2i},l+s-i,k+l+s+1-i)}[1+x-y] \\ &\qquad +s_{(1^{k+2l+2r-1+2i},l+s+1-i,k+l+s-i)}[1+x-y]\big) \end{aligned}$$

where the first summation is taken to be empty if $r = s$ *and the second term in the second summation only occurs if* $k \neq 0$*.*

Proof of Lemma 4.7. We again apply Sergeev's formula, this time with $X_2 = x_1 + x_2$ and $Y_2 = y_1 + y_2$. We will then substitute $x_1 = x$, $x_2 = y^2$, $y_1 = xy$, and $y_2 = y$. If λ is not a 2-hook then λ contains the cell $(3,3)$. So the product $\prod_{(i,j)\in\lambda}(x_j - y_i)$ in Sergeev's formula for $s_\lambda[x_1+x_2-y_1-y_2]$ is 0 since $x_3 - y_3 = 0$. This proves statement 1. Statement 1 of Theorem 4.5 follows immediately since this implies $s_\lambda[s_{(1^a,b,b)}][1+x-y] = 0$ unless λ is contained in a 2-hook.

Now, $\Delta(X_2) = x_1 - x_2$, $\Delta(Y_2) = y_1 - y_2$, $X_2^{\delta_2} = x_1$, $Y_2^{\delta_2} = y_1$, $A_2^x P(x_1,x_2) = P(x_1,x_2) - P(x_2,x_1)$, and $A_2^y P(y_1,y_2) = P(y_1,y_2) - P(y_2,y_1)$. For $\lambda = (1^n)$,

$$\prod_{(i,j)\in\lambda}(x_j - y_i) = (x_1 - y_1)(x_2 - y_1)(-y_1^{n-2})$$

So

$$
\begin{aligned}
&s_{(1^n)}[x_1 + x_2 - y_1 - y_2] \\
&\quad = \frac{1}{(x_1 - x_2)(y_1 - y_2)} A_2^x A_2^y x_1 y_1 (x_1 - y_1)(x_2 - y_1)(-y_1^{n-2}) \\
&\quad = \frac{1}{(x_1 - x_2)(y_1 - y_2)} A_2^x (-1)^{n-2} x_1 (y_1^{n-1}(x_1 - y_1)(x_2 - y_1) - y_2^{n-1}(x_1 - y_2)(x_2 - y_2)) \\
&\quad = \frac{1}{(x_1 - x_2)(y_1 - y_2)} (-1)^{n-2}(x_1 - x_2)(y_1^{n-1}(x_1 - y_1)(x_2 - y_1) - y_2^{n-1}(x_1 - y_2)(x_2 - y_2)) \\
&\quad = \frac{1}{(y_1 - y_2)} (-1)^{n-2}(y_1^{n-1}(x_1 - y_1)(x_2 - y_1) - y_2^{n-1}(x_1 - y_2)(x_2 - y_2))
\end{aligned}
$$

Substituting $x_1 = x$, $x_2 = y^2$, $y_1 = xy$, and $y_2 = y$ gives

$$
\begin{aligned}
&s_{(1^n)}[x + y^2 - xy - y] \\
&\quad = \frac{1}{(xy - y)} (-1)^{n-2}((xy)^{n-1}(x - xy)(y^2 - xy) - y^{n-1}(x - y)(y^2 - y)) \\
&\quad = \frac{1}{y(x - 1)} (-1)^{n-2}((xy)^n (1 - y)(y - x) - y^n (x - y)(y - 1)) \\
&\quad = \frac{x^n - 1}{x - 1} (-y)^{n-1}(1 - y)(x - y) \\
&\quad = (1 + x + \cdots + x^{n-1})(-y)^{n-1}(1 - y)(x - y)
\end{aligned}
$$

Comparing with the expression

$$
s_{(1^a,b,c)}[1 + x - y] = x^{b-1}(1 + x + x^2 + \cdots + x^{c-b})(-y)^a (1 - y)(x - y)
$$

we see that

$$
s_{(1^n)}[x + y^2 - xy - y] = s_{(1^{n-1},1,n)}[1 + x - y] = s_{(1^n,n)}[1 + x - y]
$$

which proves statement 2 of the lemma.

For statement 3, when $\lambda = (n)$ we have

$$
\prod_{(i,j)\in\lambda} (x_j - y_i) = (x_1 - y_1)(x_1 - y_2) x_1^{n-2}
$$

So

$$
\begin{aligned}
&s_{(n)}[x_1 + x_2 - y_1 - y_2] \\
&\quad = \frac{1}{(x_1 - x_2)(y_1 - y_2)} A_2^x A_2^y x_1 y_1 (x_1 - y_1)(x_1 - y_2) x_1^{n-2} \\
&\quad = \frac{1}{(x_1 - x_2)(y_1 - y_2)} A_2^x x_1^{n-1} (y_1 - y_2)(x_1 - y_1)(x_1 - y_2) \\
&\quad = \frac{1}{(x_1 - x_2)} (x_1^{n-1}(x_1 - y_1)(x_1 - y_2) - x_2^{n-1}(x_2 - y_1)(x_2 - y_2))
\end{aligned}
$$

Substituting $x_1 = x$, $x_2 = y^2$, $y_1 = xy$, and $y_2 = y$ gives

$$\begin{aligned}
&s_n[x + y^2 - xy - y] \\
&\quad = \frac{1}{(x - y^2)}(x^{n-1}(x - xy)(x - y) - y^{2(n-1)}(y^2 - xy)(y^2 - y)) \\
&\quad = \frac{1}{(x - y^2)}(x^n(1 - y)(x - y) - y^{2n}(1 - y)(x - y)) \\
&\quad = \frac{x^n - y^{2n}}{x - y^2}(1 - y)(x - y) \\
&\quad = (y^{2(n-1)} + xy^{2(n-2)} + \cdots + x^{n-2}y^2 + x^{n-1})(1 - y)(x - y)
\end{aligned}$$

Again comparing with the expression

$$s_{(1^a,b,c)}[1 + x - y] = x^{b-1}(1 + x + x^2 + \cdots + x^{c-b})(-y)^a(1 - y)(x - y)$$

we see that

$$\begin{aligned}
s_{(n)}[x + y^2 - xy - y] &= s_{(1^{2(n-1)},1,1)}[1 + x - y] + s_{(1^{2(n-2)},2,2)}[1 + x - y] \\
&\quad + \cdots + s_{(1^2,n-1,n-1)}[1 + x - y] + s_{(n,n)}[1 + x - y] \\
&= \sum_{i=1}^{n} s_{(1^{2(n-i)},i,i)}[1 + x - y]
\end{aligned}$$

which proves statement 3 of the lemma.

With statements 2 and 3 of Lemma 4.7 in hand we can now prove statements 2 and 3 of Theorem 4.5. For this, we return to the expression

$$s_\lambda[s_{(1^a,b,b)}][1 + x - y] = \begin{cases} x^{|\lambda|(b-1)}y^{|\lambda|a}s_\lambda[x + y^2 - y - xy] & \text{if } a \text{ is even} \\ x^{|\lambda|(b-1)}y^{|\lambda|a}(-1)^{|\lambda|}s_{\lambda'}[x + y^2 - y - xy] & \text{if } a \text{ is odd} \end{cases}$$

When a is even we just need to multiply the results of Lemma 4.7 by

$$x^{n(b-1)}y^{na} = x^{n(b-1)}(-y)^{na}.$$

Applying Lemma 4.4 (3), we see that

$$x^{n(b-1)}(-y)^{na}s_{(1^d,i,j)}[1 + x - y] = s_{(1^{na+d},n(b-1)+i,n(b-1)+j)}[1 + x - y].$$

So

$$\begin{aligned}
s_{(1^n)}[s_{(1^a,b,b)}][1 + x - y] &= x^{n(b-1)}y^{na}s_{(1^n,n)}[1 + x - y] \\
&= x^{n(b-1)}y^{na}s_{(1^{n-1},1,n)}[1 + x - y] \\
&= s_{(1^{na+n-1},n(b-1)+1,n(b-1)+n)}[1 + x - y]
\end{aligned}$$

and therefore

$$s_{(1^n)}[s_{(1^a,b,b)}]\big|_{\subseteq(\text{hook+row})} = s_{(1^{na+n-1},n(b-1)+1,n(b-1)+n)}$$

when a is even.

Similarly, again applying Lemma 4.4 (3), we have

$$\begin{aligned} s_{(n)}[s_{(1^a,b,b)}][1+x-y] &= x^{n(b-1)}y^{na}\sum_{i=1}^{n} s_{(1^{2(n-i)},i,i)}[1+x-y] \\ &= \sum_{i=1}^{n} s_{(1^{na+2(n-i)},n(b-1)+i,n(b-1)+i)}[1+x-y] \end{aligned}$$

and therefore

$$s_{(n)}\left[s_{(1^a,b,b)}\right]\Big|_{\subseteq(\text{hook+row})} = \sum_{i=1}^{n} s_{(1^{na+2(n-i)},n(b-1)+i,n(b-1)+i)}$$

when a is even.

Now, when a is odd, we need to multiply $s_{\lambda'}[x+y^2-y-xy]$ by

$$x^{|\lambda|(b-1)}y^{|\lambda|a}(-1)^{|\lambda|} = x^{|\lambda|(b-1)}(-y)^{|\lambda|a}.$$

Since $(n)'=(1^n)$, we just need to switch the above formulas. This completes the proof of statements 2 and 3 of Theorem 4.5.

We now turn to statement 4 of Lemma 4.7 and statements 4 and 5 of Theorem 4.5. For the lemma, if $\lambda=(1^k,n-k)$, we have

$$\prod_{(i,j)\in\lambda}(x_j-y_i) = (x_1-y_1)(x_1-y_2)x_1^{n-k-2}(x_2-y_1)(-y_1)^{k-1}$$

So

$$\begin{aligned} &s_{(1^k,n-k)}[x_1+x_2-y_1-y_2] \\ &\quad= \frac{1}{(x_1-x_2)(y_1-y_2)}A_2^xA_2^y x_1y_1(x_1-y_1)(x_1-y_2)x_1^{n-k-2}(x_2-y_1)(-y_1)^{k-1} \\ &\quad= \frac{1}{(x_1-x_2)(y_1-y_2)}A_2^x(-1)^{k-1}x_1^{n-k-1}(x_1-y_1)(x_1-y_2)((x_2-y_1)y_1^k-(x_2-y_2)y_2^k) \\ &\quad= \frac{1}{(x_1-x_2)(y_1-y_2)}(-1)^{k-1}\left[x_1^{n-k-1}(x_1-y_1)(x_1-y_2)((x_2-y_1)y_1^k-(x_2-y_2)y_2^k)\right. \\ &\qquad\qquad \left.-x_2^{n-k-1}(x_2-y_1)(x_2-y_2)((x_1-y_1)y_1^k-(x_1-y_2)y_2^k)\right] \end{aligned}$$

Rearranging slightly, we obtain

$$\begin{aligned}
&s_{(1^k,n-k)}[x_1+x_2-y_1-y_2]\\
&\quad=\frac{1}{(x_1-x_2)(y_1-y_2)}(-1)^{k-1}\\
&\qquad\qquad\times\big[y_1^k(x_1-y_1)(x_2-y_1)(x_1^{n-k-1}(x_1-y_2)-x_2^{n-k-1}(x_2-y_2))\\
&\qquad\qquad\quad-y_2^k(x_1-y_2)(x_2-y_2)(x_1^{n-k-1}(x_1-y_1)-x_2^{n-k-1}(x_2-y_1))\big]\\
&\quad=\frac{1}{(x_1-x_2)(y_1-y_2)}(-1)^{k-1}\\
&\qquad\qquad\times\big[y_1^k(x_1-y_1)(x_2-y_1)(x_1^{n-k}-x_2^{n-k}-y_2(x_1^{n-k-1}-x_2^{n-k-1}))\\
&\qquad\qquad\quad-y_2^k(x_1-y_2)(x_2-y_2)(x_1^{n-k}-x_2^{n-k}-y_1(x_1^{n-k-1}-x_2^{n-k-1}))\big]\\
&\quad=\frac{x_1^{n-k}-x_2^{n-k}}{x_1-x_2}(-1)^{k-1}\frac{y_1^k(x_1-y_1)(x_2-y_1)-y_2^k(x_1-y_2)(x_2-y_2)}{y_1-y_2}\\
&\qquad-\frac{x_1^{n-k-1}-x_2^{n-k-1}}{x_1-x_2}(-1)^{k-1}\frac{y_1^ky_2(x_1-y_1)(x_2-y_1)-y_1y_2^k(x_1-y_2)(x_2-y_2)}{y_1-y_2}
\end{aligned}$$

Substituting $x_1=x$, $x_2=y^2$, $y_1=xy$, and $y_2=y$ gives

$$\begin{aligned}
&s_{(1^k,n-k)}[x+y^2-xy-y]\\
&\quad=\frac{x^{n-k}-y^{2(n-k)}}{x-y^2}(-1)^{k-1}\frac{(xy)^k(x-xy)(y^2-xy)-y^k(x-y)(y^2-y)}{xy-y}\\
&\qquad-\frac{x^{n-k-1}-y^{2(n-k-1)}}{x-y^2}(-1)^{k-1}\frac{(xy)^ky(x-xy)(y^2-xy)-xyy^k(x-y)(y^2-y)}{xy-y}\\
&\quad=\frac{x^{n-k}-y^{2(n-k)}}{x-y^2}(-1)^{k-1}\frac{(xy)^{k+1}(1-y)(y-x)-y^{k+1}(x-y)(y-1)}{y(x-1)}\\
&\qquad-\frac{x^{n-k-1}-y^{2(n-k-1)}}{x-y^2}(-1)^{k-1}\frac{x^{k+1}y^{k+2}(1-y)(y-x)-xy^{k+2}(x-y)(y-1)}{y(x-1)}\\
&\quad=(-1)^{k-1}(1-y)(y-x)\Bigg(y^k\frac{x^{n-k}-y^{2(n-k)}}{x-y^2}\cdot\frac{x^{k+1}-1}{x-1}\\
&\qquad\qquad-y^{k+1}\frac{x^{n-k-1}-y^{2(n-k-1)}}{x-y^2}\cdot\frac{x(x^k-1)}{x-1}\Bigg)
\end{aligned}$$

Expanding the fractions and rearranging, we obtain

$$\begin{aligned}
&s_{(1^k,n-k)}[x+y^2-xy-y]\\
&\quad=(-y)^k(1-y)(x-y)\\
&\qquad\times(y^{2(n-k)-2}+xy^{2(n-k)-4}+\cdots+x^{n-k-2}y^2+x^{n-k-1})(1+x+\cdots+x^k)\\
&\qquad\times(y^{2(n-k)-3}+xy^{2(n-k)-5}+\cdots+x^{n-k-3}y^3+x^{n-k-2}y)x(1+x+\cdots+x^{k-1})\\
\\
&\quad=(1-y)(x-y)\\
&\qquad\times(-y)^{k+2(n-k)-2}(1+x+\cdots+x^k)+(-y)^{k+2(n-k)-3}x(1+x+\cdots+x^{k-1})\\
&\qquad\times(-y)^{k+2(n-k)-4}x(1+x+\cdots+x^k)+(-y)^{k+2(n-k)-5}x^2(1+x+\cdots+x^{k-1})\\
&\qquad\qquad\vdots\\
&\qquad\times(-y)^{k+2}x^{n-k-2}(1+x+\cdots+x^k)+(-y)^{k+1}x^{n-k-1}(1+x+\cdots+x^{k-1})\\
&\qquad\times(-y)^kx^{n-k-1}(1+x+\cdots+x^k)
\end{aligned}$$

Comparing with the expression

$$s_{(1^a,b,c)}[1+x-y]=x^{b-1}(1+x+x^2+\cdots+x^{c-b})(-y)^a(1-y)(x-y)$$

we have

$$\begin{aligned}
s_{(1^k,n-k)}[x+y^2-xy-y] &= s_{(1^{2n-k-2},1,k+1)}[1+x-y]+s_{(1^{2n-k-3},2,k+1)}[1+x-y]\\
&\quad+s_{(1^{2n-k-4},2,k+2)}[1+x-y]+s_{(1^{2n-k-5},3,k+2)}[1+x-y]\\
&\qquad\vdots\\
&\quad+s_{(1^{k+2},n-k-1,n-1)}[1+x-y]+s_{(1^{k+1},n-k,n-1)}[1+x-y]\\
&\quad+s_{(1^k,n-k,n)}[1+x-y]\\
&= \sum_{i=1}^{n-k}s_{(1^{2n-k-2i},i,k+i)}[1+x-y]\\
&\quad+\sum_{i=1}^{n-k-1}s_{(1^{2n-k-1-2i},1+i,k+i)}[1+x-y]
\end{aligned}$$

which proves statement 4 of Lemma 4.7.

To prove statement 4 of Theorem 4.5, as with the proof of statements 2 and 3, when a is even we just need to multiply the results of Lemma 4.7 by

$$x^{n(b-1)}y^{na}=x^{n(b-1)}(-y)^{na}.$$

Recalling that by Lemma 4.4 (3),

$$x^{n(b-1)}(-y)^{na}s_{(1^d,i,j)}[1+x-y]=s_{(1^{na+d},n(b-1)+i,n(b-1)+j)}[1+x-y],$$

we have

$$\begin{aligned} s_{(1^k,n-k)}[s_{(1^a,b,b)}][1+x-y] &= \sum_{i=1}^{n-k} s_{(1^{na+2n-k-2i},n(b-1)+i,n(b-1)+k+i)}[1+x-y] \\ &\quad + \sum_{i=1}^{n-k-1} s_{(1^{na+2n-k-1-2i},n(b-1)+1+i,n(b-1)+k+i)}[1+x-y], \end{aligned}$$

which proves statement 4 of Theorem 4.5.

When a is odd, we need to multiply $s_{(1^k,n-k)'}[x+y^2-y-xy]$ by

$$x^{|\lambda|(b-1)}y^{|\lambda|a}(-1)^{|\lambda|} = x^{|\lambda|(b-1)}(-y)^{|\lambda|a}.$$

Since $(1^k, n-k)' = (1^{n-k-1}, k+1)$, we just need to substitute $n-k-1$ for k and $k+1$ for $n-k$ in statement 4 of Theorem 4.5 to obtain statement 5.

Finally, we need to prove statement 5 of Lemma 4.7 and statements 6 and 7 of Theorem 4.5. Setting $\lambda = (1^k, 2^l, r, s)$ with $r \geq 2$, $s \geq 2$, we have

$$\prod_{(i,j)\in\lambda}(x_j - y_i) = (x_1-y_1)(x_1-y_2)x_1^{s-2}(x_2-y_1)(x_2-y_2)x_2^{r-2}(-y_1)^{k+l}(-y_2)^l$$

So

$$\begin{aligned} &s_{(1^k,2^l,r,s)}[x_1+x_2-y_1-y_2] \\ &= \frac{1}{(x_1-x_2)(y_1-y_2)}A_2^x A_2^y x_1 y_1 (x_1-y_1)(x_1-y_2) \\ &\qquad \times x_1^{s-2}(x_2-y_1)(x_2-y_2)x_2^{r-2}(-y_1)^{k+l}(-y_2)^l \\ &= \frac{1}{(x_1-x_2)(y_1-y_2)}A_2^x x_1^{s-1}x_2^{r-2}(x_1-y_1)(x_1-y_2)(x_2-y_1)(x_2-y_2) \\ &\qquad \times(-1)^k(y_1^{k+l+1}y_2^l - y_1^l y_2^{k+l+1}) \\ &= \frac{1}{(x_1-x_2)(y_1-y_2)}(x_1^{s-1}x_2^{r-2} - x_1^{r-2}x_2^{s-1})(x_1-y_1)(x_1-y_2)(x_2-y_1)(x_2-y_2) \\ &\qquad \times(-1)^k(y_1^{k+l+1}y_2^l - y_1^l y_2^{k+l+1}) \\ &= (x_1x_2)^{r-2}\left(\frac{x_1^{s-r+1}-x_2^{s-r+1}}{x_1-x_2}\right)(y_1y_2)^l(-1)^k\left(\frac{y_1^{k+1}-y_2^{k+1}}{y_1-y_2}\right) \\ &\qquad \times(x_1-y_1)(x_1-y_2)(x_2-y_1)(x_2-y_2) \end{aligned}$$

Substituting $x_1 = x$, $x_2 = y^2$, $y_1 = xy$, and $y_2 = y$ gives

$$\begin{aligned}
& s_{(1^k,2^l,r,s)}[x+y^2-xy-y] \\
&= (xy^2)^{r-2}\left(\frac{y^{2(s-r+1)}-x^{s-r+1}}{y^2-x}\right)(xy^2)^l(-1)^k\left(\frac{y^{k+1}-(xy)^{k+1}}{y-xy}\right) \\
&\qquad \times(x-xy)(x-y)(y^2-xy)(y^2-y) \\
&= (-1)^k x^{l+r-1}y^{k+2l+2r-2}(1-y)^2(x-y)^2\left(\frac{1-x^{k+1}}{1-x}\right)\left(\frac{y^{2(s-r+1)}-x^{s-r+1}}{y^2-x}\right) \\
&= x^{l+r-1}(-y)^{k+2l+2r-2}(1+x+\cdots+x^k)(1-y)^2(x-y)^2 \\
&\qquad \times(y^{2(s-r)}+y^{2(s-r)-2}x+\cdots+y^2x^{s-r-1}+x^{s-r}) \\
&= (1-y)(x-y)x^{l+r-1}(-y)^{k+2l+2r-2}(1+x+\cdots+x^k)(x+y^2-(1+x)y) \\
&\qquad \times(x^{s-r}+x^{s-r-1}y^2+\cdots+xy^{2(s-r)-2}+y^{2(s-r)}) \\
&= (1-y)(x-y)x^{l+r-1}(-y)^{k+2l+2r-2}(x+y^2)(1+x+\cdots+x^k) \\
&\qquad \times(x^{s-r}+x^{s-r-1}y^2+\cdots+xy^{2(s-r)-2}+y^{2(s-r)}) \\
&\quad +(1-y)(x-y)x^{l+r-1}(-y)^{k+2l+2r-1}(1+x)(1+x+\cdots+x^k) \\
&\qquad \times(x^{s-r}+x^{s-r-1}y^2+\cdots+xy^{2(s-r)-2}+y^{2(s-r)})
\end{aligned}$$

Referring to Lemma 4.4 (2) and noting that

$$(1+x)(1+x+\cdots+x^k) = 1+x+\cdots+x^{k+1}+x(1+x+\cdots+x^{k-1})$$

(where the second term only exists for $k > 0$), we can write this as

$$\begin{aligned}
& s_{(1^k,2^l,r,s)}[x+y^2-xy-y] \\
&= x^{l+r-1}(-y)^{k+2l+2r-2}(x+y^2)\sum_{i=0}^{s-r} s_{(1^{2j},s-r+1-j,k+s-r+1-j)}[1+x-y] \\
&\quad +(1-y)(x-y)x^{l+r-1}(-y)^{k+2l+2r-1} \\
&\quad \times\left(1+x+\cdots+x^{k+1}+x(1+x+\cdots+x^{k-1})\right) \\
&\quad \times(x^{s-r}+x^{s-r-1}y^2+\cdots+xy^{2(s-r)-2}+y^{2(s-r)})
\end{aligned}$$

This simplifies to

$$\begin{aligned} s_{(1^k,2^l,r,s)}[x+y^2-xy-y] &= x^{l+r}(-y)^{k+2l+2r-2}\sum_{i=0}^{s-r} s_{(1^{2i},s-r+1-i,k+s-r+1-i)}[1+x-y] \\ &\quad + x^{l+r-1}(-y)^{k+2l+2r}\sum_{i=0}^{s-r} s_{(1^{2i},s-r+1-i,k+s-r+1-i)}[1+x-y] \\ &\quad + x^{l+r-1}(-y)^{k+2l+2r-1}\sum_{i=0}^{s-r} s_{(1^{2i},s-r+1-i,k+s-r+2-i)}[1+x-y] \\ &\quad + x^{l+r}(-y)^{k+2l+2r-1}\sum_{i=0}^{s-r} s_{(1^{2i},s-r+1-i,k+s-r-i)}[1+x-y] \end{aligned}$$

where the last summation only occurs if $k > 0$. Applying Lemma 4.4 (3), this becomes

$$\begin{aligned} s_{(1^k,2^l,r,s)}[x+y^2-xy-y] &= \sum_{i=0}^{s-r} s_{(1^{k+2l+2r-2+2i},l+s+1-i,k+l+s+1-i)}[1+x-y] \\ &\quad + \sum_{i=0}^{s-r} s_{(1^{k+2l+2r+2i},l+s-i,k+l+s-i)}[1+x-y] \\ &\quad + \sum_{i=0}^{s-r} s_{(1^{k+2l+2r-1+2i},l+s-i,k+l+s+1-i)}[1+x-y] \\ &\quad + \sum_{i=0}^{s-r} s_{(1^{k+2l+2r-1+2i},l+s+1-i,k+l+s-i)}[1+x-y] \end{aligned}$$

If we peel off the $i = 0$ term in the first summation and the $i = s - r$ term in the second summation, and combine the remaining terms, we get the first three expressions in statement 5 of Lemma 4.7. The remaining two summations are precisely the fourth expression, so the proof of Lemma 4.7 is complete. To complete the proof of Theorem 4.5, when a is even we again just need to multiply the result of Lemma 4.7 by

$$x^{n(b-1)}y^{na} = x^{n(b-1)}(-y)^{na}.$$

Applying Lemma 4.4 (3) and comparing the expressions in Lemma 4.7 (5) and Theorem 4.5 (6), we see that we have precisely what we need.

Now, when a is odd, we need to multiply $s_{\lambda'}[x+y^2-y-xy]$ by

$$x^{|\lambda|(b-1)}y^{|\lambda|a}(-1)^{|\lambda|} = x^{|\lambda|(b-1)}(-y)^{|\lambda|a}.$$

Since $\lambda = (1^k, 2^l, r, s)$, we have

$$\lambda' = (1^{s-r}, 2^{r-2}, l+2, k+l+2),$$

so the result follows by substituting $s-r$ for k, $r-2$ for l, $l+2$ for r, and $k+l+2$ for s in statement 6 of the theorem. ■

We can now apply the conjugation rule to Theorem 4.5 to obtain the following result about hooks plus a column:

Theorem 4.8

1. $s_\lambda[s_{(2^a,b)}]\big|_{\subseteq(\text{hook+col})} = 0$ *unless* λ *is contained in a 2-hook.*

2. *For* $\lambda = (1^n)$,
$$s_{(1^n)}[s_{(2^a,b)}]\big|_{\subseteq(\text{hook+col})} = s_{(1^{n-1},2^{na},n(b-1)+1)}$$

3. *For* $\lambda = (n)$,
$$s_{(n)}[s_{(2^a,b)}]\big|_{\subseteq(\text{hook+col})} = \sum_{i=1}^{n} s_{(2^{na-1+i},nb+2-2i)}$$

4. *For* $\lambda = (1^k, n-k)$, *with* $k \geq 1$, $n-k > 1$, *and* b *even,*
$$\begin{aligned} s_{(1^k,n-k)}[s_{(2^a,b)}]\big|_{\subseteq(\text{hook+col})} &= \sum_{i=1}^{n-k} s_{(1^k,2^{na-1+i},nb-k+2-2i)} \\ &\quad + \sum_{i=1}^{n-k-1} s_{(1^{k-1},2^{na+i},nb-k+1-2i)} \end{aligned}$$

5. *For* $\lambda = (1^k, n-k)$, *with* $k \geq 1$, $n-k > 1$, *and* b *odd,*
$$\begin{aligned} s_{(1^k,n-k)}[s_{(2^a,b)}]\big|_{\subseteq(\text{hook+col})} &= \sum_{i=1}^{k+1} s_{(1^k,2^{na-1+i},nb-k+2-2i)} \\ &\quad + \sum_{i=1}^{k} s_{(1^{n-k-2},2^{na+i},nb-k+1-2i)} \end{aligned}$$

6. *For* $(1^k, 2^l, r, s) \vdash n$ *with* $k \geq 0$, $l \geq 0$, $r \geq 2$, $s \geq 2$, *and* b *even,*
$$\begin{aligned} s_{(1^k,2^l,r,s)}[s_{(2^a,b)}]\big|_{\subseteq(\text{hook+col})} &= s_{(1^k,2^{na+l+r-1},n(b-2)+k+2l+2s+2)} \\ &\quad +2\sum_{j=0}^{s-r-1} s_{(1^k,2^{na+l+s-1-j},n(b-2)+k+2l+2r+2+2j)} \\ &\quad +s_{(1^k,2^{na+l+s},n(b-2)+k+2l+2r)} \\ &\quad +\sum_{i=0}^{s-r}\Big(s_{(1^{k+1},2^{na+l+s-1-i},n(b-2)+k+2l+2r+1+2i)} \\ &\qquad + s_{(1^{k-1},2^{na+l+s-i},n(b-2)+k+2l+2r+1+2i)}\Big) \end{aligned}$$

7. *For* $(1^k, 2^l, r, s) \vdash n$ *with* $k \geq 0$*,* $l \geq 0$*,* $r \geq 2$*,* $s \geq 2$*, and* b *odd,*

$$\begin{aligned} s_{(1^k,2^l,r,s)}\big[s_{(2^a,b)}\big]\Big|_{\subseteq(\text{hook+col})} &= s_{(1^k,2^{na+l+r-1},n(b-2)+k+2l+2s+2)} \\ &\quad +2\sum_{j=0}^{k-1} s_{(1^k,2^{na+l+s-1-j},n(b-2)+k+2l+2r+2+2j)} \\ &\quad +s_{(1^k,2^{na+l+s},n(b-2)+k+2l+2r)} \\ &\quad +\sum_{i=0}^{k}\big(s_{(1^{k+1},2^{na+l+s-1-i},n(b-2)+k+2l+2r+1+2i)} \\ &\qquad\qquad + s_{(1^{k-1},2^{na+l+s-i},n(b-2)+k+2l+2r+1+2i)}\big) \end{aligned}$$

Proof. Applying the conjugation rule, we have

$$\begin{aligned} s_{(1^k,2^l,r,s)}\big[s_{(1^a,b,b)}\big]' &= \begin{cases} s_{(1^k,2^l,r,s)}\big[s_{(1^a,b,b)'}\big] & \text{if } a \text{ is even} \\ s_{(1^k,2^l,r,s)'}\big[s_{(1^a,b,b)'}\big] & \text{if } a \text{ is odd} \end{cases} \\ &= \begin{cases} s_{(1^k,2^l,r,s)}\big[s_{(2^{b-1},a+2)}\big] & \text{if } a \text{ is even} \\ s_{(1^{s-r},2^{r-2},l+2,k+l+2)}\big[s_{(2^{b-1},a+2)}\big] & \text{if } a \text{ is odd} \end{cases} \end{aligned}$$

So statement 1 follows immediately by conjugating statement 1 of Theorem 4.5. When a is even, statement 6 follows by conjugating the formula in statement 6 of Theorem 4.5 and then substituting a for $b-1$ and b for $a+2$. When a is odd, statement 7 follows by conjugating the formula in statement 7 of Theorem 4.5 and substituting k for $s-r$, l for $r-2$, r for $l+2$, s for $k+l+2$, a for $b-1$, and b for $a+2$.

For statements 2 and 3, we have

$$s_{(1^n)}\big[s_{(1^a,b,b)}\big]' = \begin{cases} s_{(1^n)}\big[s_{(2^{b-1},a+2)}\big] & \text{if } a \text{ is even} \\ s_{(n)}\big[s_{(2^{b-1},a+2)}\big] & \text{if } a \text{ is odd} \end{cases}$$

and

$$s_{(n)}\big[s_{(1^a,b,b)}\big]' = \begin{cases} s_{(n)}\big[s_{(2^{b-1},a+2)}\big] & \text{if } a \text{ is even} \\ s_{(1^n)}\big[s_{(2^{b-1},a+2)}\big] & \text{if } a \text{ is odd} \end{cases}$$

So statements 2 and 3 follow by substituting a for $b-1$ and b for $a+2$ into statements 2 and 3 of Theorem 4.5.

Finally, for statements 4 and 5, we have

$$s_{(1^k,n-k)}\big[s_{(1^a,b,b)}\big]' = \begin{cases} s_{(1^k,n-k)}\big[s_{(2^{b-1},a+2)}\big] & \text{if } a \text{ is even} \\ s_{(1^{n-k-1},k+1)}\big[s_{(2^{b-1},a+2)}\big] & \text{if } a \text{ is odd} \end{cases}$$

So statement 4 follows by substituting a for $b-1$ and b for $a+2$ in statement 4 of Theorem 4.5 and statement 5 follows by substituting a for $b-1$, b for $a+2$, k for $n-k-1$, and $n-k$ for $k+1$ in statement 5 of Theorem 4.5. ■

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