

Reconstructing integer sets from their representation functions

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Abstract

We give a simple common proof to recent results by Dombi and by Chen and Wang concerning the number of representations of an integer in the form $a_1 + a_2$, where a_1 and a_2 are elements of a given infinite set of integers. Considering the similar problem for differences, we show that there exists a partition $\mathbb{N} = \cup_{k=1}^{\infty} A_k$ of the set of positive integers such that each A_k is a perfect difference set (meaning that any non-zero integer has a unique representation as $a_1 - a_2$ with $a_1, a_2 \in A_k$). A number of open problems are presented.

1 Introduction and summary

For a set $A \subseteq \mathbb{Z}$ and an integer $n \in \mathbb{Z}$ consider the representation functions

$$R_A^{(1)}(n) = \{(a_1, a_2) \in A \times A : a_1 + a_2 = n\},$$
$$R_A^{(2)}(n) = \{(a_1, a_2) \in A \times A : a_1 + a_2 = n, a_1 < a_2\},$$

and

$$R_A^{(3)}(n) = \{(a_1, a_2) \in A \times A : a_1 + a_2 = n, a_1 \leq a_2\}.$$

To what extent do $R_A^{(j)}(n)$ determine the set A ? Problems of this sort were, to our knowledge, first studied by Nathanson in [N78]. Let $\mathbb{N}$ denote the set of all positive integers. In his research talks and private communications, Sárközy has raised the following question: do there exist $A, B \subseteq \mathbb{N}$ with the infinite symmetric difference such that $R_A^{(j)}(n) = R_B^{(j)}(n)$ for all, but finitely many $n \in \mathbb{N}$?

Dombi noticed in [D02] that the answer is negative for $j = 1$, by the simple observation that $R_A^{(1)}(n)$ is odd if and only if $n = 2a$ for some $a \in A$. On the other hand, he has shown that for $j = 2$ the answer is positive and indeed, there is a partition $\mathbb{N} = A \cup B$ such that $R_A^{(2)}(n) = R_B^{(2)}(n)$ for all $n \in \mathbb{N}$.

Theorem 1 (Dombi [D02]) Define the mapping $T: \mathbb{N} \rightarrow \{1, -1\}$ by

$$T(1) = 1, \quad T(2n) = -T(2n-1), \quad T(2n+1) = T(n+1); \quad n \in \mathbb{N}$$

and let

$$A = \{n \in \mathbb{N}: T(n) = 1\}, \quad B = \{n \in \mathbb{N}: T(n) = -1\}.$$

Then $R_A^{(2)}(n) = R_B^{(2)}(n)$ for all $n \in \mathbb{N}$.

For the function $R_A^{(3)}(n)$ the problem was solved by Chen and Wang in [CW03].

Theorem 2 (Chen and Wang [CW03])¹ Define the mapping $T: \mathbb{N} \rightarrow \{1, -1\}$ by

$$T(1) = 1, \quad T(2n) = -T(2n-1), \quad T(2n+1) = -T(n+1); \quad n \in \mathbb{N}$$

and let

$$A = \{n \in \mathbb{N}: T(n) = 1\}, \quad B = \{n \in \mathbb{N}: T(n) = -1\}.$$

Then $R_A^{(3)}(n) = R_B^{(3)}(n)$ for all integer $n \geq 3$.

Below we give Theorems 1 and 2 a new simple proof, establishing both results through one common argument which also shows that the constructions of Dombi and Chen-Wang are, essentially, unique. We then proceed to investigate the parallel problem for differences.

Let $r_A(n)$ denote the number of representations of the integer n as a difference of two elements of the set $A \subseteq \mathbb{Z}$:

$$r_A(n) = \{(a', a'') \in A \times A: a'' - a' = n\}.$$

It is not difficult to see that for any finite partition of $\mathbb{N}$ one can find a partition set, say A , such that there are arbitrarily large integer n with $r_A(n) = \infty$. On the other hand, we were able to partition $\mathbb{N}$ into the *infinite* number of subsets with identical finite difference representation functions. Indeed, our subsets are perfect difference sets. (Recall, that $A \subseteq \mathbb{Z}$ is a *perfect difference set* if any non-zero integer has a unique representation as a difference of two elements of A ; in our terms, $r_A(n) = 1$ for any $n \in \mathbb{N}$.) Moreover, one can arrange it so that the subsets in question have completely different structure.

Theorem 3 *There is a partition $\mathbb{N} = \cup_{k=1}^{\infty} A_k$ of the set of all positive integers such that each A_k is a perfect difference set and $|A_i \cap (A_j + z)| \leq 2$ for any $i, j, z \in \mathbb{N}$.*

¹The way we present Theorems 1 and 2 emphasizes the striking similarity between the partitions $\mathbb{N} = A \cup B$ considered in these theorems. Ironically, Dombi conjectured that sets $A, B \subseteq \mathbb{N}$ with the infinite symmetric difference satisfying $R_A^{(3)}(n) = R_B^{(3)}(n)$ (for n large enough) do *not* exist. The result of Chen and Wang shows, however, that such sets do exist and can be obtained by a very minor modification of Dombi's original construction.

2 The proofs

Proof of Theorems 1 and 2. Let A_2, B_2 , and T_2 denote the two sets and the mapping of Theorem 1, and let A_3, B_3 , and T_3 denote the two sets and the mapping of Theorem 2. For $j \in \{2, 3\}$ define

$$\alpha_j(x) = \sum_{a \in A_j} x^a, \quad \beta_j(x) = \sum_{b \in B_j} x^b, \quad \text{and} \quad \tau_j(x) = \sum_{n \in \mathbb{N}} T_j(n) x^n.$$

Thus $\alpha_j(x), \beta_j(x)$, and $\tau_j(x)$ converge absolutely for $|x| < 1$ and satisfy

$$\alpha_j(x) + \beta_j(x) = \frac{x}{1-x}, \quad \alpha_j(x) - \beta_j(x) = \tau_j(x). \quad (1)$$

Moreover, it is easily seen that

$$2 \sum_{n \in \mathbb{N}} R_A^{(j)}(n) x^n = (\alpha_j(x))^2 + (-1)^{j+1} \alpha_j(x^2) \quad (2)$$

and similar identity holds with B and β substituted for A and α , respectively. Taking into account that the sum $T_j(1) + \cdots + T_j(n-1)$ vanishes for n odd and equals $-T_j(n)$ for n even, we derive from (1) and (2) that

$$\begin{aligned} 2 \sum_{n \in \mathbb{N}} (R_A^{(j)}(n) - R_B^{(j)}(n)) x^n &= ((\alpha_j(x))^2 - (\beta_j(x))^2) + (-1)^{j+1} (\alpha_j(x^2) - \beta_j(x^2)) \\ &= \frac{x}{1-x} \tau_j(x) + (-1)^{j+1} \tau_j(x^2) \\ &= \sum_{n \in \mathbb{N}} \left(\sum_{1 \leq i \leq n-1} T_j(i) \right) x^n + (-1)^{j+1} \sum_{n \in \mathbb{N}} T_j(n) x^{2n} \\ &= \sum_{n \in \mathbb{N}} (-T_j(2n) + (-1)^{j+1} T_j(n)) x^{2n} \end{aligned}$$

for $|x| < 1$ and $j \in \{2, 3\}$. It remains to observe that $T_j(2n) = (-1)^{j+1} T_j(n)$, except if $j = 3$ and $n = 1$. $\square$

Suppose that $\mathbb{N} = A \cup B$ is a partition of the set of positive integers and let $T(n) = 1$ if $n \in A$ and $T(n) = -1$ if $n \in B$. Our proof of Theorems 1 and 2 shows that then $R_A^{(j)}(n) = R_B^{(j)}(n)$ for all sufficiently large n if and only if $T(1) + \cdots + T(2n) = 0$ and $T(2n) = (-1)^{j+1} T(n)$, for all but finitely many $n \in \mathbb{N}$. The reader will easily check that this is equivalent to the assertion that there exists $n_0 \in \mathbb{N}$ such that $T(2n) = -T(2n-1)$ and $T(2n-1) = (-1)^j T(n)$ for $n \geq n_0$, and $T(1) + \cdots + T(2n_0) = 0$. That is, any partition $\mathbb{N} = A \cup B$ satisfying $R_A^{(j)}(n) = R_B^{(j)}(n)$ for all sufficiently large n is obtained essentially as in Theorems 1 and 2.

Proof of Theorem 3. Fix a function $f: \mathbb{N} \rightarrow \mathbb{N}$ satisfying

$$f(2m-1) \leq m, \quad f(2m) = m+1; \quad m = 1, 2, \dots \quad (3)$$

and such that for any $k \in \mathbb{N}$ the inverse image $f^{-1}(k) = \{n \in \mathbb{N} : k = f(n)\}$ is infinite. Set $A_k = \emptyset$ for all $k \in \mathbb{N}$. Our construction involves infinitely many steps which we enumerate by positive integers. At the n th step we add one or two elements to the set $A_{f(n)}$ so that (i) every positive integer is added to some A_k at certain step; (ii) no positive integer is added to several distinct A_k at different steps; (iii) for any $d, k \in \mathbb{N}$ there is a step such that the element(s) added at this step to A_k produce(s) a pair $(a_1, a_2) \in A_k \times A_k$ with $a_2 - a_1 = d$; (iv) the element(s) added to A_k at any step produce(s) no non-trivial equality of the form $a_1 - a_2 = a_3 - a_4$ with $a_1, a_2, a_3, a_4 \in A_k$; (v) the element(s) added to A_k at any step produce(s) no triple $(a_1, a_2, a_3) \in A_k \times A_k \times A_k$ which is a shift of another triple $(b_1, b_2, b_3) \in A_l \times A_l \times A_l$ with some $l \neq k$. Once we manage to satisfy (i)–(v), our proof is over; we now proceed to describe exactly how the elements to be added to $A_{f(n)}$ at the n th step are chosen.

If $n = 2m$ is even then it follows from (3) that the set $A_{f(n)} = A_{m+1}$ was not affected by steps $1, \dots, n-1$. This set, therefore, remains empty by the beginning of the n th step, and we initialize it inserting to it the smallest positive integer not contained in $\cup_{l=1}^m A_l$.

Suppose now that n is odd and write for brevity $k = f(n)$. Let d be the smallest positive integer, not representable as $a_1 - a_2$ with $a_1, a_2 \in A_k$. We insert to A_k two numbers z and $z + d$, where z is to satisfy the following conditions:

- (a) $\{z, z + d\} \cap (\cup_{l=1}^{\infty} A_l) = \emptyset$;
- (b) equality $a_1 - a_2 = a_3 - a_4$ with $a_1, a_2, a_3, a_4 \in A_k \cup \{z, z + d\}$ holds only trivially; that is, if and only if either $a_1 = a_3$ and $a_2 = a_4$, or $a_1 = a_2$ and $a_3 = a_4$;
- (c) none of the triples $(a_1, z, z + d)$, (a_1, a_2, z) , $(a_1, a_2, z + d)$ with $a_1, a_2 \in A_k$ are translates of a triple (b_1, b_2, b_3) with $b_1, b_2, b_3 \in A_l$, $l \neq k$.

Clearly, condition (a) excludes only a finite number of possible values of z , and a little meditation shows that this is the case also with condition (b). Concentrating on condition (c), we notice that the actual number of values of l to be taken into account is finite, as all but $(n+1)/2$ sets A_l are empty by the beginning of the n th step. Furthermore, for any fixed l the number of triples (b_1, b_2, b_3) with $b_1, b_2, b_3 \in A_l$ is finite, and the number of possible values of $a_1 \in A_k$ is finite, too. It follows that condition (c) also excludes only finite number of z . Thus choosing z is always possible, and this concludes the proof. $\square$

3 Open problems

We list below some related problems.

The proof of Theorem 3 can be simplified if we wish to construct just one perfect difference set $A \subseteq \mathbb{N}$. In this case we can start with the empty set $A^{(0)} = \emptyset$ and define at the n th step $A^{(n)} = A^{(n-1)} \cup \{z_n, z_n + d_n\}$, where d_n is the smallest non-negative integer not representable as $a_1 - a_2$ with $a_1, a_2 \in A^{(n-1)}$, and z_n is to be so chosen that $z_n, z_n + d_n \notin A^{(n-1)}$, and no non-trivial equality $a_1 - a_2 = a_3 - a_4$ with $a_1, a_2, a_3, a_4 \in$

$A^{(n-1)} \cup \{z_n, z_n + d_n\}$ is created. The number of choices of z_n excluded by these conditions is $O(n^3)$ and since $d_n = O(n^2)$, the n th element of the resulting set A is $O(n^3)$. It follows that the counting function $A(x) = |A \cap [1, x]|$ satisfies $A(x) \gg x^{1/3}$. On the other hand, it is easily seen that for any perfect difference set $A \subseteq \mathbb{N}$ we have $A(x) \ll x^{1/2}$.

Problem 1 Does there exist a perfect difference set $A \subseteq \mathbb{N}$ with the counting function $A(x) \gg x^{1/2}$? If not, is it true that for any $\varepsilon > 0$ there exists a perfect difference set $A \subseteq \mathbb{N}$ with $A(x) \gg x^{1/2-\varepsilon}$? If not, how large can $\liminf_{x \rightarrow \infty} \ln A(x) / \ln x$ for a perfect difference set $A \subseteq \mathbb{N}$ be?

The definition of $R_A^{(1)}(n)$ extends readily onto the case where A is a subset of an arbitrary abelian group G and n is an element of the group. Suppose that G is finite and for a group character χ let $\hat{A}(\chi) = |G|^{-1} \sum_{a \in A} \overline{\chi}(a)$, the Fourier coefficient of the indicator function of A . The identity $R_A^{(1)} = R_B^{(1)}$ translates easily into the requirement that either $\hat{A}(\chi) = \hat{B}(\chi)$ or $\hat{A}(\chi) = -\hat{B}(\chi)$ hold for any character χ . Though this seems to be a rather strong condition, numerical computations show that for certain groups pairs (A, B) such that $R_A^{(1)} = R_B^{(1)}$ are not that rare as one could expect. Quite likely, these pairs are not limited to simple special cases as for instance $|A| = |G|/2$, $B = G \setminus A$, or $B = \{a + d : a \in A\}$ with a fixed element $d \in G$ of order two. Nevertheless we state our next problem in the most general form.

Problem 2 For any finite abelian group G determine all pairs of subsets $A, B \subseteq G$ such that $R_A^{(1)} = R_B^{(1)}$.

We note that if G is of odd order then no non-trivial pairs exist, as in this case the values of n for which $R_A^{(1)}(n)$ is odd determine the set A uniquely. On the other hand, if G is an elementary 2-group then any two perfect difference sets $A, B \subseteq G$ satisfy $R_A^{(1)} = R_B^{(1)}$.

As a common generalization of the representation functions $R_A^{(1)}$ and r_A , one can consider two potentially different sets $A, B \subseteq \mathbb{Z}$ and for $n \in \mathbb{Z}$ define

$$r_{A,B}(n) = \#\{(a, b) \in A \times B : a + b = n\}.$$

An unpublished observation due to Freiman, Yudin, and the present author is as follows. Suppose that A and B are finite and non-empty, and for $k \in \mathbb{N}$ let ν_k denote the k th largest value attained by $r_{A,B}$. Thus $\{\nu_k\}$ is the spectrum of the function $r_{A,B}$ and we have $\nu_1 \geq \nu_2 \geq \dots$, $\nu_1 + \nu_2 + \dots = |A||B|$, and $\nu_k = 0$ for all k large enough. Then

$$\nu_k^2 \leq \nu_k + \nu_{k+1} + \nu_{k+2} + \dots \quad (4)$$

for any $k \in \mathbb{N}$.

For the proof we write $A = \{a_1, \dots, a_l\}$ and $B = \{b_1, \dots, b_m\}$ where the elements are numbered in the increasing order, and notice first that

$$r_{A,B}(a_i + b_j) \leq \min\{i + j - 1, l + m - (i + j - 1)\}; \quad 1 \leq i \leq l, 1 \leq j \leq m.$$

For if $a_i + b_j = a_u + b_v$ then either $u \leq i$, or $v \leq j$; since there are at most i such representations with $u \leq i$ and at most j representations with $v \leq j$, and one representation

satisfies both $u \leq i$ and $v \leq j$, we conclude that $r_{A,B}(a_i + b_j) \leq i + j - 1$. The proof of the estimate $r_{A,B}(a_i + b_j) \leq l + m - (i + j - 1)$ is almost identical; just notice that if $a_i + b_j = a_u + b_v$ then either $u \geq i$ or $v \geq j$. Now we have

$$\begin{aligned}
\nu_1 + \cdots + \nu_k &\leq \#\{(i, j): r_{A,B}(a_i + b_j) \geq \nu_k\} \\
&\leq \#\{(i, j): \min\{i + j - 1, l + m - (i + j - 1)\} \geq \nu_k\} \\
&= \#\{(i, j): \nu_k \leq i + j - 1 \leq l + m - \nu_k\} \\
&= lm - 2 \#\{(i, j): i + j \leq \nu_k\} \\
&= lm - \nu_k(\nu_k - 1)
\end{aligned}$$

and (4) follows from $lm = \nu_1 + \cdots + \nu_k + \nu_{k+1} + \cdots$.

Problem 3 What are the general properties shared by the functions $r_{A,B}(n)$ (for all finite non-empty $A, B \subseteq \mathbb{Z}$), other than that reflected by (4)?

Since the spectrum $\{\nu_k\}$ defines a partition of the integer $|A||B|$, it can be visualized with a Ferrers diagram corresponding to this partition; that is, an arrangement of $|A||B|$ square boxes in bottom-aligned columns such that the leftmost column is of height ν_1 , the next column is of height ν_2 , and so on. For any $t \in \mathbb{N}$, the length of the t th row of this diagram (counting the rows from the bottom) is then $N_t = \#\{n: r_{A,B}(n) \geq t\}$. We notice that from a well-known result of Pollard [P75] it follows that $N_1 + \cdots + N_t \geq t(|A| + |B| - t)$ for any $t \leq \min\{|A|, |B|\}$; one can derive this inequality as a corollary of (4), too.

We conclude our note with two problems due to Gowers and Konyagin, presented here from their kind permission. Both problems pertain to the group $\mathbb{Z}/p\mathbb{Z}$ of residue classes modulo a prime p .

Problem 4 (Gowers, personal communication) For a prime p , let $A \subseteq \mathbb{Z}/p\mathbb{Z}$ be a subset of cardinality $|A| = (p + 1)/2$. The average value of $R_A^{(1)}$ is then $(p + 1)^2/(4p) = p/4 + O(1)$. Is it true that for any positive constant ε and any sufficiently large p , there exists $n \in \mathbb{Z}/p\mathbb{Z}$ satisfying $|R_A^{(1)}(n) - p/4| < \varepsilon p$?

Problem 5 (Konyagin, personal communication) Do there exist positive constants ε and C such that for any sufficiently large prime p and any non-empty subset $A \subseteq \mathbb{Z}/p\mathbb{Z}$ of cardinality $|A| < \sqrt{p}$, there is $n \in \mathbb{Z}/p\mathbb{Z}$ satisfying $1 \leq R_A^{(1)}(n) \leq C|A|^{1-\varepsilon}$?

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