# MacMahon-type Identities for Signed Even Permutations

Dan Bernstein

Department of Mathematics
The Weizmann Institute of Science, Rehovot 76100, Israel
dan.bernstein@weizmann.ac.il

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**Abstract**

MacMahon's classic theorem states that the *length* and *major index* statistics are equidistributed on the symmetric group $S_n$. By defining natural analogues or generalizations of those statistics, similar equidistribution results have been obtained for the alternating group $A_n$ by Regev and Roichman, for the hyperoctahedral group $B_n$ by Adin, Brenti and Roichman, and for the group of even-signed permutations $D_n$ by Biagioli. We prove analogues of MacMahon's equidistribution theorem for the group of signed even permutations and for its subgroup of even-signed even permutations.

## 1 Introduction

A classic theorem by MacMahon [6] states that two *permutation statistics*, namely the *length* (or *inversion number*) and the *major index*, are equidistributed on the symmetric group $S_n$. Many refinements and generalizations of this theorem are known today (see [8] for a brief review). In [8], Regev and Roichman gave an analogue of MacMahon's theorem for the alternating group $A_n \subseteq S_n$, and in [1], Adin, Brenti and Roichman gave an analogue for the hyperoctahedral group $B_n = C_2 \wr S_n$. Both results involve natural generalizations of the $S_n$ statistics having the equidistribution property.

Our main result here (Proposition 4.1) is an analogue of MacMahon's equidistribution theorem for *the group of signed even permutations* $L_n = C_2 \wr A_n \subseteq B_n$. Namely, we define two statistics on $L_n$, the *L-length* and the *negative alternating reverse major index*, and show that they have the same generating function, hence they are equidistributed. Our Main Lemma (Lemma 4.6) shows that every element of $L_n$ has a unique decomposition into a descent-free factor and a signless even factor.

In [3], Biagioli proved an analogue of MacMahon's theorem for *the group of even-signed permutations* $D_n$ (signed permutations with an even number of sign changes). Using our main result, we prove an analogue for *the group of even-signed even permutations* $(L \cap D)_n = L_n \cap D_n$ (see Proposition 5.2).

The rest of this paper is organized as follows: Section 2 contains a review of wreath products and known results concerning generators and canonical presentations in $S_n$, $B_n$ and $A_n$. In Section 3 we define the group $L_n$, introduce a canonical presentation in $L_n$, and define the statistics we use. In Section 4 we prove the equidistribution property for $L_n$, and in Section 5 we prove the equidistribution property for $(L \cap D)_n$. Finally, in Section 6, we note three open problems.

# 2 Preliminaries

## 2.1 Notation

For an integer $a \geq 0$ we let $[a] = \{1, 2, \ldots, a\}$ (where $[0] = \emptyset$).

Let $C_a$ be the cyclic group of order $a$.

Let $S_n$ be the symmetric group on $1, \ldots, n$ and let $A_n \subset S_n$ denote the alternating group.

## 2.2 Wreath Products

Let $G$ be a group and let $A$ be a subgroup of $S_n$. Recall that the wreath product $G \wr A$ is the group $\{ \big((g_1, \ldots, g_n), v\big) \mid g_i \in G, \;\; v \in A \}$ with multiplication given by

$$\big((g_1, \ldots, g_n), v\big)\big((h_1, \ldots, h_n), w\big) = \big((g_1 h_{v^{-1}(1)}, \ldots, g_n h_{v^{-1}(n)}), vw\big).$$

The order of $G \wr A$ is $|G|^n |A|$.

Let $X = G \times [n]$. For $\big((g_1, \ldots, g_n), v\big) \in G \wr A$, define $f_{((g_1,\ldots,g_n),v)} : X \to X$ by

$$f_{((g_1,\ldots,g_n),v)}(h, i) := (hg_{v(i)}, v(i)).$$

One can verify that if $G$ is Abelian, then function composition is compatible with multiplication in $G \wr A$, that is $f_{((g_1,\ldots,g_n),v)} f_{((h_1,\ldots,h_n),w)} = f_{((g_1,\ldots,g_n),v)((h_1,\ldots,h_n),w)}$. Thus, if $G$ is Abelian we can identify $\big((g_1, \ldots, g_n), v\big)$ with $f_{((g_1,\ldots,g_n),v)}$ and we can write $\pi = \big((g_1, \ldots, g_n), v\big) \in G \wr A$ as

$$\pi = [f_\pi(1, 1), f_\pi(1, 2), \ldots, f_\pi(1, n)] = [(g_{v(1)}, v(1)), \ldots, (g_{v(n)}, v(n))].$$

Call this the *window notation* of $\pi$.

### 2.2.1 The Group of Signed Permutations

If $G = C_2 = \{-1, 1\}$, then we write $X$ simply as $\{\pm 1, \pm 2, \ldots, \pm n\}$ and identify every $\sigma \in C_2 \wr A$ with a bijection of $X$ onto itself satisfying $\sigma(-i) = -\sigma(i)$ for all $i \in [n]$. We write $\sigma = [\sigma_1, \ldots, \sigma_n]$ to mean that $\sigma(i) = \sigma_i$ for $i \in [n]$.

In particular, *the hyperoctahedral group* $B_n := C_2 \wr S_n$ is the group of all bijections of $\{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying the above condition. It is also known as the group of *signed permutations*.

## 2.3 Generators and Canonical Presentation

In this subsection we review generators and canonical presentations in the groups $S_n$, $B_n$ and $A_{n+1}$.

### 2.3.1 $S_n$

**The Coxeter System of $S_n$.** $S_n$ is a Coxeter group of type $A$. The Coxeter generators are the adjacent transpositions $\{ s_i \}_{i=1}^{n-1}$ where $s_i := (i, i+1)$. The defining relations are the Moore-Coxeter relations:

$$\begin{aligned} (s_i s_{i+1})^3 &= 1 \quad (1 \le i < n), \\ (s_i s_j)^2 &= 1 \quad (|i-j| > 1), \\ s_i^2 &= 1 \quad (1 \le i < n). \end{aligned}$$

**The $S$ Canonical Presentation.** The following presentation of elements in $S_n$ by Coxeter generators is well known (see for example [5, pp. 61–62]).

For each $1 \le j \le n-1$ define

$$R_j^S := \{1,\, s_j,\, s_j s_{j-1},\, \ldots,\, s_j s_{j-1} \cdots s_1\},$$

and note that $R_1^S, \ldots, R_{n-1}^S \subseteq S_n$.

**Theorem 2.1 (see [5, pp. 61–62]).** *Let $w \in S_n$. Then there exist unique elements $w_j \in R_j^S$, $1 \le j \le n-1$, such that $w = w_1 \ldots w_{n-1}$. Thus, the presentation $w = w_1 \ldots w_{n-1}$ is unique.*

For a proof, see for example [8, Section 3.1].

**Definition 2.2 (see [8, Definition 3.2]).** Call $w = w_1 \ldots w_{n-1}$ in the above theorem *the $S$ canonical presentation of $w \in S_n$.*

### 2.3.2 $B_n$

**The Coxeter System of $B_n$.** $B_n$ is a Coxeter group of type $B$, generated by $s_1, \ldots, s_{n-1}$ together with an exceptional generator $s_0 := [-1, 2, 3, \ldots, n]$, whose action is as follows:

$$\begin{aligned} [\sigma_1, \sigma_2, \ldots, \sigma_n] s_0 &= [-\sigma_1, \sigma_2, \ldots, \sigma_n] \\ s_0 [\sigma_1, \ldots, \pm 1, \ldots, \sigma_n] &= [\sigma_1, \ldots, \mp 1, \ldots, \sigma_n] \end{aligned}$$

(see [4, §8.1]). The additional relations are: $s_0^2 = 1$, $(s_0 s_1)^4 = 1$, and $s_0 s_i = s_i s_0$ for all $1 < i < n$.

**The $B$ Canonical Presentation.** For each $0 \le j \le n-1$ define

$$\begin{aligned} R_j^B := \{1,\, s_j,\, s_j s_{j-1},\, \ldots,\, s_j s_{j-1} \cdots s_1, s_j s_{j-1} \cdots s_1 s_0, \\ s_j s_{j-1} \cdots s_1 s_0 s_1,\, \ldots,\, s_j s_{j-1} \cdots s_1 s_0 s_1 \cdots s_j\}, \end{aligned}$$

and note that $R_0^B, \ldots, R_{n-1}^B \subseteq B_n$.

The following theorem is the case $a = 2$ of [9, Propositions 3.1 and 3.3]. For a proof of the general case, see for example [2, Ch. 3.3].

**Theorem 2.3.** *Let $\sigma \in B_n$. Then there exist unique elements $\sigma_j \in R_j^B$, $0 \le j \le n-1$, such that $\sigma = \sigma_0 \dots \sigma_{n-1}$. Moreover, written explicitly $\sigma_0 \dots \sigma_{n-1} = s_{i_1} s_{i_2} \dots s_{i_r}$ is a reduced expression for $\sigma$, that is $r$ is the minimum length of an expression of $\sigma$ as a product of elements in $\{s_i\}_{i=0}^{n-1}$.*

**Definition 2.4.** Call $\sigma = \sigma_0 \dots \sigma_{n-1}$ in the above theorem *the $B$ canonical presentation of $\sigma \in B_n$.*

**Remark 2.5.** For $\sigma \in S_n$, the $B$ canonical presentation of $\sigma$ coincides with its $S$ canonical presentation.

**Example 2.6.** Let $\sigma = [5, -1, 2, -3, 4]$, then $\sigma_4 = s_4 s_3 s_2 s_1$; $\sigma\sigma_4^{-1} = [-1, 2, -3, 4, 5]$, therefore $\sigma_3 = 1$ and $\sigma_2 = s_2 s_1 s_0 s_1 s_2$; and finally $\sigma\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1} = [-1, 2, 3, 4, 5]$ so $\sigma_1 = 1$ and $\sigma_0 = s_0$. Thus $\sigma = \sigma_0\sigma_1\sigma_2\sigma_3\sigma_4 = (s_0)(1)(s_2 s_1 s_0 s_1 s_2)(1)(s_4 s_3 s_2 s_1)$.

### 2.3.3 $A_{n+1}$

**A Generating Set for $A_{n+1}$.** Let

$$a_i := s_1 s_{i+1} \quad (1 \le i \le n-1).$$

The set $A = \{\, a_i \,\}_{i=1}^{n-1}$ generates $A_{n+1}$. This set has appeared in [7], where it is shown that the generators satisfy the relations

$$\begin{aligned} (a_i a_j)^2 &= 1 \quad (|i-j| > 1), \\ (a_i a_{i+1})^3 &= 1 \quad (1 \le i < n-1), \\ a_i^2 &= 1 \quad (1 < i \le n-1), \\ a_1^3 &= 1 \end{aligned}$$

(see [7, Proposition 2.5]).

Note that $(A_{n+1}, A)$ is not a Coxeter system (in fact, $A_{n+1}$ is not a Coxeter group) as $a_1^2 \neq 1$.

**The $A$ Canonical Presentation.** The following presentation of elements in $A_{n+1}$ by generators from $A$ has appeared in [8, Section 3.3].

For each $1 \le j \le n-1$ define

$$R_j^A := \{1,\, a_j,\, a_j a_{j-1},\, \dots,\, a_j \cdots a_2,\, a_j \cdots a_2 a_1,\, a_j \cdots a_2 a_1^{-1}\},$$

and note that $R_1^A, \dots, R_{n-1}^A \subseteq A_{n+1}$.

**Theorem 2.7 (see [8, Theorem 3.4]).** *Let $v \in A_{n+1}$. Then there exist unique elements $v_j \in R_j^A$, $1 \le j \le n-1$, such that $v = v_1 \dots v_{n-1}$, and this presentation is unique.*

**Definition 2.8 (see [8, Definition 3.5]).** Call $v = v_1 \dots v_{n-1}$ in the above theorem *the $A$ canonical presentation of $v \in A_{n+1}$.*

# 3 The Group of Signed Even Permutations

Our main object of interest in this paper is the group $L_n := C_2 \wr A_n$. It is the subgroup of $B_n$ of index 2 containing the *signed even permutations* (which is not to be confused with the group of even-signed permutations mentioned in Section 5). The order of $L_n$ is $|C_2|^n|A_n| = 2^{n-1}n!$.

**Example 3.1 ($L_3$).** Table 1 lists all the elements of $L_3$ (in window notation) with their $B$ and $L$ canonical presentation and $B$- and $L$-length (defined in the sequel).

| $\pi$ | $B$ canonical presentation | $\ell_B(\pi)$ | $L$ canonical presentation | $\ell_L(\pi)$ |
|---|---|---|---|---|
| $[+1,+2,+3]$ | $1$ | 0 | $1$ | 0 |
| $[-1,+2,+3]$ | $(s_0)$ | 1 | $(a_0)$ | 1 |
| $[+1,-2,+3]$ | $(s_1s_0s_1)$ | 3 | $(a_1a_0a_1^{-1})$ | 2 |
| $[-1,-2,+3]$ | $(s_0)(s_1s_0s_1)$ | 4 | $(a_0a_1a_0a_1^{-1})$ | 3 |
| $[+1,+2,-3]$ | $(s_2s_1s_0s_1s_2)$ | 5 | $(a_1^{-1}a_0a_1)$ | 4 |
| $[-1,+2,-3]$ | $(s_0)(s_2s_1s_0s_1s_2)$ | 6 | $(a_0)(a_1^{-1}a_0a_1)$ | 5 |
| $[+1,-2,-3]$ | $(s_1s_0s_1)(s_2s_1s_0s_1s_2)$ | 8 | $(a_1a_0a_1^{-1})(a_1^{-1}a_0a_1)$ | 6 |
| $[-1,-2,-3]$ | $(s_0)(s_1s_0s_1)(s_2s_1s_0s_1s_2)$ | 9 | $(a_0a_1a_0a_1^{-1})(a_1^{-1}a_0a_1)$ | 7 |
| $[+2,+3,+1]$ | $(s_1)(s_2)$ | 2 | $(a_1)$ | 1 |
| $[-2,+3,+1]$ | $(s_1s_0)(s_2)$ | 3 | $(a_1a_0a_1^{-1})(a_1)$ | 3 |
| $[+2,-3,+1]$ | $(s_1)(s_2s_1s_0s_1)$ | 5 | $(a_1^{-1}a_0a_1^{-1})$ | 4 |
| $[-2,-3,+1]$ | $(s_1s_0)(s_2s_1s_0s_1)$ | 6 | $(a_1a_0a_1^{-1})(a_1^{-1}a_0a_1^{-1})$ | 5 |
| $[+2,+3,-1]$ | $(s_0)(s_1)(s_2)$ | 3 | $(a_0)(a_1)$ | 2 |
| $[-2,+3,-1]$ | $(s_0)(s_1s_0)(s_2)$ | 4 | $(a_0a_1a_0a_1^{-1})(a_1)$ | 4 |
| $[+2,-3,-1]$ | $(s_0)(s_1)(s_2s_1s_0s_1)$ | 6 | $(a_0)(a_1^{-1}a_0a_1^{-1})$ | 5 |
| $[-2,-3,-1]$ | $(s_0)(s_1s_0)(s_2s_1s_0s_1)$ | 7 | $(a_0a_1a_0a_1^{-1})(a_1^{-1}a_0a_1^{-1})$ | 6 |
| $[+3,+1,+2]$ | $(s_2s_1)$ | 2 | $(a_1^{-1})$ | 1 |
| $[-3,+1,+2]$ | $(s_2s_1s_0)$ | 3 | $(a_1^{-1}a_0)$ | 3 |
| $[+3,-1,+2]$ | $(s_0)(s_2s_1)$ | 3 | $(a_0)(a_1^{-1})$ | 2 |
| $[-3,-1,+2]$ | $(s_0)(s_2s_1s_0)$ | 4 | $(a_0)(a_1^{-1}a_0)$ | 4 |
| $[+3,+1,-2]$ | $(s_1s_0s_1)(s_2s_1)$ | 5 | $(a_1a_0a_1^{-1})(a_1^{-1})$ | 3 |
| $[-3,+1,-2]$ | $(s_1s_0s_1)(s_2s_1s_0)$ | 6 | $(a_1a_0a_1^{-1})(a_1^{-1}a_0)$ | 6 |
| $[+3,-1,-2]$ | $(s_0)(s_1s_0s_1)(s_2s_1)$ | 6 | $(a_0a_1a_0a_1^{-1})(a_1^{-1})$ | 4 |
| $[-3,-1,-2]$ | $(s_0)(s_1s_0s_1)(s_2s_1s_0)$ | 7 | $(a_0a_1a_0a_1^{-1})(a_1^{-1}a_0)$ | 7 |

Table 1: $L_3$

## 3.1 Characterization in Terms of the $B$ Canonical Presentation

Define the group homomorphism abs : $C_2 \wr S_n \to S_n$ by $((\epsilon_1, \ldots, \epsilon_n), \sigma) \mapsto \sigma$, or equivalently, in terms of our representation of elements of $C_2 \wr S_n$ as bijections of $\{\pm 1, \ldots, \pm n\}$ onto itself, $\text{abs}(\sigma)(i) := |\sigma(i)|$.

From this formulation one sees immediately that for any $\sigma \in B_n$, $\mathrm{abs}(\sigma s_0) = \mathrm{abs}(\sigma)$. Thus if $\sigma = s_{i_1} \ldots s_{i_k}$, then deleting all occurrences of $s_0$ from $s_{i_1} \ldots s_{i_k}$ what remains is an expression for $\mathrm{abs}(\sigma)$. Since by definition $\mathrm{abs}(L_n) = A_n$, we have the following proposition.

**Proposition 3.2.**

$$L_n = \left\{ \sigma \in B_n \mid \sigma = s_{i_1} \ldots s_{i_k},\ \ \#\{ j \mid i_j \neq 0 \} \textit{ is even} \right\}.$$

## 3.2 Generators and Canonical Presentation

### 3.2.1 A Generating Set for $L_{n+1}$

$L_{n+1}$ is generated by $a_1, \ldots, a_{n-1}$ together with the generator $a_0 := s_0 = [-1, 2, 3, \ldots, n, n+1]$. The additional relations are $a_0^2 = 1$, $(a_0 a_1)^6 = (a_0 a_1^{-1})^6 = 1$, and $(a_0 a_i)^4 = 1$ for all $1 < i \leq n-1$.

### 3.2.2 The $L$ Canonical Presentation

Let $R_0^L := \{1,\, a_0,\, a_1 a_0 a_1^{-1},\, a_0 a_1 a_0 a_1^{-1}\}$ and for each $1 \leq j \leq n-1$ define

$$\begin{aligned} R_j^L := & R_j^A \cup \{a_j a_{j-1} \cdots a_2 a_1^{-1} a_0,\, a_j a_{j-1} \cdots a_2 a_1^{-1} a_0 a_1^{-1}\} \\ & \cup \{a_j a_{j-1} \cdots a_2 a_1^{-1} a_0 a_1,\, \ldots,\, a_j a_{j-1} \cdots a_2 a_1^{-1} a_0 a_1 a_2 \cdots a_j\}. \end{aligned}$$

For example,

$$R_2^L = \{1,\, a_2,\, a_2 a_1,\, a_2 a_1^{-1},\, a_2 a_1^{-1} a_0,\, a_2 a_1^{-1} a_0 a_1^{-1},\, a_2 a_1^{-1} a_0 a_1,\, a_2 a_1^{-1} a_0 a_1 a_2\}.$$

Note that $R_0^L, \ldots, R_{n-1}^L \subseteq L_{n+1}$.

**Theorem 3.3.** *Let $\pi \in L_{n+1}$. Then there exist unique elements $\pi_j \in R_j^L$, $0 \leq j \leq n-1$, such that $\pi = \pi_0 \ldots \pi_{n-1}$, and this presentation is unique.*

A proof is given below.

**Definition 3.4.** Call $\pi = \pi_0 \ldots \pi_{n-1}$ in the above theorem *the $L$ canonical presentation of $\pi \in L_{n+1}$.*

The following recursive **L-Procedure** is a way to calculate the $L$ canonical presentation:

First note that $R_0^L = L_2$ so $R_0^L$ gives the canonical presentations of all $\pi \in L_2$.

For $n > 1$, let $\pi \in L_{n+1}$, $|\pi(r)| = n+1$.

If $\pi(r) = n+1$, 'pull $n+1$ to its place on the right' by

$$\begin{aligned} [\ldots, n+1, \ldots] a_{r-1} a_r \cdots a_{n-1} &= [\ldots\ldots, n+1] \quad \text{if } r > 2\ , \\ [k, n+1, \ldots] a_1^{-1} a_2 \cdots a_{n-1} &= [\ldots\ldots, n+1] \quad \text{if } r = 2\ , \\ (*) \qquad [n+1, \ldots] a_1 a_2 \cdots a_{n-1} &= [\ldots\ldots, n+1] \quad \text{if } r = 1\ ; \end{aligned}$$

and if $\pi(r) = -(n+1)$, 'correct the sign' by

$$\begin{aligned} [\dots, -(n+1), \dots] a_{r-2} \cdots a_1^{-1} a_0 &= [n+1, \dots] \quad \text{if } r > 3\ , \\ [\ell, k, -(n+1), \dots] a_1^{-1} a_0 &= [n+1, \dots] \quad \text{if } r = 3\ , \\ [k, -(n+1), \dots] a_1 a_0 &= [n+1, \dots] \quad \text{if } r = 2\ , \\ [-(n+1), \dots] a_0 &= [n+1, \dots] \quad \text{if } r = 1\ , \end{aligned}$$

and then 'pull to the right' using $(*)$.

This gives $\pi_{n-1} \in R^L_{n-1}$ and $\pi\pi_{n-1}^{-1} \in L_n$. Therefore by induction $\pi = \pi_0 \dots \pi_{n-2}\pi_{n-1}$ with $\pi_j \in R^L_j$ for all $0 \le j \le n-1$.

For example, let $\pi = [3, 5, -4, 2, -1]$, then $\pi_3 = a_3a_2a_1$; $\pi\pi_3^{-1} = [-4, 3, 2, -1, 5]$, therefore $\pi_2 = a_2a_1^{-1}a_0$; next $\pi\pi_3^{-1}\pi_2^{-1} = [2, 3, -1, 4, 5]$ so $\pi_1 = a_1$; and finally $\pi\pi_3^{-1}\pi_2^{-1}\pi_1^{-1} = [-1, 2, 3, 4, 5]$ so $\pi_0 = a_0$. Thus

$$\pi = \pi_0\pi_1\pi_2\pi_3 = (a_0)(a_1)(a_2a_1^{-1}a_0)(a_3a_2a_1).$$

Table 1 gives the $L$ canonical presentation of $L_3$.

*Proof of Theorem 3.3.* The L-Procedure proves the existence of such a presentation, and the uniqueness follows by a counting argument:

$$\prod_{j=0}^{n-1} |R^L_j| = \prod_{j=0}^{n-1} 2(j+2) = 2^n(n+1)! = 2^{n+1}|A_{n+1}| = |L_{n+1}|. \qquad \square$$

**Remark 3.5.** For $\pi \in A_{n+1}$, the $L$ canonical presentation of $\pi$ coincides with its $A$ canonical presentation.

**Remark 3.6.** The canonical presentation of $\pi \in L_{n+1}$ is not necessarily a reduced expression. For example, the canonical presentation of $\pi = [-3, 1, -2] \in L_3$ is $\pi = (a_1a_0a_1^{-1})(a_1^{-1}a_0)$ which is not reduced ($\pi = a_1a_0a_1a_0$).

## 3.3 $B_n$ and $L_{n+1}$ Statistics

**Definition 3.7.** Let $w = [w_1, w_2, \dots, w_n]$ be a word on $\mathbb{Z}$. The *inversion number* of $w$ is defined as $\mathrm{inv}(w) := \#\{\, 1 \le i < j \le n \mid w_i > w_j \,\}$.

For example, $\mathrm{inv}([5, -1, 2, -3, 4]) = 6$.

**Definition 3.8.** 1. Let $\sigma \in B_n$, then $j \ge 2$ is a *l.t.r.min* (left-to-right minimum) of $\sigma$ if $\sigma(i) > \sigma(j)$ for all $1 \le i < j$.

2. Define $\mathrm{del}_B(\sigma) := \#\,\mathrm{ltrm}(\sigma) = \#\{\, 2 \le j \le n \mid j \text{ is a l.t.r.min of } \sigma \,\}$.

For example, the left-to-right minima of $\sigma = [5, -1, 2, -3, 4]$ are $\{2, 4\}$ so $\mathrm{del}_B(\sigma) = 2$.

**Remark 3.9.** The implicit definition of $\mathrm{del}_S(w)$ for $w \in S_n$ in [8, Proposition 7.2] is similar to the above definition of $\mathrm{del}_B$. In particular, if $w \in S_n$ then $\mathrm{del}_S(w) = \mathrm{del}_B(w)$.

**Definition 3.10.** Let $\sigma \in B_n$. Define

$$\mathrm{Neg}(\sigma) := \{\, i \in [n] \mid \sigma(i) < 0 \,\}.$$

**Remark 3.11.** 1. If $v \in S_n$ and $\sigma \in B_n$ then

$$\begin{aligned} \mathrm{Neg}(v\sigma) &= \{\, i \in [n] \mid v(\sigma(i)) < 0 \,\} \\ &= \{\, i \in [n] \mid \sigma(i) < 0 \,\} \\ &= \mathrm{Neg}(\sigma). \end{aligned}$$

2. $\mathrm{Neg}(\sigma^{-1}) = \{\, |\sigma(i)| \mid i \in \mathrm{Neg}(\sigma) \,\}$.

**Definition 3.12.** Let $\sigma \in B_n$. Define the *B-length* of $\sigma$ in the usual way, i.e., $\ell_B(\sigma)$ is the length of $\sigma$ with respect to the Coxeter generators of $B_n$.

For example,

$$\ell_B([5,-1,2,-3,4]) = \ell_B(s_0 s_2 s_1 s_0 s_1 s_2 s_4 s_3 s_2 s_1) = 10$$

(see Example 2.6).

**Lemma 3.13 (see [4, §8.1]).** *Let $\sigma \in B_n$. Then*

$$\ell_B(\sigma) = \mathrm{inv}(\sigma) + \sum_{i \in \mathrm{Neg}(\sigma^{-1})} i. \tag{1}$$

In [8], the *A-length* of $w \in A_n$, $\ell_A(w)$ was defined as the length of $w$'s $A$ canonical presentation, and it was shown to have the following property.

**Proposition 3.14 (see [8, Proposition 4.4]).** *Let $w \in A_n$, then*

$$\ell_A(w) = \ell_S(w) - \mathrm{del}_S(w),$$

*where $\ell_S(w)$ is the length of $w$ with respect to the Coxeter generators of $S_n$.*

This serves as motivation for the following definition.

**Definition 3.15.** Let $\sigma \in B_n$. Define the *L-length* of $\sigma$ as

$$\ell_L(\sigma) := \ell_B(\sigma) - \mathrm{del}_B(\sigma) = \mathrm{inv}(\sigma) - \mathrm{del}_B(\sigma) + \sum_{i \in \mathrm{Neg}(\sigma^{-1})} i. \tag{2}$$

**Remark 3.16.** 1. The function $\ell_L$ is *not* a length function with respect to any set of generators, that is for every set of generators of $L_n$, there exists $\pi \in L_n$ such that $\ell_L(\pi)$ is in not the length of a reduced expression for $\pi$ using those generators. For example, in $L_3$ we have $\ell_L([3,1,2]) = \ell_L([-1,2,3]) = 1$ but $\ell_L([3,1,2][-1,2,3]) = \ell_L([-3,1,2]) = 3$.

2. If $w \in A_n$ then, according to Proposition 3.14 and the above remarks, $\ell_A(w) = \ell_L(w)$.

**Definition 3.17.** 1. The *S-descent set* of $\sigma \in B_n$ is defined by

$$\mathrm{Des}_S(\sigma) := \{\, 1 \le i \le n-1 \mid \sigma(i) > \sigma(i+1) \,\}.$$

2. Define the *major index* of $\sigma \in B_n$ by

$$\mathrm{maj}_B(\sigma) := \sum_{i \in \mathrm{Des}_S(\sigma)} i.$$

3. Define the *reverse major index* of $\sigma \in B_n$ by

$$\mathrm{rmaj}_{B_n}(\sigma) := \sum_{i \in \mathrm{Des}_S(\sigma)} (n-i).$$

For example, if $\sigma = [5,-1,2,-3,4]$ then $\mathrm{Des}_S(\sigma) = \{1,3\}$, $\mathrm{maj}_B(\sigma) = 4$ and $\mathrm{rmaj}_{B_5}(\sigma) = 6$.

**Remark 3.18.** $\mathrm{Des}_S(\sigma) = \{\, 1 \le i \le n-1 \mid \ell_B(\sigma s_i) < \ell_B(\sigma) \,\}$. Indeed, by Remark 3.11 and the definition of inv, for $1 \le i \le n-1$

$$\begin{aligned}
\ell_B(\sigma s_i) - \ell_B(\sigma) &= \big(\mathrm{inv}(\sigma s_i) + \sum_{i \in \mathrm{Neg}((\sigma s_i)^{-1})} i\big) - \big(\mathrm{inv}(\sigma) + \sum_{i \in \mathrm{Neg}(\sigma^{-1})} i\big) \\
&= \mathrm{inv}(\sigma s_i) - \mathrm{inv}(\sigma) \\
&= \begin{cases} +1 & \text{if } \sigma(i) < \sigma(i+1), \\ -1 & \text{if } \sigma(i) > \sigma(i+1). \end{cases}
\end{aligned}$$

The $\mathrm{maj}_B$ and $\mathrm{rmaj}_{B_n}$ statistics are equidistributed on $B_n$, as the following lemma shows.

**Lemma 3.19.** *There exists an involution $\phi$ of $B_n$ satisfying the conditions*

$$\mathrm{maj}_B(\sigma) = \mathrm{rmaj}_{B_n}(\phi(\sigma))$$

*and*

$$\mathrm{Neg}(\sigma^{-1}) = \mathrm{Neg}((\phi(\sigma))^{-1}).. \tag{3}$$

*Proof.* Given $\sigma = [\sigma_1, \ldots, \sigma_n] \in B_n$, $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_n}$, let $\rho_\sigma$ be the order-reversing permutation on $\{\sigma_1, \ldots, \sigma_n\}$, that is $\rho_\sigma(\sigma_{i_k}) = \sigma_{i_{n+1-k}}$, and define

$$\phi(\sigma) = [\rho_\sigma(\sigma_n), \rho_\sigma(\sigma_{n-1}), \ldots, \rho_\sigma(\sigma_1)].$$

Since $\rho_\sigma$ is a permutation, the letters in the window notation of $\phi(\sigma)$ are again $\sigma_1, \ldots, \sigma_n$, so $\rho_{\phi(\sigma)} = \rho_\sigma$. Thus

$$\begin{aligned}
\phi^2(\sigma) &= [\rho_{\phi(\sigma)}(\rho_\sigma(\sigma_1)), \ldots, \rho_{\phi(\sigma)}(\rho_\sigma(\sigma_n))] \\
&= [\rho_\sigma^2(\sigma_1), \ldots, \rho_\sigma^2(\sigma_n)] \\
&= \sigma,
\end{aligned}$$

and by Remark 3.11, $\operatorname{Neg}(\sigma^{-1}) = \operatorname{Neg}(\phi(\sigma)^{-1})$.

Finally,

$$\begin{aligned} i \in \operatorname{Des}_S(\phi(\sigma)) &\iff \phi(\sigma)(i) > \phi(\sigma)(i+1) \\ &\iff \rho_\sigma(\sigma_{n+1-i}) > \rho_\sigma(\sigma_{n-i}) \\ &\iff \sigma_{n+1-i} < \sigma_{n-i} \\ &\iff n-i \in \operatorname{Des}_S(\sigma), \end{aligned}$$

So

$$\operatorname{rmaj}_{B_n}(\phi(\sigma)) = \sum_{i \in \operatorname{Des}_S(\phi(\sigma))} n-i = \sum_{i \in \operatorname{Des}_S(\sigma)} i = \operatorname{maj}_B(\sigma). \qquad \square$$

**Example 3.20.** Let $\sigma = [5, -1, 2, -3, 4]$. To compute $\phi(\sigma)$, we first reverse $\sigma$ to get $[4, -3, 2, -1, 5]$, then apply the order-reversing permutation on $\{-3, -1, 2, 4, 5\}$ to get $\phi(\sigma) = [-1, 5, 2, 4, -3]$. Indeed we have $\operatorname{maj}_B(\sigma) = 4 = \operatorname{rmaj}_{B_5}(\phi(\sigma))$ and $\operatorname{Neg}(\sigma^{-1}) = \{1, 3\} = \operatorname{Neg}(\phi(\sigma)^{-1})$.

**Definition 3.21.** 1. The *A-descent set* of $\pi \in L_{n+1}$ is defined by

$$\operatorname{Des}_A(\pi) := \{\, 1 \le i \le n-1 \mid \ell_L(\pi a_i) \le \ell_L(\pi) \,\},$$

and the *A-descent number* of $\pi \in L_{n+1}$ is defined by $\operatorname{des}_A(\pi) := |\operatorname{Des}_A \pi|$.

2. Define the *alternating reverse major index* of $\pi \in L_{n+1}$ by

$$\operatorname{rmaj}_{L_{n+1}}(\pi) := \sum_{i \in \operatorname{Des}_A(\pi)} (n-i).$$

3. Define the *negative alternating reverse major index* of $\pi \in L_{n+1}$ by

$$\operatorname{nrmaj}_{L_{n+1}}(\pi) := \operatorname{rmaj}_{L_{n+1}}(\pi) + \sum_{i \in \operatorname{Neg}(\pi^{-1})} i.$$

For example, if $\pi = [5, -1, 2, -3, 4]$ then $\operatorname{Des}_A(\pi) = \{1, 2\}$, $\operatorname{rmaj}_{L_5}(\pi) = 5$, and $\operatorname{nrmaj}_{L_5}(\pi) = 5 + 1 + 3 = 9$.

**Remark 3.22.** 1. For $w \in A_{n+1}$, the above definitions agree with [8, Definition 1.5].

2. In general, $\operatorname{Des}_A(\pi) \ne \{\, 1 \le i \le n-1 \mid \pi(i) > \pi(i+1) \,\}$.

# 4 Equidistribution on $L_{n+1}$

The following is our main result.

**Proposition 4.1.** *For every $B \subseteq [n+1]$*

$$\begin{aligned} \sum_{\{\, \pi \in L_{n+1} \mid \operatorname{Neg}(\pi^{-1}) \subseteq B \,\}} q^{\operatorname{nrmaj}_{L_{n+1}}(\pi)} &= \sum_{\{\, \pi \in L_{n+1} \mid \operatorname{Neg}(\pi^{-1}) \subseteq B \,\}} q^{\ell_L(\pi)} \\ &= \prod_{i \in B} (1 + q^i) \prod_{i=1}^{n-1} (1 + q + \cdots + q^{i-1} + 2q^i). \end{aligned}$$

**Example 4.2.** For $n = 3$ and $B = \{2\}$ we have

$$\sum_{\{\pi \in L_4 | \mathrm{Neg}(\pi^{-1}) \subseteq \{2\}\}} q^{\mathrm{nrmaj}_{L_4}(\pi)} = \sum_{\{\pi \in L_4 | \mathrm{Neg}(\pi^{-1}) \subseteq \{2\}\}} q^{\ell_L(\pi)}$$
$$= (1+q^2)(1+2q)(1+q+2q^2) = 1 + 3q + 5q^2 + 7q^3 + 4q^4 + 4q^5,$$

as one may verify using Table 2.

| $\pi$ | $\mathrm{nrmaj}_{L_4}(\pi)$ | $\ell_L(\pi)$ |
|---|---|---|
| $[+1,+2,+3,+4]$ | 0 | 0 |
| $[+1,+3,+4,+2]$ | 1 | 2 |
| $[+1,+4,+2,+3]$ | 2 | 2 |
| $[+2,+1,+4,+3]$ | 1 | 1 |
| $[+2,+3,+1,+4]$ | 2 | 1 |
| $[+2,+4,+3,+1]$ | 3 | 3 |
| $[+3,+1,+2,+4]$ | 2 | 1 |
| $[+3,+2,+4,+1]$ | 1 | 2 |
| $[+3,+4,+1,+2]$ | 3 | 3 |
| $[+4,+1,+3,+2]$ | 3 | 3 |
| $[+4,+2,+1,+3]$ | 2 | 2 |
| $[+4,+3,+2,+1]$ | 3 | 3 |
| $[+1,-2,+3,+4]$ | 2 | 2 |
| $[+1,+3,+4,-2]$ | 3 | 4 |
| $[+1,+4,-2,+3]$ | 4 | 4 |
| $[-2,+1,+4,+3]$ | 3 | 3 |
| $[-2,+3,+1,+4]$ | 4 | 3 |
| $[-2,+4,+3,+1]$ | 5 | 5 |
| $[+3,+1,-2,+4]$ | 4 | 3 |
| $[+3,-2,+4,+1]$ | 3 | 4 |
| $[+3,+4,+1,-2]$ | 5 | 5 |
| $[+4,+1,+3,-2]$ | 5 | 5 |
| $[+4,-2,+1,+3]$ | 4 | 4 |
| $[+4,+3,-2,+1]$ | 5 | 5 |

Table 2: $\{\pi \in L_4 \mid \mathrm{Neg}(\pi^{-1}) \subseteq \{2\}\}$

By the Inclusion-Exclusion Principle we have:

**Corollary 4.3.** *For every $B \subseteq [n+1]$*

$$\sum_{\{\pi \in L_{n+1} | \mathrm{Neg}(\pi^{-1}) = B\}} q^{\mathrm{nrmaj}_{L_{n+1}}(\pi)} = \sum_{\{\pi \in L_{n+1} | \mathrm{Neg}(\pi^{-1}) = B\}} q^{\ell_L(\pi)}.$$

Note that the case $B = \emptyset$ of Proposition 4.1 is just the case $t = 1$ of the following theorem.

**Theorem 4.4 (see [8, Theorem 6.1(2)]).**

$$\sum_{w \in A_{n+1}} q^{\ell_A(w)} t^{\mathrm{del}_A(w)} = \sum_{w \in A_{n+1}} q^{\mathrm{rmaj}_{A_{n+1}}(w)} t^{\mathrm{del}_A(w)}$$
$$= (1 + 2qt)(1 + q + 2q^2 t) \cdots (1 + q + \cdots + q^{n-2} + 2q^{n-1} t).$$

The proof of Proposition 4.1 uses the decomposition of

$$\{\, \pi \in L_{n+1} \mid \mathrm{Neg}(\pi^{-1}) \subseteq B \,\}$$

into left cosets of $A_{n+1}$, and a set of distinguished coset representatives.

**Lemma 4.5.** *Let $w \in S_{n+1}$. Then*

$$\ell_L(w) = \ell_L(s_1 w).$$

*Proof.*

$$\mathrm{inv}(s_1 w) = \begin{cases} \mathrm{inv}(w) + 1 & \text{if } w^{-1}(1) < w^{-1}(2); \\ \mathrm{inv}(w) - 1 & \text{if } w^{-1}(1) > w^{-1}(2) \end{cases}$$

and

$$\mathrm{del}_B(s_1 w) = \begin{cases} \mathrm{del}_B(w) + 1 & \text{if } w^{-1}(1) < w^{-1}(2); \\ \mathrm{del}_B(w) - 1 & \text{if } w^{-1}(1) > w^{-1}(2), \end{cases}$$

therefore

$$\ell_L(w) = \mathrm{inv}(w) - \mathrm{del}_B(w) = \mathrm{inv}(s_1 w) - \mathrm{del}_B(s_1 w) = \ell_L(s_1 w). \qquad \square$$

**Lemma 4.6 (Main Lemma).** *Let $\pi \in L_{n+1}$. Then there exists a unique $\sigma \in L_{n+1}$ such that $u = \sigma^{-1}\pi \in A_{n+1}$ and $\mathrm{des}_A(\sigma) = 0$. Moreover, $\mathrm{Des}_A(u) = \mathrm{Des}_A(\pi)$, $\mathrm{inv}(u) - \mathrm{del}_S(u) = \mathrm{inv}(\pi) - \mathrm{del}_B(\pi)$, and $\mathrm{Neg}(\pi^{-1}) = \mathrm{Neg}(\sigma^{-1})$.*

*Proof.* Let $\sigma' \in B_{n+1}$ be the increasing word with the letters of $\pi$. Clearly $\mathrm{inv}(\sigma') = \mathrm{del}_B(\sigma') = 0$ so by (2), $\ell_L(\sigma') = \sum_{i \in \mathrm{Neg}(\sigma'^{-1})} i$.

For every $v \in S_{n+1}$ and $i, j \in [n+1]$,

$$v(i) < v(j) \iff (\sigma' v)(i) < (\sigma' v)(j),$$

thus

$$\mathrm{inv}(\sigma' v) = \mathrm{inv}(v) \tag{4}$$

and

$$\mathrm{del}_B(\sigma' v) = \mathrm{del}_S(v). \tag{5}$$

By Remark 3.11, $\mathrm{Neg}((\sigma' v)^{-1}) = \mathrm{Neg}(v^{-1}\sigma'^{-1}) = \mathrm{Neg}(\sigma'^{-1})$. Therefore for every $v \in S_{n+1}$,

$$\begin{aligned}
\ell_L(\sigma' v) &= \mathrm{inv}(\sigma' v) + \sum_{i \in \mathrm{Neg}((\sigma' v)^{-1})} i \quad - \mathrm{del}_B(\sigma' v) \\
&= \sum_{i \in \mathrm{Neg}(\sigma'^{-1})} i \quad + \mathrm{inv}(v) - \mathrm{del}_B(v) \qquad (6) \\
&= \ell_L(\sigma') + \ell_L(v).
\end{aligned}$$

There are two possible cases to consider:
Case 1: $\sigma' \in L_{n+1}$. Let $\sigma = \sigma'$ and let $u = \sigma'^{-1}\pi$.
Using (6) we have for $1 \le i \le n-1$,

$$\begin{aligned}
\ell_L(\sigma a_i) &= \ell_L(\sigma' a_i) \\
&= \ell_L(\sigma') + \ell_L(a_i) \\
&> \ell_L(\sigma') \\
&= \ell_L(\sigma)
\end{aligned}$$

and

$$\begin{aligned}
\ell_L(\pi) - \ell_L(\pi a_i) &= \ell_L(\sigma u) - \ell_L(\sigma(u a_i)) \\
&= \ell_L(\sigma' u) - \ell_L(\sigma'(u a_i)) \\
&= \ell_L(\sigma') + \ell_L(u) - \ell_L(\sigma') - \ell_L(u a_i) \\
&= \ell_L(u) - \ell_L(u a_i).
\end{aligned}$$

Therefore $\mathrm{des}_A(\sigma) = 0$ and $\mathrm{Des}_A(u) = \mathrm{Des}_A(\pi)$ as desired. From (4) and (5) we also get that

$$\begin{aligned}
\mathrm{inv}(\pi) - \mathrm{del}_B(\pi) &= \mathrm{inv}(\sigma u) - \mathrm{del}_B(\sigma u) \\
&= \mathrm{inv}(\sigma' u) - \mathrm{del}_B(\sigma' u) \\
&= \mathrm{inv}(u) - \mathrm{del}_S(u).
\end{aligned}$$

Case 2: $\sigma' s_1 \in L_{n+1}$. Let $\sigma = \sigma' s_1$ and let $u = s_1 \sigma'^{-1}\pi$.
Using (6) we have for $1 \le i \le n-1$,

$$\begin{aligned}
\ell_L(\sigma a_i) &= \ell_L(\sigma' s_{i+1}) \\
&= \ell_L(\sigma') + \ell_L(s_{i+1}) \\
&> \ell_L(\sigma') \\
&= \ell_L(\sigma') + \ell_L(s_1) && (\ell_L(s_1) = 0) \\
&= \ell_L(\sigma' s_1) \\
&= \ell_L(\sigma)
\end{aligned}$$

and, using also Lemma 4.5,

$$\begin{aligned}\ell_L(\pi) - \ell_L(\pi a_i) &= \ell_L(\sigma' s_1 u) - \ell_L(\sigma'(s_1 u a_i)) \\ &= \ell_L(\sigma') + \ell_L(s_1 u) - \ell_L(\sigma') - \ell_L(s_1 u a_i) \\ &= \ell_L(s_1 u) - \ell_L(s_1(u a_i)) \\ &= \ell_L(u) - \ell_L(u a_i).\end{aligned}$$

Therefore $\mathrm{des}_A(\sigma) = 0$ and $\mathrm{Des}_A(u) = \mathrm{Des}_A(\pi)$ as desired. From (4) and (5) and Lemma 4.5,

$$\begin{aligned}\mathrm{inv}(\pi) - \mathrm{del}_B(\pi) &= \mathrm{inv}(\sigma' s_1 u) - \mathrm{del}_B(\sigma' s_1 u) \\ &= \mathrm{inv}(s_1 u) - \mathrm{del}_S(s_1 u) \\ &= \mathrm{inv}(u) - \mathrm{del}_S(u).\end{aligned}$$

In both cases, the fact that $\mathrm{Neg}(\pi^{-1}) = \mathrm{Neg}(\sigma^{-1})$ follows by Remark 3.11 from the fact that $\pi^{-1} = u^{-1}\sigma^{-1}$ and $u \in A_{n+1}$.

To see that $\sigma$ is unique, suppose $\tilde{\sigma} \in L_{n+1}$ satisfies $\mathrm{des}_A(\tilde{\sigma}) = 0$ and $\tilde{u} = \tilde{\sigma}^{-1}\pi \in A_{n+1}$. Then $0 = \mathrm{des}_A(\tilde{\sigma}) = \mathrm{des}_A(\sigma u \tilde{u}^{-1})$ (since $\tilde{\sigma} = \sigma u \tilde{u}^{-1}$), so for $1 \le i \le n-1$,

$$\begin{aligned}0 &\le \ell_L(\sigma u \tilde{u}^{-1} a_i) - \ell_L(\sigma u \tilde{u}^{-1}) \\ &= \ell_L(\sigma) + \ell_L(u \tilde{u}^{-1} a_i) - \ell_L(\sigma) - \ell_L(u \tilde{u}^{-1}) \\ &= \ell_L(u \tilde{u}^{-1} a_i) - \ell_L(u \tilde{u}^{-1}) \\ &= \ell_A(u \tilde{u}^{-1} a_i) - \ell_A(u \tilde{u}^{-1}),\end{aligned}$$

whence $u\tilde{u}^{-1} = 1$, i.e. $\sigma = \tilde{\sigma}$. □

Let $T = \{\, \sigma \in L_{n+1} \mid \mathrm{des}_A(\sigma) = 0 \,\}$.

**Corollary 4.7.** *1. For every $B \subseteq [n+1]$ there exists a unique $\sigma \in T$ such that $B = \mathrm{Neg}(\sigma^{-1})$.*

*2. For every $B \subseteq [n+1]$,*

$$\{\, \pi \in L_n \mid \mathrm{Neg}(\pi^{-1}) \subseteq B \,\} = \biguplus_{u \in A_{n+1}} \{\, \sigma u \mid \sigma \in T,\ \ \mathrm{Neg}(\sigma^{-1}) \subseteq B \,\}, \tag{7}$$

*where $\uplus$ denotes disjoint union.*

**Corollary 4.8.** *Let $\pi \in L_{n+1}$, and write $\pi = \sigma u$ with $\sigma$ and $u$ like in Lemma 4.6. Then $\ell_L(\pi) = \ell_A(u) + \sum_{i \in \mathrm{Neg}(\pi^{-1})} i$ .*

*Proof.* By (2), Lemma 4.6 and Proposition 3.14,

$$\begin{aligned}\ell_L(\pi) &= \mathrm{inv}(\pi) - \mathrm{del}_B(\pi) + \sum_{i \in \mathrm{Neg}(\pi^{-1})} i \\ &= \mathrm{inv}(u) - \mathrm{del}_S(u) + \sum_{i \in \mathrm{Neg}(\pi^{-1})} i \\ &= \ell_A(u) + \sum_{i \in \mathrm{Neg}(\pi^{-1})} i.\end{aligned}$$

□

*Proof of Proposition 4.1.* From Corollary 4.7, Lemma 4.6 and Theorem 4.4,

$$\begin{aligned}
\sum_{\substack{\pi\in L_{n+1}\\ \mathrm{Neg}(\pi^{-1})\subseteq B}} q^{\mathrm{nrmaj}_{L_{n+1}}(\pi)} &= \sum_{\substack{\sigma\in T\\ \mathrm{Neg}(\sigma^{-1})\subseteq B}} \sum_{u\in A_{n+1}} q^{\mathrm{nrmaj}_{L_{n+1}}(\sigma u)} \\
&= \sum_{\substack{\sigma\in T\\ \mathrm{Neg}(\sigma^{-1})\subseteq B}} \sum_{u\in A_{n+1}} q^{\mathrm{rmaj}_L(\sigma u)+\sum_{i\in\mathrm{Neg}((\sigma u)^{-1})} i} \\
&= \sum_{\substack{\sigma\in T\\ \mathrm{Neg}(\sigma^{-1})\subseteq B}} q^{\sum_{i\in\mathrm{Neg}(\sigma^{-1})} i} \sum_{u\in A_{n+1}} q^{\mathrm{rmaj}_A(u)} \\
&= \sum_{C\subseteq B} q^{\sum_{i\in C} i} \sum_{u\in A_{n+1}} q^{\mathrm{rmaj}_A(u)} \\
&= \prod_{i\in B}(1+q^i)\prod_{i=1}^{n-1}(1+q+\cdots+q^{i-1}+2q^i).
\end{aligned}$$

By similar considerations, this time invoking the other equality in Theorem 4.4,

$$\begin{aligned}
\sum_{\substack{\pi\in L_{n+1}\\ \mathrm{Neg}(\pi^{-1})\subseteq B}} q^{\ell_L(\pi)} &= \sum_{\substack{\sigma\in T\\ \mathrm{Neg}(\sigma^{-1})\subseteq B}} \sum_{u\in A_{n+1}} q^{\ell_L(\sigma u)} \\
&= \sum_{\substack{\sigma\in T\\ \mathrm{Neg}(\sigma^{-1})\subseteq B}} \sum_{u\in A_{n+1}} q^{\mathrm{inv}(\sigma u)+\sum_{i\in\mathrm{Neg}((\sigma u)^{-1})} i-\mathrm{del}_B(\sigma u)} \\
&= \sum_{\substack{\sigma\in T\\ \mathrm{Neg}(\sigma^{-1})\subseteq B}} q^{\sum_{i\in\mathrm{Neg}(\sigma^{-1})} i} \sum_{u\in A_{n+1}} q^{\mathrm{inv}(u)-\mathrm{del}_S(u)} \\
&= \sum_{C\subseteq B} q^{\sum_{i\in C} i} \sum_{u\in A_{n+1}} q^{\ell_A(u)} \\
&= \prod_{i\in B}(1+q^i)\prod_{i=1}^{n-1}(1+q+\cdots+q^{i-1}+2q^i). \qquad \square
\end{aligned}$$

# 5 Even-signed Even Permutations

We denote by $D_n$ the group of *even-signed permutations*, that is the subgroup of $B_n$ consisting of all the signed permutations having an even number of negative entries in their window notation. Equivalently,

$$D_n = \{\, \sigma \in B_n \mid \#\,\mathrm{Neg}(\sigma^{-1}) \text{ is even}\,\}.$$

$D_n$ is a Coxeter group of type $D$, generated by $\tilde{s}_0, s_1, \ldots, s_{n-1}$, where $\tilde{s}_0 = s_0 s_1 s_0 = [-2, -1, 3, \ldots, n]$ (see, for example, [4, §8.2]).

Following Biagioli [3], we define the *D-length* of $\sigma \in D_n$ by

$$\ell_D(\sigma) = \ell_B(\sigma) - \#\operatorname{Neg}(\sigma),$$

which is also the length of a reduced expression for $\sigma$ in the above generators (see [4, §8.2] for a proof), and we let

$$\operatorname{dmaj}(\sigma) = \operatorname{maj}_B(\sigma) - \#\operatorname{Neg}(\sigma) + \sum_{i \in \operatorname{Neg}(\sigma^{-1})} i.$$

Biagioli proved the following $D_n$-analogue of MacMahon's theorem.

**Proposition 5.1 (see [3, Proposition 3.1]).**

$$\sum_{\sigma \in D_n} q^{\operatorname{dmaj}(\sigma)} = \sum_{\sigma \in D_n} q^{\ell_D(\sigma)}.$$

Let

$$\operatorname{drmaj}_n(\sigma) = \operatorname{rmaj}_{B_n}(\sigma) - \#\operatorname{Neg}(\sigma) + \sum_{i \in \operatorname{Neg}(\sigma^{-1})} i.$$

Since the involution $\phi$ from Lemma 3.19 satisfies the condition (3), dmaj and $\operatorname{drmaj}_n$ are equidistributed on $D_n$, hence we can replace dmaj with $\operatorname{drmaj}_n$ in Proposition 5.1.

Let $(L \cap D)_{n+1} = L_{n+1} \cap D_{n+1}$, the group of even-signed even permutations on $\pm 1, \ldots, \pm(n+1)$, and let

$$\ell_{(L \cap D)}(\pi) = \ell_D(\pi) - \operatorname{del}_B(\pi)$$

and

$$\operatorname{drmaj}_{(L\cap D)_{n+1}}(\pi) = \operatorname{rmaj}_{L_{n+1}}(\pi) - \#\operatorname{Neg}(\pi) + \sum_{i \in \operatorname{Neg}(\pi^{-1})} i.$$

**Proposition 5.2.**

$$\sum_{\pi \in (L\cap D)_{n+1}} q^{\operatorname{drmaj}_{(L\cap D)_{n+1}}(\pi)} = \sum_{\pi \in (L\cap D)_{n+1}} q^{\ell_{(L\cap D)}(\pi)}.$$

*Proof.* From the definitions and from Corollary 4.3 we have for every $i$

$$\begin{aligned}
\sum_{\substack{\pi \in L_{n+1} \\ \#\operatorname{Neg}(\pi^{-1})=2i}} q^{\operatorname{drmaj}_{(L\cap D)_{n+1}}(\pi)} &= \sum_{\substack{\pi \in L_{n+1} \\ \#\operatorname{Neg}(\pi^{-1})=2i}} q^{\operatorname{nrmaj}_{L_{n+1}}(\pi) - \#\operatorname{Neg}(\pi)} \\
&= q^{-2i} \sum_{\substack{B \subseteq [n+1] \\ |B|=2i}} \sum_{\substack{\pi \in L_{n+1} \\ \operatorname{Neg}(\pi^{-1})=B}} q^{\operatorname{nrmaj}_{L_{n+1}}(\pi)} \\
&= q^{-2i} \sum_{\substack{B \subseteq [n+1] \\ |B|=2i}} \sum_{\substack{\pi \in L_{n+1} \\ \operatorname{Neg}(\pi^{-1})=B}} q^{\ell_L(\pi)} \\
&= \sum_{\substack{\pi \in L_{n+1} \\ \#\operatorname{Neg}(\pi^{-1})=2i}} q^{\ell_L(\pi) - \#\operatorname{Neg}(\pi)} \\
&= \sum_{\substack{\pi \in L_{n+1} \\ \#\operatorname{Neg}(\pi^{-1})=2i}} q^{\ell_{(L\cap D)}(\pi)}.
\end{aligned}$$

Taking the sum over all $i$ we get the desired equality. $\square$

# 6 Open Problems

The following questions arise quite naturally when considering what is known for $S_n$ and $B_n$ and comparing our results for $L_{n+1}$ with the results for $A_{n+1}$ from [8]. However, they remain open.

1. Is it possible to define a descent number $\mathrm{des}_L$ on $L_{n+1}$ for which a theorem like Corollary 1.11 in [8], that is

$$\sum_{\pi \in L_{n+1}} q_1^{\mathrm{nrmaj}_{L_{n+1}}(\pi)} q_2^{\mathrm{des}_L(\pi^{-1})} = \sum_{\pi \in L_{n+1}} q_1^{\ell_L(\pi)} q_2^{\mathrm{des}_L(\pi^{-1})}$$

   holds?

2. The statistic $\mathrm{del}_S$ (resp. $\mathrm{del}_A$), as defined in [8], has an algebraic interpretation as the number of occurrences of $s_1$ (resp. $a_1^{\pm 1}$) in the canonical presentation of an element. Is there an interpretation of $\mathrm{del}_B(\sigma)$ based on counting occurrences of generators in the $B$ canonical presentation of $\sigma$? Alternatively, is there another canonical presentation of $B_n$ for which $\mathrm{del}_B$ has such a meaning?

3. For $\pi \in L_{n+1}$ one can define *length*$(\pi)$, the length of $\pi$ with respect to the set of generators $\{a_0, a_1, \ldots, a_{n-1}\}$, and then proceed to define a notion of descent. Is there a closed formula for *length*$(\pi)$? How does it relate to $\ell_L(\pi)$?


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