# Minuscule Heaps Over Dynkin diagrams of type $\tilde{A}$

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Submitted: Oct 18, 2002; Accepted: Dec 12, 2003; Published: Jan 2, 2004

**Abstract**

A minuscule heap is a partially ordered set, together with a labeling of its elements by the nodes of a Dynkin diagram, satisfying certain conditions derived by J. Stembridge. This paper classifies the minuscule heaps over the Dynkin diagram of type $\tilde{A}$.

## 1 Introduction

The aim of this paper is to classify the minuscule heaps over a Dynkin diagram of type $\tilde{A}$.

Let $A$ be a symmetrizable generalized Cartan matrix, and let $\mathfrak{g}$ be a corresponding Kac-Moody Lie algebra. Let $\Gamma$ be a Dynkin diagram which is an encoding of $A$. Minuscule heaps arose in connection with $\lambda$-minuscule elements of the Weyl group $W$ of $\mathfrak{g}$. According to Proctor [9] and Stembridge [12] the notion of *$\lambda$-minuscule elements* of $W$ was defined by Peterson in his unpublished work in the 1980's. For an integral weight $\lambda$, an element $w$ of $W$ is said to be a *$\lambda$-minuscule element* if it has a reduced decomposition $s_{i_1} s_{i_2} \dots s_{i_r}$ such that

$$s_{i_j} s_{i_{j+1}} \dots s_{i_r} \lambda = s_{i_{j+1}} \dots s_{i_r} \lambda - \alpha_{i_j} \ (1 \leq \forall j \leq r),$$

and it is called *minuscule* if $w$ is $\lambda$-minuscule for some integral weight $\lambda$. Here $\alpha_i$ is the simple root corresponding to $s_i$. It is known that a minuscule element is *fully commutative*, namely any reduced decomposition can be converted into any other by exchanging adjacent commuting generators several times (see [9, §15], [10, Theorem A] and [11, Theorem 2.2], or [12, Proposition 2.1]). To a fully commutative element $w$, one can associate a $\Gamma$-labeled poset called its *heap*.

A $\Gamma$-*labeled poset* is a triple $(P, \leq, \phi)$ in which $(P, \leq)$ is a poset and $\phi : P \to N(\Gamma)$ is any map (called the *labeling map*). A linear extension of a $\Gamma$-labeled poset naturally

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determines a word in the generators of $W$. The heap of the fully commutative element $w$ is a $\Gamma$-labeled poset whose linear extensions determine all reduced decompositions of $w$. A *minuscule heap* is the heap of a minuscule element of $W$.

Stembridge obtained certain structural conditions for a finite $\Gamma$-labeled poset to be a minuscule heap ([11, Proposition 3.1]). In this paper a minuscule heap is defined by a finite $\Gamma$-labeled poset which satisfies the conditions (H1-1), (H1-2) and (H2a) (see §2). In the following, we state a relation between minuscule elements and minuscule heaps.

Let $(P, \leq, \phi)$ be a minuscule heap. Put $r := \#P$. Let $\mu : P \to [1, r]$ be a *linear extension* of $P$, namely $\mu$ is a bijection and if $p \leq q$ then $\mu(p) \leq \mu(q)$. For $\mu$, we associate a minuscule heap $(P, \leq, \phi)$ to $w \in W$ by the following expression

$$w := s_{\phi\circ\mu^{-1}(1)} s_{\phi\circ\mu^{-1}(2)} \dots s_{\phi\circ\mu^{-1}(r)}.$$

We note that an element obtained from a minuscule heap, by the relation above, is a minuscule element. Conversely for any minuscule element $w$ there exists a unique minuscule heap which determines $w$.

In [11], there is an important condition which states that "the labels that occur in $P$ index an acyclic subset of the Dynkin diagram". A nice consequence of this condition is that if it holds then $P$ is a *ranked poset*. However Dynkin diagrams $\Gamma$ of type $\tilde{A}$ are cyclic, and most of minuscule heaps over $\Gamma$ are not ranked. In this paper, we introduce an analogy of slant lattices [6] (here called $L_k$) (§8) and use it to prove that a subset $P$ of an extended slant lattice $L_k$ is a convex subset if and only if $P$ is a minuscule heap over $\Gamma$ up to isomorphic. Slant lattices $L$ were also used to classify the minuscule heaps over simply-laced, star-shaped Dynkin diagrams in [6].

It is known that the affine permutation group $\tilde{S}_{n+1}$ is isomorphic to an affine Weyl group $W(\tilde{A}_n)$. In [5], Green showed that the 321-avoiding permutations of affine permutations coincide with the fully commutative elements of $W(\tilde{A}_n)$. He showed also that the fully commutative elements of $W(\tilde{A}_n)$ form a union of Kazhdan-Lusztig cells. Here we show that the fully commutative elements of $W(\tilde{A}_n)$ coincide with its minuscule elements [Theorem 5.1].

The paper is organized as follows. In §2 we recall and provide some basic terminology. In §§3, 4 we collect some general facts on poset and on $\Gamma$-labeled poset with a general Dynkin diagram. In §5, we show that the fully commutative elements of $W(\tilde{A})$ coincide with the minuscule elements of $W(\tilde{A})$. From §4, we characterize the minuscule heaps over a Dynkin diagram of type $\tilde{A}$. In §6, we characterize the totally ordered minuscule heaps over a Dynkin diagram of type $\tilde{A}$. In §7, we determine the structure of a subposet which we call *v-interval*. In §8, we characterize the minuscule heaps over a Dynkin diagram of type $\tilde{A}$ up to isomorphism and introduce the notion of *extended slant lattices* $L_k$. In §9, we show that any minuscule heap over a Dynkin diagram of type $\tilde{A}$ is isomorphic to a convex subset of an extended slant lattice $L_k$.

## 2 Definitions

We start with the definition of general terms associated to a partially ordered set. We denote the number of elements of a set $P$ by $\#P$.

Let $(P, \leq)$ be a poset (*partially ordered set*). For $p, q \in P$, we say that $q$ *covers* $p$ *(or* $p$ *is covered by* $q$*)* if $p < q$ and $(p, q) = \emptyset$, and denote by $p \to q$. We say that $p, q$ are a *covering pair* if $p \to q$ or $q \to p$. In this paper, we assume that $P$ is completely determined by the covering relations, namely

(*) If $p, q \in P$ and $p \leq q$ then there exists a finite sequence of elements of $P$, say $p_0, p_1, \ldots, p_r$ such that $p_0 = p, p_r = q$ and $p_i$ covers $p_{i-1}$ for $1 \leq i \leq r$.

We call such a sequence $p_0, p_1, \ldots, p_r$ a *saturated chain from* $p$ *to* $q$.

We denote ordering relations on posets as follows. Let $P$ be a poset and let $Q$ be its subset. For $x, y \in Q$, we write $[x, y]_P = \{z \in P | x \leq_P z \leq_P y\}$ and write $[x, y]_Q = \{z \in Q | x \leq_Q z \leq_Q y\}$. In general, a maximal connected subposet of $P$ is called a *connected component* of $P$. A subset $Q$ of $P$ is said to be convex in $P$ if whenever $p, q \in Q$ and $p \leq q$ we have $[p, q]_P \subset Q$.

Let $\Gamma$ be a Dynkin diagram and let $N(\Gamma)$ be the node set of $\Gamma$. By an abuse of language, we sometimes identify $N(\Gamma)$ with $\Gamma$. We say that a triple $(P, \leq, \phi)$ (or simply $P$) is a $\Gamma$-*labeled* poset if $(P, \leq)$ is a partially ordered set, and $\phi$ is any map from $P$ to $N(\Gamma)$. We call $\phi$ the *labeling map* and call $\phi(p)$ the *label* of $p$. We denote $\mathrm{Im}\,\phi$ by $\mathrm{supp}\,P$, and call it the *support* of $P$. For each $v \in N(\Gamma)$, we put

$$P_v := \{p \in P | \phi(p) = v\}.$$

For $v \in N(\Gamma)$ and $p, q \in P$ satisfying $p < q$, we say that $[p, q]$ is a $v$-*interval* if $p, q \in P_v$ and $(p, q) \cap P_v = \emptyset$.

Let $(P, \leq, \phi), (Q, \preceq, \psi)$ be $\Gamma$-labeled posets. We say that $P$ and $Q$ are *isomorphic* as $\Gamma$-labeled poset if there exists a poset isomorphism $\Phi : P \to Q$ such that $\phi(p) = \psi(\Phi(p)) (\forall p \in P)$.

Let $\Gamma$ and $\Gamma'$ be Dynkin diagrams, and let $(a_{i,j})_{i,j \in I}, (a'_{i,j})_{i,j \in I'}$ be the corresponding generalized Cartan matrices. Let $P$ be a $\Gamma$-labeled poset and let $Q$ be a $\Gamma'$-labeled poset. We say that $P$ and $Q$ are *abstractly isomorphic* (or isomorphic if no confusion arises) if there is a poset isomorphism $\alpha : P \to Q$ and an isomorphism of subdiagrams $\beta : \mathrm{supp}\,P \to \mathrm{supp}\,Q$ (namely a bijection $\mathrm{supp}\,P \to \mathrm{supp}\,Q$ such that $a_{i,j} = a'_{\beta(i),\beta(j)}$ for all $i, j \in \mathrm{supp}\,P$) such that $\beta$ maps the label of $p$ to the label of $\alpha(p)$ for every $p \in P$.

Let $D(3) := \{w, x, y, z\}$ be a set and let $\to$ be a binary relation on $D(3)$ with $w \to x, x \to z, w \to y, y \to z$. Let $\leq$ be an ordering on $D(3)$ which is the reflective, transitive closure of $\to$. Let $\Gamma$ be the Dynkin diagram of type $A_3$ and let $N(\Gamma) = \{1, 2, 3\}$ be the node set of $\Gamma$. (Put $1, 2, 3$ on $N(\Gamma)$ from an edge node to another one.) Define a map $\phi : D(3) \to N(\Gamma)$ by putting $\phi(w) = \phi(z) := 2, \phi(x) := 1$ and $\phi(y) := 3$. We regard $D(3)$ as a $\Gamma$-labeled poset with $\phi$. Let $\Gamma'$ be a Dynkin diagram. We say that $\Gamma'$-labeled poset $Q$ is a *diamond* if $Q$ and $(D(3), \leq, \phi)$ are abstractly isomorphic.

Let $A = (a_{i,j})_{i,j \in N(\Gamma)}$ be a Cartan matrix corresponding to $\Gamma$. We say that a $\Gamma$-labeled poset $(P, \leq, \phi)$ is a *minuscule heap* if $P$ is finite and satisfies the conditions (H1-1),(H1-2) and (H2a).

**(H1-1)** For $p, q \in P$, if $p \to q$, then $\phi(p)$ and $\phi(q)$ are either equal or adjacent in $\Gamma$.

**(H1-2)** For $p, q \in P$, if $\phi(p)$ and $\phi(q)$ are either equal or adjacent in $\Gamma$ then $p$ and $q$ are comparable.

**(H2a)** For $p, q \in P$, if $\phi(p) = \phi(q)$ and $p \leq q$ then $\sum_{x \in [p,q]} a_{\phi(x),\phi(p)} = 2$.

In particular we regard an empty set as a minuscule heap.

**Remark 2.1.** In [11], Stembridge obtained two structural conditions, which he called (H1) and (H2), for a finite $\Gamma$-labeled poset to be a minuscule heap. In this paper we separate (H1) to two conditions (H1-1),(H1-2). And we use the condition (H2a) instead of (H2) which are equivalent. (Proposition 4.4.)

# 3 Basic Properties on Poset

In this section we provide some general facts on posets. We omit the proofs below since they are straightforward.

**Proposition 3.1.** *Let $S$ be a set and let $\rightsquigarrow$ be a binary relation on $S$. Let $\preceq$ be the reflexive, transitive closure of $\rightsquigarrow$.*

*Then $(S, \preceq)$ is a poset if and only if $s = s_0 \rightsquigarrow s_1 \rightsquigarrow \cdots \rightsquigarrow s_r = s$ implies $s_0 = s_1 = \cdots = s_r = s$ for some $r \geq 0$, where $s, s_0, s_1, \ldots, s_r \in S$.*

**Proposition 3.2.** *Let $S, \rightsquigarrow, \preceq$ be the same as Proposition 3.1. Assume that $(S, \preceq)$ be a poset.*

*Then $\rightsquigarrow$ is the covering relation on $(S, \preceq)$ if and only if $p \rightsquigarrow q$ and $p = p_0 \rightsquigarrow p_1 \rightsquigarrow \cdots \rightsquigarrow p_r = q$ implies $r = 1$.*

**Proposition 3.3.** *Let $(P, \leq)$ be a poset and let $G$ be a group which acts on $P$ as a poset automorphism, namely $p \leq q$ if and only if $g(p) \leq g(q)$ for $g \in G$ and $p, q \in P$.*

*Assume that $G$ satisfies the following condition,*

- *for $p \in P$ and $g \in G$, if $p$ and $g(p)$ are comparable then $p = g(p)$.*

*Put $P/G := \{\overline{p} | p \in P\}$, where $\overline{p} = \{g(p) | g \in G\}$, and put a relation $\preceq$ on $P/G$ as following,*

$$\overline{p} \preceq \overline{q} \text{ if and only if } p \leq g(q) \text{ for some } g \in G \ (\overline{p}, \overline{q} \in G/P).$$

*Then it follows,*

- *$\preceq$ is well-defined,*
- *$(P/G, \preceq)$ is a poset.*

**Proposition 3.4.** *Let $P$ be a poset and let $p, q \in P$ satisfying $p \leq q$. Then $[p, q]$ is a convex subset of $P$.*

**Proposition 3.5.** *Let $P$ be a poset and $Q$ be a convex subset. If $z$ is a minimal or maximal element of $Q$ then $Q \setminus \{z\}$ is a convex subset of $P$.*

# 4 Basic Properties on Γ-labeled Posets

In this section we provide some general facts on Γ-labeled posets over a general Dynkin diagram. (See [7] or [8] for the definition of Dynkin diagrams.)

**Proposition 4.1.** *Let $(P, \leq, \phi)$ be a $\Gamma$-labeled poset. If $(P, \leq, \phi)$ satisfies (H1-2) then $P_v$ is a totally ordered set for each $v \in N(\Gamma)$.*

*Proof.* By (H1-2), $p$ and $q$ are comparable, where $p, q \in P_v$. □

**Proposition 4.2.** *Let $(P, \leq, \phi)$ be a $\Gamma$-labeled poset which satisfies (H1-1) and (H1-2). Then $P$ is connected if and only if* supp $P$ *is connected.*

*Proof.* Assume that $P$ is connected. For $u, v \in \text{supp } P$, there exists $p \in P_v$ and $q \in P_u$. Now we can take a sequence $p = p_0, p_1, \ldots, p_r = q$ such that $p_{i-1}, p_i$ are a covering pair. Then $\phi(p_0), \phi(p_1), \ldots, \phi(p_r)$ consists of a connected subdiagram of $\Gamma$ by (H1-1). Hence supp $P$ is connected.

Conversely assume that supp $P$ is connected. Let $p, q \in P$ and put $v := \phi(p), u := \phi(q)$. Since $\Gamma$ is connected, we can take a sequence $v = v_0, v_1, \ldots, v_r = u \in \text{supp } P$ such that $v_{i-1}$ and $v_i$ are adjacent in $\Gamma$. Take some $p_i \in P_{v_i}$ $(1 \leq i \leq r-1)$ and put $p_0 = p, p_r = q$. Then $p_{i-1}, p_i$ are comparable by (H1-2). So $P$ is connected. □

**Proposition 4.3.** *Let $(P, \leq, \phi)$ be a $\Gamma$-labeled poset satisfying (H1-1) and (H1-2). Let $P_1, P_2, \ldots, P_r$ be the connected components of $P$. Then* $\text{supp } P = \sqcup_{i=1}^r \text{supp } P_i$. *In particular $v$ and $u$ are distinct and non-adjacent, where $v \in \text{supp } P_i, u \in \text{supp } P_j$ and $i \neq j$.*

*Proof.* If there exists $v \in \text{supp } P_i \cap \text{supp } P_j$ then we can take $p \in P_i \cap P_v$ and $q \in \text{supp } P_j \cap P_v$. By (H1-2), $p$ and $q$ are comparable. This implies $i = j$ because $P_i, P_j$ are connected components.

If there exists $v \in \text{supp } P_i$ and $u \in \text{supp } P_j$ such that $v$ and $u$ are adjacent in $\Gamma$ then we obtain a contradiction by a similar argument. □

**Proposition 4.4.** *Let $(P, \leq, \phi)$ be a finite $\Gamma$-labeled poset. If $(P, \leq, \phi)$ satisfies (H1-1) and (H1-2) then each of the following each conditions are equivalent to (H2a).*

*(H2) For any $v$-interval $[p, q]$, we have $\sum_{x \in [p,q]} a_{\phi(x), \phi(p)} = 2$.*

*(H2b) For any $v$-interval $[p, q]$, we have $\sum_{x \in (p,q)} a_{\phi(x), \phi(p)} = -2$.*

*Proof.* By $a_{v,v} = 2$, it is obvious that (H2) and (H2b) are equivalent for any $v \in N(\Gamma)$. It is also obvious that (H2a) implies (H2).

We assume that (H2) holds. By (H1-2), $P_v$ is a totally ordered set. Let $p, q$ be elements of $P_v$ which satisfy $p \leq q$. If we have $p = q$ then (H2a) trivially holds, since $a_{v,v} = 2$ for $v \in N(\Gamma)$. So we assume $p < q$. Let $p = p_0, p_1, \ldots, p_r = q$ be the elements of $P_v \cap [p, q]$ by an increasing ordering. For $x \in [p, q] \setminus (\cup_{1 \leq i \leq r}[p_{i-1}, p_i])$, $x$ and $p_j$ are incomparable for some $0 \leq j \leq r$. By (H1-2), $\phi(x)$ is different from $v$ and not adjacent to $v$. Thus we have $a_{\phi(x),v} = 0$. This implies

$$\sum_{x \in [p,q]} a_{\phi(x),v} = \sum_{1 \leq i \leq r} \sum_{x \in [p_{i-1}, p_i)} a_{\phi(x),v} + a_{\phi(p_r),v} = 0 + 2 = 2.$$

□

**Remark 4.5.** Let $\Gamma$ be a simply-laced Dynkin diagram and let $v \in N(\Gamma)$. Let $[p, q]$ be a $v$-interval. (H2b) requires that there exists just two elements of $(p, q)$ whose labels are adjacent to $v$ in $\Gamma$. This fact is very important since the Dynkin diagram of type $\tilde{A}_n$ $(n \geq 2)$ is simply-laced. However we investigate a minuscule heap over the Dynkin diagram of type $\tilde{A}_1$. In this case, (H2b) requires that there exists only one element of $(p, q)$ whose labels are adjacent to $v$ in $\Gamma$. These facts are used to prove Propositions 4.6, 6.1 and 6.2.

**Proposition 4.6.** *Let $\Gamma$ be a simply-laced Dynkin diagram and let $(P, \leq, \phi)$ be a minuscule heap over $\Gamma$. Let $p, q \in P$ such that $\phi(p) = \phi(q)$. If there exists an element $x \in P$ such that $p \to x \to q$ then $[p, q]$ is a $\phi(p)$-interval. In particular $[p, q]$ is a diamond.*

*Proof.* We note that $\phi(x)$ and $\phi(p)$ are adjacent in $\Gamma$. If there exists $y \in (p, q)$ such that $\phi(y) = \phi(p)$ then $x$ and $y$ are comparable by (H1-2). This implies that we have either $p < x < y$ or $y < x < q$. It contradicts $p \to x \to q$. So we have $(p, q) \cap P_{\phi(p)} = \emptyset$. However there exists $y \in (p, q)$ such that $\phi(y)$ is adjacent to $\phi(p)$ by Remark 4.5. Let $p = p_0, p_1, \ldots, p_r = q$ be a saturated chain from $p$ to $q$ which contains $y$. We note that this saturated chain does not contain $x$. In fact a sequence $p, x, q$ is a unique saturated chain which contains $x$.

For $p = p_0, p_1, \ldots, p_r = q$, $y$ is the only element which can cover $p$, and $y$ is the only element which can be covered by $q$. Thus this saturated chain consists of $p, y, q$. Hence we have $[p, q] = \{p, x, y, q\}$. By (H1-1) and (H2), $[p, q]$ is a diamond. □

**Proposition 4.7.** *Let $\Gamma$ be a Dynkin diagram and let $\Phi$ be a graph automorphism on $\Gamma$. If a $\Gamma$-labeled poset $(P, \leq, \phi)$ is a minuscule heap then $(P, \leq, \Phi \circ \phi)$ is a minuscule heap. Furthermore these minuscule heaps are abstractly isomorphic.*

*Proof.* Since $\Phi$ is a graph automorphism, it is obvious that (H1-1) and (H1-2) hold. As $a_{v,u} = a_{\Phi(v),\Phi(u)}$, (H2a) holds. □

# 5 Relation Between Fully Commutative Elements and Minuscule Elements

First we show that the fully commutative elements of $W(\tilde{A})$ coincide with its minuscule elements.

**Theorem 5.1.** *Let $\Gamma$ be a simply-laced Dynkin diagram with a finite node set. The fully commutative elements of $W(\Gamma)$ coincide with its minuscule elements if and only if $\Gamma$ is of type $A$ or $\tilde{A}$.*

*Proof.* It is well known that a minuscule element is fully commutative.

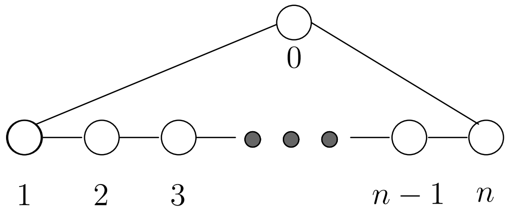
Assume $\Gamma$ branches off. Then there exists a node $v \in N(\Gamma)$ such that the number of the adjacent nodes to $v$ is larger than two. Let $x, y, z$ be adjacent nodes to $v$. We can verify that $s_v s_x s_y s_z s_v$ is fully commutative, where $s_u$ is a generator associated to $u \in N(\Gamma)$. But it violates (H2) and so it would not be minuscule. Hence $\Gamma$ cannot have a junction. Thus $\Gamma$ must be of type $A$ or $\tilde{A}$.

If $\Gamma$ is of type $A$, then it is well-known that a fully commutative element is minuscule. The remaining case is when $\Gamma$ is of type $\tilde{A}$. Let $w$ be a fully commutative element of $W(\tilde{A})$ and let $s_1 s_2 \ldots s_r$ be a reduced expression of $w$. By the commutativity of $w$, if $s_i, s_j$ are consecutive occurrences of the generator $s$ (meaning that $s_i = s_j = s$ for some generator $s$ ($i$ <j) and $s_i \neq s$ for $i < h < j$), then there are at least two generators $s_{h_1}, s_{h_2}$ such that $s_i$ and $s_{h_1}$ (or $s_{h_2}$) are non-commutative $i < h_1, h_2 < j$. If there is no consecutive occurrences of any generators, then $w$ is minuscule.

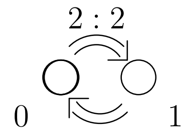
Remember that any heap of a fully commutative element satisfies (H1). For proving that $w$ is minuscule, it is sufficient that there exists just two non-commutative generators $s_{h_1}, s_{h_2}$. If not, we can take three non-commutative generators from $s_{i+1}, \ldots, s_{j-1}$. Now, as $\Gamma$ is of type $\tilde{A}$, each node has only two adjacent nodes. Thus we can take consecutive occurrences $s_{i'}, s_{j'}$ of $s'$. By the commutativity of $w$, we can take two generators $s_{h'_1}, s_{h'_2}$ from $s_{i'+1}, \ldots, s_{j'-1}$ which are non-commutative to $s'$. The nodes associated to $s_{i'+1}, s_{j'+1}$ are adjacent to the node associated to $s'$ and they are different from $s_i$ because $s_i \neq s_h$ for $i < h < j$. This implies $s_{i'+1} = s_{j'+1}$, in other words they are consecutive occurrences. By using a similar argument, the length of $w$ must be infinite. It cannot happen. ☐

# 6 Totally Ordered Minuscule Heaps over Dynkin Diagrams of Type $\tilde{A}$

In this section we determine the structure of totally ordered minuscule heaps over Dynkin diagrams of type $\tilde{A}$. From this section on we assume that $\Gamma$ is a Dynkin diagram of type $\tilde{A}_n$ with the node set $N(\Gamma) := \{0, 1, \ldots, n\}$. (see Figure 1 for the definition of Dynkin diagram of type $\tilde{A}_n$.) We associate $i \in \mathbb{Z}$ to $j \in N(\Gamma) = \{0, 1, \ldots, n+1\}$ by the following rule $j = i \mod (n+1)$. We note that the Dynkin diagram of type $\tilde{A}_1$ and its Cartan matrix $A := A(\Gamma)$ are different from others of type $\tilde{A}_n (n \geq 2)$. First we classify the minuscule heaps over $\tilde{A}_1$. In fact a minuscule heap over $\Gamma(\tilde{A}_1)$ is a totally ordered set.




Dynkin diagram of type $\tilde{A}_n$ $(n \geq 2)$




Dynkin diagram of type $\tilde{A}_1$

Figure 1: The Dynkin diagram of type $\tilde{A}_n$

The Cartan matrix $A = A(\Gamma(\tilde{A}_1)) = (a_{i,j})$ is

$$A := \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

**Proposition 6.1.** *Let $\Gamma$ be the Dynkin diagram of type $\tilde{A}_1$. A minuscule heap over $\Gamma$ is a totally ordered set and is characterized by $r := \#P$ and the label $v$ of its smallest element if $r \neq 0$. Namely, define a $\Gamma$-labeled poset $(P, \leq, \phi)$ by putting*

- $P = \{p_1, p_2, \ldots, p_r\}$,
- $p_{i-1} \to p_i$ *($1 < i \leq r$),*
- $\leq$ *is the transitive and reflective closure of* $\to$,
- $\phi(p_i) \equiv v + i - 1 \pmod 2$.

*Then $(P, \leq, \phi)$ is a minuscule heap over $\Gamma$. Conversely a minuscule heap over $\Gamma$ is abstractly isomorphic to a minuscule heap as defined above.*

*Proof.* Define $P$ as above. Then it is obvious that (H1-1) and (H1-2) hold on $P$. By the definition of $P$, a $v$-interval has the form $[p_i, p_{i+2}]$ for $v \in N(\Gamma)$. So we have

$$\sum_{x \in (p_i, p_{i+2})} a_{\phi(x),v} = \sum_{x \in \{p_{i+1}\}} a_{\phi(x),v} = -2.$$

This implies that $P$ is a minuscule heap.

Conversely assume that $(P, \leq, \phi)$ is a minuscule heap. Then it is obvious that $P$ is a totally ordered set by the shape of $\Gamma$ and (H1-2). Hence we can write $P = \{p_1, p_2, \ldots, p_r\}$ with $p_{i-1} \to p_i$ $(1 < i \leq r)$. By (H2b), we have $\phi(p_{i-1}) \neq \phi(p_i)$. Since the labels of $p_i$ are binary values ($\{0, 1\}$), two nodes always alternatively appear. Thus we have $\phi(p_i) \equiv v + i - 1 \pmod 2$. □

By a similar argument above, we can determine the structure of a totally ordered minuscule heap over $\Gamma(\tilde{A}_n)$ $(n \geq 2)$.

**Proposition 6.2.** *Let $(P, \leq, \phi)$ be a totally ordered minuscule heap over $\Gamma$. Put $P = \{p_0, p_1, \ldots, p_r\}$ with $p_{i-1} \to p_i (1 \leq i \leq r)$. Then the labels of the elements of $P$ are either of type (T1) or (T2),*

**(T1)** $\phi(p_i) \equiv \phi(p_0) + i \pmod{n+1}$

**(T2)** $\phi(p_i) \equiv \phi(p_0) - i \pmod{n+1}$

*Proof.* By (H2b), we have $\phi(p_0) \neq \phi(p_1)$. By (H1-1), we have either $\phi(p_1) = \phi(p_0) + 1$ or $\phi(p_1) = \phi(p_0) - 1$.

Assume $\phi(p_1) = \phi(p_0)+1$. Then we have either $\phi(p_2) = \phi(p_1)+1$ or $\phi(p_2) = \phi(p_1)-1$. If we have $\phi(p_2) = \phi(p_1) - 1$ then $\phi(p_2) = \phi(p_0)$. By Proposition 4.6, $[p_0, p_2]$ is a $\phi(p_0)$-interval. However we have

$$\sum_{x \in (p_0,p_2)} a_{\phi(x),\phi(p_0)} = \sum_{x \in \{p_1\}} a_{\phi(x),\phi(p_0)} = -1 \neq -2.$$

It contradicts (H2b). Thus we have $\phi(p_2) = \phi(p_1) + 1$. By repeating an argument above, the labels of the elements are of type (T1).

By a similar argument, we obtain the case (T2) from the assumption $\phi(p_1) = \phi(p_0) - 1$. □

We say that $P$ is of type (T1) (resp. (T2)) if the labels of $P$ are of type (T1) (resp. (T2)).

# 7 The Structure of $v$-intervals

In this section we investigate the structure of $v$-intervals for a minuscule heap over $\Gamma(\tilde{A}_n)$ with $n \geq 2$. To determine $v$-intervals for $v \in N(\Gamma)$ is useful to determine the structure of a minuscule heap.

By the symmetry of the shape of the Dynkin diagram of type $\tilde{A}$, to determine the structure of all of 1-intervals is equivalent to determine that of all of $v$-intervals for any $v \in N(\Gamma)$. Hence we investigate the structure of the 1-intervals.

**Lemma 7.1.** *Let $P$ be a minuscule heap over $\Gamma(\tilde{A}_n)$ and let $[p_1, q_1] \subset P$ be a 1-interval. Let $p_2, q_2$ be elements of $(p_1, q_1)$ whose labels are adjacent to 1 in $\Gamma$. Then we have $\phi(p_2) \neq \phi(q_2)$.*

*Proof.* We note that $\phi(p_2), \phi(q_2)$ must be 0 or 2. Our claim is that $\phi(p_2) = \phi(q_2)$ is impossible.

If we have $\phi(p_2) = \phi(q_2) = 2$ then $p_2$ and $q_2$ are comparable. Let us assume that $p_2 < q_2$ then $[p_2, q_2]$ is a 2-interval by (H2b). Hence there exist $p_3, q_3 \in (p_2, q_2)$ such that $p_2 \to p_3, q_3 \to q_2$ and $\phi(p_3), \phi(q_3)$ are adjacent to 2. Now $\phi(p_3), \phi(q_3)$ can be only equal to 1 or 3. However they cannot be equal to 1 since $[p_1, q_1]$ is a 1-interval. So we have $\phi(p_3) = \phi(q_3) = 3$. By repeating the arguments above, we can take a 0-interval $[p_{n+1}, q_{n+1}]$ from $(p_1, q_1)$ and we know that $[p_{n+1}, q_{n+1}]$ must contain an element whose label is 1. It contradicts that $[p_1, q_1]$ is a 1-interval. □

For $i = 1, 2, 3$, we say that a 1-interval $[p, q]$ is *of type (Vi)* if $[p, q]$ satisfies the following:

**(V1)** $[p, q]$ is a totally ordered set and consists of $n + 2$ elements. The labels of the elements in the increasing order are $1, 2, 3, \ldots, n-1, n, 0, 1$ respectively;

**(V2)** $[p, q]$ is a totally ordered set and consists of $n + 2$ elements; The labels of the elements in the increasing order are $1, 0, n, n-1, \ldots, 3, 2, 1$ respectively.

**(V3)** $[p, q]$ is a diamond.

**Proposition 7.2.** *Any* 1*-interval* $[p, q]$ *is either of type (V1), (V2) or (V3).*

*Proof.* By Lemma 7.1, $(p, q)$ contains a unique pair of elements $x, y$ whose labels are $0, 2$ respectively. By (H1-1), only $x$ or $y$ can cover $p$ and only $x$ or $y$ can be covered by $q$.

Assume that both $x$ and $y$ cover $p$ then we claim that $[p, q]$ is of type (V3). By the assumption that $x$ and $y$ are incomparable, there is a saturated chain $x = p_1, p_2, \ldots, p_r = q$ from $x$ to $q$ which does not contain $y$. We note that $\phi(p_{r-1})$ is either 0 or 2, namely $p_{r-1}$ is either $x$ or $y$. This implies $p_{r-1} = x$ and $r = 2$. Thus we have $x \to q$. By a similar argument, we have $y \to q$. So $[p, q] = \{p, x, y, q\}$ is of type (V3).

Next assume that only $x$ covers $p$. We claim that $[p, q]$ is of type (V1). Let $p = p_0, x = p_1, p_2, \ldots, p_r = q$ be a saturated chain from $p$ to $q$. We note that $\phi(p_{i-1}) - \phi(p_i)$ is either 1 or $-1$ because $\phi(p_{i-1}), \phi(p_i)$ are adjacent. If these labels are all different then they are of type (V1) or (V2). If there are repeated labels then we can take a pair $p_i, p_{i+2}$ such that $\phi(p_i) = \phi(p_{i+2})$. Let us choose such a minimal $i$. By Prop. 4.6, $[p_i, p_{i+2}]$ is a diamond. Thus there exists $p'_{i+1} \in [p_i, p_{i+2}]$ such that $\phi(p'_{i+1})$ is not equal to $\phi(p_i)$ and $\phi(p_{i+1})$. If we change $p_{i+1}$ to $p'_{i+1}$ then we take another saturated chain $p_0, p_1, \ldots, p'_{i+1}, \ldots, p_r$ from $p$ to $q$ such that $\phi(p_{i-1}) = \phi(p'_{i+1})$. By using the same argument, there exists a saturated chain $p = p_0, x = p_1, p_2, \ldots, p_r = q$ with $\phi(p_1) = \phi(p_3)$. It contradicts that $[p, q]$ is a 1-interval. So each labels are different. There exists only one saturated chain from $p$ to $q$ is only $p_0, p_1, \ldots, p_r$. Assume that there exists another saturated chain $p = q_0, x = q_1, \ldots, q_r = q$. So, there exists $q_i$ such that $q_i \neq p_i$ and $q_j = p_j$ $(0 \leq j < i)$. By the above argument, we have $\phi(p_i) = \phi(q_i)$. By (H1-2), $p_i$ and $q_i$ are comparable. If $p_i < q_i$ then we have $q_{i-1} < p_i < q_i$. If $q_i < p_i$ then we have $p_{i-1} < q_i < p_i$. These are contradictions. So $[p, q]$ is of type (V1).

By using a similar argument, if we assume that only $y$ covers $p$ then we obtain that $[p, q]$ is of type (V2). □

For not only 1-intervals but also for a $v$-interval $[p, q]$, we say that $[p, q]$ is *of type (Vi)* if $[p, q]$ satisfies the following:

**(V1)** $[p, q]$ is a totally ordered set and consists of $n + 2$ elements. The labels of the elements in the increasing order are $v, v+1, v+2, \ldots, v+n-1, v+n, v$ respectively;

**(V2)** $[p, q]$ is a totally ordered set and consists of $n + 2$ elements. The labels of the elements in the increasing order are $v, v+n, v+n-1, \ldots, v+2, v+1, v$ respectively;

**(V3)** $[p, q]$ is a diamond.

Define a graph automorphism $\Phi : \Gamma \to \Gamma$ by $\Phi(i) := i+1 \pmod{n+1}$. If we apply $\Phi$ and Prop. 4.7 to Prop. 7.2 then we can characterize all the 2-intervals. Thus we characterize all the $v$-intervals for all $v \in N(\Gamma)$.

**Proposition 7.3.** *Any $v$-interval $[p, q]$ is either of type (V1), (V2) or (V3).*

**Proposition 7.4.** *If $P$ contains a $v$-interval of type (V1) (resp. (V2)) for some $v \in N(\Gamma)$ then $P$ is a totally ordered set and is of type (V1) (resp (V2)).*

*Proof.* Let $(P, \leq, \phi)$ be a minuscule heap over $\Gamma(\tilde{A}_n)$ which contains a $v$-interval $[p, q]$ of type (V1). It is sufficient to show that every element of $P$ covers and is covered by at most one of its element. If we assume that there exists $x \in P$ which covers or is covered by two (or more) elements, then we get a contradiction as follows. By supp $[p, q] = N(\Gamma)$, $x$ is comparable to an element $p \in [p, q]$. Thus we have $p < x$ or $x < q$. Now we assume $p < x$ holds. (If we can consider the dual poset $P^*$, we can verify the case $x < q$.)

Assume $x \in P$ is covered by two (or more) elements $x_1, x_2$ of $P$. We can take a saturated chain $p = p_0 \to p_1 \to \cdots \to p_i = x$. By the shape of $\Gamma$, $\phi(x_1)$ or $\phi(x_2)$ is equal to $\phi(p_{i-1})$. Without loss of generality, we can assume that $\phi(x_1) = \phi(p_{i-1})$. By Prop. 4.6, $[p_{i-1}, x_1]$ is a diamond. Thus $p_{i-1}$ must be covered by two (or more) elements. By repeating this argument, $p$ is covered by two (or more) elements. We denote these elements by $p_1, p_1'$. Since $\phi(p_1')$ is next to $\phi(p)$, we have $\phi(p_1') = \phi(q)$. So $p_1', q$ are comparable. Because $p_1'$ covers $p$, we have $p_1' < q$. On the other hand, $p_1'$ is not contained in $[p, q]$, a contradiction.

Assume $x \in P$ covers two (or more) elements $x_1, x_2$ of $P$. We can take two saturated chains $p = p_0 \to \cdots \to p_{i-1} = x_1 \to p_i = x$ and $p = q_0 \to \cdots \to q_{i-1} = x_2 \to q_i = x$. Thus there is an element $p_k = q_k$ in the saturated chain such that $p_0 = q_0, p_1 = q_1, \ldots, p_k = q_k$ and $p_{k+1} \neq q_{k+1}$. Hence $p_k$ is covered by two elements, a contradiction.

The case for $v$-interval of type (V2) is proved in a similar way. □

The following corollary immediately follows from the above Proposition.

**Corollary 7.5.** *Let $P$ be a minuscule heap over $\Gamma$. If $P$ contains a $v$-interval of type (Vi) ($1 \leq i \leq 3$) for some $v \in N(\Gamma)$ then any $u$-interval is of type (Vi) for any $u \in N(\Gamma)$.*

We should make a remark about minuscule heaps $(P, \leq, \phi)$ with supp $P \neq \Gamma$. Let $v \in N(\Gamma) \setminus$ supp $P$. If we choose a graph automorphism $\Phi : \Gamma \to \Gamma$ such that $\Phi(v) = 0$ then we can regard $(P, \leq, \Phi \circ \phi)$ as a minuscule heap over the Dynkin diagram $\Gamma'$ of type $A_n$. And the minuscule heaps over $\Gamma'$ are already classified. To summarize, we now know the following:,

- If $P$ is a totally ordered set then $P$ is either of type (T1) or (T2).
- If $P$ is not a totally ordered set with supp $P \neq \Gamma$ then $P$ is a minuscule heap over a Dynkin diagram of type $A$.

The remaining case is that when $P$ is not totally ordered and supp $P = \Gamma$. We know that any $v$-interval of $P$ is a diamond for any $v \in N(\Gamma)$. In §9 we study such minuscule heaps.

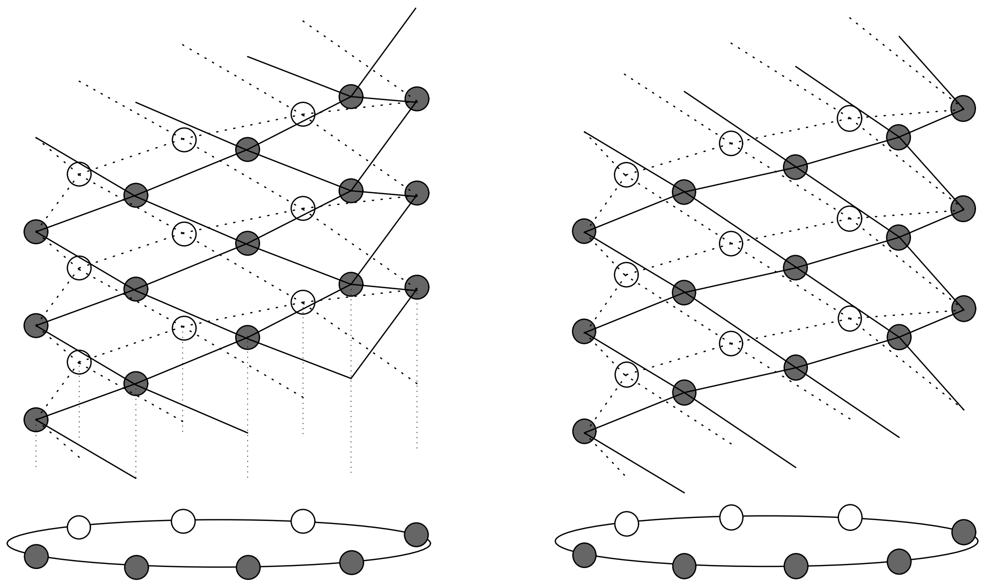


Figure 2: extended slant lattices $L_3$ (left figure) and $L_4$ (right figure) over $\Gamma(\tilde{A}_7)$

# 8 An Extended Slant Lattice

In this section we introduce the notion of extended slant lattices $L_k$ $(1 \leq k \leq n)$ (see figure 2) which are Γ-labeled posets such that any minuscule heaps over $\Gamma(\tilde{A}_n)$ are isomorphic to those convex subsets.

We put $L_0 := \{(a,b) \in \mathbb{Z} \times \mathbb{Z} | a \equiv b \pmod 2\}$ and define a relation $\to$ on $L_0$ by $(v,m) \to (v,m)+(1,-1)$ or $(v,m) \to (v,m)+(-1,-1)$. Let $\leq$ be the reflective, transitive closure of $\to$. We note that $(L_0, \leq)$ is a poset. Then the following lemma is obtained immediately.

**Lemma 8.1.** *Let $Q$ be an order ideal generated by $(0,0)$ of $L_0$. Then $Q = \{(a,b) | b \geq |a|\}$, where $||$ is the absolute value symbol.*

For $1 \leq k \leq n$, we define a map $\psi_k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by putting $\psi_k(v,m) := (v+n+1, m-2k+n+1)$. Then $\psi_k$ is a poset automorphism of $(\mathbb{Z} \times \mathbb{Z}, \leq)$. Let $G_k$ be the group generated by $\psi_k$. Then $G_k$ satisfies the conditions of Prop. 3.3. Hence we define a poset $L_k$ by putting

$$L_k := (\mathbb{Z} \times \mathbb{Z}/G_k, \leq_{L_k}),$$

and call it *an extended slant lattice.* We note that the sum $+$, defined by putting $\overline{(v,m)} + \overline{(u,m')} := \overline{(v+u, m+m')}$ for $\overline{(v,m)}$ and $\overline{(u,m')}$, is well-defined on $L_k$. $\{(v,m) \in [0,n] \times \mathbb{Z} | v \equiv m \pmod 2\}$ is a complete representative system of $L_k$. Thus we can define a map $\phi : L_k \to [0,n]$ by putting $\phi(\overline{(v,m)}) := v$, where $0 \leq v \leq n$. Now we regard $L_k = (\mathbb{Z} \times \mathbb{Z}/G_k, \leq_{L_k}, \phi)$ as a Γ-labeled poset by this map.

Let us define a binary relation $\rightsquigarrow$ by $\overline{(v,m)} \rightsquigarrow \overline{(v+1,m-1)}, \overline{(v,m)} \rightsquigarrow \overline{(v-1,m-1)}$ for $\overline{(v,m)}, \overline{(v+1,m-1)}, \overline{(v-1,m-1)} \in L_k$ and put $\preceq$ the reflective and transitive closure of $\rightsquigarrow$.

**Lemma 8.2.** *$\preceq$ is equivalent to $\leq$.*

*Proof.* Let $p,q$ be elements of $L_k$ which satisfy $p \preceq q$. Then we can take a sequence $p = p_0 \rightsquigarrow p_1 \rightsquigarrow \cdots \rightsquigarrow p_r = q$. Now we have $p_i = p_{i-1} + \overline{(\pm 1, -1)}$. If we put $p_{i-1} = \overline{(v,m)}$ then we can write $p_i = \overline{(v \pm 1, m-1)}$. By the definition of $\leq$, we have $(v,m) \leq_{L_0} (v \pm 1, m-1)$. Thus we have $\overline{(v,m)} \leq_{L_k} \overline{(v \pm 1, m-1)}$. This implies $p \leq_{L_k} q$.

Conversely assume that $\overline{(v,m)} \leq_{L_k} \overline{(u,l)}$. Then we have $(v,m) \leq_{L_0} (u,l) + j(n+1, n+1-2k)$ for some $j \in \mathbb{Z}$. Let $p_0, p_1, \ldots, p_r$ be a saturated chain from $(v,m)$ to $(u + j(n+1), l + j(n+1-2k))$. Then we have $\overline{p_0} \rightsquigarrow \overline{p_1} \rightsquigarrow \cdots \rightsquigarrow \overline{p_r}$. Hence $\overline{(v,m)} \preceq \overline{(u,l)}$. □

Although the following proposition maybe seems to be obvious, it does not hold on $L_1, L_n$.

**Proposition 8.3.** *For $2 \leq k \leq n-1$, $\overline{(v,m)} \in L_k$ covers $\overline{(v \pm 1, m+1)}$. $\overline{(v,m)}$ is covered by $\overline{(v \pm 1, m-1)}$.*

*Proof.* It is sufficient to show that $\rightsquigarrow$ is a cover relation on $L_k$. To do so, we show that $\overline{(a,b)} = s_0 \rightsquigarrow s_1 \rightsquigarrow \cdots \rightsquigarrow s_r = \overline{(a \pm 1, b-1)}$ implies $r = 1$ by Lemma 3.2.

First we prove in the case $s_r = \overline{(a+1, b-1)}$. Put $\alpha := \#\{1 \leq i \leq r | s_i = s_{i-1} + \overline{(1,-1)}\}$ and put $\beta := \#\{1 \leq i \leq r | s_i = s_{i-1} + \overline{(-1,-1)}\}$. (We note that $\alpha + \beta = r$ and $\alpha, \beta \geq 0$.) Then we have $s_r = \overline{(a + \alpha - \beta, b - \alpha - \beta)} = \overline{(a+1, b-1)}$. This implies that

$$\begin{aligned} a + \alpha - \beta &= a + 1 + \gamma(n+1), \\ b - \alpha - \beta &= b - 1 + \gamma(n+1-2k), \end{aligned}$$

for some $\gamma \in \mathbb{Z}$. The solution of above equations is $\alpha = 1 + k\gamma$, $\beta = \gamma(k-n-1)$. By the assumption, we have $2 \leq k \leq n-1$ and $\alpha \geq 0$. Thus, $\gamma \geq 0$. On the other hand, $k - n - 1 < 0$ and $\beta \geq 0$. So, $\gamma \leq 0$. This implies $\gamma = 0$. Then we have $\alpha = 1$ and $\beta = 0$. So we conclude that $r = \alpha + \beta = 1 + 0 = 1$ holds.

By using a similar argument, we have $\alpha = 0$ and $\beta = 1$ from the case $s_r = \overline{(a-1, b-1)}$. □

**Remark 8.4.** In the proof above, there exists a solution $\alpha = 0, \beta = n$ for $s_r = \overline{(a+1, b-1)}$, $k = 1$ and $\gamma = -1$. Hence $\rightsquigarrow$ is not a covering relation on $L_1$.

**Proposition 8.5.** *$L_1, L_n$ are totally ordered sets.*

*Proof.* We give only proof for $L_1$. (We can similarly prove it for $L_n$.)

We know that $\{\overline{(v,m)} | 0 \leq v \leq n, m \equiv v \pmod 2\}$ is a complete representative system. For $\overline{(0,2m)}$, we have $\overline{(0,2m)} = \overline{(m(n+1), m(n+1))}$. So we can write $\overline{(i, i+2m)} = \overline{(i + m(n+1), i + m(n+1))}$ for $0 \leq i \leq n$. Thus $L_1$ is isomorphic to $\mathbb{Z}$ as poset. (We note that $-j$ is an element on $\mathbb{Z}$ corresponding to $\overline{(j,j)}$. For example, we have $\overline{(0,0)} > \overline{(1,1)}$ in the ordering $\leq$ on $L_1$.) □

**Corollary 8.6.** *For $1 \leq k \leq n$, we have that (H1-1),(H1-2) and (H2a) hold on $L_k$, and their finite convex subsets are minuscule. Moreover their $v$-intervals are diamonds when $2 \leq k \leq n-1$.*

*Proof.* By Propositions 8.3, 8.5, (H1-1) holds on $L_k$. By the definition of $\preceq$ and by the fact that (H1-2) holds on $L_0$, (H1-2) holds on $L_k$.

Next we show that (H2a) holds. If $k = 1$ or $k = n$ then $P$ has the structure in Prop. 6.2. So (H2a) holds. By the definition of $L_k$, a $v$-interval is a diamond when $2 \leq k \leq n-1$. So (H2a) holds.

Thus their finite convex subsets are minuscule. □

For a finite subset $Q$ of $L_k$ and $v \in N(\Gamma)$, we can define a unique maximal (resp. minimal) element $t_v$ (resp. $b_v$) of $Q_v$ since (H1-2) holds on $L_k$

**Lemma 8.7.** *Let $y_0 \in \mathbb{Z}$ and put $Q := \{p \in L_k | p \leq \overline{(k, y_0)}\}$. Then we have*

$$Q = \{\overline{(a,b)} | 0 \leq a \leq k, b \geq y_0 + k - a\} \cup \{\overline{(a,b)} | k < a \leq n, b \geq y_0 + a - k\}.$$

*Let $x_0 \in \mathbb{Z}$ and put $Q' := \{p \in L_k | p \geq \overline{(n+1-k, x_0)}\}$. Then we have*

$$Q' = \{\overline{(a,b)} | 0 \leq a \leq n+1-k, b \leq x_0 + k + a\} \cup \{\overline{(a,b)} | n+1-k < a \leq n, b \leq x_0 + 2n + 2 - k - a\}.$$

*Proof.* We can take a complete representative system $\{\overline{(v,m)} | 0 \leq v \leq n, m \equiv v \pmod 2\}$. For $0 \leq a \leq n$, we have

$$\begin{aligned} & \overline{(a,b)} \leq \overline{(k, y_0)} \\ \iff & (a,b) + j(n+1, n+1-2k) \leq (k, y_0) \ \exists j \in \mathbb{Z} \\ \iff & (a-k+j(n+1), b-y_0+j(n+1-2k)) \leq (0,0) \ \exists j \in \mathbb{Z} \\ & \text{(By Lemma 8.1)} \\ \iff & b - y_0 + j(n+1-2k) \geq |a-k+j(n+1)| \ \exists j \in \mathbb{Z}. \end{aligned}$$

Assume $0 \leq a \leq k$. Then we have $a-k+j(n+1) \leq 0 \iff j \leq 0$ or equivalently we have $a-k+j(n+1) \geq 0 \iff j \geq 1$. For $j \leq 0$, we have

$$\begin{aligned} b & \geq y_0 - j(n+1-2k) - a + k - j(n+1) \\ & = y_0 + k - a - 2j(n+1-k). \end{aligned}$$

Hence the necessary and sufficient condition for $(a-k+j(n+1), b-y_0+k+j(n+1-2k)) \leq (0,0)$ (for some $j \leq 0$) is

$$b \geq y_0 + k - a.$$

For $j \geq 1$, we have

$$\begin{aligned} b & \geq y_0 - j(n+1-2k) + a - k + j(n+1) \\ & = y_0 + a - k + 2jk. \end{aligned}$$

Hence the necessary and sufficient condition for $(a-k+j(n+1), b-y_0+j(n+1-2k)) \leq (0,0)$ (for some $j \geq 1$) is

$$b \geq y_0 + a.$$

Thus we have $b \geq y_0 + k - a$. Assume $k < a \leq n$. We have $a - k + j(n+1) \geq 0 \iff j \geq 0$ or equivalently $a - k + j(n+1) \leq 0 \iff j \leq -1$. For $j \geq 0$ we have

$$b \ \geq \ y_0 + a - k.$$

For $j \leq -1$ we have

$$b \ \geq \ y_0 - a + 2n + 2 - k.$$

On the other hand, we have $y_0 - a + 2n + 2 - k \geq y_0 + a - k$. Thus we have $b \geq y_0 + a - k$.

We can prove for the case for $Q'$ in a similar way. □

Let $x_0, y_0 \in \mathbb{Z}$ satisfying $x_0 \geq y_0$. For $v \in N(\Gamma)(= [0,n])$, we define $y_v$ by putting $y_v := y_0 + k - v$ if $0 \leq v \leq k$ and $y_v := y_0 + a - k$ if $k < v \leq n$. We define $x_v$ by putting $x_v := x_0 + k + v$ if $0 \leq v \leq n+1-k$ and $x_v := x_0 + 2n + 2 - k - v$ if $n+1-k < v \leq n$. Then we have the following,

**Proposition 8.8.**

$$[\overline{(n+1-k, x_0+n+1-k)}, \overline{(k, y_0-k)}] = \{(v,m) \in L_k | y_v \leq m \leq x_v\}$$

*Proof.* It is obvious if we take $Q \cap Q'$ in Lemma 8.7. □

**Proposition 8.9.** *Let $2 \leq k \leq n-1$ and let $Q$ be a finite subposet of $L_k$ satisfying* $\mathrm{supp}\, Q = \Gamma$. *For $v \in N(\Gamma)$, put $b_v := \overline{(v, x_v)}$ and $t_v := \overline{(v, y_v)}$, where $t_v$ (resp $b_v$) is a maximal (resp. minimal) element of $P_v$.*

*$Q$ is a convex subset if and only if $Q$ satisfies the followings,*

- $Q_v = \{\overline{(v,m)} \in L_k | y_v \leq m \leq x_v\}$,
- *$t_{v-1}, t_v$ are a covering pair,*
- *$b_{v-1}, b_v$ are a covering pair,*

*for each $1 \leq v \leq n+1$.*

*Proof.* Let $Q$ be a convex subset. Since (H1-2) holds on $L_k$, we have $Q_v = \{\overline{(v,m)} \in L_k | y_v \leq m \leq x_v\}$.

Assume that $\overline{(v-1,m)}, \overline{(v,l)} \in Q$ for some $1 \leq v \leq n$. If we have $m < l$ then there exists a saturated chain $\overline{(v,l)} \to \overline{(v-1,l-1)} \to \overline{(v,l-2)} \to \cdots \to \overline{(v,m-1)} \to \overline{(v-1,m)}$ on $L_k$. Since $Q$ is a convex subset, this saturated chain is contained in $Q$. This implies that we have $-1 \leq y_{v-1} - y_v$. Next we assume $m > l$. Then we have $y_{v-1} - y_v \leq 1$ by a similar argument. So we have $-1 \leq y_{v-1} - y_v \leq 1$. This implies that $t_{v-1}$ and $t_v$ are a covering pair. By a similar argument, $b_{v-1}$ and $b_v$ are a covering pair.

$\overline{(0,m)} = \overline{(n+1, m-2k+n+1)}$ implies that we have $|y_n - y_0 + 2k - n - 1| = |x_n - x_0 + 2k - n - 1| = 1$. This implies that $t_0$ and $t_n$ (or $b_0$ and $b_n$) are a covering pair.

Conversely let $Q$ be a subset satisfying the condition. We claim that $Q$ is obtained from $[\overline{(n+1-k, x_0+n+1-k)}, \overline{(k, y_0-k)}]$ by deleting its maximal or minimal elements. Thus it follows that $P$ is a convex subset by Propositions 3.4, 3.5.

For $1 \le v \le n$ if we have $t_{v-1} \leftarrow t_v$ and $t_v \rightarrow t_{v+1}$ then we construct another poset $Q' = Q_v \cup \{\overline{(v, y_v - 2)}\}$. We note that $Q'$ satisfies the condition. We regard $Q$ as $Q' \setminus \{\overline{(c, y_v - 2)}\}$. We also note that $Q_0 = Q'_0$. By repeating this operation, we obtain a poset $Q'$ such that $t_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_k$ and $t_k \leftarrow t_{k+1} \leftarrow \cdots \leftarrow t_n \leftarrow t_{n+1} = t_0$. For $1 \le v \le n$ if we have $b_{v-1} \rightarrow b_v$ and $b_v \leftarrow b_{v+1}$ then we obtain another poset $Q' \cup \{\overline{(v, x_v + 2)}\}$. By repeating this operation, we obtain a poset $Q'$ such that $b_0 \leftarrow b_1 \leftarrow \cdots \leftarrow b_{n+1-k}$ and $b_{n+1-k} \rightarrow \cdots \rightarrow b_n \rightarrow b_{n+1} = b_0$. By Prop. 8.8, we have $Q' = [\overline{(n+1-k, x_0+n+1-k)}, \overline{(k, y_0-k)}]$. In particular $Q'$ is a convex subset. Thus $Q$ is a convex subset. $\square$

# 9 Classification of Minuscule Heaps over $\Gamma(\tilde{A})$

In this section, we classify the minuscule heaps over $\Gamma = \Gamma(\tilde{A}_n)$ (Theorem 9.8). In previous sections, we classified some minuscule heaps. The remaining case is that of minuscule heap $P$ over $\Gamma$ such that $n \ge 2$, supp $P = \Gamma$, and its $v$-intervals are of type (V3).

Let $P$ be a minuscule heap over $\Gamma$ which is not totally ordered. First we observe the structure of $P_i \cup P_{i+1}$, where $P_i = \{p \in P | \phi(p) = i\}$. By (H1-2), $P_i \cup P_{i+1}$ is a totally ordered set. We denote the number of element of $P_i$ (resp. $P_{i+1}$) by $r_i$ (resp. $r_{i+1}$). Therefore there are $(r_i - 1)$ $i$-intervals in $P_i \cup P_{i+1}$. We note that each $i$-interval contains one element of $P_{i+1}$. Thus we have $r_{i+1} \ge r_i - 1$. By a similar argument, we have $r_i \ge r_{i+1} - 1$. Hence we have $r_i + 1 \ge r_{i+1} \ge r_i - 1$.

Let us assume $r_{i+1} = r_i - 1$. Now there are $r_i - 1 (= r_{i+1})$ $i$-intervals. Each $i$-interval contains one element of $P_{i+1}$. Since an $i$-interval is a diamond, if we take an element of $P_i$ and an element of $P_{i+1}$ which is contained in the interval then they are a covering pair. Let $p_1, p_2, \ldots, p_{r_i}$ be the elements of $P_i$ in the increasing order and let $q_1, q_2, \ldots, q_{r_{i+1}}$ be the elements of $P_{i+1}$ in the increasing order. By the argument above, we have

$$p_1 \rightarrow q_1 \rightarrow p_2 \rightarrow q_2 \rightarrow \cdots \rightarrow q_{r_{i+1}} \rightarrow p_{r_i}.$$

By a similar argument if we assume $r_{i+1} = r_i + 1$ then we have

$$q_1 \rightarrow p_1 \rightarrow q_2 \rightarrow p_2 \rightarrow \cdots \rightarrow p_{r_i} \rightarrow q_{r_{i+1}}.$$

Next let us assume $r_i = r_{i+1}$. By a similar argument, we have either

$$q_1 \rightarrow p_1 \rightarrow q_2 \rightarrow p_2 \rightarrow \cdots \rightarrow q_{r_{i+1}} < p_{r_i}$$

or

$$p_1 < q_1 \rightarrow p_2 \rightarrow q_2 \rightarrow \cdots \rightarrow q_{r_i} \rightarrow p_{r_i} \rightarrow q_{r_{i+1}}.$$

In the former case, $[p_{r_i-1}, p_{r_i}]$ is an $i$-interval. Thus we have $q_{r_{i+1}} \to p_{r_i}$. In the latter case, we have $p_1 \to q_1$. At this point, we have completed to determine the structure of $P_i \cup P_{i+1}$. We record these results in the following corollaries 9.1, 9.2,

**Corollary 9.1.** *Let $P$ be a minuscule heap with* $\operatorname{supp} P = \Gamma$ *such that $P$ is not a totally ordered set. Then $t_i, t_{i+1}$ ($b_i, b_{i+1}$) are a covering pair for each $i \in N(\Gamma)$.*

**Corollary 9.2.** *For $v \in N(\Gamma)$, put $r_v := \#P_v$. Then we have,*

- *"$r_{i+1} = r_i$" is equivalent to either "$t_i \to t_{i+1}$ and $b_i \to b_{i+1}$" or "$t_i \leftarrow t_{i+1}$ and $b_i \leftarrow b_{i+1}$",*
- *$r_{i+1} =$ "$r_i + 1$" is equivalent to "$t_i \to t_{i+1}$ and $b_i \leftarrow b_{i+1}$",*
- *$r_{i+1} =$ "$r_i - 1$" is equivalent to "$t_i \leftarrow t_{i+1}$ and $b_i \to b_{i+1}$".*

We call the number of covering pairs satisfying $t_v \to t_{v+1}$ $(1 \le v \le n+1)$ the *gradient* of $P$.

**Proposition 9.3.** *Let $P$ be a minuscule heap over $\Gamma$ which is not totally ordered and* $\operatorname{supp} = \Gamma$. *Let $k$ be a gradient of $k$. Then we have $2 \le k \le n-1$.*

*Proof.* If $k = 1$ then there is a unique covering pair $t_{i-1}, t_i$ such that $t_{i-1} \to t_i$. On the other hand, we have $t_{i-1} \to t_{i-2} \to \ldots t_{i+1} \to t_i$. It contradicts that $t_{i-1}$, $t_i$ are a covering pair.

We can prove, for the case $k = n$, by using a similar argument. $\square$

Let $P$ be a not totally ordered minuscule heap with $\operatorname{supp} P = \Gamma$ and let $k$ be the gradient of $P$. Define a map $\nu : P \to L_k$ by putting,

$$\nu(t_0) := \overline{(0,0)},$$

$$\nu(t_i) := \begin{cases} \nu(t_{i-1}) + \overline{(1,-1)} & \text{if } t_{i-1} \to t_i, \\ \nu(t_{i-1}) + \overline{(1,1)} & \text{if } t_{i-1} \leftarrow t_i, \end{cases}$$

and if $p$ is $j$-th largest element in $P_v$ then

$$\nu(p) := \nu(t_v) + \overline{(0, 2j-2)}.$$

We note that $\nu$ is an injection.

**Proposition 9.4.** *Let $t_n$ (resp. $t_0$) be the unique maximal element of $P_n$ (resp. $P_0$). If $t_n \to t_0$ then we have $\nu(t_n) \to \nu(t_0)$. If $t_n \leftarrow t_0$ then we have $\nu(t_n) \leftarrow \nu(t_0)$.*

*Proof.* Assume $t_n \to t_0$. Now we have $\#\{1 \le i \le n | t_{i-1} \to t_i\} = k-1$. Thus,

$$\begin{aligned} \nu(t_n) &= \nu(t_0) + (k-1)\overline{(1,-1)} + (n-k+1)\overline{(1,1)} \\ &= \nu(t_0) + \overline{(n, n-2k+2)} \\ &= \nu(t_0) + \overline{(-1,1)} \\ &\to \nu(t_0). \end{aligned}$$

Assume $t_n \leftarrow t_0$. Now we have $\#\{1 \leq i \leq n | t_{i-1} \rightarrow t_i\} = k$. Thus we have

$$\begin{aligned} \nu(t_n) &= \nu(t_0) + k\overline{(1,-1)} + (n-k)\overline{(1,1)} \\ &= \nu(t_0) + \overline{(n, n-2k)} \\ &= \nu(t_0) + \overline{(-1,-1)} \\ &\leftarrow \nu(t_0). \end{aligned}$$

□

We note that $\nu$ preserves the cover-relations on $t_0, t_1, \ldots, t_n$.

**Proposition 9.5.** *Let $P$ be a minuscule heap over $\Gamma$ which is not totally ordered and let $k$ be the gradient of $P$. Let $\nu$ be a map defined as above. Then* Im $\nu$ *is a convex subset in $L_k$.*

*Proof.* Put $Q := \text{Im } \nu$. For $0 \leq v \leq n$, $Q_v$ satisfies the condition in Prop. 8.9 by the definition of $\nu$. Let $t'_v$ be a maximal element of $Q_v$.

Then $t'_v$ and $t'_{v+1}$ are a covering pair since $\nu$ preserves the cover-relations on $t_0, t_1, \ldots, t_n$. Finally we show that $b'_v$ and $b'_{v+1}$ are a covering pair, where $b'_v$ is a minimal element of $Q_v$. Assume $t_v \rightarrow t_{v+1}$. By Corollary 9.2, $b_v \rightarrow b_{v+1}$ implies $r_v = r_{v+1}$. Thus we have $b'_{v+1} = \nu(b_{v+1}) = \nu(t_{v+1}) + \overline{(0, 2(r_{v+1}-1))} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2(r_v - 1))} = \nu(b_v) + \overline{(1,-1)} \leftarrow \nu(b_v) = b'_v$. $b_v \leftarrow b_{v+1}$ implies $r_{v+1} = r_v + 1$. Now we have $b'_{v+1} = \nu(b_{v+1}) = \nu(t_v) + \overline{(0, 2(r_{v+1}-1))} = \nu(t_{v+1}) + \overline{(0, 2r_v)} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2r_v)} = \nu(t_v) + \overline{(1,1)} + \overline{(0, 2(r_v-1))} = \nu(b_v) + \overline{(1,1)} \rightarrow \nu(b_v) = b'_v$.

We can prove in the case $t_v \leftarrow t_{v+1}$ by a similar argument. □

**Proposition 9.6.** *Let $P$ be a minuscule heap over $\Gamma$ which is not totally ordered and let $k$ be the gradient of $P$. Then $P$ is isomorphic to* Im $\nu$ *as a $\Gamma$-labeled poset.*

*Proof.* Let $p, q$ be elements of $P$ which satisfy $p \rightarrow q$. Then we show that $\nu(p) < \nu(q)$. Assume $p \in P_v$ and $q \in P_{v+1}$. (We can similarly argue in the case $p \in P_{v+1}, q \in P_v$.) And we assume $t_v \rightarrow t_{v+1}$. If $p$ is the $r$-th element in $P_v$ by an increasing order then $q$ is the $r$-th element in $P_{v+1}$ by an argument in the preceding Corollary 9.1. Thus we have $\nu(q) = \nu(t_{v+1}) + \overline{(0, 2(r-1))} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2(r-1))} = \nu(p) + \overline{(1,-1)} + \overline{(0, 2(r-1))} = \nu(p) + \overline{(1,-1)} \leftarrow \nu(p)$. Hence we have $\nu(q) \geq \nu(p)$. Assume $t_v \leftarrow t_{v+1}$. If $p$ is the $r$-th element in $P_v$ then $q$ is the $(r-1)$-th element in $P_{v+1}$. Thus we have $\nu(q) = \nu(t_{v+1}) + \overline{(0, 2(r-2))} = \nu(t_v) + \overline{(1,1)} + \overline{(0, 2(r-2))} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2(r-1))} = \nu(p) + \overline{(1,-1)} \leftarrow \nu(p)$. So we have $\nu(q) \geq \nu(p)$.

The arguments above show that $p \leq q$ implies $\nu(p) \leq \nu(q)$.

Conversely we assume $\nu(p) \rightarrow \nu(q)$. Since Im $\nu$ is a convex subset, we have either $\nu(q) = \nu(p) + \overline{(1,-1)}$ or $\nu(q) = \nu(p) + \overline{(-1,-1)}$. Assume that $p \in P_v, q \in P_{v+1}$. (We can prove in the case $p \in P_{v+1}, q \in P_v$ by a similar argument.) If $t_v \rightarrow t_{v+1}$ then we have $\nu(q) = \nu(p) + \overline{(1,-1)} = \nu(t_v) + \overline{(0, 2(r-1))} + \overline{(1,-1)} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2(r-1))} = \nu(t_{v+1}) + \overline{(0, 2(r-1))}$. Thus $q$ is the $r$-th element in $P_{v+1}$. By an argument in the preceding Corollary 9.1, we have $p \rightarrow q$. If $t_v \leftarrow t_{v+1}$ then $q$ is the $(r-1)$-th element in $P_{v+1}$ by a similar argument. Thus we have $p \to q$. Now Im $\nu$ is a convex subset. Hence if $\nu(p) \leq \nu(q)$ then we can take a saturated chain from $\nu(p)$ to $\nu(q)$. Thus we have $p \leq q$. □

By Propositions 9.5, 9.6, it follows that

**Corollary 9.7.** *Let $P$ be a minuscule heap with gradient $k$ ($2 \leq k \leq n-1$) which is not totally ordered. Then $P$ is isomorphic to a convex subset of $L_k$.*

Our main theorem gives the converse claim (Theorem 9.8).

In fact, a totally ordered minuscule heap is also isomorphic to a convex subset of $L_k$ for $k = 1, n$. Let $Q$ be a totally ordered minuscule heap of type (T1) (resp. (T2)). Define a map $\nu : Q \to L_n$ by the following. If the label of a maximal element of $Q$ is $i$ then $\nu(q) = \overline{(i-j+1, i-j+1)}$ (resp. $\overline{(i+j-1, i-j+1)}$ where $q$ is the $j$-th element in $Q$ by an increasing ordering. It is obvious that $Q$ is isomorphic to Im $\nu$ as a $\Gamma$-labeled poset. Now Im $Q$ is a convex subset of $L_n$ (resp. $L_1$).

We note that a minuscule heap with supp $P \neq \Gamma$ can be identified with a minuscule heap over a Dynkin diagram of type $A$. Then it is isomorphic to a convex subset of $L_k$ for some $1 \leq k \leq n$.

**Theorem 9.8.** *Let $n \geq 2$. Let $\Gamma$ be a Dynkin diagram of type $\tilde{A}_n$. A minuscule heap $P$ over $\Gamma$ is isomorphic to a convex subset of $L_k$, where $1 \leq k \leq n$ is determined by $P$. Conversely a finite convex subset of $L_k$ is a minuscule heap.*

**Remark 9.9.** If a minuscule heap $P$ satisfies supp $P \neq \Gamma$ then $k$ in Theorem 9.8 is not unique.

**Acknowledgments.** The author would like to thank Professors I. Terada and K. Koike for helpful discussions.

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