On a combinatorial problem of Asmus Schmidt

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Abstract

For any integer $r \ge 2$, define a sequence of numbers $\{c_k^{(r)}\}_{k=0,1,\dots}$, independent of the parameter n, by

$$\sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} c_k^{(r)}, \qquad n = 0, 1, 2, \dots .$$

We prove that all the numbers $c_k^{(r)}$ are integers.

1 Stating the problem

The following curious problem was stated by A. L. Schmidt in [5] in 1992.

Problem 1. *For any integer $r \ge 2$, define a sequence of numbers $\{c_k^{(r)}\}_{k=0,1,\dots}$, independent of the parameter n, by*

$$\sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} c_k^{(r)}, \qquad n = 0, 1, 2, \dots . \tag{1}$$

Is it then true that all the numbers $c_k^{(r)}$ are integers?

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An affirmative answer for $r = 2$ was given in 1992 (but published a little bit later), independently, by Schmidt himself [6] and by V. Strehl [7]. They both proved the following explicit expression:

$$c_n^{(2)} = \sum_{j=0}^{n} \binom{n}{j}^3 = \sum_j \binom{n}{j}^2 \binom{2j}{n}, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

which was observed experimentally by W. Deuber, W. Thumser and B. Voigt. In fact, Strehl used in [7] the corresponding identity as a model for demonstrating various proof techniques for binomial identities. He also proved an explicit expression for the sequence $c_n^{(3)}$, thus answering Problem 1 affirmatively in the case $r = 3$. But for this case Strehl had only one proof based on Zeilberger's algorithm of creative telescoping. Problem 1 was restated in [3], Exercise (!) 114 on p. 256, with an indication (on p. 549) that H. Wilf had shown the desired integrality of $c_n^{(r)}$ for any r but only for any $n \le 9$.

We recall that the first non-trivial case $r = 2$ is deeply related to the famous Apéry numbers $\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$, the denominators of rational approximations to $\zeta(3)$. These numbers satisfy a 2nd-order polynomial recursion discovered by R. Apéry in 1978, while an analogous recursion (also 2nd-order and polynomial) for the numbers (2) was indicated by J. Franel already in 1894.

The aim of this paper is to give an answer in the affirmative to Problem 1 (Theorem 1) by deriving explicit expressions for the numbers $c_n^{(r)}$, and also to prove a stronger result (Theorem 2) conjectured in [7], Section 4.2.

Theorem 1. *The answer to Problem* 1 *is affirmative. In particular, we have the explicit expressions*

$$c_n^{(4)} = \sum_j \binom{2j}{j}^3 \binom{n}{j} \sum_k \binom{k+j}{k-j} \binom{j}{n-k} \binom{k}{j} \binom{2j}{k-j}, \tag{3}$$

$$c_n^{(5)} = \sum_j \binom{2j}{j}^4 \binom{n}{j}^2 \sum_k \binom{k+j}{k-j}^2 \binom{2j}{n-k} \binom{2j}{k-j}, \tag{4}$$

and in general for $s = 1, 2, \ldots$

$$c_n^{(2s)} = \sum_j \binom{2j}{j}^{2s-1} \binom{n}{j} \sum_{k_1} \binom{j}{n-k_1} \binom{k_1}{j} \binom{k_1+j}{k_1-j} \sum_{k_2} \binom{2j}{k_1-k_2} \binom{k_2+j}{k_2-j}^2 \cdots$$
$$\times \sum_{k_{s-1}} \binom{2j}{k_{s-2}-k_{s-1}} \binom{k_{s-1}+j}{k_{s-1}-j}^2 \binom{2j}{k_{s-1}-j},$$

$$c_n^{(2s+1)} = \sum_j \binom{2j}{j}^{2s} \binom{n}{j}^2 \sum_{k_1} \binom{2j}{n-k_1} \binom{k_1+j}{k_1-j}^2 \sum_{k_2} \binom{2j}{k_1-k_2} \binom{k_2+j}{k_2-j}^2 \cdots$$
$$\times \sum_{k_{s-1}} \binom{2j}{k_{s-2}-k_{s-1}} \binom{k_{s-1}+j}{k_{s-1}-j}^2 \binom{2j}{k_{s-1}-j},$$

where $n = 0, 1, 2, \ldots$.

2 Very-well-poised preliminaries

The right-hand side of (1) defines the so-called *Legendre transform* of the sequence $\{c_k^{(r)}\}_{k=0,1,\dots}$. In general, if

$$a_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} c_k = \sum_{k=0}^{n} \binom{2k}{k}\binom{n+k}{n-k} c_k,$$

then by the well-known relation for inverse Legendre pairs one has

$$\binom{2n}{n} c_n = \sum_k (-1)^{n-k} d_{n,k} a_k,$$

where

$$d_{n,k} = \binom{2n}{n-k} - \binom{2n}{n-k-1} = \frac{2k+1}{n+k+1}\binom{2n}{n-k}.$$

Therefore, putting

$$t_{n,j}^{(r)} = \sum_{k=j}^{n} (-1)^{n-k} d_{n,k} \binom{k+j}{k-j}^r, \tag{5}$$

we obtain

$$\binom{2n}{n} c_n^{(r)} = \sum_{j=0}^{n} \binom{2j}{j}^r t_{n,j}^{(r)}. \tag{6}$$

The case $r = 1$ of Problem 1 is trivial (that is why it is not included in the statement of the problem), while the cases $r = 2$ and $r = 3$ are treated in [6], [7] using the fact that $t_{n,j}^{(2)}$ and $t_{n,j}^{(3)}$ have a *closed form*. Namely, it is easy to show by Zeilberger's algorithm of creative telescoping [4] that the latter sequences, indexed by either n or j, satisfy simple 1st-order polynomial recursions. Unfortunately, this argument does not exist for $r \ge 4$.

V. Strehl observed in [7], Section 4.2, that the desired integrality would be a consequence of the divisibility of the product $\binom{2j}{j}^r \cdot t_{n,j}^{(r)}$ by $\binom{2n}{n}$ for all j, $0 \le j \le n$. He conjectured a much stronger property, which we are now able to prove.

Theorem 2. *The numbers* $\binom{2n}{n}^{-1}\binom{2j}{j} t_{n,j}^{(r)}$ *are integers.*

Our general strategy for proving Theorem 2 (and hence Theorem 1) is as follows: rewrite (5) in a hypergeometric form and apply suitable summation and transformation formulae (Propositions 1 and 2 below).

Changing l to $n-k$ in (5) we obtain

$$t_{n,j}^{(r)} = \sum_{l \ge 0} (-1)^l \frac{2n-2l+1}{2n-l+1} \binom{2n}{l} \binom{n-l+j}{n-l-j}^r,$$

where the series on the right terminates. It is convenient to write all such terminating sums simply as $\sum_l$, which is, in fact, a standard convention (see, e.g., [4]). The ratio of two consecutive terms in the latter sum is equal to

$$\frac{-(2n+1)+l}{1+l}\cdot\frac{-\frac12(2n-1)+l}{-\frac12(2n+1)+l}\cdot\left(\frac{-(n-j)+l}{-(n+j)+l}\right)^r,$$

hence

$$t_{n,j}^{(r)}=\binom{n+j}{n-j}^r\cdot{}_{r+2}F_{r+1}\left(\left.\begin{matrix}-(2n+1),\,-\frac12(2n-1),\,-(n-j),\,\dots,\,-(n-j)\\ -\frac12(2n+1),\,-(n+j),\,\dots,\,-(n+j)\end{matrix}\,\right|\,1\right)$$

is a very-well-poised hypergeometric series. (We refer the reader to the book [2] for all necessary hypergeometric definitions. We will omit the argument $z=1$ in further discussions.)

The following two classical results—Dougall's summation of a ${}_5F_4(1)$-series (proved in 1907) and Whipple's transformation of a ${}_7F_6(1)$-series (proved in 1926)—will be required to treat the cases $r=3,4,5$ of Theorems 1 and 2.

Proposition 1 ([2], Section 4.3)**.** *We have*

$$ {}_5F_4\left(\begin{matrix}a,\,1+\frac12a, & c, & d, & -m\\ \frac12a, & 1+a-c, & 1+a-d, & 1+a+m\end{matrix}\right)=\frac{(1+a)_m\,(1+a-c-d)_m}{(1+a-c)_m\,(1+a-d)_m} \tag{7}$$

and

$$\begin{aligned}&{}_7F_6\left(\begin{matrix}a,\,1+\frac12a, & b, & c, & d, & e, & -m\\ \frac12a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+m\end{matrix}\right)\\ &\quad=\frac{(1+a)_m\,(1+a-d-e)_m}{(1+a-d)_m\,(1+a-e)_m}\cdot{}_4F_3\left(\begin{matrix}1+a-b-c,\,d,\,e,\,-m\\ 1+a-b,\,1+a-c,\,d+e-a-m\end{matrix}\right),\end{aligned} \tag{8}$$

where m is a non-negative integer, and $(\,\cdot\,)$ denotes Pochhammer's symbol.

An application of (7) gives (without creative telescoping)

$$t_{n,j}^{(3)}=\binom{n+j}{n-j}^3\cdot\frac{(-2n)_{n-j}(-2n+2(n-j))_{n-j}}{(-2n+(n-j))_{n-j}^2}=\frac{(2n)!}{(3j-n)!\,(n-j)!^3},$$

which is exactly the expression obtained in [7], Section 4.2. Therefore, from (6) we have the explicit expression

$$c_n^{(3)}=\binom{2n}{n}^{-1}\sum_j\binom{2j}{j}^3\frac{(2n)!}{(3j-n)!\,(n-j)!^3}=\sum_j\binom{2j}{j}^2\binom{2j}{n-j}\binom{n}{j}^2.$$

For the case $r=5$, we are able to apply the transformation (8):

$$
\begin{aligned}
t_{n,j}^{(5)} &= \binom{n+j}{n-j}^5 \cdot \frac{(-2n)_{n-j}(-2n+2(n-j))_{n-j}}{(-2n+(n-j))_{n-j}^2} \\
&\qquad \times {}_4F_3\left(\begin{matrix} -2j,\ -(n-j),\ -(n-j),\ -(n-j) \\ -(n+j),\ -(n+j),\ 3j-n+1 \end{matrix}\right) \\
&= \binom{n+j}{n-j}^2 \frac{(2n)!}{(3j-n)!\,(n-j)!^3} \sum_l \frac{(-2j)_l\,(-(n-j))_l^3}{l!\,(-(n+j))_l^2(3j-n+1)_l} \\
&= \frac{(2n)!}{(2j)!\,(n-j)!^2} \sum_l \binom{n-l+j}{n-l-j}^2 \binom{2j}{l} \binom{2j}{n-l-j} \\
&= \frac{(2n)!}{(2j)!\,(n-j)!^2} \sum_k \binom{k+j}{k-j}^2 \binom{2j}{n-k} \binom{2j}{k-j},
\end{aligned}
$$

hence

$$
\binom{2n}{n}^{-1} \binom{2j}{j} t_{n,j}^{(5)} = \binom{n}{j}^2 \sum_k \binom{k+j}{k-j}^2 \binom{2j}{n-k} \binom{2j}{k-j}
$$

are integers and from (6) we derive formula (4).

To proceed in the case $r=4$, we apply the version of formula (8) with $b=(1+a)/2$ (so that the series on the left reduces to a ${}_6F_5(1)$-very-well-poised series):

$$
\begin{aligned}
t_{n,j}^{(4)} &= \binom{n+j}{n-j}^4 \cdot \frac{(-2n)_{n-j}(-2n+2(n-j))_{n-j}}{(-2n+(n-j))_{n-j}^2} \\
&\qquad \times {}_4F_3\left(\begin{matrix} -j,\ -(n-j),\ -(n-j),\ -(n-j) \\ -n,\ -(n+j),\ 3j-n+1 \end{matrix}\right) \\
&= \binom{n+j}{n-j} \frac{(2n)!}{(3j-n)!\,(n-j)!^3} \sum_l \frac{(-j)_l\,(-(n-j))_l^3}{l!\,(-n)_l\,(-(n+j))_l(3j-n+1)_l} \\
&= \frac{(2n)!\,j!}{n!\,(n-j)!\,(2j)!} \sum_l \binom{n-l+j}{n-l-j} \binom{j}{l} \binom{n-l}{j} \binom{2j}{n-l-j} \\
&= \frac{(2n)!\,j!}{n!\,(n-j)!\,(2j)!} \sum_k \binom{k+j}{k-j} \binom{j}{n-k} \binom{k}{j} \binom{2j}{k-j},
\end{aligned}
$$

from which, again, $\binom{2n}{n}^{-1}\binom{2j}{j} t_{n,j}^{(4)} \in \mathbb{Z}$ and we arrive at formula (3).

3 Andrews's multiple transformation

It seems that 'classical' hypergeometric identities can cover only the cases[1] $r=2,3,4,5$ of Theorems 1 and 2. In order to prove the theorems in full generality, we will require

[1] This is not really true since Andrews's 'non-classical' identity below is a consequence of very classical Whipple's transformation and the Pfaff–Saalschütz formula.

a multiple generalization of Whipple's transformation (8). The required generalization is given by G. E. Andrews in [1], Theorem 4. After making the passage $q \to 1$ in Andrews's theorem, we arrive at the following result.

Proposition 2. *For $s \geq 1$ and m a non-negative integer,*

$$
\begin{aligned}
&{}_{2s+3}F_{2s+2}\left(\begin{matrix} a, 1+\frac{1}{2}a, & b_1, & c_1, & b_2, & c_2, & \dots \\ \frac{1}{2}a, & 1+a-b_1, & 1+a-c_1, & 1+a-b_2, & 1+a-c_2, & \dots \end{matrix}\right. \\
&\qquad\qquad\qquad \left.\begin{matrix} \dots, & b_s, & c_s, & -m \\ \dots, & 1+a-b_s, & 1+a-c_s, & 1+a+m \end{matrix}\right) \\
&\quad = \frac{(1+a)_m(1+a-b_s-c_s)_m}{(1+a-b_s)_m(1+a-c_s)_m} \sum_{l_1 \geq 0} \frac{(1+a-b_1-c_1)_{l_1}(b_2)_{l_1}(c_2)_{l_1}}{l_1!\,(1+a-b_1)_{l_1}(1+a-c_1)_{l_1}} \\
&\qquad \times \sum_{l_2 \geq 0} \frac{(1+a-b_2-c_2)_{l_2}(b_3)_{l_1+l_2}(c_3)_{l_1+l_2}}{l_2!\,(1+a-b_2)_{l_1+l_2}(1+a-c_2)_{l_1+l_2}} \cdots \\
&\qquad \times \sum_{l_{s-1} \geq 0} \frac{(1+a-b_{s-1}-c_{s-1})_{l_{s-1}}(b_s)_{l_1+\cdots+l_{s-1}}(c_s)_{l_1+\cdots+l_{s-1}}}{l_{s-1}!\,(1+a-b_{s-1})_{l_1+\cdots+l_{s-1}}(1+a-c_{s-1})_{l_1+\cdots+l_{s-1}}} \\
&\qquad\quad \times \frac{(-m)_{l_1+\cdots+l_{s-1}}}{(b_s+c_s-a-m)_{l_1+\cdots+l_{s-1}}}.
\end{aligned}
$$

Proof of Theorem 2. As in Section 2, we will distinguish the cases corresponding to the parity of r.

If $r = 2s+1$, then setting $a = -(2n+1)$ and $b_1 = c_1 = \cdots = b_s = c_s = -m = -(n-j)$ in Proposition 2 we obtain

$$
\begin{aligned}
t_{n,j}^{(2s+1)} &= \binom{n+j}{n-j}^{2s-2} \frac{(2n)!}{(3j-n)!\,(n-j)!^3} \sum_{l_1} \binom{2j}{l_1} \left(\frac{(-(n-j))_{l_1}}{(-(n+j))_{l_1}}\right)^2 \\
&\quad \times \sum_{l_2} \binom{2j}{l_2} \left(\frac{(-(n-j))_{l_1+l_2}}{(-(n+j))_{l_1+l_2}}\right)^2 \cdots \\
&\quad \times \sum_{l_{s-1}} \binom{2j}{l_{s-1}} \left(\frac{(-(n-j))_{l_1+\cdots+l_{s-1}}}{(-(n+j))_{l_1+\cdots+l_{s-1}}}\right)^2 \\
&\qquad \times \frac{(-1)^{l_1+\cdots+l_{s-1}}(-(n-j))_{l_1+\cdots+l_{s-1}}}{(3j-n+1)_{l_1+\cdots+l_{s-1}}} \\
&= \frac{(2n)!}{(2j)!\,(n-j)!^2} \sum_{l_1} \binom{2j}{l_1} \binom{n-l_1+j}{n-l_1-j}^2 \sum_{l_2} \binom{2j}{l_2} \binom{n-l_1-l_2+j}{n-l_1-l_2-j}^2 \cdots \\
&\quad \times \sum_{l_{s-1}} \binom{2j}{l_{s-1}} \binom{n-l_1-\cdots-l_{s-1}+j}{n-l_1-\cdots-l_{s-1}-j}^2 \cdot \binom{2j}{n-l_1-\cdots-l_{s-1}-j}.
\end{aligned}
$$

If $r = 2s$, we apply Proposition 2 with the choice $a = -(2n+1)$, $b_1 = (a+1)/2 = -n$

and $c_1 = b_2 = \cdots = b_s = c_s = -m = -(n-j)$:

$$\begin{aligned}
t_{n,j}^{(2s)} &= \binom{n+j}{n-j}^{2s-3} \frac{(2n)!}{(3j-n)!\,(n-j)!^3} \sum_{l_1} \binom{j}{l_1} \frac{(-(n-j))_{l_1}}{(-n)_{l_1}} \frac{(-(n-j))_{l_1}}{(-(n+j))_{l_1}} \\
&\qquad \times \sum_{l_2} \binom{2j}{l_2} \left(\frac{(-(n-j))_{l_1+l_2}}{(-(n+j))_{l_1+l_2}} \right)^2 \cdots \\
&\qquad \times \sum_{l_{s-1}} \binom{2j}{l_{s-1}} \left(\frac{(-(n-j))_{l_1+\cdots+l_{s-1}}}{(-(n+j))_{l_1+\cdots+l_{s-1}}} \right)^2 \\
&\qquad\qquad \times \frac{(-1)^{l_1+\cdots+l_{s-1}}(-(n-j))_{l_1+\cdots+l_{s-1}}}{(3j-n+1)_{l_1+\cdots+l_{s-1}}} \\
&= \frac{(2n)!\,j!}{n!\,(n-j)!\,(2j)!} \sum_{l_1} \binom{j}{l_1} \binom{n-l_1}{j} \binom{n-l_1+j}{n-l_1-j} \\
&\qquad \times \sum_{l_2} \binom{2j}{l_2} \binom{n-l_1-l_2+j}{n-l_1-l_2-j}^2 \cdots \\
&\qquad \times \sum_{l_{s-1}} \binom{2j}{l_{s-1}} \binom{n-l_1-\cdots-l_{s-1}+j}{n-l_1-\cdots-l_{s-1}-j}^2 \cdot \binom{2j}{n-l_1-\cdots-l_{s-1}-j}.
\end{aligned}$$

In both cases, the desired integrality

$$\binom{2n}{n}^{-1} \binom{2j}{j} t_{n,j}^{(r)} \in \mathbb{Z}, \qquad j = 0, 1, \ldots, n,$$

clearly holds, and Theorem 2 follows. □

Theorem 1 was actually proved during the proof of Theorem 2 with explicit expressions being obtained for $c_n^{(4)}$, $c_n^{(5)}$ and general $c_n^{(r)}$, $r \ge 2$.

We would like to conclude the paper by the following q-question.

Problem 2. *Find and solve an appropriate q-analogue of Problem* 1.

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