Partition Identities I
Sandwich Theorems and Logical 0–1 Laws

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Abstract

The *Sandwich Theorems* proved in this paper give a new method to show that the partition function $a(n)$ of a partition identity

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

satisfies the condition RT_1

$$\lim_{n\to\infty} \frac{a(n-1)}{a(n)} \ = \ 1\,.$$

This leads to numerous examples of naturally occuring classes of relational structures whose finite members enjoy a logical 0–1 law.

1 Introduction

Partition identities

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)} \tag{1}$$

have been a staple in combinatorics and additive number theory since the pioneering work of Hardy and Ramanujan into the number of partitions of a positive integer n, that is,

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the number of ways to write n as a sum of positive integers. Unless explicitly stated otherwise, it is assumed that the $p(n)$, and hence the $a(n)$, are nonnegative integers. When a partition identity is mentioned without a specific reference then the reader can assume (1) above is meant, using the two counting functions $p(n)$ and $a(n)$.

The nomenclature for the anatomy of a partition identity used here is:[1]

symbol	name	abbreviation
$a(n)$	*partition (count) function*	
$p(n)$	*component (count) function*	
$\mathbf{A}(x) := \sum a(n)x^n$	*partition generating function*	PGF
$\mathbf{P}(x) := \sum p(n)x^n$	*component generating function*	
$\mathsf{rank}(p) := \sum p(n)$	*rank of the partition identity.*	

We adopt the following convention throughout this paper:

> ($\star$) $\mathbf{A}(x)$, $\mathbf{P}(x)$, $a(n)$ and $p(n)$, possibly with subscripts or other modifiers, will exclusively refer to the partition identity functions described in the previous table.

In the study of the multiplicative theory of the natural numbers, or of the integers of an algebraic number field, the total count function is readily accessible whereas the prime count function is quite difficult to pin down. Just the opposite tends to be the case in additive number theory, combinatorics and algebra. For example in the partition problems considered by Bateman and Erdős one starts with a set M of natural numbers and asks how many ways one can partition a natural number n into summands from M. In this case $p(n) = \chi_M(n)$, the characteristic function of M; the investigative effort goes into understanding properties of $a(n)$. To enumerate a class of finite functional digraphs one starts with an enumeration of the components. In algebra, to enumerate the finite Abelian groups one starts with the fact that the indecomposables are the cyclic p-groups, one of size p^k for each prime number p and each positive integer k. The reader should therefore not be surprised that we start with hypotheses on $p(n)$ and deduce information about $a(n)$.

[1] We adopt the convention of [7] that upper case bold letters name (formal) power series whose coefficients are given by the corresponding lower case italic letters, for example $\mathbf{F}(x) = \sum f(n)x^n$. By this convention $\mathbf{F}_1(x)$ is the power series $\sum f_1(n)x^n$, etc. It will be convenient to define coefficients $f(n)$ of a power series $\mathbf{F}(x)$ to be 0 for negative values of n.

Our choice of the letters $\mathbf{A}(x), \mathbf{P}(x), a(n), p(n)$ for working with partition identities follows [7] where the goal is to develop, in parallel, results for additive number systems and multiplicative number systems. These two subject areas had developed somewhat independently and consequently there is no commonly accepted uniform notation scheme. $p(n)$ traditionally refers to the partition count function (which is our $a(n)$) in additive systems, and to the prime count function in multiplicative systems. The uniform notation adopted in [7] uses $p(n)$ to count indecomposable objects (the components/**primes**) and $a(n)$ to count **aggregrate** objects (sums of components/products of primes).

2 The Property RT_1

The property

$$\frac{f(n-1)}{f(n)} \to 1\,,$$

where $f(n)$ is eventually positive, is called RT_1 because it is the condition used in the well known limit form of the *Ratio Test* for convergence of the power series $\sum f(n)x^n$; if RT_1 holds then the radius of convergence of $\sum f(n)x^n$ is 1 .

When dealing with partition functions $a(n)$ it is convenient to use interchangeably any of the phrases:

(i) $a(n)$ *satisfies* RT_1,

(ii) $\mathbf{A}(x)$ *satisfies* RT_1, or

(iii) *the partition identity satisfies* RT_1.

The true significance of knowing that $f(n)$ satisfies RT_1 is not merely that it yields a radius of 1, but it has much more to do with the fact that the values of $f(n)$ vary slowly as n increases, as expressed by

$$(1-\varepsilon)\cdot f(n-1) \;<\; f(n) \;<\; (1+\varepsilon)\cdot f(n-1)$$

for n sufficiently large. The property RT_1 plays a significant role in the results of Bateman and Erdős and is essential to Compton's approach to proving logical 0–1 laws.[2]

There are three main results concerning when a partition function $a(n)$ satisfies RT_1 , that is, when $a(n-1)/a(n) \to 1$ as $n \to \infty$. But first some definitions. A partition identity is *reduced* if

$$\gcd\big\{n : p(n) > 0\big\} \;=\; 1\,.$$

It is well known that $a(n)$ is eventually positive iff the partition identity is reduced—see, for example, p. 43 of [7]. Given a partition identity let

$$\begin{aligned} d &:= \gcd\big\{n : p(n) > 0\big\} \\ p^\star(n) &:= p(nd) \\ a^\star(n) &:= a(nd)\,. \end{aligned}$$

Then

$$\mathbf{A}^\star(x) \;:=\; \sum_{n=0}^{\infty} a^\star(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p^\star(n)}\,. \qquad (2)$$

This is the *reduced form* of the partition identity (1). The reduced form of a partition identity is reduced; and a reduced partition identity is the same as its reduced form.

Here are the three principal theorems concerning conditions on a partition identity that guarantee $a(n)$ satisfies RT_1:

[2]The property RT_ρ , meaning $a(n-1)/a(n) \to \rho$, is called *smoothly growing* by Compton [10]; RT_ρ is the additive number theory analog of the property RV_α , *regular variation of index* α , in multiplicative number theory. RT_1 is the analog of RV_0 , *slowly varying at infinity.*

- **Theorem A.** (Bell [3]) *Given a reduced partition identity* (1)*, if* $p(n)$ *is polynomially bounded, that is,* $p(n) = \mathrm{O}\big(n^\gamma\big)$ *for some* $\gamma \in \mathbb{R}$ *, then* $a(n)$ *satisfies* RT_1 *.* This generalizes a result of Bateman and Erdős [2] that says if $p(n) \in \{0,1\}$ then RT_1 holds.

- **Theorem B.** (Bell and Burris [5]) *Suppose* $\dfrac{p(n-1)}{p(n)} \to 1$ *as* $n \to \infty$*. Then the partition function* $a(n)$ *satisfies* RT_1 *.*

- **Theorem C.** (Stewart's Sum Theorem: see [7], p. 85) *If*

$$\sum_{n=0}^{\infty} a_j(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p_j(n)} \qquad (j=1,2)$$

and each $a_j^\star(n)$ *satisfies* RT_1 *then* $a^\star(n)$ *also satisfies* RT_1 *, where*

$$\sum_{n=0}^{\infty} a(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

with $p(n) = p_1(n) + p_2(n)$ *.*

The goals of this paper are:

- To considerably extend the collection of partition identities for which it is known that $a(n)$ satisfies RT_1 ; and to show that this extension is, in a natural sense, best possible.

- To give a new proof of Bell's Theorem A: if $p(n)$ is polynomially bounded then $a(n)$ satisfies RT_1 .

- To show that the new techniques for proving $a(n)$ satisfies RT_1 lead to new examples of natural classes of finite structures which have a logical 0–1 law.

3 Background requirements

In addition to the results on RT_1 already mentioned we need the following two well known results (see [7]):

- **Theorem D.** Finite rank implies polynomial growth for $a(n)$: if (1) is reduced and $r := \mathsf{rank}(p) < \infty$ then $a(n) \sim C \cdot n^{r-1}$ for some positive C.

- **Theorem E.** Infinite rank implies superpolynomial growth for $a(n)$: if (1) is reduced and $\mathsf{rank}(p) = \infty$ then for all k we have $a(n)/n^k \to \infty$ as $n \to \infty$.

Also a Tauberian theorem is needed:

Theorem 3.1 (Schur). *With $0 \le \rho < \infty$ suppose that*

(i) $f(n)$ *satisfies* RT_ρ ,

(ii) $\mathbf{G}(x)$ *has radius of convergence greater than* ρ , *and*

(iii) $\mathbf{G}(\rho) > 0$.

Let $\mathbf{H}(x) \;=\; \mathbf{F}(x) \cdot \mathbf{G}(x)$. *Then* $h(n) \;\sim\; \mathbf{G}(\rho) \cdot f(n)$.

Proof. (See [7], p. 62.) □

The notation $f(n) \preceq g(n)$ means that $f(n)$ *is eventually less or equal to* $g(n)$.

4 The Sandwich Theorem

There has long been interest in studying the partial sums $\sum_{j \le n} f(j)$ of the coefficients of a power series $\mathbf{F}(x)$, but here the fixed length tails of these partial sums are of particular interest. For L a nonnegative integer let

$$f^L(n) \;:=\; f(n) \;+\; \cdots \;+\; f(n-L)\,.$$

For a PGF $\mathbf{A}(x)$ whose coefficients are eventually positive, the least nonnegative integer L such that $a(n) > 0$ for $n \ge L$ is called the *conductor*[3] of $\mathbf{A}(x)$; designate it by $L_{\mathbf{A}}$. As the following lemma shows, the coefficients of such a PGF enjoy a weak form of monotonicity that leads to monotonicity for $a^L(n)$ for $L \ge L_{\mathbf{A}}$. Furthermore, the study of $a^L(n)$ leads to powerful methods for showing that $a(n)$ satisfies RT_1.

Lemma 4.1. *Let* $\mathbf{A}(x)$ *be a* PGF *whose coefficients are eventually positive. Then for any* $L \ge L_{\mathbf{A}}$,

(a) $a(n) \ge a(m) \quad$ *if* $\; n - m \ge L$;

(b) $a^L(n)$ *is nondecreasing for all* n ;

(c) $a^L(n)$ *is positive for* $n \ge L_{\mathbf{A}}$;

(d) $a^{mL}(n) \;\le\; m \cdot a^L(n)$ *for* $m = 1, 2, \cdots$ *and* $n \ge 0$.

Proof. $\mathbf{A}(x)$ satisfies (1), so let $\ell_1, \ell_2, \ldots$ be the (possibly finite) nondecreasing sequence of positive integers consisting of exactly $p(n)$ occurrences of each $n \ge 1$. For $m \ge 1$ let V_m be the set of nonnegative integer solutions of the equation $\sum \ell_i x_i \;=\; m$. Then (1) gives $a(m) \;=\; |V_m|$ for $m \ge 1$.

[3] Wilf [14], p. 97, uses this name in the case that $p(n) \in \{0, 1\}$. He mentions that given such a $p(n)$ that is eventually 0, the *Frobenius Problem* of computing $L_{\mathbf{A}}$ seems to be a difficult problem.

Suppose $a(j) > 0$. This means $V_j \neq \emptyset$, so choose a $\vec{d} \in V_j$. Then for any $m \geq 1$ and $\vec{c} \in V_m$ one has $\vec{c} + \vec{d} \in V_{m+j}$. This shows that $|V_m| \leq |V_{m+j}|$ since $\vec{c} \mapsto \vec{c} + \vec{d}$ is an injection from V_m to V_{m+j}. Consequently we have proved:

$$a(j) > 0 \quad \text{implies} \quad a(m) \leq a(m+j) \qquad \text{for } m \geq 1\,. \tag{3}$$

For $n - m \geq L_{\mathbf{A}}$ one has $a(n-m) > 0$ from the definition of $L_{\mathbf{A}}$; then (3) gives $a(m) \leq a\big(m + (n-m)\big) = a(n)$. This proves (a). For (b) note that

$$a^L(n+1) \; - \; a^L(n) \;=\; a(n+1) \; - \; a(n-L) \;\geq\; 0$$

by part (a). For $n \geq L_{\mathbf{A}}$ clearly $a^L(n) \geq a(n) > 0$; this is (c). Finally for (d) one has

$$\begin{aligned} a^{mL}(n) \;&\leq\; \sum_{j=0}^{m-1} \sum_{i=0}^{L} a(n - jL - i) \;=\; \sum_{j=0}^{m-1} a^L(n - jL) \\ &\leq\; \sum_{j=0}^{m-1} a^L(n) \;=\; m \cdot a^L(mL)\,. \end{aligned}$$

□

Lemma 4.2. *Let $\mathbf{A}(x)$ be a* PGF *with $a(n)$ eventually positive, and suppose $L \geq L_{\mathbf{A}}$ is an integer such that*

$$\big|a(n) - a(n-1)\big| \;=\; \mathrm{o}\big(a^L(n)\big)\,.$$

Then

$$\frac{a(n-1)}{a(n)} \;\to\; 1 \qquad \text{as } \; n \to \infty\,.$$

Proof. Let $\varepsilon > 0$ be given and choose a positive integer M such that for $n \geq M$

$$\big|a(n) - a(n-1)\big| \;\leq\; \frac{\varepsilon}{L(L+1)} a^L(n).$$

For $n \geq M + L$ choose $\widetilde{n}$ and $\widehat{n}$ from $\{n - L, \ldots, n\}$ such that

$$a(\widetilde{n}) \;\leq\; a(j) \;\leq\; a(\widehat{n}) \qquad \text{for } \; n - L \leq j \leq n\,.$$

Then for $n \geq M + L$

$$\begin{aligned} 0 \;\leq\; a(\widehat{n}) - a(\widetilde{n}) \;&\leq\; \sum_{j=n-L+1}^{n} \big|a(j) \; - \; a(j-1)\big| \\ &\leq\; \frac{\varepsilon}{L(L+1)} \sum_{j=n-L+1}^{n} a^L(j) \\ &\leq\; \frac{\varepsilon}{L+1} a^L(n) \end{aligned}$$

$$\leq \quad \varepsilon a(\widehat{n}),$$

and thus, as $0 < a(\widetilde{n}) \leq a(n) \leq a(\widehat{n})$ for $n \geq M + L$,

$$1 - \varepsilon \ \leq \ \frac{a(\widetilde{n})}{a(\widehat{n})} \ \leq \ \frac{a(n-1)}{a(n)} \ \leq \ \frac{a(\widehat{n})}{a(\widetilde{n})} \ \leq \ (1-\varepsilon)^{-1}.$$

From this it follows that $a(n-1)/a(n) \to 1$ as $n \to \infty$. $\square$

Lemma 4.3. *Suppose* $\mathbf{A}_1(x)$ *and* $\mathbf{A}_2(x)$ *are two* PGF*s and* $L \geq L_{\mathbf{A}}$ *a positive integer such that, with* $\mathbf{A}(x) = \mathbf{A}_1(x) \cdot \mathbf{A}_2(x)$,

(i) $\dfrac{a_1(n-1)}{a_1(n)} \ \to \ 1$;

(ii) $a_2^L(n) \ = \ \mathrm{o}\big(a^L(n)\big)$.

Then as $n \to \infty$,

$$\frac{a(n-1)}{a(n)} \ \to \ 1.$$

Proof. Given $\varepsilon > 0$ choose a positive integer M that is a multiple of L and such that for $n \geq M$,

$$\big|a_1(n) - a_1(n-1)\big| \ \leq \ \frac{\varepsilon}{2} a_1(n).$$

Then there are positive constants C_1, C_2 such that for $n \geq M$,

$$\begin{aligned}
\big|a(n) - a(n-1)\big| \ &= \ \Big|\sum_{j=0}^{n} \big(a_1(j) - a_1(j-1)\big) a_2(n-j)\Big| \\
&\leq \ \sum_{j=M}^{n} \big|a_1(j) - a_1(j-1)\big| a_2(n-j) \\
&\qquad + \sum_{j<M} \big|a_1(j) - a_1(j-1)\big| \cdot a_2(n-j) \\
&\leq \ \frac{\varepsilon}{2} \sum_{j=M}^{n} a_1(j) a_2(n-j) \ + \ C_1 \sum_{j<M} a_2(n-j) \\
&\leq \ \frac{\varepsilon}{2} \sum_{j=0}^{n} a_1(j) a_2(n-j) \ + \ C_2 \sum_{j \leq M} a_2(n-j) \\
&= \ \frac{\varepsilon}{2} a(n) \ + \ C_2 a_2^M(n) \\
&\leq \ \frac{\varepsilon}{2} a(n) \ + \ C_2 \frac{M}{L} a_2^L(n).
\end{aligned}$$

Now choose $N \geq M$ such that for $n \geq N$

$$C_2 \frac{M}{L} a_2^L(n) \ \leq \ \frac{\varepsilon}{2} a^L(n)\,.$$

Then for $n \geq N$,

$$\big|a(n) - a(n-1)\big| \ \leq \ \varepsilon a^L(n)\,.$$

Thus $\big|a(n) - a(n-1)\big| \ = \ \mathrm{o}\big(a^L(n)\big)$, so by Lemma 4.2 it follows that $a(n)$ satisfies RT_1. □

Theorem 4.4 (Sandwich Theorem). *Suppose*

$$\dot{\mathbf{A}}(x) \ := \ \sum_{n=0}^{\infty} \dot{a}(n) x^n \ = \ \prod_{n=1}^{\infty} (1 - x^n)^{-\dot{p}(n)}$$

is a reduced partition identity with

$$\frac{\dot{a}(n-1)}{\dot{a}(n)} \ \to \ 1 \qquad \textit{as} \ \ n \to \infty\,.$$

Then any partition identity

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n) x^n \ = \ \prod_{n=1}^{\infty} (1 - x^n)^{-p(n)} \tag{4}$$

satisfying

$$\dot{p}(n) \ \leq \ p(n) \ = \ \mathrm{O}\big(\dot{a}(n)\big) \tag{5}$$

will be such that

$$\frac{a(n-1)}{a(n)} \ \to \ 1 \qquad \textit{as} \ \ n \to \infty\,.$$

Proof. Clearly

$$\mathbf{A}(x) \ = \ \dot{\mathbf{A}}(x) \cdot \ddot{\mathbf{A}}(x)$$

where

$$\begin{aligned} \ddot{p}(n) \ &:= \ p(n) - \dot{p}(n) \\ \ddot{\mathbf{A}}(x) \ &= \ \prod_{n=1}^{\infty} (1 - x^n)^{-\ddot{p}(n)}\,. \end{aligned}$$

If $\ddot{p}(n)$ is eventually 0 then Theorem C gives the conclusion, for in this case the reduced form of $\ddot{\mathbf{A}}(x)$ satisfies RT_1 by Theorem D.

So assume $\ddot{p}(n)$ is not eventually 0. Choose positive integers $d_1 > d_2 > 1$ such that $\ddot{p}(d_1)$ and $\ddot{p}(d_2)$ are positive. Let

$$\mathbf{A}_1(x) \ := \ (1 - x^{d_1})^{-1} (1 - x^{d_2})^{-1} \dot{\mathbf{A}}(x) \tag{6}$$

$$
\begin{aligned}
\mathbf{A}_2(x) &:= \big(1-x^{d_1}\big)\big(1-x^{d_2}\big)\ddot{\mathbf{A}}(x) \\
\mathbf{P}_2(x) &:= -x^{d_1} - x^{d_2} + \sum_{n=1}^{\infty} \ddot{p}(n)x^n \\
\mathbf{H}_2(x) &:= \mathbf{P}_2(x) + \mathbf{P}_2(x^2)/2 + \cdots \\
\mathbf{B}_j(x) &:= (1-x)^{-j}\dot{\mathbf{A}}(x) \qquad \text{for } j=1,2\,. \qquad (7)
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{A}(x) &= \mathbf{A}_1(x)\mathbf{A}_2(x) \qquad (8)\\
\mathbf{A}_2(x) &= \exp\big(\mathbf{H}_2(x)\big)\,. \qquad (9)
\end{aligned}
$$

Our goal is to show that $\mathbf{A}_1(x)$ and $\mathbf{A}_2(x)$ satisfy the conditions of Lemma 4.3. Applying Theorem C to (6) one has

$$\frac{a_1(n-1)}{a_1(n)} \rightarrow 1\,;$$

so Schur's Tauberian Theorem applied to (6) gives

$$d_1d_2\cdot a_1(n) \sim [x^n](1-x)^{-2}\dot{\mathbf{A}}(x) = b_2(n)\,. \qquad (10)$$

From (7) one readily sees that

$$
\begin{aligned}
b_2(n) &= b_1(0) + \cdots + b_1(n) \\
\frac{b_1(n-1)}{b_1(n)} &\rightarrow 1 \qquad \text{as } n\rightarrow\infty\,;
\end{aligned}
$$

so

$$\frac{b_1(n)}{b_2(n)} \rightarrow 0 \qquad \text{as } n\rightarrow\infty\,.$$

This, with (10), shows $b_1(n) = \mathrm{o}\big(a_1(n)\big)$, that is

$$\sum_{j=0}^{n} \dot{a}(j) = \mathrm{o}\big(a_1(n)\big)\,. \qquad (11)$$

Differentiating both sides of (9) with respect to x and equating coefficients gives

$$na_2(n) = \sum_{j=1}^{n} jh_2(j)\cdot a_2(n-j)\,. \qquad (12)$$

By (5), since $p_2(n)\le p(n)$ one has

$$p_2(n) = \mathrm{O}\big(\dot{a}(n)\big)\,.$$

From this and the fact that $\dot{a}(0)=1$ it follows that there is a $C>0$ such that for $n\ge 1$,

$$\sum_{j=1}^{n} p_2(j) \le C\sum_{j=0}^{n}\dot{a}(j)\,. \qquad (13)$$

The definition of $h_2(n)$ and items (11), (13) yield

$$\begin{aligned} nh_2(n) &= \sum_{j|n} jp_2(j) \ \le\ n\sum_{j=1}^{n} p_2(j) \\ &\le\ Cn\sum_{j=0}^{n} \dot{a}(j) \ =\ \mathrm{o}\big(na_1(n)\big)\,. \end{aligned}$$

Let $L = L_{\mathbf{A}}$. Given $\varepsilon > 0$ choose M to be a multiple of L such that

$$nh_2(n) \ <\ \frac{\varepsilon}{2}\cdot na_1(n) \qquad \text{for}\ \ n \ge M\,. \tag{14}$$

By (8) one has, for all n,

$$a_2(n) \ \le\ a(n)\,. \tag{15}$$

There is a positive constant K such that, for $n \ge M$,

$$\begin{aligned} na_2(n) &= \sum_{j=M}^{n} jh_2(j)a_2(n-j) \ +\ \sum_{j<M} jh_2(j)a_2(n-j) && \text{by (12)} \\ &\le \sum_{j=M}^{n} \frac{\varepsilon}{2}na_1(j)a_2(n-j) \ +\ \sum_{j<M} a_2(n-j)jh_2(j) && \text{by (14)} \\ &\le n\frac{\varepsilon}{2}a(n) \ +\ Ka_2^M(n) && \text{by (8)} \\ &\le n\frac{\varepsilon}{2}a(n) \ +\ Ka^M(n) && \text{by (15)}\,; \end{aligned}$$

so

$$a_2(n) \ \le\ \frac{\varepsilon}{2}a(n) \ +\ \frac{KM}{L}\frac{a^L(n)}{n}\,.$$

Thus for $n \ge M$, using Lemma 4.1 (b), (d),

$$\begin{aligned} a_2^L(n) &\le \frac{\varepsilon}{2}a^L(n) \ +\ \frac{KM}{L}\sum_{j=0}^{L}\frac{a^L(n-j)}{n-j} \\ &\le \frac{\varepsilon}{2}a^L(n) \ +\ \frac{KM}{L(n-L)}\sum_{j=0}^{L} a^L(n) \\ &= \frac{\varepsilon}{2}a^L(n) \ +\ \frac{KM(L+1)}{L(n-L)}a^L(n)\,; \end{aligned}$$

so by choosing $N \ge M$ such that for $n \ge N$,

$$\frac{KM(L+1)}{L(n-L)} \ \le\ \frac{\varepsilon}{2}\,,$$

one has, for $n \ge N$,

$$a_2^L(n) \ \le\ \varepsilon a^L(n)\,.$$

Thus $a_2^L(n) \ = \ \mathrm{o}\big(a^L(n)\big)$, so $a(n-1)/a(n) \to 1$ as $n \to \infty$, by Lemma 4.3. □

4.1 A New Proof of Bell's Polynomial Bound Theorem

The following theorem is one of our favorites for proving the RT_1 property; it is at the very heart of our considerable generalization of the Bateman and Erdős results in [6].

Theorem 4.5 (Bell [3]). *For a reduced partition identity*

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

with $p(n) = \mathrm{O}\big(n^\gamma\big)$ *one has*

$$\frac{a(n-1)}{a(n)} \ \to \ 1 \qquad as \ \ n \to \infty\,.$$

Proof. For the case that $\mathsf{rank}(p) < \infty$ simply apply Theorem D. Now suppose that $\mathsf{rank}(p) = \infty$. Since the partition identity is reduced we have $\gcd\big(n : p(n) > 0\big) = 1$. Choose a positive integer M such that $\gcd\big(n \le M : p(n) > 0\big) = 1$ and such that there are at least $\gamma + 2$ positive integers $n \le M$ with $p(n) > 0$. Let

$$\dot{p}(n) \ := \ \begin{cases} p(n) & \text{if } n \le M \\ 0 & \text{if } n > M. \end{cases}$$

Then $0 \le \dot{p}(n) \le p(n)$ holds for $n \ge 1$; also

(i) $\dot{p}(n)$ is eventually 0,

(ii) $\dot{p}(n)$ is equal to $p(n)$ on at least $\gamma + 2$ values of n for which $p(n)$ does not vanish, so the rank of $\dot{p}(n)$ is at least $\gamma + 1$; and

(iii) the gcd of the n for which $\dot{p}(n)$ does not vanish is 1.

Condition (iii) says that the partition identity determined by $\dot{p}(n)$, namely

$$\sum_{n=0}^{\infty} \dot{a}(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-\dot{p}(n)}$$

is reduced. Clearly $\dot{p}(n) \le p(n)$. By Theorem D there is a positive constant C such that $\dot{a}(n) \sim C \cdot n^{r-1}$ where $r := \mathsf{rank}(\dot{p}(n)) \ge \gamma + 2$. This shows that $\dot{a}(n)$ satisfies RT_1 and, in view of the polynomial bound $\mathrm{O}(n^\gamma)$ on the growth of $p(n)$, $p(n) = \mathrm{O}\big(\dot{a}(n)\big)$. Now that we have $\dot{a}(n)$ satisfying RT_1 and

$$\dot{p}(n) \ \le \ p(n) \ = \ \mathrm{O}\big(\dot{a}(n)\big)\,,$$

the Sandwich Theorem gives the conclusion. ∎

4.2 Showing $p(n) = \mathrm{O}\big(\dot{a}(n)\big)$ is best possible

An example is given to show that the upper bound condition on the Sandwich Theorem does not allow for any obvious improvement such as $p(n) = \mathrm{O}\big(n^k \cdot \dot{a}(n)\big)$.

Let $f(n) \geq 1$ be a positive nondecreasing unbounded function. An example is constructed of a PGF $\dot{\mathbf{A}}(x)$ satisfying RT_1 for which one can find a $p(n)$ satisfying

$$\dot{p}(n) \;\leq\; p(n) \;=\; \mathrm{O}\big(f(n)\dot{a}(n)\big)$$

but $a(n)$ fails to satisfy RT_1. This shows that Theorem 4.4 is, in an important sense, the best possible. $f(n)$ can be replaced by a function which is unbounded (but not necessarily nondecreasing) and the result will still be true, but it requires a little more detail.

The construction of $\dot{\mathbf{A}}(x)$ proceeds by recursion, essentially by defining $\dot{p}(n)$ on longer and longer initial segments of the natural numbers. Let

$$\begin{aligned}
m_1 &:= 1\\
\alpha_j &:= 1 + \frac{1}{1+j} \qquad \text{for } \; j \geq 0\\
\dot{p}_0(n) &:= 1 \qquad \text{for } \; n \geq 1\\
\dot{p}_1(n) &:= 2^{n-1} \qquad \text{for } \; n \geq 1\,.
\end{aligned}$$

Given a $\dot{p}_k(n)$, of define $\dot{a}_k(n)$ by

$$\dot{\mathbf{A}}_k(x) \;:=\; \sum_{n\geq 0} \dot{a}_k(n)x^n \;=\; \prod_{n\geq 1}\big(1-x^n\big)^{-\dot{p}_k(n)}.$$

Let $\Phi(k)$ be the conjunction of the following three assertions:

(a) $\dot{p}_{k-1}(m_k) \;>\; m_k$

(b) $[x^n](1-x)^{-k}\cdot\dot{\mathbf{A}}_{k-1}(x) \;<\; \alpha_{k-1}\cdot f(n) \qquad \text{for } \; n > m_k$

(c) $\dot{p}_k(n) \;=\; \begin{cases} \dot{p}_{k-1}(n) & \text{if } 1 \leq n \leq m_k\\ \lfloor \alpha_{k-1}^{n-m_k}\cdot\dot{p}_{k-1}(m_k)\rfloor & \text{if } n > m_k\,. \end{cases}$

It is easy to check that $\Phi(1)$ holds. We claim:

> Given m_j, $\dot{p}_j(n)$, and $\dot{a}_j(n)$ for $1 \leq j \leq k$, such that each of the conditions
>
> $$\Phi(1),\; \cdots,\; \Phi(k)$$
>
> holds, one can find m_{k+1}, $\dot{p}_{k+1}(n)$, and $\dot{a}_{k+1}(n)$ such that $\Phi(k+1)$ holds.

To do this one only needs to find an m_{k+1} that satisfies (a) and (b) as one can use (c) to define $\dot{p}_{k+1}(n)$. One can find such an m_{k+1} because Theorem B leads to

$$\frac{\dot{a}_k(n-1)}{\dot{a}_k(n)} \ \rightarrow\ \frac{1}{\alpha_k}\,,$$

and this in turn allows us to invoke Schur's Tauberian Theorem to obtain

$$\frac{[x^n](1-x)^{-k}\dot{\mathbf{A}}_k(x)}{\dot{a}_k(n)} \ \rightarrow\ \Big(\frac{\alpha_k}{\alpha_k-1}\Big)^k \ <\ \infty.$$

Note that for k any positive integer one has $\dot{p}_{k-1}(n)$ agreeing with $\dot{p}_k(n)$ on the interval $1 \leq n \leq m_k$. One arrives at $\dot{p}(n)$ by letting

$$\dot{p}(n) \ := \ \dot{p}_k(n) \qquad \text{for any } k \text{ such that } n \leq m_{k+1}\,.$$

Then $\dot{p}(n)$ satisfies RT_1 since for $n \geq m_k$ one has

$$\begin{array}{rcl} \dot{p}(n) & \leq & \alpha_k \cdot \big(1+\dot{p}(n-1)\big) \\ \dot{p}(n) & \rightarrow & \infty \\ \alpha_k & \rightarrow & 1. \end{array}$$

Thus by Theorem B, $\dot{a}(n)$ satisfies RT_1.

Now define a $p(n)$ that lies between $\dot{p}(n)$ and $3f(n)\dot{a}(n)$ for which $a(n)$ does not satisfy RT_1. Put $n_1 = 1$ and let $\Psi(k)$ be the conjunction of the following two assertions:

$$\text{(a)}\quad p_k(n) \ = \ \begin{cases} 2\lfloor f(n_k)\dot{a}(n_k)\rfloor + 1 & \text{if } n = n_k \\ \dot{p}(n) & \text{otherwise}\,, \end{cases}$$

$$\text{(b)}\quad a_k(n_k) \ < \ f_k(n_k)\dot{a}(n_k)\,.$$

As before, given a $p_k(n)$ define the corresponding $a_k(n)$ by

$$\mathbf{A}_k(x) \ := \ \sum_{n\geq 0} a_k(n)x^n \ = \ \prod_{n\geq 1}\big(1-x^n\big)^{-p_k(n)}\,.$$

Clearly $\Psi(1)$ holds, and we claim:

> Given m_j, $p_j(n)$, and $a_j(n)$ for $1 \leq j \leq k$, such that each of the conditions
>
> $$\Psi(1),\ \cdots,\ \Psi(k)$$
>
> holds, one can find m_{k+1}, $p_{k+1}(n)$ and $a_{k+1}(n)$ such that $\Psi(k+1)$ holds.

One only needs to find an n_{k+1} that satisfies (b), and this is possible since

$$a(n) \ \leq \ [x^n](1-x)^{-\dot{p}(n_1)-\cdots-\dot{p}(n_k)} \cdot \dot{\mathbf{A}}(x)$$

holds, by construction, for infinitely many n.

Note that for k any positive integer one has $p_{k-1}(n)$ agreeing with $p_k(n)$ on the interval $1 \le n < n_k$. One arrives at $p(n)$ by letting

$$p(n) \ := \ p_k(n) \qquad \text{for any } k \text{ such that } n \le n_k\,.$$

Now we want to show that $a(n)$ does not satisfy RT_1. Notice that $a(n)$ is nondecreasing (as $p(1) > 0$) and

$$a(n_k) \ \ge \ p(n_k) \ = \ 2\lfloor f(n_k)\dot{a}(n_k)\rfloor \ + \ 1\,.$$

Let $n \in [n_k, n_{k+1}]$. Then

$$a_{k-1}(n) \ < \ f(n)\dot{a}(n)\,,$$

so

$$a(n_k - 1) \ < \ f(n_k)\dot{a}(n_k)$$

and thus

$$\frac{a(n_k)}{a(n_k-1)} \ge 2\,.$$

One has

$$\dot{p}(n) \ \le \ p(n) \ = \ \mathrm{O}(f(n)\dot{a}(n))$$

and an infinite sequence n_k with

$$a(n_k)/a(n_k-1) \ge 2\,,$$

so $a(n)$ certainly does not satisfy RT_1.

5 The Eventual Sandwich Theorem

A partition function $a(n)$, satisfying a partition identity

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}\,,$$

can be notoriously sensitive to changes in $p(n)$. However if $p(n)$ satisfies RT_1 then the situation is much more stable. The next two lemmas show that if one removes any finite number of factors from the product expression in a partition identity and puts back the same number of factors, but possibly with different powers of x involved, then the resulting partition function is asymptotic to a positive constant times the original partition function. But first some definitions.

Given a function $f(n)$ let $F(x) := \sum_{n\le x} f(n)$, the *partial sum* function of the $f(n)$. We say that $g(n)$ is a *shuffle* of $f(n)$ if $G(x)$ is eventually equal to $F(x)$. This is the same thing as saying that $f(n)$ is eventually equal to $g(n)$ and $\sum_n f(n) - g(n) \,=\, 0$.

The notation $\delta_{n=k}$ means the Kronecker function that takes the value 1 if $n = k$ and otherwise it is zero. Given integers $c \ne d$ the shuffle $g(n) \,:=\, f(n) \,-\, \delta_{n=c} \,+\, \delta_{n=d}$ of $f(n)$

is called the *(c,d)-exchange* of $f(n)$. Note that any shuffle $g(n)$ of $f(n)$ can be obtained as a sequence of exchanges starting with $f(n)$.

Given $f(n)$ as a function on the nonnegative integers one can visualize a shuffle of $f(n)$ by picturing $f(n)$ as a collection of urns labelled by the positive integers with $f(n)$ marbles in urn n. To carry out a (c,d)-exchange you take exactly one marble from urn c and move it to urn d. A shuffle consists of taking a finite number of marbles from the collection of urns and putting them back in the urns in any way desired. Clearly any such shuffle of the contents of the urns can be achieved by finitely many exchanges, moving one marble at a time. The next two lemmas say that if we shuffle a component count function $p(n)$ that satisfies RT_1 then the impact on the partition count function $a(n)$ is merely to change the asymptotics by a positive constant factor.

Lemma 5.1 (The Exchange Lemma). *Let $p_1(n)$ satisfy RT_1. If $p_1(d_1) > 0$ and d_2 is a positive integer distinct from d_1 let $p_2(n)$ be the (d_1, d_2)-exchange of $p_1(n)$. Then*

$$\frac{a_1(n)}{a_2(n)} \ \sim\ \frac{d_2}{d_1}\,.$$

Proof. We are given that $p_1(n)$ satisfies RT_1, and since $p_2(n)$ is eventually equal to $p_1(n)$ it must also satisfy RT_1. Thus by Theorem B the corresponding partition count functions $a_1(n)$ and $a_2(n)$ satisfy RT_1.

Now from $p_2(n) := p_1(n) - \delta_{n=d_1} + \delta_{n=d_2}$ we have

$$p_1(n) + \delta_{n=d_2} \ =\ p_2(n) + \delta_{n=d_1}\,,$$

which means that the corresponding PGFs $\mathbf{A}_1$ and $\mathbf{A}_2$ are related by

$$(1-x^{d_1})\cdot\ \mathbf{A}_1(x) \ =\ (1-x^{d_2})\cdot\ \mathbf{A}_2(x)\,.$$

Multiplying both sides by $(1-x)^{-1}$ gives

$$(1+x+\cdots+x^{d_1-1})\cdot\ \mathbf{A}_1(x) \ =\ (1+x+\cdots+x^{d_2-1})\cdot\ \mathbf{A}_2(x)\,,$$

and thus

$$a_1(n)+a_1(n-1)+\cdots+a_1(n-d_1+1) \ =\ a_2(n)+a_2(n-1)+\cdots+a_2(n-d_2+1).$$

As both $a_1(n)$ and $a_2(n)$ satisfy RT_1 we have, for any j, $a_i(n-j) \sim a_i(n)$, so

$$d_1\cdot a_1(n) \ \sim\ d_2\cdot a_2(n),$$

proving the lemma. ∎

Lemma 5.2 (Shuffle Lemma). *Let $p_1(n)$ satisfy RT_1 and suppose $p_2(n)$ is a shuffle of $p_1(n)$. Then for some constant $c > 0$*

$$a_2(n) \ \sim\ c\cdot a_1(n)\,.$$

Proof. One can transform $p_1(n)$ into $p_2(n)$ by a finite sequence of exchanges, so Lemma 5.1 gives the proof. □

One of the difficulties in applying the Sandwich Theorem is that one needs to have $\dot{p}(n) \le p(n)$ for *all* $n \ge 1$, but often one only has the 'eventual' result $\dot{p}(n) \preceq p(n)$. The next theorem shows that if $\dot{p}(n)$ satisfies RT_1 then one has some much appreciated leeway, namely given $\dot{p}(n) \preceq p(n) = \mathrm{O}\big(\dot{a}(n)\big)$ one can often turn the '$\preceq$' into a '$\le$' by applying a suitable shuffle to $\dot{p}(n)$.

Theorem 5.3 (The Eventual Sandwich Theorem). *Suppose*

(i) $\dot{p}(n)$ *satisfies* RT_1

(ii) $\dot{p}(n) \;\preceq\; p(n) \;=\; \mathrm{O}\big(\dot{a}(n)\big)$

(iii) $\sum_n \big(p(n) - \dot{p}(n)\big) \;\ge\; 0\,.$

Then

$$\frac{a(n-1)}{a(n)} \;\to\; 1 \qquad \textit{as} \;\; n \to \infty\,.$$

Proof. First we want to show that there is a shuffle $\widehat{p}(n)$ of $\dot{p}(n)$ with $\widehat{p}(n) \;\le\; p(n)$ for $n \ge 1$. The condition $\dot{p}(n) \;\preceq\; p(n)$ from (ii) and condition (iii) are certainly necessary for this to be possible. They are also sufficient. To see this note that they guarantee the existence of an N such that $p(n) \;\ge\; \dot{p}(n)$ for $n \ge N$ and

$$\sum_{n=1}^{N} \big(p(n) - \dot{p}(n)\big) \;\ge\; 0. \tag{16}$$

Then turning to our 'labelled urns with marbles' modelling of the functions $p(n)$ and $\dot{p}(n)$ the inequality (16) says that $p(n)$ has at least as many marbles in its first N urns as $\dot{p}(n)$ does. Consequently one can shuffle (just the contents of the first N urns of) $\dot{p}(n)$ and obtain a function $\widehat{p}(n)$ such that $\widehat{p}(n) \;\le\; p(n)$ for $n \ge 1$.

Now $p(n) \;=\; \mathrm{O}\big(\widehat{a}(n)\big)$ since $p(n) \;=\; \mathrm{O}\big(\dot{a}(n)\big)$ by (ii); and since $\dot{a}(n) \;=\; \mathrm{O}\big(\widehat{a}(n)\big)$ by Lemma 5.2. This means we are in a position to apply the Sandwich Theorem since

$$\widehat{p}(n) \;\le\; p(n) \;=\; \mathrm{O}\big(\widehat{a}(n)\big),$$

and since $\widehat{p}(n)$ satisfies RT_1 (note that Lemma 5.2 shows that RT_1 is preserved by shuffles). □

6 The Classical Partition Function Heirarchy

Thanks to Theorem B that shows RT_1 is preserved in the passage from $p(n)$ to $a(n)$, one can start with a favorite function satisfying RT_1 and, by iterating this procedure, create an infinite heirarchy of "intervals" $[p(n), \mathrm{O}\big(a(n)\big)]$ to use to prove that partition functions satisfy RT_1; and thus to prove logical 0–1 laws.

Our favorite heirarchy we call the *Classical Partition Function* heirarchy, and it is defined recursively as follows:

$$\begin{aligned}\mathsf{part}_0(n) &:= 1 \qquad \text{for } \; n \geq 1\\ \sum_{n=0}^{\infty} \mathsf{part}_{k+1}(n) x^n &= \prod_{n=1}^{\infty} \big(1 - x^n\big)^{-\mathsf{part}_k(n)}.\end{aligned}$$

Clearly the original partition function $\mathsf{part}(n)$ is $\mathsf{part}_1(n)$ in this heirarchy. Fortunately the asymptotics of this heirarchy have been well-studied using the tools of analytic number theory. One could use these results to see that each of the functions $\mathsf{part}_k(n)$ indeed satisfies RT_1; but invoking Theorem B seems much simpler. However these asymptotics allow us to draw other conclusions that strengthen our use of Sandwich Theorems. Thus they are given here in detail, following Petrogradsky's presentation.

Theorem 6.1 (See Petrogradsky [13], Theorem 2.1). *In the following the 'input' $p(n)$ to a partition identity is in the left column, the 'output' $a(n)$ is in the right column, where $\alpha \geq 1$ and $k \geq 1$, and the constants θ and κ are defined after the table:*

$p(n)$	$a(n)$
$\big(\sigma + \mathrm{o}(1)\big) \cdot n^{\alpha-1}$	$\exp\Big(\big(\theta + \mathrm{o}(1)\big) \cdot n^{\alpha/(\alpha+1)}\Big)$
$\exp\Big(\big(\sigma + \mathrm{o}(1)\big) \cdot n^{\alpha/(\alpha+1)}\Big)$	$\exp\Big(\big(\kappa + \mathrm{o}(1)\big) \cdot \dfrac{n}{(\log n)^{1/\alpha}}\Big)$
$\exp\Big(\big(\sigma + \mathrm{o}(1)\big) \cdot \dfrac{n}{(\log^{(k)} n)^{1/\alpha}}\Big)$	$\exp\Big(\big(\sigma + \mathrm{o}(1)\big) \cdot \dfrac{n}{(\log^{(k+1)} n)^{1/\alpha}}\Big)$

where

$$\begin{aligned}\theta &= \big(1 + 1/\alpha\big) \cdot \Big(\sigma\zeta(\alpha+1) \cdot \Gamma(\alpha+1)\Big)^{1/(\alpha+1)}\\ \kappa &= \alpha \cdot \Big(\frac{\sigma}{\alpha+1}\Big)^{1+(1/\alpha)}.\end{aligned}$$

It is easy to verify that the $p(n)$ discussed in Theorem 6.1 do indeed satisfy RT_1, and the usual Hardy-Ramanujan asymptotics for $\mathsf{part}(n)$ fit into the above table:

$$\begin{aligned}\mathsf{part}(n) &\sim \frac{\exp\big(\pi\sqrt{2n/3}\big)}{4\sqrt{3}n}\\ &= \exp\Big(\big(\pi\sqrt{2n/3}\big) - \big(\log\big(4\sqrt{3}n\big)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\left(\pi\sqrt{2/3} - \frac{\log\left(4\sqrt{3}n\right)}{\sqrt{n}}\right)\cdot\sqrt{n}\right)\\
&= \exp\left(\left(\pi\sqrt{2/3} + \mathrm{o}(1)\right)\cdot\sqrt{n}\right)\\
&= \exp\left(\left(\sigma + \mathrm{o}(1)\right)\cdot\sqrt{n}\right),
\end{aligned}
$$

where

$$\sigma \;=\; \pi\sqrt{2/3}\,.$$

Starting with this one can apply Theorem 6.1 to find the asymptotics for the classical partition heirarchy.

Corollary 6.2. *Defining* $\log^{(0)}(n) = 1$ *one has*

$$\mathsf{part}_k(n) \;=\; \exp\left(\left(C_k + \mathrm{o}(1)\right)\cdot\frac{n}{(\log^{(k-1)} n)^{1/2}}\right),$$

for $k \geq 1$ *and for suitable positive constants* C_k *.*

Corollary 6.3. *For* $k, r \geq 1$ *and* $\varepsilon > 0$ *one has*

$$
\begin{aligned}
n^{n^{1-\varepsilon}}\cdot\mathsf{part}_k(n) &= \mathrm{o}\big(\mathsf{part}_{k+1}(n)\big)\\
\mathsf{part}_k(n)^r &= \mathrm{o}\big(\mathsf{part}_{k+1}(n)\big)\,.
\end{aligned}
$$

Corollary 6.4. *Given* $\varepsilon > 0$ *and* $k, r \geq 1$*, suppose* $p(n)$ *is a component function that satisfies one of the conditions:*

$$
\begin{aligned}
\mathsf{part}_k(n) \;&\preceq\; p(n) \;=\; \mathrm{O}\big(\mathsf{part}_k(n)^r\big)\\
\frac{\mathsf{part}_m(n)}{n^{n^{1-\varepsilon}}} \;&\preceq\; p(n) \;=\; \mathrm{O}\big(\mathsf{part}_m(n)\big)\,.
\end{aligned}
$$

Then $a(n)$ *satisfies* RT_1*.*

Proof. Apply the Eventual Sandwich Theorem to Corollary 6.3. □

Theorem 6.1 offers further concrete examples of function intervals which we can use to prove $a(n)$ satisfies RT_1. These will be featured in the examples in [6].

Corollary 6.5. *Suppose a partition identity satisfies one of the following conditions on* $p(n)$*, where* $C_1 > 0$*,* $\varepsilon > 0$*,* $k \geq 1$*, and* $\alpha \geq 1$*:*

(1) $1 \;\preceq\; p(n) \;=\; \mathrm{O}\left(e^{\pi\sqrt{\frac{2}{3}n}}\Big/ n\right)$

(2) $C_1 \;\preceq\; p(n) \;=\; \mathrm{O}\left(e^{\left(\pi\sqrt{\frac{2}{3}C_1}-\varepsilon\right)\sqrt{n}}\Big/ n\right)$

(3) $C_1 n^{\alpha-1} \;\preceq\; p(n) \;=\; \mathrm{O}\left(e^{C_2 n^{\alpha/(\alpha+1)}}\right)$,

where $C_2 \;=\; \left(1 + \dfrac{1}{\alpha}\right)\cdot\left(C_1\zeta(\alpha+1)\Gamma(\alpha+1)\right)^{1/(\alpha+1)} \;-\; \varepsilon$

(4) $e^{C_1 n^{\alpha/(\alpha+1)}} \preceq p(n) = \mathrm{O}\Big(e^{C_2 n/(\log n)^{1/\alpha}}\Big)$,

where $C_2 = \alpha \cdot \Big(\dfrac{C_1}{\alpha+1}\Big)^{1+1/\alpha} - \varepsilon$,

(5) $e^{C_1 n/(\log^{(k)} n)^{1/\alpha}} \preceq p(n) = \mathrm{O}\Big(e^{(C_1-\varepsilon)\cdot n/\big(\log^{(k+1)} n\big)^{1/\alpha}}\Big)$.

Then

$$\frac{a(n-1)}{a(n)} \to 1 \qquad as \;\; n \to \infty .$$

Proof. Apply the Eventual Sandwich Theorem to Theorem 6.1. □

7 Logical 0–1 Laws

If we want a really expressive logic for studying relational structures (like graphs) then we can choose *higher order logic* where one can quantify over *elements* of the structures, over *subsets* of the structures, over *functions* on and between the structures, etc. In other words, the kind of logic that we use in everyday mathematics. This powerful kind of logic was introduced by Frege in 1879 and further developed in the Principia Mathematica of Whitehead and Russell (1910–1913). Unfortunately at this level of generality there are no tools from the logic that we can apply to prove mathematical theorems. By accepting the limitations of a restricted language like monadic second order logic we forfeit many interesting topics that are beyond the expressive power of the logic; but there are also many that we can access, and for such we have special tools (in particular the Ehrenfeucht-Fraisse games) to give uniform proofs for results over diverse collections of structures.

7.1 MSO Logic

Monadic second order logic[4] for relational structures is just the usual first order logic augmented with variables and quantifiers for unary predicates. Thus one can "talk about" arbitrary subsets U of a structure as well as the elements x of the structure. We will give a precise description of MSO logic for the relational language with one binary relation symbol as it is fairly simple to present, and it captures the essential details of MSO logic when working with any number of relations of any number of arguments.

7.2 The Syntax of MSO Logic for One Binary Relation

One starts with

- a *binary relation symbol* E;
- a symbol $=$ for *equality*;

[4]See Chap. 6 of [7] for a much more detailed introduction to MSO.

- symbols for *propositional connectives*, say $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\leftrightarrow$ (iff) ;
- the *quantifier* symbols $\forall$ (for all) and $\exists$ (there exists) ;
- a set $\mathcal{X}$ of *first order variables* ;
- a set $\mathcal{U}$ of *monadic second order variables.*

The MSO *formulas* are defined as follows, by induction:

- the *atomic formulas* are expressions of the form

  $E(x,y)\,,\ x=y\,,$ and $U(x)\,;$

- if φ and ψ are MSO formulas then so are

$$(\neg\,\varphi)\quad(\varphi\vee\psi)\quad(\varphi\wedge\psi)\quad(\varphi\rightarrow\psi)\quad(\varphi\leftrightarrow\psi);$$

- if φ is a MSO formula then so are $(\forall x\,\varphi)$, $(\exists x\,\varphi)$, $(\forall U\,\varphi)$ and $(\exists U\,\varphi)$.

The MSO *sentences* are the MSO formulas with no free occurrences of variables.

7.3 The Semantics of MSO Logic for One Binary Relation

The sentences of MSO logic for one binary relation are used to express properties of *relational structures* $\mathbf{G}=(G,E)$ consisting of a set G (the *universe* of $\mathbf{G}$) equipped with a *binary relation* E between the elements of G. Such a structure $\mathbf{G}$ is commonly called a *digraph*, G its set of *vertices* and E the *edge relation* of $\mathbf{G}$. If a MSO sentence φ is true in a structure $\mathbf{G}$ we say $\mathbf{G}$ *satisfies* φ as well as $\mathbf{G}$ is a *model* of φ.[5]

By a MSO *class* (of relational structures) we will always mean the *finite* models of a MSO sentence. For a basic example of an MSO class of digraphs we have:

- k**-colorable digraphs** where k is a fixed positive integer. To show that this is a MSO class we need a MSO sentence that describes this class—just say that there exist k predicates $U_1,\ldots,U_k$ (the *colors*) such that for every vertex x of the digraph exactly one of the assertions $U_i(x)$ holds, that is, x satisfies exactly one of the properties U_i (x has exactly one of the colors U_i) ; and if $E(x,y)$ holds then x and y satisfy distinct U_i (have distinct colors). Here is a MSO sentence that defines

[5]Much of modern mathematical logic studies connections between the form of sentences and the properties of the structures that satisfy those sentences. For example if φ is a *universal sentence*, that is, a sentence φ of the form $(\forall x_1)\cdots(\forall x_k)\psi(x_1,\ldots,x_k)$ with no quantifiers in ψ, then given any model $\mathbf{G}$ of φ we know that every (induced) subdigraph of $\mathbf{G}$ is a model of φ.

(axiomatizes) 3-colorable digraphs.

$$
\begin{aligned}
&(\exists U_1)(\exists U_2)(\exists U_3)\Big[\\
&\quad (\forall x)\Big[\big(U_1(x) \vee U_2(x) \vee U_3(x)\big) \quad \wedge \; \big(U_1(x) \;\rightarrow\; \neg\, U_2(x) \wedge \neg\, U_3(x)\big)\\
&\quad \wedge \; \big(U_2(x) \;\rightarrow\; \neg\, U_1(x) \wedge \neg\, U_3(x)\big)\Big]\\
&\wedge \quad (\forall x)(\forall y)\Big[E(x,y) \;\rightarrow\\
&\quad \Big(\big(U_1(x) \rightarrow \neg\, U_1(y)\big) \;\wedge\; \big(U_2(x) \rightarrow \neg\, U_2(y)\big) \;\wedge\; \big(U_3(x) \rightarrow \neg\, U_3(y)\big)\Big)\Big]\;.
\end{aligned}
$$

For the reader who has not worked with formal logic systems it is worth noting that it is not so easy to give a definitive quick snapshot of the kinds of mathematical concepts that can be expressed in MSO; one learns this by accumulating experience with examples. But perhaps the sense that there are genuine limitations can be conveyed by saying that classes of relational structures whose definition involves an infinite number of parameters (for example, saying that each vertex of a graph has a prime degree) usually cannot be defined by a sentence in MSO logic.

7.4 Adequate Classes with a MSO 0–1 Law

Given a class $\mathcal{A}$ of finite relational structures let $\mathcal{P}$ denote the subclass of connected structures. $\mathcal{A}$ is *adequate* if it is closed under *disjoint union* and *extracting components*. $\mathcal{A}$ being adequate simply guarantees that the generating function $\mathbf{A}(x)$ is a partition generating function (satisfying a partition identity). Well known examples include: graphs, regular graphs, functional digraphs, permutations, forests, posets and equivalence relations.

A perfectly general way to construct an adequate class is to start with a collection $\mathcal{P}$ of finite connected structures and let $\mathcal{A}$ be the class of finite structures with components from $\mathcal{P}$. For example if we choose chains as the components then the adequate class is linear forests.

A class $\mathcal{A}$ of finite relational structures has a MSO *0–1 law*[6] if for every monadic second order sentence φ the probability that φ holds in a randomly chosen member of $\mathcal{A}$ is either 0 or 1. More precisely, the proportion of structures in $\mathcal{A}$ of size n that satisfy φ tends either to 0 or 1 as $n \to \infty$. A good example is the class of *free trees* (McColm [12]).

Define the count functions $a_{\mathcal{A}}(n)$ for $\mathcal{A}$ and $p_{\mathcal{A}}(n)$ for $\mathcal{P}$ as follows (counting up to isomorphism):

- $a_{\mathcal{A}}(n)$ is the number of members of $\mathcal{A}$ that have exactly n elements in their universe.

[6]The original study of logical 0–1 laws in the 1970s was for *first-order* logic, the main examples being the classes *Graphs* and *Digraphs*. It turns out that these clases do not have a MSO 0–1 law. Clearly these classes are *fast growing*, that is, the radius of convergence of the generating function $\mathbf{A}(x)$ is 0. In the 1970s Compton introduced his RT_1 test for slowly growing adequate classes to have a logical 0–1 law. At first he proved this for first-order logic; then for MSO logic. See [10] for a comprehensive summary.

- $p_{\mathcal{A}}(n)$ is the number of members of $\mathcal{P}$ that have exactly n elements in their universe.

Compton ([8], [9]) showed:

> if $\mathcal{A}$ is an adequate class and $a_{\mathcal{A}}(n)$ satisfies RT_1 then $\mathcal{A}$ has a monadic second-order 0–1 law.

Theorem 4.4 shows us how to find a vast array of partition identities satisfying RT_1, and thus one has a correspondingly vast array of classes of relational structures[7] with a monadic second-order 0–1 law. Such examples are of course custom made, and may appear artificial—it is more satisfying to prove MSO 0–1 laws for *naturally* occurring classes of structures. We will conclude this section with three such examples,

- Forests of bounded height
- Varieties of MonoUnary Algebras
- Acyclic Graphs of bounded diameter,

to illustrate the power of our new results. Before discussing these examples let it be mentioned that prior to this paper the techniques for proving a logical 0–1 law for adequate classes $\mathcal{A}$ (based solely on knowledge of $a_{\mathcal{A}}(n)$) relied on (A1)–(A4).

7.5 Forests of Bounded Height

In this example one can view a forest as either a poset or as graph with rooted trees. In the poset case, the height of a forest is one less than the maximum number of vertices in a chain in the forest. Each of the classes is defined by finitely many universally quantified sentences. For example in the poset case (where the tree roots are at the top) one can use

$$\begin{aligned}
&(\forall x)\,\big(x \le x\big)\\
&(\forall x \forall y)\,\big(x \le y \,\&\, y \le z \;\rightarrow x \le z\big)\\
&(\forall x \forall y)\,\Big(\big(x \le y \,\&\, x \le z\big) \;\rightarrow\; \big(y \le z\big) \vee \big(z \le y\big)\Big).
\end{aligned}$$

Let $\mathcal{F}_m$ be the collection of forests of height at most m, and let $p_m(n)$ and $a_m(n)$ be its counting functions. For $m = 0$ one has $p_0(1) = 1$, and otherwise $p_0(n) = 0$; and $a_0(n) = 1$ for all n. For $m = 1$ clearly $p_1(n) = 1$ for all $n \ge 1$, so $a_1(n) = \mathsf{part}(n)$. For $m \ge 1$ it is

[7] Given *any* partition function $a(n)$ satisfying RT_1 there is a simple way to create an adequate class $\mathcal{A}$ with $a_{\mathcal{A}}(n) = a(n)$. Start with the class of graphs $\mathcal{G}$. The number $p_{\mathcal{G}}(n)$ of connected graphs of size n grows exponentially, certainly faster than $p(n)$ as the radius of convergence of $\sum p_{\mathcal{G}}(n)x^n$ is 0. Of course $p(n)$ may exceed $p_{\mathcal{G}}(n)$ for a finite number of values, so add enough coloring predicates $Red(x)$, $Blue(x)$ etc., to the language of graphs so that the number of connected colored graphs of size n exceeds $p(n)$ for all $n \ge 1$. Now let $\mathcal{P}$ be a subclass of this class $\mathcal{G}_c$ consisting of connected colored graphs with exactly $p(n)$ members of size n. Then let $\mathcal{A}$ be the class of all finite colored graphs whose components come from $\mathcal{P}$. The partition function $a_{\mathcal{A}}(n)$ of $\mathcal{A}$ will be precisely the original $a(n)$.

easy to see that removing the root from a tree in $\mathcal{F}_m$ gives a forest in $\mathcal{F}_{m-1}$, and indeed this operation is a bijection between the trees of $\mathcal{F}_m$ and all of $\mathcal{F}_{m-1}$. Thus for $m \geq 1$

$$p_m(n) \ = \ a_{m-1}(n-1)\,.$$

By Theorem B and induction on m we see that $a_m(n)$ satisfies RT_1; consequently each $\mathcal{F}_m$ has a monadic second-order 0–1 law.

The proof in this example did not require our new results, just Theorem B. But it is a new result, and it is needed to establish the ground step in the next example which does use the Sandwich Theorem.

7.6 Varieties of MonoUnary Algebras

A monounary algebra $\mathbf{S} = (S, f)$ is a set with a unary operation. It has long been known that every variety of monounary algebras can be defined by a single equation, either one of the form $f^m(x) = f^m(y)$ or $f^{m+k}(x) = f^m(x)$. Only the trivial variety defined by $x = y$ has unique factorization. However one can view any class of algebras as relational structures by simply converting n-ary operations into $n+1$-ary relations. (Historically this is how logic developed, with function symbols being added later.) Although this is not the usual practice in algebra, for the purpose of logical properties it can be considered an equivalent formulation. If one treats the operation f of a monounary algebra as a binary relation then one obtains a digraph with the defining characteristics of a function, namely each vertex has a unique outdirected edge (possibly to itself). This formulation does not help with varieties defined by an equation of the form $f^m(x) = f^m(y)$ as such a variety is not closed under disjoint union. However for the variety of monounary algebras $\mathcal{M}_{m,k}$ defined by the equation $f^{m+k}(x) = f^m(x)$, the relational formulation gives an adequate class of relational structures.

The connected models of the identity $f^{m+k}(x) = f^m(x)$ look like a directed cycle of d trees of height at most m, where $d|k$. Let the count functions for $\mathcal{M}_{m,k}$ be $a_{m,k}(n)$ and $p_{m,k}(n)$.

Case $k = 1$**:** The unary functions satisfying an identity $f^{m+1}(x) = f^m(x)$ can be identified with the forests in $\mathcal{F}_m$. By our previous analysis of $\mathcal{F}_m$ it follows that $p_{m,1}(n)$ and $a_{m,1}(n)$ satisfy RT_1; thus the variety $\mathcal{M}_{m,1}$ has a monadic second-order 0–1 law, for any $m \geq 1$.

Case $k > 1$**:** Let $p_{m,1,d}(n)$ count the number of directed cycle arrangements of d components from $\mathcal{M}_{m,1}$. Then it is quite straightforward to see that

$$\begin{aligned} p_{m,1}(n) \ \leq \ p_{m,k}(n) \ &= \ \sum_{d|k} p_{m,1,d}(n) \\ &\leq \ \sum_{d|k} d! \cdot p_{m,1}(n) \ \leq \ k \cdot k! \cdot p_{m,1}(n) \ = \ \mathrm{O}\big(a_{m,1}(n)\big)\,. \end{aligned}$$

By the Sandwich Theorem and the Case $k = 1$ it follows that $a_{m,k}(n)$ satisfies RT_1; so the class $\mathcal{M}_{m,k}$ of monounary algebras has a monadic second-order 0–1 law.

7.7 Acyclic Graphs of Bounded Diameter

Let $\mathcal{G}_d$ be the class of acyclic graphs of diameter at most d, meaning that the distance[8] between any two vertices is at most d. Given a connected member of this class there is a vertex v such that the distance from v to any other vertex is at most $\lceil d/2 \rceil + 1$. Such a vertex is in the *center* of the graph. Let

$$\begin{aligned} c(d) &= \lceil d/2 \rceil \\ f(d) &= \lfloor d/2 \rfloor . \end{aligned}$$

We claim that for all $n \geq 1$

$$\frac{p_{\mathcal{F}_{f(d)}}(n)}{n} \;\leq\; p_{\mathcal{G}_d}(n) \;\leq\; p_{\mathcal{F}_{f(d)}}(n) \;+\; n \cdot p_{\mathcal{F}_{f(d)-1}}(n-1)\,, \tag{17}$$

where $p_{\mathcal{F}_k}(n)$ is the component count function of forests of height k from the first example. For the lower bound note that any connected member of $\mathcal{F}_{f(d)}$ becomes a connected member of $\mathcal{G}_d$ by ignoring the root; and this map is at most n to 1. This gives the first inequality in (17).

Given any class $\mathcal{K}$ of structures let $\mathcal{K}(n)$ denote the members of $\mathcal{K}$ of size n. For the upper bound in (17) note that any connected member of $\mathcal{G}_d(n)$ turns into a connected member of $\mathcal{F}(n)$ by simply designating a vertex to be the root. By choosing the root vertex in the center one obtains a member of $\mathcal{F}_{c(d)}(n)$. By snipping at most one leaf (and only if one has to) from the tree one has a member of $\mathcal{F}_{f(d)-1}(n) \cup \mathcal{F}_{f(d)}(n)$. This mapping is an injection for the part that maps into $\mathcal{F}_{f(d)}(n)$, and at most n to one for the part mapping into $\mathcal{F}_{f(d)-1}(n-1)$. Then using Corollary 6.3 this gives

$$\frac{p_{\mathcal{F}_{f(d)}}(n)}{n} \;\leq\; p_{\mathcal{G}_d}(n) \;=\; \mathrm{O}\big(p_{\mathcal{F}_{f(d)}}(n)\big)\,.$$

From our first example the component function for each $\mathcal{F}_k$ satisfies RT_1 , so by Corollary 6.4 every $a_{\mathcal{G}_k}(n)$ satisfies RT_1. Thus with the help of the Sandwich Theorem we have proved that the class of acyclic graphs of diameter at most d has a monadic second-order 0–1 law.

8 Generalized Partition Identities

Generalized partition identities allow $p(n)$ to take on nonnegative real values. Essentially everything that has been presented goes through in this setting. The reason for restricting attention to the case that the $p(n)$ have nonnegative integer values is simply that this is where the applications to combinatorics, additive number theory, and logical limit laws are to be found. The modification of the previous results to apply to generalized partition identities is quite straightforward.

[8] The distance between two vertices is the length of a shortest path connecting them, where the length of a path with j vertices is $j-1$.

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