Directed subgraph complexes

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Abstract

Let G be a directed graph, and let Δ_G^{ACY} be the simplicial complex whose simplices are the edge sets of acyclic subgraphs of G. Similarly, we define Δ_G^{NSC} to be the simplicial complex with the edge sets of not strongly connected subgraphs of G as simplices. We show that Δ_G^{ACY} is homotopy equivalent to the $(n-1-k)$-dimensional sphere if G is a disjoint union of k strongly connected graphs. Otherwise, it is contractible. If G belongs to a certain class of graphs, the homotopy type of Δ_G^{NSC} is shown to be a wedge of $(2n-4)$-dimensional spheres. The number of spheres can easily be read off the chromatic polynomial of a certain associated undirected graph.

We also consider some consequences related to finite topologies and hyperplane arrangements.

1 Introduction

A *monotone* property of a (directed or undirected) graph is one which is preserved under deletion of edges. Hence, the set of all graphs on a particular vertex set, $[n]$ say, that satisfy a monotone property form a simplicial complex whose vertex set is the set of edges of the graphs. In numerous recent papers, see e.g. [1, 5, 6, 10, 11, 14, 15], the topological properties of such complexes of graphs have been studied. Although most papers have dealt with complexes of all graphs having a particular property P, it is indeed natural to study the complex of all subgraphs of a given graph that satisfy P. The purpose of this paper is to study directed graph complexes of this type. The properties that we

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focus on are acyclicity and strong non-connectivity. Both were studied by Björner and Welker [5] in the case of all graphs. We adapt their techniques in order to generalize their results. We also consider some consequences related to topics such as finite topologies and hyperplane arrangements.

In Section 3 we study acyclic graphs. The homotopy type of the complex of acyclic subgraphs of any given directed graph is determined. It is either a homotopy sphere or contractible. Thereafter, in Section 4, we focus on not strongly connected graphs. More precisely, we compute the homotopy type of the complex of not strongly connected subgraphs of a directed graph, if the graph belongs to a particular class, which we call 2-*dense* graphs.

We begin, however, with a brief survey in the next section of the more or less standard tools from topological combinatorics that will be made use of later.

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2 Basic topological combinatorics

Here, we briefly review the parts of the topological combinatorics machinery that we will use later. For more details we refer to the survey [4].

To any poset P, we associate the *order complex* $\Delta(P)$. It is the simplicial complex whose faces are the chains of P. Similarly, to any simplicial complex Σ, we associate its *face poset* $P(\Sigma)$ which consists of the nonempty faces of Σ ordered by inclusion. The complex $\Delta(P(\Sigma))$ is the barycentric subdivision of Σ, hence it is homeomorphic to Σ. (We do not distinguish notationally between a complex and its underlying topological space.)

Our first two tools are due to Quillen [12]. In a poset P with an element $x \in P$, we write $P_{\leq x} = \{y \in P \mid y \leq x\}$.

Lemma 2.1 (Quillen Fiber Lemma). *Let P and Q be posets, and suppose we have an order-preserving map $f : P \to Q$ such that $\Delta(f^{-1}(Q_{\leq q}))$ is contractible for all $q \in Q$. Then $\Delta(P)$ is homotopy equivalent to $\Delta(Q)$.*

Lemma 2.2 (Closure Lemma). *Let P be a poset, and suppose that $f : P \to P$ is a closure operator (i.e. $f(p) \geq p$ and $f^2(p) = f(p)$ for all $p \in P$). Then $\Delta(P)$ is homotopy equivalent to $\Delta(f(P))$.*

If a poset P has unique minimal and maximal elements, denoted by $\hat{0}$ and $\hat{1}$, respectively, then its *proper part* is $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$.

The next result can be found e.g. in [4].

Lemma 2.3. *Let L be a lattice. Form a simplicial complex Σ whose faces are those subsets of the atoms of L whose joins are not the top element. Then $\Delta(\overline{L})$ and Σ are homotopy equivalent.*

For simplicial complexes Δ_1 and Δ_2, let $\Delta_1 * \Delta_2$ denote their join. The following lemma is well-known.

Lemma 2.4. *Suppose Δ_i is homotopy equivalent to a wedge of n_i spheres of dimension d_i, $i = 1, 2$. Then $\Delta_1 * \Delta_2$ is homotopy equivalent to a wedge of $n_1 n_2$ spheres of dimension $d_1 + d_2 + 1$.*

Finally, we need a convenient collapsibility lemma stated by Björner and Welker [5]. Let $2^{[n]}$ denote the set of subsets of $[n]$. Given $i \in [n]$, define a map $2^{[n]} \to 2^{[n]}$ by

$$F \mapsto F \pm i = \begin{cases} F \cup \{i\} & \text{if } i \notin F, \\ F \setminus \{i\} & \text{if } i \in F. \end{cases}$$

Lemma 2.5. *If $\Delta_1 \subseteq \Delta_2$ are simplicial complexes on the vertex set $[n]$ and there exist vertices $i, j \in [n]$ such that $F \pm i$ maps $\Delta_2 \setminus \Delta_1$ to itself and $F \pm j$ maps Δ_1 to itself, then Δ_2 is contractible (and so is Δ_1).*

3 Acyclic subgraphs

From now on, let G be a fixed directed graph on vertex set $[n]$. Like all graphs in this paper (directed and undirected), G will be assumed to have no loops or multiple edges. Our first object of study is the complex Δ_G^{ACY} of all acyclic subgraphs of G. More precisely, with $E(G)$ denoting the edge set of G, we define

$$\Delta_G^{ACY} = \{F \subseteq E(G) \mid ([n], F) \text{ has no directed cycle}\}.$$

Let $\mathrm{Tr}(\cdot)$ denote transitive closure. Define Pos_G to be the following subset of all partial orders on $[n]$: a poset belongs to Pos_G iff its comparability graph is $\mathrm{Tr}(H)$ for some subgraph H of G. Under inclusion, Pos_G is a poset. We denote its unique minimal element, the empty relation, by $\hat{0}$. In the following lemma, the case of G being the complete graph is [5, Lemma 2.1].

Lemma 3.1. *The complexes $\Delta(\mathsf{Pos}_G \setminus \{\hat{0}\})$ and Δ_G^{ACY} are homotopy equivalent.*

Proof. The map $H \mapsto \mathrm{Tr}(H) \cap G$ is a closure operator on $P(\Delta_G^{ACY})$. We claim that its image is isomorphic to $\mathsf{Pos}_G \setminus \{\hat{0}\}$. To show this, it suffices to check that $\mathrm{Tr}(H)$ can be reconstructed from $\mathrm{Tr}(H) \cap G$; it can, since $\mathrm{Tr}(H) = \mathrm{Tr}(\mathrm{Tr}(H) \cap (G))$. Thus, by Lemma 2.2, the barycentric subdivision of Δ_G^{ACY} is homotopy equivalent to $\Delta(\mathsf{Pos}_G \setminus \{\hat{0}\})$. □

Recall that the vertices of any directed graph can be partitioned into *strongly connected components*: x and y belong to the same component iff there exist directed paths from x to y and from y to x. If every vertex belongs to the same component, then the graph is *strongly connected*.

Björner and Welker stated the following theorem in the case of G being the complete graph only. However, it is straightforward to check that their proof goes through in the more general case, too.

Theorem 3.2 (See Theorem 2.2 in [5]). *If G is strongly connected, then $\Delta(\mathsf{Pos}_G \setminus \{\hat{0}\})$ is homotopy equivalent to the $(n-2)$-sphere.*

It is now straightforward to prove the main result of this section.

Theorem 3.3. *If G is a disjoint union of k strongly connected components, then Δ_G^{ACY} is homotopy equivalent to the $(n-1-k)$-sphere. Otherwise, Δ_G^{ACY} is contractible.*

Proof. If G is not a disjoint union of strongly connected components, then G has an edge e which is not included in any cycle. Thus, Δ_G^{ACY} is a cone with apex e.

Now suppose that G is a disjoint union of k strongly connected components. If $k = 1$, then we are done by Theorem 3.2 and Lemma 3.1. Otherwise, Δ_G^{ACY} is a join of k complexes of this type. Applying Lemma 2.4 $(k-1)$ times, we conclude that Δ_G^{ACY} is homotopy equivalent to the sphere of dimension

$$\sum_{i=1}^{k}(a_i - 2) + k - 1 = n - k - 1,$$

where a_i is the number of vertices in the ith component of G. $\square$

Recall that a *quasiorder* is a reflexive and transitive relation. The poset (actually a lattice) of quasiorders on $[n]$ is a well-studied object (see e.g. [7]), mainly since quasiorders on $[n]$ correspond in a 1-1 fashion to topologies on $[n]$. The subposet Pos_n of partial orders on $[n]$ then corresponds to the topologies that satisfy the T_0 separation axiom. Thus, the next corollary can be thought of as a statement about finite topologies.

For a quasiorder R on $[n]$, let Pos_n^R be the poset of all posets that are contained in R.

Corollary 3.4. *Let R be a quasiorder on $[n]$. If R is in fact an equivalence relation with k equivalence classes, then $\Delta(\mathsf{Pos}_n^R \setminus \{\hat{0}\})$ is homotopy equivalent to the $(n-1-k)$-sphere. Otherwise, $\Delta(\mathsf{Pos}_n^R \setminus \{\hat{0}\})$ is contractible.*

Proof. There is an obvious correspondence between quasiorders and transitively closed directed graphs. Applying Theorem 3.3 with the graph corresponding to R yields the result via Lemma 3.1. $\square$

4 Not strongly connected subgraphs

In this section we turn our attention to Δ_G^{NSC}, the complex of subgraphs of G that are not strongly connected. More precisely,

$$\Delta_G^{NSC} = \{F \subseteq E(G) \mid ([n], F) \text{ is not strongly connected}\}.$$

Again, the case of G being the complete graph was analysed in [5].

Let Π_G be the subposet of the partition lattice Π_n consisting of the possible partitions into strongly connected components of subgraphs of G. The partition corresponding to a graph H is denoted by $\pi(H)$. Clearly, if $\pi(H), \pi(H') \in \Pi_G$, then their join (in Π_n) belongs to Π_G. Since $\hat{0} \in \Pi_G$, we conclude that Π_G is a lattice, although it is easy to construct an example showing that Π_G is not a sublattice of Π_n.

By a *minimal cyclic set* of G, we mean an inclusion-minimal subset $S \subseteq [n]$ with the property that some directed G-cycle has S as vertex set. Clearly, such sets correspond to atoms of Π_G. We let $\widehat{G}$ denote the hypergraph on $[n]$ whose edges are precisely the minimal cyclic sets of G.

Directed graphs whose minimal cyclic sets all have cardinality two will be important to us. We call such graphs 2-*dense*. Thus, G is 2-dense iff every cycle contains two vertices that themselves form a cycle in G, i.e. iff $\widehat{G}$ is an ordinary graph.

Recall that to any (undirected) graph $H = ([n], E)$, one associates the *graphical arrangement* $\mathcal{A}_H$. This is a hyperplane arrangement in $\mathbb{R}^n$ containing $|E|$ different hyperplanes, each given by a coordinate equation $x_i = x_j$ for $\{i, j\} \in E$. Its *intersection lattice*, $L(\mathcal{A}_H)$, is the lattice of all possible intersections of collections of such hyperplanes, ordered by reverse inclusion.

Theorem 4.1. *Suppose that G is 2-dense. If $\widehat{G}$ is connected, the order complexes $\Delta(\overline{\Pi_G})$ and $\Delta(\overline{L(\mathcal{A}_{\widehat{G}})})$ are homotopy equivalent. If $\widehat{G}$ is disconnected, then $\Delta(\overline{\Pi_G})$ is contractible.*

Proof. Suppose that G is 2-dense and denote the edge set of $\widehat{G}$ by $E(\widehat{G})$. Let Σ denote the simplicial complex on the vertex set $E(\widehat{G})$ whose simplices are given by the disconnected subgraphs of $\widehat{G}$. By Lemma 2.3, we have $\Delta(\overline{\Pi_G}) \simeq \Sigma$.

If $\widehat{G}$ is disconnected, then Σ is just a simplex, and therefore contractible.

Now suppose that $\widehat{G}$ is connected. Taking transitive closure and then intersecting with $\widehat{G}$ yields a closure operator on $P(\Sigma)$. Its image is isomorphic to the poset of all partitions of $[n]$ that arise as sets of connected components in nonempty disconnected subgraphs of $\widehat{G}$. Clearly, this poset is isomorphic to $\overline{L(\mathcal{A}_{\widehat{G}})}$. By Lemma 2.2, the theorem follows. $\square$

Remark. Requiring G to be 2-dense is not necessary in the above theorem. If G is not 2-dense, then $\mathcal{A}_{\widehat{G}}$ should be interpreted as the hypergraph subspace arrangement given by $\widehat{G}$. This generalization will not, however, be useful to us later in this paper. For more on hypergraph arrangements and subspace arrangements in general, we refer to the survey [2].

Björner's and Welker's proof of [5, Lemma 3.1] goes through to prove the more general statement below. We state it here to be able to point out where the 2-density assumption is being used. Below, $P \oplus Q$ denotes ordinal sum of posets.

Lemma 4.2 (See Lemma 3.1 in [5]). *If G is 2-dense, then Δ_G^{NSC} and $\Delta(\mathsf{Pos}_G \setminus \{\hat{0}\} \oplus \overline{\Pi_G})$ are homotopy equivalent.*

Proof. For convenience, let $Q = \mathsf{Pos}_G \setminus \{\hat{0}\} \oplus \overline{\Pi_G}$. Consider the natural order-preserving surjection $\varphi : P(\Delta_G^{NSC}) \to Q$ given by

$$\varphi(H) = \begin{cases} \mathrm{Tr}(H) \in \mathsf{Pos}_G \setminus \{\hat{0}\} & \text{if } H \text{ is acyclic,} \\ \pi(H) \in \overline{\Pi_G} & \text{otherwise.} \end{cases}$$

In order to use Lemma 2.1, we study the inverse images of φ.

To begin with, we pick $p \in \mathsf{Pos}_G \setminus \{\hat{0}\}$. Clearly, $\varphi^{-1}(Q_{\leq p})$ has a unique maximal element, namely the intersection of G and (the comparability graph of) p. This element is a cone point, and $\Delta(\varphi^{-1}(Q_{\leq p}))$ is contractible.

Now choose $\tau \in \overline{\Pi_G}$. Since G is 2-dense, any non-singleton block of τ contains a directed G-cycle of length two. Without loss of generality, suppose that 1 and 2 form such a cycle. Let $\Delta_2 = \Delta(\varphi^{-1}(Q_{\leq \tau}))$ and let $\Delta_1 \subseteq \Delta_2$ be the subcomplex comprising the graphs that contain no directed path from 1 to 2 except possibly the edge $(1,2)$. Now observe that adding the edge $(2,1)$ to $H \in \Delta_1$ affects the partition into strongly connected components at worst by merging the part which contains 1 with that which contains 2. This shows that $H \mapsto H \pm (2,1)$ maps Δ_1 into itself. Similarly, $H \mapsto H \pm (1,2)$ maps $\Delta_2 \setminus \Delta_1$ into itself. Thus, by Lemma 2.5, $\Delta(\varphi^{-1}(Q_{\leq \tau}))$ is contractible, and we are done. $\square$

Using $\chi_{\widehat{G}}(t)$ to denote the chromatic polynomial of $\widehat{G}$, we are now in position to state the main theorem. Note that the case of G being not strongly connected is uninteresting since Δ_G^{NSC} is just a simplex in this case.

Theorem 4.3. *If G is 2-dense and strongly connected, then Δ_G^{NSC} is homotopy equivalent to a wedge of $(2n-4)$-dimensional spheres. The number of spheres is $|\chi'_{\widehat{G}}(0)|$.*

Proof. By Lemma 4.2 and the definition of ordinal sums, $\Delta_G^{NSC} \simeq \Delta(\mathsf{Pos}_G \setminus \{\hat{0}\}) * \Delta(\overline{\Pi_G})$.

If $\widehat{G}$ is disconnected, Δ_G^{NSC} is contractible by Theorem 4.1. In this case, the linear coefficient of $\chi_{\widehat{G}}(t)$, and thus its absolute value $|\chi'_{\widehat{G}}(0)|$, vanishes as desired. We may therefore assume that $\widehat{G}$ is connected.

It is well-known, see e.g. Rota [13], that the characteristic polynomial of $L(\mathcal{A}_{\widehat{G}})$ and the chromatic polynomial of $\widehat{G}$ coincide, i.e.

$$\chi_{\widehat{G}}(t) = \sum_{x \in L(\mathcal{A}_{\widehat{G}})} \mu(\hat{0}, x) t^{\dim(x)},$$

where μ is the Möbius function of $L(\mathcal{A}_{\widehat{G}})$. Moreover, by a theorem of Björner [3], $\Delta(\overline{L(\mathcal{A}_{\widehat{G}})})$ has the homotopy type of a wedge of $|\mu(\hat{0}, \hat{1})|$ spheres of dimension $\operatorname{codim}(\hat{1}) - 2$. Since the top element has dimension one in our case, we conclude that $\Delta(\overline{L(\mathcal{A}_{\widehat{G}})})$, and therefore $\Delta(\overline{\Pi_G})$, has the homotopy type of a wedge of $(n-3)$-dimensional spheres and that the number of spheres is the absolute value of the linear coefficient of $\chi_{\widehat{G}}(t)$.

Theorem 3.3 shows that $\Delta(\mathsf{Pos}_G \setminus \{\hat{0}\}) \simeq S^{n-2}$, so, by Lemma 2.4, we are done. $\square$

Remark. The number of spheres above, i.e. the absolute value of the linear coefficient of the chromatic polynomial of $\widehat{G}$, has a nice interpretation due to Greene and Zaslavsky [9]. It is the number of acyclic orientations of $\widehat{G}$ having a unique fixed sink. See also [8].

Corollary 4.4 (Theorem 1.2 in [5]). *The complex of all not strongly connected directed graphs on $[n]$ is homotopy equivalent to a wedge of $(n-1)!$ spheres of dimension $2n-4$.*

Proof. If G is the complete directed graph, then $\widehat{G}$ is the complete undirected graph. The linear coefficient in its chromatic polynomial is $(-1)^{n-1}(n-1)!$. $\square$

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