Graphic and Protographic Lists of Integers

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Abstract

A positive list (list of positive integers) is *protographic* if its merger with all but finitely many positive graphic lists is graphic. Define the family $\mathcal{P}_s$ of s-*protogaphic* lists by letting $\mathcal{P}_0$ be the family of positive graphic lists and letting $\mathcal{P}_s$ for $s > 0$ be the family of positive lists whose merger with all but finitely many lists in $\mathcal{P}_{s-1}$ is in $\mathcal{P}_{s-1}$.

The main result is that $X \in \mathcal{P}_s$ if and only if $t(X) \in \mathcal{P}_{s-1}$, where $t(X)$ is the list obtained from X by subtracting one from each term of X (deleting those that become 0) and appending a 1 for each term of X. A corollary is that the maximum number of iterations to reach a graphic list from an n-term even list with sum $2k$ is $k-n+1$ (when $k \geq n$), achieved by the unique such list having one term larger than 1.

1 Introduction

An integer list of length n is an n-tuple of integers. A *graphic list* is a list whose entries are the degrees of the vertices in a simple graph. Whether a list is graphic is determined by the multiset of entries; the order of the entries is irrelevant. Because entries equal to 0

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do not affect whether a list is graphic, we consider only lists of positive integers. (We use "list" rather than "sequence" since a sequence is a function whose domain is infinite.)

Many characterizations of graphic lists are known: Sierksma and Hoogeveen [6] state seven. A well-known explicit characterization due to Erdős and Gallai [2] is that, when the entries $d_1, \ldots, d_n$ are written in nonincreasing order, the inequalities $\sum_{i=1}^{k} d_i \le k(k+1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ hold for every k (see Aigner and Triesch [1] for an elegant proof).

This note introduces a measure of how far a list is from being graphic. Let a positive list X be *protographic* if there are only finitely many positive graphic lists Y such that the merger $X \cup Y$ is not graphic, where the *merger* of two lists is obtained by summing the multiplicities of their elements.

More generally, define a sequence of families of lists recursively as follows. Let $\mathcal{P}_0$ be the set of positive graphic lists. For $s > 0$, let $\mathcal{P}_s$ be the set of positive lists X such that $X \cup Y \in \mathcal{P}_{s-1}$ for all but finitely many $Y \in \mathcal{P}_{s-1}$. The lists in $\mathcal{P}_s$ are the *s-protographic* lists. Thus the positive graphic lists are the 0-protographic lists, and the protographic lists are the 1-protographic lists.

As might be expected, every s-protographic list is also $(s+1)$-protographic. This follows from the fact that $X \cup Y \in \mathcal{P}_s$ when $X \in \mathcal{P}_s$ and $Y \in \mathcal{P}_s$. To see this latter fact, write $(X \cup Y) \cup Z$ as $X \cup (Y \cup Z)$ for $Z \in \mathcal{P}_{s-1}$. We have $Y \cup Z \in \mathcal{P}_{s-1}$ for all but finitely many such Z. Excluding the finitely many Z such that $Y \cup Z \notin \mathcal{P}_{s-1}$ and the finitely many Z such that $Y \cup Z \in \mathcal{P}_{s-1}$ but $X \cup (Y \cup Z) \notin \mathcal{P}_{s-1}$, we have that $(X \cup Y) \cup Z \in \mathcal{P}_{s-1}$ for all but finitely many $Z \in \mathcal{P}_{s-1}$.

We use $\max(X)$ for the largest entry and $\ell(X)$ for the length (number of terms) of a list X. Our main result is the characterization of s-protographic lists using a special operation on lists. Let $t(X)$ denote the list obtained from X by subtracting 1 from each element of X (discarding terms that reach 0) and then appending $\ell(X)$ entries equal to 1.

We prove that X is protographic if and only if $t(X)$ is graphic. This serves as the basis step for an induction to prove the characterization in general:

Theorem 1 *If X is a positive list of integers, and s is a positive integer, then $X \in \mathcal{P}_s$ if and only if $t(X) \in \mathcal{P}_{s-1}$.*

The definition implies inductively that only lists with even sum can be s-protographic. We define an *even list* to be a positive list with even sum. An even list with all entries equal to 1 is graphic. Since $\max(t(X)) = \max(X) - 1$ when $\max(X) > 1$, our theorem thus proves inductively that every list with even sum belongs to $\mathcal{P}_s$ for some s.

For an even list X, we define the *non-graphicality* $\gamma(X)$ to be the minimum s such that $X \in \mathcal{P}_s$. A corollary of our theorem shows that the maximum non-graphicality among even lists with sum $2k$ is k, achieved by the list consisting of a single term equal to $2k$. More generally, the maximum non-graphicality among n-term even lists with sum $2k$ is $k - n + 1$ (when $k > n - 1$), achieved by the unique such list having one term larger than 1. The non-graphicality of n-term lists with unbounded sum is unbounded.

2 The Proofs

Let $n(G)$ denote the number of vertices of a graph G. The *boundary* ∂S of a set S of vertices in a graph G is the set of vertices outside S whose neighborhoods intersect S. A *dominating set* for G is a set $S \subseteq V(G)$ such that $\partial S = V(G) - S$. Ore [5] observed that every graph G without isolated vertices has a dominating set of size at most $n(G)/2$. A simple proof is that for every minimal dominating set, the remaining vertices also form a dominating set.

Lemma 2 *If G is a simple graph without isolated vertices such that $|\partial S| < k$ for all $S \subseteq V(G)$, then $n(G) \le 2k-2$.*

Proof. Let S be a smallest dominating set of G. By Ore's observation [5], $n(G) = |S| + |\partial S| \le n(G)/2 + k - 1$. Thus $n(G) \le 2k-2$. ∎

A list X with $\max(X) \ge \ell(X)$ is not graphic. The Havel–Hakimi Theorem ([3, 4]) states that a positive list X is graphic if and only if the list obtained from X by deleting the element $\max(X)$ and subtracting 1 from $\max(X)$ of the next largest elements is graphic. Let X' denote the positive list obtained from a list X by doing this and also dropping any elements that thus become 0. We use 1^k to denote k entries equal to 1.

We will need an operation on simple graphs that also is used in inductive proofs of the Havel–Hakimi Theorem. Given vertices w, x, y, z in a simple graph G such $wx, yz \in E(G)$ and $xy, wz \notin E(G)$, the operation of deleting wx, yz and adding xy, wz to $E(G)$ is a *2-switch*; it produces another simple graph with the same vertex degrees.

Theorem 3 *When X is a positive list, $X \in \mathcal{P}_1$ if and only if $t(X)$ is graphic.*

Proof. Let $k = \ell(X)$.

Necessity. Let Y_n be the degree list of the star with n leaves. By the definition of $\mathcal{P}_1$, the list $X \cup Y_n$ is graphic for sufficiently large n, say $n > n_0$. Take n such that $n > \max\{n_0, \max(X), k\}$. By the Havel–Hakimi Theorem, $(X \cup Y_n)'$ is graphic. Since $n = \max(X \cup Y_n)$, and the next k largest elements are those of X, and $n > k$, we have $(X \cup Y_n)' = (x_1 - 1, \ldots, x_k - 1, 1^k) = t(X)$ (discarding terms such that $x_i - 1 = 0$).

Sufficiency. Assume that $t(X)$ is graphic. We claim that if Y is a positive graphic list of length at least $2k-1$, then $X \cup Y$ is graphic. Since there are finitely many graphic lists of length at most $2k-2$, this will yield $X \in \mathcal{P}_1$.

Among all graphs with degree list $t(X)$, choose one whose set of vertices of degree 1 induces the fewest edges. Let H be the graph with $2k$ vertices obtained from it by adding an isolated vertex for each 1 in X. Let $w_1, \ldots, w_k$ be k vertices of degree 1 in H that induce the fewest edges among all sets of k vertices of degree 1. The remaining vertices are $u_1, \ldots, u_k$, indexed so that $d_H(u_i) = x_i - 1$ (this includes all the added isolated vertices).

Let $W = \{w_1, \ldots, w_k\}$ and $U = \{u_1, \ldots, u_k\}$. We reduce the problem to the case where W is an independent set in H. If W induces an edge, then its endpoints have degree 1, so if there is an edge induced by U we can perform a 2-switch to reduce the number of edges within W. Hence if W induces an edge, then we may assume that U is an independent set in H. Now $\sum d_H(u_i) < k$, because the only edges incident to U are also incident to W, and fewer than k such edges are incident to W.

Thus X consists of k positive numbers summing to $k+j$, where $j<k$. In this case we show that X is graphic, by induction on k. If all entries are 1, then X is realized by a matching. Otherwise, the pigeonhole principle implies that X contains a 1. Form X' by deleting this 1 and subtracting 1 from some larger element of X. Now X' has length $k-1$ and sum $k-1+j-1$, with $j-1<k-1$. By the induction hypothesis, X' is graphic, and we add a pendant edge to a realization of it to obtain a realization of X. Since every s-protographic list is $(s+1)$-protographic, this yields $X \in P_1$.

Hence we may assume that W is an independent set in H. Now let G be a graph with degree list Y. By Lemma 2, there exists $S \subseteq V(G)$ with $|\partial S| \geq k$; give the names $w_1, \ldots, w_k$ to distinct vertices in ∂S. Let $z_1, \ldots, z_j$ (not necessarily distinct) be vertices of S such that $z_i w_i \in E(G)$ for each i.

Because W is an independent set in H, the union $G \cup H$ is a simple graph with $k+\ell(Y)$ vertices. In $G \cup H$, replace the edge $z_i w_i$ with the edge $z_i u_i$ for $1 \leq i \leq j$. This increases the degree of u_i to x_i and decreases the degree of w_i to $d_G(w_i)$. Hence the modified graph F is a simple graph with degree list $X \cup Y$. ■

Let $B_{s,n}$ denote the list of length n consisting of one entry equal to $n-1+2s$ and $n-1$ entries equal to 1. Note that $B_{0,n}$ is the degree list of a star with n vertices. By construction, it is immediate that $t(B_{s,n}) = B_{s-1,n+1}$. The proof of the main result (Theorem 1) involves a statement about $B_{s,n}$ equivalent to the other two.

The application of 2-switches in the proof of the Havel–Hakimi Theorem is a statement that we will need here: for every graphic list X, there is a simple graph G whose degree list is X in which a vertex of highest degree is adjacent only to vertices of the highest degrees among the remaining vertices. If w has maximum degree, and w is adjacent to z but not to x among the highest-degree vertices, then there exists $y \in N(x) - N(z)$ since $d(x) \geq d(z)$, and the 2-switch that replaces wz and xy with wx and zy reduces the number of missing desired neighbors of w.

Theorem 4 *For a positive list X and nonnegative integer s, the following are equivalent:*

A) $X \in \mathcal{P}_{s+1}$;

B) $X \cup B_{s,n} \in \mathcal{P}_s$ for sufficiently large n;

C) $t(X) \in \mathcal{P}_s$.

Furthermore, $B_{s+1,n} \in \mathcal{P}_{s+1}$, and there are finitely many lists in $\mathcal{P}_{s+1}$ of a given length.

Proof. We prove all claims simultaneously by induction on s. Theorem 3 states the equivalence of A and C for $s=0$. The definition of $\mathcal{P}_1$ yields A $\Rightarrow$ B for $s=0$. Note that $B_{1,n} \in \mathcal{P}_1$, because $t(B_{1,n}) = B_{0,n+1} \in \mathcal{P}_0$. For every $X \in \mathcal{P}_1$ of length k, the list $t(X)$ has length at most $2k$. The finiteness of $\{X \in \mathcal{P}_1 \colon \ell(X) = k\}$ thus follows from the finiteness of the set of graphic lists of length at most $2k$.

To complete the basis step, it remains only to show B $\Rightarrow$ C when $s=0$. Choose some n with $n > \ell(X)$ such that $X \cup B_{0,n}$ is graphic. Note that $n-1$ is the largest value in this list. Choose a graph G with degree list $X \cup B_{0,n}$ such that the vertex w of degree $n-1$ is adjacent to vertices of the next highest degrees, as in the proof of the Havel–Hakimi Theorem. Now $G - w$ has degree list $t(X)$.

For the induction step, consider $s > 0$.

A $\Rightarrow$ B. This follows from the definition of $\mathcal{P}_{s+1}$, since part of the final statement of the induction hypothesis is that $B_{s,n} \in \mathcal{P}_s$ for all n.

B $\Rightarrow$ C. For sufficiently large n, we are given $X \cup B_{s,n} \in \mathcal{P}_s$. By the induction hypothesis, $t(X \cup B_{s,n}) \in \mathcal{P}_{s-1}$. Since $t(X \cup Y) = t(X) \cup t(Y)$ for all X and Y, we have $t(X) \cup t(B_{s,n}) \in \mathcal{P}_{s-1}$. Thus $t(X) \cup B_{s-1,n+1} \in \mathcal{P}_{s-1}$. Since this holds for sufficiently large n, the induction hypothesis for B $\Rightarrow$ A yields $t(X) \in \mathcal{P}_s$.

C $\Rightarrow$ A. Suppose that $t(X) \in \mathcal{P}_s$. By the definition of $\mathcal{P}_s$, there exists n_0 such that $t(X) \cup W \in \mathcal{P}_{s-1}$ whenever $W \in \mathcal{P}_{s-1}$ and $\ell(W) > n_0$. Consider $Z = X \cup Y$ for $Y \in \mathcal{P}_s$ with $\ell(Y) > n_0$. From the final statement of the induction hypothesis, $\ell(Y) > n_0$ excludes only finitely many candidates for Y from $\mathcal{P}_s$; thus $X \in \mathcal{P}_{s+1}$ will follow from $Z \in \mathcal{P}_s$.

We have $t(Z) = t(X) \cup t(Y)$. The induction hypothesis for A $\Rightarrow$ C yields $t(Y) \in \mathcal{P}_{s-1}$. Also, $\ell(t(Y)) \geq \ell(Y) > n_0$. By the choice of n_0, $t(X) \cup t(Y) \in \mathcal{P}_{s-1}$, and hence $t(Z) \in \mathcal{P}_{s-1}$. Now the induction hypothesis for C $\Rightarrow$ A implies $Z \in \mathcal{P}_s$.

Finally, consider the last statement. We have $t(B_{s+1,n}) = B_{s,n+1}$, which by the induction hypothesis for this statement belongs to $\mathcal{P}_s$. Since we have now proved C $\Rightarrow$ A, we conclude that $B_{s+1,n} \in \mathcal{P}_{s+1}$. Also, A $\Rightarrow$ C and the induction hypothesis for the last statement implies that $\mathcal{P}_{s+1}$ has finitely many lists of a given length. ■

Recall that the non-graphicality $\gamma(X)$ of an even list X is the least s with $X \in \mathcal{P}_s$.

Corollary 5 *If X is an even list, then $\gamma(X) \leq \max\{0, \frac{\max(X)-\ell(X)+1}{2}\}$, with equality for non-graphic lists only when X has only one element larger than 1. In particular, for $k \geq n-1$ the non-graphicality among n-term lists with sum $2k$ is maximized only by the unique list having just one entry larger than 1, where it equals $k-n+1$.*

Proof. When X is graphic, $\max(X) \leq \ell(X) - 1$, so the claim holds when $\gamma(X) = 0$. We proceed by induction on $\gamma(X)$.

If X is not graphic, then $\max(X) > 1$, and by Theorem 4 we have $\gamma(X) = 1 + \gamma(t(X))$. Also $\max(t(X)) = \max(X) - 1$ and $\ell(t(X)) \geq \ell(X) + 1$, with equality only when X has exactly one element larger than 1. By the induction hypothesis, $\gamma(X) \leq \max\{1, 1 + \frac{\max(X)-1-(\ell(X)+1)}{2}\}$, with equality only when X and $t(X)$ each have exactly one element larger than 1. Since X is not graphic, this implies that $\max(X) > \ell(X) - 1$, and hence the desired bound and condition for equality follow.

When X has exactly one element larger than 1, the same is true of $t(X)$ (unless the largest element in X is 2), and by induction the bound on $\gamma(X)$ holds with equality. In this case $\max(X) = 2k - n + 1$, so the bound $\frac{\max(X)-\ell(X)+1}{2}$ equals $k - n + 1$. ■

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