On regular factors in regular graphs with small radius

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Abstract

In this note we examine the connection between vertices of high eccentricity and the existence of k-factors in regular graphs. This leads to new results in the case that the radius of the graph is small (≤ 3), namely that a d-regular graph G has all k-factors, for $k|V(G)|$ even and $k \leq d$, if it has at most $2d+2$ vertices of eccentricity > 3. In particular, each regular graph G of diameter ≤ 3 has every k-factor, for $k|V(G)|$ even and $k \leq d$.

1 Introduction

All graphs considered are finite and simple. We use standard graph terminology. For vertices $u, v \in V(G)$ let $d(u,v)$ be the number of edges in a shortest path from u to v, called the distance between u and v. Let further $e(v) := \max\{d(v,x) : \ x \in V(G)\}$ denote the eccentricity of x. The radius $r(G)$ and the diameter $\mathrm{dm}(G)$ of a graph G are the minimum and maximum eccentricity, respectively. If a graph G is disconnected, then $e(v) := \infty$ for all vertices v in G.

The complete graph with n vertices is denoted by K_n. For a set $S \subseteq V(G)$ let $G[S]$ be the subgraph induced by S. In an r-almost regular graph the degrees of any two vertices differ by at most r. For $b \geq a > 0$ we call a subgraph F of G an $[a,b]$-factor, if $V(F) = V(G)$ and the degrees of all vertices in F are between a and b. We call a $[k,k]$-factor simply a k-factor. If we do not say otherwise, we quietly assume that $k < d$ if G is a d-regular graph.

Many sufficient conditions for the existence of a k-factor in a regular graph are known today. Good surveys can be found in Akiyama and Kano [1] as well as Volkmann [8]. As far as we know, none of these conditions have taken the eccentricity of vertices into

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account. It is an easy exercise to show that every regular graph G with $\mathrm{dm}(G) = 1$ has a k-factor if $k|V(G)|$ is even. For $\mathrm{dm}(G) \geq 2$ the case becomes more involved. The main result of this note is the following theorem, which provides a connection between vertices x with $e(x) > 3$ and the existence of a k-factor.

Theorem 1.1 *For $d \geq 3$ let G be a connected d-regular graph. For an integer $1 \leq k < d$ with $k|V(G)|$ even G has a k-factor if*

- *d and k are even;*
- *d is even, k is odd and G has at most $(d+1) \cdot \min\{k+1, d-k+1\}$ vertices of eccentricity ≥ 4;*
- *d and k are odd and G has at most $1 + (d+2)(k+1)$ vertices of eccentricity ≥ 4;*
- *d is odd and k is even and G has at most $1+(d+2)(d-k+1)$ vertices of eccentricity ≥ 4.*

Theorem 1.1 implies the following two results as corollaries.

Theorem 1.2 *A connected d-regular graph, $d \geq 2$, with at most $2d+2$ vertices of eccentricity ≥ 4 has every k-factor for $k|V(G)|$ even.*

Theorem 1.3 *A connected d-regular graph, $d \geq 2$, with diameter ≤ 3 has every k-factor for $k|V(G)|$ even.*

Theorem 1.1 is in the following way best possible: Let d be even and let k be odd with $d \geq 2k+4$. Take $k+1$ copies of $K_{d+1} - uv$ and a copy of $K_{d+1} - M$, where M denotes a matching of cardinality $\frac{d-2(k+1)}{2}$, as well as a vertex x. Connect x to all vertices u, v of degree $d-1$. The resulting graph G is d-regular and has

$$(k+1)(d-1) + 2k + 3 = (d+1)(k+1) + 1$$

vertices of eccentricity 4. It further has no k-factor since $\Theta_G(\{x\}, \emptyset, k) = -2$ (see Theorem 2.1). Now let d and k be odd with $d \geq 3k+6$. For an odd integer $0 < p < d$ define $K_{d+2}(p) := K_{d+2} - F(p)$, where $F(p)$ denotes a $[1,2]$-factor such that p vertices of K^p are of degree $d-1$ and the remaining vertices are of degree d. Take $k+1$ copies of $K_{d+2}(3)$, one copy of $K_{d+2}(d-3(k+1))$ as well as a vertex x. Connect x with all vertices of degree $d-1$. The resulting graph H is d-regular and has $2+(k+1)(d+2)$ vertices of eccentricity 4. It further has no k-factor since $\Theta_H(\{x\}, \emptyset, k) = -2$.

Quite some results on factors in regular graphs have been generalized to almost regular graphs (cf. [1], [8]). Theorem 1.1, however, cannot be easily generalized to r-almost regular graphs:

The complete bipartite graph $K_{p,p+r}$, $r > 0$, is r-almost regular and of diameter 2 but obviously has no k-factor.

For complete multipartite graphs, which are r-almost regular and of diameter 2, a result of Hoffman and Rodger [4] shows, that a k-factor only exists, if certain necessary and sufficient conditions are met.

The conditions in Theorem 1.1 are closely related to those given in the following result of Niessen and Randerath [5] on regular graphs.

Theorem 1.4 *Let n, d and k be integers with $n > d > k \geq 1$ such that nd and nk are even. A d-regular graph of order n has a k-factor in the following cases:*

- *d and k are even;*
- *d is even and k is odd and $n < 2(d+1)$;*
- *d and k are odd and $n < 1 + (k+2)(d+2)$;*
- *d is odd and k is even and $n < 1 + (d-k+2)(d+2)$.*

In all other cases there exists a d-regular graph of order n without a k-factor.

For a regular graph with radius ≤ 3, Theorem 1.1 provides conditions for the existence of a k-factor, which allow for a higher order than Theorem 1.4.

2 Proof of the Main Theorem

The proof of Theorem 1.1 uses the k-factor Theorem of Belck [2] and Tutte [7], which we cite in its version for regular graphs.

Theorem 2.1 *The d-regular graph G has a k-factor if and only if*

$$\Theta_G(D,S,k) := k|D| - k|S| + d|S| - e_G(D,S) - q_G(D,S,k) \geq 0 \tag{1}$$

for all disjoint subsets D, S of $V(G)$. Here $q_G(D,S,k)$ denotes the number of components C of $G - (D \cup S)$ satisfying

$$e_G(S, V(C)) + k|V(C)| \equiv 1 \pmod 2.$$

We simply call these components odd.

It always holds $\Theta_G(D,S,k) \equiv k|V(G)| \pmod 2$ for all disjoint subsets D, S of $V(G)$, whether G has a k-factor or not.

In 1985, Enomoto, Jackson, Katerinis and Saito [3] proved the following result.

Lemma 2.2 *Let G be a graph and k a positive integer with $k|V(G)|$ even. If $D, S \subset V(G)$ such that $\Theta_G(D,S,k) \leq -2$ with $|S|$ minimum over all such pairs, then $S = \emptyset$ or $\Delta(G[S]) \leq k-2$.*

For regular graphs without a k-factor, for odd k, we can give the following result on the subsets D and S.

Lemma 2.3 *Let n, k, d be integers such that n is even and k is odd with $n > d > k > 0$. Let further $2k \leq d$ if d is even. If a connected d-regular graph G of order n has no k-factor, then for all disjoint subsets D, S of $V(G)$ with $\Theta_G(D, S, k) \leq -2$ it holds $|D| > |S|$.*

Proof. If G does not have a k-factor, then, since kn is even, there exist disjoint subsets D, S of $V(G)$ with $\Theta_G(D, S, k) \leq -2$. Since G is connected, $D \cup S \neq \emptyset$. Let $q := q_G(D, S, k)$ and $W := G - (D \cup S)$.

Case 1: Let d be even. The graph G is connected and of even degree d, thus at least 2-edge-connected, and we get

$$e_G(D \cup S, V(W)) \geq 2q. \tag{2}$$

Since $e_G(D, S) \leq \min\{d|D| - e_G(D, V(W)), d|S| - e_G(S, V(W))\}$, we have

$$2e_G(D, S) \leq d(|D| + |S|) - e_G(D \cup S, V(W)), \tag{3}$$

which together with (2) results in $2q \leq d(|D| + |S|) - 2e_G(D, S)$. Taking (1) into account leads to $(d - 2k)(|D| - |S|) \geq 4$, giving us the desired result.

Case 2: Let d be odd. We get for every odd component C of W

$$\begin{aligned} e_G(D, V(C)) &= d|V(C)| - e_G(S, V(C)) - 2|E(C)| \\ &\equiv k|V(C)| + e_G(S, V(C)) - 2|E(C)| \equiv 1 \pmod{2}. \end{aligned}$$

Thus $e_G(D, S) \leq d|D| - q$ which gives us in (1)

$$k(|D| - |S|) + d|S| - q + 2 \leq e_G(D, S) \leq d|D| - q,$$

leading to

$$(d - k)(|D| - |S|) \geq 2. \ \Box$$

Proof of Theorem 1.1. The first case follows from the well-known Theorem of Petersen [6].

In the remaining cases let, without loss of generality, k be odd and furthermore $2k \leq d$ if d is even, as the graph G has a k-factor if and only if G has a $(d - k)$-factor. We are only going to prove the case that d and k are both odd. The proof to the case d even and k odd only differs in the number of vertices of eccentricity ≥ 4 and uses analogous argumentation.

Assume that G does not have a k-factor. With Theorem 2.1 there exist disjoint subsets D, S of $V(G)$ such that $\Theta_G(D, S, k) \leq -2$. From Lemma 2.3 we know that $|D| > |S|$ and $q \geq k(|D| - |S|) + 2 \geq k + 2$.

Let $X := \{v \in V(G) :\ e(v) \geq 4\}$ and $C^X := V(C) \cap X$ for every odd component C of W. By the hypothesis we have $r := |X| \leq 1 + (d+2)(k+1)$. Call an odd component C an A-component, if $|C| \leq d$ and let a denote the number of A-components. For every A-component C it holds $e_G(D \cup S, V(C)) \geq d$.

Case 1: There exist at most two odd components which have a vertex x such that $e_G(x, D \cup S) = 0$. Let l, $0 \leq l \leq 2$, be the number of such odd components of W. Then these are not A-components, giving us $a \leq q - l$, and it holds $e_G(V(C), D \cup S) \geq |V(C)|$ for all other odd components. This results in

$$\begin{aligned} e_G(V(W), D \cup S) &\geq ad + (q - a - l)(d+1) + l \\ &= q(d+1) - a - ld \\ &\geq q(d+1) - (q - l) - ld \\ &= d(q - l) + l > d(q-2). \end{aligned}$$

This together with (3) results in

$$d(|D| + |S|) - 2e_G(D, S) > d(q-2). \tag{4}$$

Inequality (4) and $\Theta_G(D, S, k) \leq -2$ lead to

$$(d - 2k)(|D| - |S|) > (d-2)q - 2d + 4.$$

If we now use $q \geq 2 + k(|D| - |S|)$, we get

$$(d - 2k)(|D| - |S|) > (d-2)(2 + k(|D| - |S|)) - 2d + 4,$$

giving us the contradiction

$$0 \ \geq \ d(1-k)(|D| - |S|) \ > \ 2(d-2) + 4 - 2d = 0. \tag{5}$$

Case 2: There exist at least three odd components having a vertex x such that $e_G(x, D \cup S) = 0$. Assume that one of these vertices is not a member of X. Then $e(x) \leq 3$ for this vertex and we have $e_G(V(C), D \cup S) \geq |V(C)|$ for all other odd components. Analogously to $l = 1$ in Case 1 we can then show $e_G(V(W), D \cup S) > (q-2)d$ and arrive at the contradiction (5). Thus each vertex x with $e_G(x, D \cup S) = 0$ is a member of X. Let $\mathcal{B}$ denote the set of all odd components of W which are not A-components. Then $|\mathcal{B}| \geq 3$ and $a \leq q - 3$ and it holds

$$\begin{aligned} e_G(V(W), D \cup S) &\geq ad + \sum_{C \in \mathcal{B}} (|V(C)| - |C^X|) \\ &\geq ad - r + \sum_{C \in \mathcal{B}} |V(C)| \\ &\geq ad - r + (q - a)(d+1) \\ &= q(d+1) - a - r. \end{aligned}$$

This combined with (3) and $\Theta_G(D, S, k) \leq -2$ leads to

$$(d-2k)(|D|-|S|) \geq q(d-1)+4-a-r. \tag{6}$$

Since $a \leq q-3$, $q \geq k(|D|-|S|)+2$ and $r \leq 1+(d+2)(k+1)$, we can deduce the inequality

$$d(1-k)(|D|-|S|) \geq 2d+2-(d+2)(k+1), \tag{7}$$

which does not give us any information in the case $k = 1$. Let us first consider $k \geq 3$. Then inequality (7) can be rewritten as

$$|D|-|S| \leq \frac{(d+1)(k+1)-2d-3}{d(k-1)} = 1+\frac{k-2}{d(k-1)} < 2.$$

By Lemma 2.3 it follows that $|D| = |S|+1$. Let now $q = k+2+\eta$ with a non-negative integer η. With (6) and $|D| = |S|+1$ we get

$$\begin{aligned} a &\geq (k+2+\eta)(d-1)-d+2k+4-1-(d+2)(k+1) \\ &= \eta(d-1)-k-1. \end{aligned} \tag{8}$$

Since $q \geq a+3$ we get $k+\eta-1 \geq \eta(d-1)-k-1$, or $2k \geq \eta(d-2)$. Thus $\eta \leq 2$ with equality if and only if $k = d-2$. Since $q \leq k+4$, the inequality $\Theta_G(D, S, k) \leq -2$ yields $d|S| - e_G(D, S) \leq 2$ and thus $e_G(V(W), D \cup S) \leq d+2$. For $a \geq 1$ there are at most 2 edges leading to non-A-components, which together with $q \geq a+3$ and the connectivity of G yields a contradiction.

For $\eta \geq 1$, we have $a \geq 1$, so it remains the case $\eta = 0$ and $a = 0$, giving us $|S| = 0$ or $e_G(D, S) = d|S|$ and hence $e_G(V(W), D) \leq d$. Since $a = 0$ and from the definition of the odd components in Theorem 2.1, every odd component of $G-(D \cup S)$ has at least $d+2$ vertices. Thus W has at least $(k+2)(d+2)$ vertices, of whom at most $r \leq 1+(d+2)(k+1)$ are not connected to D with an edge. This means

$$e_G(V(W), D) \geq (k+2)(d+2)-1-(d+2)(k+1) = d+1,$$

which yields a contradiction.

It remains the case that $k = 1$. According to Lemma 2.2, we have $|S| = 0$, if we take D and S such that S is of minimum order. Thus $q \geq |D|+2$. From the definition of odd components we have $|V(C)| \geq d+2$ for every non–A–component C. This gives us

$$\begin{aligned} e_G(V(W), D) &\geq ad+(q-a)(d+2)-r \\ &\geq q(d+2)-2a-1-2(d+2) \\ &\geq qd-2d+1 \\ &\geq (|D|+2)d-2d+1 \\ &\geq d|D|+1, \end{aligned}$$

which contradicts $e_G(V(W), D) \leq d|D|$. □

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