q-Exponential Families

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Abstract

We develop an analog of the exponential families of Wilf in which the label sets are finite dimensional vector spaces over a finite field rather than finite sets of positive integers. The essential features of exponential families are preserved, including the exponential formula relating the deck enumerator and the hand enumerator.

1 Introduction and Definitions

In this paper we analogize Wilf's approach to labelled counting in [9] based on exponential families. Notation and definitions follow his as closely as possible. This work is further elaboration of the subset-subspace analogy that has long been a rich source in enumerative combinatorics. See Kung [4] for an historical survey.

Let $\mathbf{F}_q$ the finite field of order q and ${\mathbf{F}_q}^{(\mathbf{N})}$ the vector space of countable dimension over $\mathbf{F}_q$ whose elements are infinite sequences $(a_1, a_2, \ldots)$ with a finite number of non-zero entries. Let $e_1, e_2, \ldots$ be the standard basis and define E_n to be the span of $e_1, \ldots, e_n$. Let P be an abstract set of 'pictures.'

Definition 1 *A* ***q-card*** *$Q(V, p)$ is a pair consisting of a subspace $V \subset {\mathbf{F}_q}^{(\mathbf{N})}$ and a picture $p \in P$. V is the* ***label space*** *space and the* ***dimension*** *of Q is $\dim V$. A card is* ***standard*** *if its label space is E_n.*

Definition 2 *A* ***q-hand*** *is a finite set of q-cards whose label spaces form a direct sum decomposition of E_n for some n. The* ***dimension*** *of the hand is n.*

Definition 3 *A q-card $Q(V', p)$ is a* ***relabeling*** *of $Q(V, p)$ if $\dim V' = \dim V$. (Thus, the pictures must be the same.) The* ***standard relabeling*** *of $Q(V, p)$ is $Q(E_{\dim V}, p)$.*

Definition 4 *A **q-deck** $\mathcal{D}$ is a finite set of standard q-cards whose dimensions are the same and whose pictures are different. The dimension of the deck* $\dim \mathcal{D}$ *is the common dimension of the cards.*

Generally, we will omit the q-prefix and refer to "cards, hands, and decks" when the context makes it clear.

Definition 5 *A **q-exponential family** $\mathcal{F}$ is a collection of q-decks $\mathcal{D}_1, \mathcal{D}_2, \ldots$ where $\mathcal{D}_n$ is a q-deck of dimension n, possibly empty.*

Definition 6 *The **merger** of two q-exponential families $\mathcal{F}'$ and $\mathcal{F}''$ is the q-exponential family $\mathcal{F}' \oplus \mathcal{F}''$ whose q-deck of dimension n is the disjoint union of the q-decks $\mathcal{D}'_n$ and $\mathcal{D}''_n$.*

For notational simplicity we define $\gamma_n = |\mathrm{GL}_n(\mathbf{F}_q)|$ to be the order of the general linear group, which is given by

$$\gamma_n = \prod_{0 \le i \le n-1} (q^n - q^i).$$

Define the *hand enumerator*

$$\mathcal{H}(x, y) = \sum_{n,k} h(n, k) \frac{x^n}{\gamma_n} y^k$$

and the *deck enumerator*

$$\mathcal{D}(x) = \sum_n \frac{d_n}{\gamma_n} x^n,$$

where $h(n, k)$ is the number of hands of dimension n with k cards and d_n is the number of cards in $\mathcal{D}_n$. We define $h(0, 0)$ to be 1 and will see later why that is necessary. However, $h(n, 0) = h(0, k) = 0$ for $n > 0$ and $k > 0$.

2 Counting

The one variable hand enumerator

$$\mathcal{H}(x) = \mathcal{H}(x, 1) = \sum_n h(n) \frac{x^n}{\gamma_n}$$

counts the hands without regard to the number of cards. Here, $h(n) = \sum_k h(n, k)$ is the number of hands of dimension n. The fundamental lemma of q-labeled counting is key in deriving the analog of the exponential formula.

Lemma 7 *Let $\mathcal{F}'$ and $\mathcal{F}''$ be q-exponential families and $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ the merger of the two. Then the two-variable hand enumerators are related by*

$$\mathcal{H}(x, y) = \mathcal{H}'(x, y)\mathcal{H}''(x, y).$$

Proof A hand of dimension n with k cards has label spaces that give a splitting of E_n. Those cards that come from decks in $\mathcal{F}'$, say k' of them have label spaces that form a subspace V' of dimension n', and those that come from decks of $\mathcal{F}''$, the remaining $k-k''$ cards, have label spaces that form a complementary subspace V'' of dimension $n-n'$. Thus, we select a hand from $\mathcal{F}'$ and a hand from $\mathcal{F}''$ and relabel the cards by selecting a splitting $E_n = V' \oplus V''$. The number of ways to select V' and V'' of dimensions n' and $n-n'$ is $\frac{\gamma_n}{\gamma_{n'}\gamma_{n-n'}}$, because $\mathrm{GL}_n(\mathbf{F}_q)$ acts transitively on the set of pairs (V', V'') such that $E_n = V' \oplus V''$ and $\dim V' = n'$, $\dim V'' = n - n''$. The stabilizer subgroup is isomorphic to $\mathrm{GL}_{n'}(\mathbf{F}_q) \times \mathrm{GL}_{n-n'}(\mathbf{F}_q)$. Therefore

$$h(n,k) = \sum_{n',k'} \frac{\gamma_n}{\gamma_{n'}\gamma_{n-n'}} h'(n',k')h''(n-n',k-k').$$

Here we need $h(0,0) = h'(0,0) = h''(0,0) = 1$ to allow for the possibility of making up a hand in the merger by choosing no cards from one of the families. We might also point out that $\gamma_0 = 1$ because the zero map on the vector space of dimension zero is invertible.

In the product

$$\mathcal{H}'(x,y)\mathcal{H}''(x,y) = \sum_{n',k'} h'(n',k')\frac{x^{n'}}{\gamma_{n'}}y^{k'} \sum_{n'',k''} h''(n'',k'')\frac{x^{n''}}{\gamma_{n''}}y^{k''}$$

the coefficient of $x^n y^k$ is

$$\sum_{n'+n''=n,k'+k''=k} h'(n',k')h''(n'',k'')\frac{1}{\gamma_{n'}\gamma_{n''}}.$$

Substituting $n'' = n - n'$ and $k'' = k - k'$ shows that this is exactly $h(n,k)/\gamma_n$. $\square$

We continue on to prove the exponential formula using the three steps that Wilf calls *the trickle, the flow, and the flood.*

For the first step, we consider a family consisting of one deck $\mathcal{D}_r$ with one card. Thus, $d_r = 1$ and all other $d_i = 0$. A hand with k cards has dimension $n = kr$. The label spaces for a hand form a splitting of E_n into k subspaces each of dimension r. Thus the number of hands of dimension n is the number of such splittings $E_n = V_1 \oplus \cdots \oplus V_k$. Changing the order of the factors does not change the splitting, so we can count the number of ordered splittings and divide by $k!$. The group $\mathrm{GL}_n(\mathbf{F}_q)$ acts transitively on the set of ordered splittings with stabilizer subgroup isomorphic to $\mathrm{GL}_r(\mathbf{F}_q)^k$, and so the number of n-dimensional hands (where $n = kr$) is

$$h(kr,k) = \frac{1}{k!}\frac{\gamma_{kr}}{\gamma_r^k}.$$

The hand enumerator is

$$\mathcal{H}(x,y) \quad = \quad \sum_k \frac{1}{k!}\frac{\gamma_{kr}}{\gamma_r^k}\frac{x^{kr}}{\gamma_{kr}}y^k$$

$$
\begin{aligned}
&= \sum_k \frac{1}{k!}\left(\frac{yx^r}{\gamma_r}\right)^k \\
&= \exp\left(\frac{yx^r}{\gamma_r}\right) \\
&= \exp(y\mathcal{D}(x)).
\end{aligned}
$$

The second step handles the case of a single non-empty deck $\mathcal{D}_r$ with d_r cards. If we denote the family in step one by $\mathcal{F}_1$, then we are now considering the repeated merger $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_1$, in which there are d_r factors. Therefore, the hand enumerator is the d_r-th power of the hand enumerator in step one, namely

$$
\mathcal{H}(x,y) = \left(\exp\left(\frac{yx^r}{\gamma_r}\right)\right)^{d_r} = \exp\left(\frac{yd_rx^r}{\gamma_r}\right).
$$

The deck enumerator is $\mathcal{D}(x) = \frac{d_rx^r}{\gamma_r}$, and so we have $\mathcal{H}(x,y) = \exp(y\mathcal{D}(x))$.

In the third step we look at a full q-exponential family as a merger of $\mathcal{F}_r$, $r = 1, 2, \ldots$, where $\mathcal{F}_r$ is the family consisting of just one non-empty deck $\mathcal{D}_r$. Then the hand enumerator of the infinite merger is the infinite product of the hand enumerators

$$
\begin{aligned}
\mathcal{H}(x,y) &= \prod_r \exp\left(\frac{yd_rx^r}{\gamma_r}\right) \\
&= \exp\left(\sum_r \frac{yd_rx^r}{\gamma_r}\right).
\end{aligned}
$$

The sum in the last line is the deck enumerator of the family, and so we have proved

Theorem 8 (Analog of the exponential formula) *Let $\mathcal{F}$ be a q-exponential family with hand enumerator $\mathcal{H}(x,y)$ and deck enumerator $\mathcal{D}(x)$. Then*

$$
\mathcal{H}(x,y) = \exp(y\mathcal{D}(x)).
$$

In [9, §3.13] Wilf defines the polynomials $\phi_n(y) = \sum_k h(n,k)y^k$. The exponential formula can be written

$$
e^{y\mathcal{D}(x)} = \sum_{n\geq 0} \frac{\phi_n(y)}{n!}x^n,
$$

from which it follows that it follows that these polynomials satisfy

$$
\phi_n(u+v) = \sum_m \binom{n}{m}\phi_m(u)\phi_{n-m}(v)
$$

so that they are *polynomials of binomial type* as defined by Mullin and Rota [6].

For q-exponential families there is an analog: for the polynomials $\phi_n(y)$ defined as above, the q-exponential formula just proved can be written

$$
e^{y\mathcal{D}(x)} = \sum_{n\geq 0} \frac{\phi_n(y)}{\gamma_n}x^n.
$$

It follows that

$$
\phi_n(u+v) = \sum_m \frac{\gamma_n}{\gamma_m\gamma_{n-m}}\phi_m(u)\phi_{n-m}(v).
$$

3 q-Analog of the Stirling subset numbers

The Stirling subset number $\left\{ {n \atop k} \right\}$, or Stirling number of the second kind, counts the number of partitions of a set of size n into k non-empty subsets. The direct sum of subspaces is analogous to the disjoint union of subsets, and so we define $\left\{ {n \atop k} \right\}_q$ to be the number of splittings of E_n into k non-zero subspaces. We regard a splitting as a hand of dimension n with k cards in the q-exponential family $\mathcal{F}$ with decks $\mathcal{D}_r$ consisting of just one card each. (The picture set is irrelevant. We can assume there is just one picture and that it is the same on each card.) Then $h(n,k) = \left\{ {n \atop k} \right\}_q$ and

$$\begin{aligned} \mathcal{H}(x,y) &= \sum_{n,k} \left\{ {n \atop k} \right\}_q \frac{x^n}{\gamma_n} y^k \\ \mathcal{D}(x) &= \sum_r \frac{x^r}{\gamma_r}. \end{aligned}$$

From the exponential formula

$$\begin{aligned} \left\{ {n \atop k} \right\}_q &= \gamma_n [x^n y^k] \mathcal{H}(x,y) \\ &= \gamma_n [x^n y^k] \sum_j \frac{y^j \mathcal{D}(x)^j}{j!} \\ &= \gamma_n [x^n] \frac{\mathcal{D}(x)^k}{k!} \\ &= \gamma_n [x^n] \frac{1}{k!} \left(\sum_r \frac{x^r}{\gamma_r} \right)^k \\ &= \gamma_n \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i \geq 1}} \frac{1}{\gamma_{n_1} \cdots \gamma_{n_k}}. \end{aligned}$$

Thus we see that

$$\left\{ {n \atop k} \right\}_q = \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i \geq 1}} \frac{\gamma_n}{\gamma_{n_1} \cdots \gamma_{n_k}}.$$

Compare this with the exponential formula for the Stirling subset numbers

$$\sum_{n,k} \left\{ {n \atop k} \right\} \frac{x^n}{n!} y^k = \exp(y(e^x - 1))$$

to see that we have a perfect analog of the formula

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i \geq 1}} \frac{n!}{n_1! \cdots n_k!}.$$

The Bell number $b(n) = \sum_k \left\{ {n \atop k} \right\}$ counts the number of set partitions of a set of size n and therefore has a q-analog $b_q(n) = \sum_k \left\{ {n \atop k} \right\}_q$, which counts the number of direct sum decompositions of an n-dimensional vector space over $\mathbf{F}_q$. These are the coefficients in the one-variable hand enumerator $\mathcal{H}(x) = \mathcal{H}(x,1) = \sum_n b_q(n) \frac{x^n}{\gamma_n}$. By setting $y = 1$ in the q-exponential formula we get

$$\sum_n b_q(n) \frac{x^n}{\gamma_n} = \exp\left(\sum_{r \geq 1} \frac{x^r}{\gamma_r} \right).$$

This formula for the exponential generating function of the q-Bell numbers first appears as Example 11 in [1]. It can also be derived from the exponential formula in [7]. (Look at Example 2.2 and then use Corollary 3.3 with $f(n) = 1$ for $n > 0$. Notice that $n!M(n) = \gamma_n/(q-1)^n$ and substitute $x/(q-1)$ for x.)

4 Diagonalizations

Consider the q-exponential family with just one non-empty deck $\mathcal{D}_1$ with q cards. A card in $\mathcal{D}_1$ has the label E_1 and a picture α, which is an element of $\mathbf{F}_q$. Then a hand of dimension n must consist of n cards whose label spaces form a splitting of $E_n = V_1 \oplus \cdots \oplus V_n$ into one-dimensional subspaces and whose pictures $\alpha_1, \ldots, \alpha_n$ represent a diagonal matrix with respect to that splitting. Thus, each hand of dimension n represents a diagonalization of a diagonalizable $n \times n$ matrix.

The deck enumerator is

$$\mathcal{D}(x) = \frac{q}{\gamma_1} x = \frac{q}{q-1} x,$$

and the one variable hand enumerator is

$$\mathcal{H}(x) = \exp\left(\frac{q}{q-1} x \right).$$

Since every hand of dimension n has n cards, there is no more information in the two variable hand enumerator. Then, $h(n)$, the number of diagonalizations of $n \times n$ matrices, is

$$h(n) = \frac{\gamma_n}{n!} \left(\frac{q}{q-1} \right)^n.$$

A variation of this family changes the picture set slightly so that only non-zero α in $\mathbf{F}_q$ are used. In this case a hand corresponds to a diagonalization of an invertible matrix. Since $d_1 = q - 1$, the deck and hand enumerators become $\mathcal{D}(x) = x$ and $\mathcal{H}(x) = e^x$. Thus, the number of diagonalizations of invertible $n \times n$ matrices is

$$h(n) = \frac{\gamma_n}{n!}.$$

A direct proof without generating functions can be given for this result, but we leave that as an exercise for the reader.

In order to count diagonalizable matrices rather than diagonalizations we have to go beyond exponential families. The problem can be seen in the simplest of cases. The 2×2 identity matrix has as many diagonalizations as there are splittings $E_2 = V_1 \oplus V_2$ into two one-dimensional subspaces. However, a 2×2 matrix with distinct eigenvalues has only one diagonalization. We take this up in the next section.

5 Beyond exponential families

5.1 Diagonalizable matrices

An $n \times n$ diagonalizable matrix gives a unique decomposition of E_n into a direct sum of eigenspaces, one for each eigenvalue. However, in general it does not give a unique decomposition of E_n into a sum of one-dimensional spaces. This fact forces us to go beyond q-exponential families. We would like a hand of dimension n to correspond to the eigenspace decomposition of a diagonalizable $n \times n$ matrix. Thus, a card needs a label space $V \subset E_n$ and an eigenvalue 'picture' $\alpha \in \mathbf{F}_q$. However, a picture cannot be used more than once and a hand cannot contain more than q cards. In spite of this difficulty, we can see the family of diagonalizable matrices as the merger of q families, one for each possible eigenvalue. The fundamental counting lemma still holds for the merger of families and this will allow us to count.

For each α in $\mathbf{F}_q$ we define the family whose hands consist of single cards with label space E_n and picture α, $n \geq 0$. We define the hand enumerator $\mathcal{H}_\alpha(x, y)$ to be the generating function

$$\mathcal{H}_\alpha(x, y) = 1 + y \sum_{n \geq 1} \frac{x^n}{\gamma_n}.$$

Notice that y appears only with exponent 1 because every hand has only one card. Now we can form the product of these hand enumerators as α ranges over $\mathbf{F}_q$, and by the fundamental of q-labeled counting, that product will be the hand enumerator for the family of diagonalizable matrices:

$$\mathcal{H}_{\text{diag}}(x, y) = \left(1 + y \sum_{n \geq 1} \frac{x^n}{\gamma_n}\right)^q.$$

Here the coefficient of $x^n y^k$, when multiplied by γ_n is the number of $n \times n$ diagonalizable matrices with k distinct eigenvalues. Setting $y = 1$ gives the one variable enumerator $\mathcal{H}_{\text{diag}}(x) = \sum_{n \geq 0} \frac{h_{\text{diag}}(n)}{\gamma_n} x^n$. Hence,

$$\begin{aligned} h_{\text{diag}}(n) &= \gamma_n [x^n] \left(\sum_{m \geq 0} \frac{x^m}{\gamma_m}\right)^q \\ &= \sum_{n_1 + \cdots + n_q = n} \frac{\gamma_n}{\gamma_{n_1} \cdots \gamma_{n_q}}. \end{aligned}$$

5.2 Projections

A projection is a matrix P such that $P^2 = P$. A projection is completely determined by the pair of complementary subspaces, the kernel of P and the image of P. Thus, the number of $n \times n$ projections can be obtained from the q-Stirling number $\left\{ {n \atop 2} \right\}_q$, which counts the number of decompositions into two non-trivial subspaces. First we have to multiply that by two because the order of the two subspaces now matters and then we have to add two to count the trivial decompositions corresponding to the identity matrix and to the zero matrix. Thus, the number of $n \times n$ projections is $2\left\{ {n \atop 2} \right\}_q + 2$. Recalling the formula from section 3 for the q-Stirling numbers we see that the number of projections is

$$\begin{aligned} 2\left\{ {n \atop 2} \right\}_q + 2 &= \sum_{\substack{n_1+n_2=n \\ n_i \geq 1}} \frac{\gamma_n}{\gamma_{n_1}\gamma_{n_2}} + 2 \\ &= \sum_{\substack{n_1+n_2=n \\ n_i \geq 0}} \frac{\gamma_n}{\gamma_{n_1}\gamma_{n_2}}. \end{aligned}$$

The 2 on the right is incorporated into the sum by allowing n_1 and n_2 to be 0.

An alternative approach is to use a variation of the family of diagonalizable matrices. A projection is a diagonalizable matrix whose only eigenvalues are 0 or 1. The family of projections is the merger of the two families with $\alpha = 0$ and $\alpha = 1$. Then the one variable enumerator for the family of projections is

$$\mathcal{H}_{\mathrm{pr}}(x) = \left(\sum_{m \geq 0} \frac{x^m}{\gamma_m} \right)^2,$$

and the number of projections within the $n \times n$ matrices is

$$h_{\mathrm{pr}}(n) = \sum_{j=0}^{n} \frac{\gamma_n}{\gamma_j \gamma_{n-j}}.$$

5.3 A q-analog of the Stirling cycle numbers

The Stirling cycle number $\left[{n \atop k} \right]$, or Stirling number of the first kind, is the number of permutations on n letters having k cycles. Replacing a set of size n by a vector space of dimension n over $\mathbf{F}_q$ and the permutation group S_n by the general linear group $\mathrm{GL}_n(\mathbf{F}_q)$, a reasonable q-analog of the Stirling cycle numbers defines $\left[{n \atop k} \right]_q$ to be the number of invertible n by n matrices that decompose into the direct sum of k cyclic summands. In order to make this definition precise we specify that we are counting the number of summands in the primary rational canonical form. Recall, that each endomorphism of a finite dimensional vector space decomposes uniquely into cyclic summands whose matrix representations are those of companion matrices of powers of irreducible polynomials. (Without that specification we have the ambiguity, for example, of a 2×2 matrix with distinct eigenvalues being itself cyclic and also decomposing as the direct sum of two 1×1 matrices. In this case the number of primary cyclic summands is two.)

We construct the appropriate family as the infinite merger of families $\mathcal{F}_\phi$, where ϕ is an irreducible monic polynomial over $\mathbf{F}_q$. A hand in $\mathcal{F}_\phi$ is a matrix whose characteristic polynomial is a power of ϕ. The conjugacy class of such a matrix is determined by the multiplicity data (b_i), where b_i is the number of copies of the companion matrix $C(\phi^i)$ occurring. The dimension of the hand is $\sum_i ib_i(\deg\phi)$, because $C(\phi^i)$ is a matrix of dimension $i(\deg\phi)$. Note that the multiplicity data is a partition of the integer $\sum_i ib_i$. The number of parts in the partition b is the sum $k=\sum_i b_i$, and this is the number of cyclic summands in the rational primary canonical form.

To describe the hand enumerator for $\mathcal{F}_\phi$ we need the number of matrices whose conjugacy class is determined by the multiplicity data b. The cardinality of this conjugacy class is $\gamma_n/c_\phi(b)$, where $n=\sum_i ib_i(\deg\phi)$ and $c_\phi(b)$ is the order of the stabilizer subgroup under the conjugation action of the rational canonical form associated to ϕ and b. Summing over all partitions b we get the hand enumerator

$$\mathcal{H}_\phi(x,y)=\sum_b \frac{x^{\sum ib_i\deg\phi}y^{\sum b_i}}{c_\phi(b)}.$$

Note that the exponent in the x variable is the dimension and the exponent in the y variable is the number of cyclic summands. The proof of the formula for $c_\phi(b)$ can be found in [3]. We give the formula here for completeness, although we make no further use of it. For the partition $b=(b_1,b_2,\ldots)$ define

$$d_i=b_1+2b_2+\cdots+(i-1)b_{i-1}+i(b_i+b_{i+1}+\cdots).$$

In the Ferrers diagram of b consisting or b_i rows with i boxes, d_i is the number of boxes in the first i columns. Then

$$c_\phi(b)=\prod_i\prod_{j=1}^{b_i}(q^{d_i\deg\phi}-q^{(d_i-j)\deg\phi}).$$

Forming the merger of the families $\mathcal{F}_\phi$ as ϕ ranges over the irreducible monic polynomials in $\mathbf{F}_q[t]$ results in the family of all square matrices over $\mathbf{F}_q$ decomposed into their rational primary components. In order to have only the invertible matrices, we simply omit the polynomial $\phi(t)=t$ in forming the merger, because a matrix is invertible if and only if its characteristic polynomial is not divisible by t. With this family we have a factorization of the generating function for the q-analog of the Stirling cycle numbers

$$\sum \begin{bmatrix} n\\ k\end{bmatrix}_q \frac{x^n}{\gamma_n}y^k=\prod_{\phi\neq t}\left(1+\sum_b\frac{x^{\sum ib_i\deg\phi}y^{\sum b_i}}{c_\phi(b)}\right)$$

Note the constant 1 in each factor to allow for ϕ not occurring in the primary decomposition of a matrix. This dictates that $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_q$ should be 1 just as it is for the ordinary Stirling numbers.

This factorization of the generating function for these q-Stirling numbers was first obtained by Kung [3, p. 148]. (To compare the formulas replace the x and y in this paper with u and x in Kung's paper, break the sum over all partitions into a sum over $j \geq 1$ and an inner sum over partitions of j, and group the factors for all irreducible polynomials of the same degree.) In that important paper he defined the cycle index for groups of automorphisms of finite dimensional vector spaces over finite fields and found its basic properties. The ordinary generating function for the cycle index of the full general linear group has a factorization [3, Lemma 1] that specializes to the factorization above for the q-Stirling cycle numbers. Extensive use of the vector space cycle index is made in later papers of Stong [8] and Fulman [2] to study the asymptotic combinatorics of the canonical form as the matrix size goes to infinity. Some of this work is also summarized in [5].

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