On the non-holonomic character of logarithms, powers, and the nth prime function

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Es ist eine Tatsache, daß die genauere Kenntnis
des Verhaltens einer analytischen Funktion
in der Nähe ihrer singulären Stellen
eine Quelle von arithmetischen Sätzen ist.[1]

— Erich Hecke [27, Kap. VIII]

Abstract

We establish that the sequences formed by logarithms and by "fractional" powers of integers, as well as the sequence of prime numbers, are non-holonomic, thereby answering three open problems of Gerhold [*El. J. Comb.* **11** (2004), R87]. Our proofs depend on basic complex analysis, namely a conjunction of the Structure Theorem for singularities of solutions to linear differential equations and of an Abelian theorem. A brief discussion is offered regarding the scope of singularity-based methods and several naturally occurring sequences are proved to be non-holonomic.

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[1] *"It is a fact that the precise knowledge of the behaviour of an analytic function in the vicinity of its singular points is a source of arithmetic properties."*

Introduction

A sequence $(f_n)_{n\geq 0}$ of complex numbers is said to be *holonomic* (or *P-recursive*) if it satisfies a linear recurrence with coefficients that are polynomial in the index n, that is,

$$p_0(n)f_{n+d} + p_1(n)f_{n+d-1} + \cdots + p_d(n)f_n = 0, \qquad n \geq 0, \tag{1}$$

for some polynomials $p_j(X) \in \mathbb{C}[X]$. A formal power series $f(z) = \sum_{n\geq 0} f_n z^n$ is *holonomic* (or *∂-finite*) if it satisfies a linear differential equation with coefficients that are polynomial in the variable z, that is,

$$q_0(z)\frac{d^e}{dz^e}f(z) + q_1(z)\frac{d^{e-1}}{dz^{e-1}}f(z) + \cdots + q_e(z)f(z) = 0, \tag{2}$$

for some polynomials $q_k(X) \in \mathbb{C}[X]$. It is well known that a sequence is holonomic if and only if its generating series is holonomic. (See Stanley's book [41] for the basic properties of these sequences and series.) By extension, a function analytic at 0 is called holonomic if its power series representation is itself holonomic.

Holonomic sequences encapsulate many of the combinatorial sequences of common interest, for instance, a wide class of sums involving binomial coefficients. At the same time, they enjoy a varied set of closure properties and several formal mechanisms have been recognised to lead systematically to holonomic sequences—what we have in mind here includes finite state models and regular grammars leading to rational (hence, holonomic) functions, context-free specifications leading to algebraic (hence, holonomic) functions, the multivariate holonomic framework initiated by Lipshitz and Zeilberger [32, 33, 47], a wide class of problems endowed with symmetry conducive to holonomic functions (via Gessel's theory [24]). On these aspects, we may rely on general references like [20, 41] as well as on many works of Zeilberger, who is to be held accountable for unearthing the power of the holonomic framework accross combinatorics; see [36, 47]. Thus, in a way, a non-holonomicity result represents some sort of a structural complexity lower bound.

This note answers three problems described as open in an article of Stefan Gerhold [23] very recently published in the *Electronic Journal of Combinatorics.*

Proposition 1. *The sequence $f_n = \log n$ is not holonomic.*

(For definiteness, we agree that $\log 0 \equiv 0$ here.)

For α an integer, the sequence

$$h_n = n^\alpha$$

is clearly holonomic. (As a matter of fact, the generating function is rational if $\alpha \in \mathbb{Z}_{\geq 0}$ and of polylogarithmic type if $\alpha \in \mathbb{Z}_{<0}$.) Gerhold [23] proved that for any α that is rational but not integral, h_n fails to be holonomic. For instance, $h_n = \sqrt{n}$ fails to be holonomic because, in essence, $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \ldots)$ is not a finite extension of $\mathbb{Q}$.

Proposition 2. *For $\alpha \in \mathbb{C}$, the sequence of powers $h_n = n^\alpha$ is holonomic if and only if $\alpha \in \mathbb{Z}$.*

(We agree that $h_0 = 0$.)

Proposition 3. *The sequence g_n defined by the fact that g_n is the nth prime is non-holonomic.*

(We agree that $g_0 = 1$, $g_1 = 2$, $g_2 = 3$, $g_3 = 5$, and so on.)

Proposition 1, conjectured by Gerhold in [23] was only proved under the assumption that a difficult conjecture of number theory (Schanuel's conjecture) holds. The author of [23] describes the statement of our Proposition 2 as a *"natural conjecture"*. Proposition 3 answers an explicit question of Gerhold who writes: *"we do not know of any proof that the sequence of primes is non-holonomic"*.

Our proofs are plainly based on the combination of two facts. First, holonomic objects satisfy rich closure properties. In particular, we make use of closure of under sum, product and composition with an algebraic function. Second, the asymptotic behaviour of holonomic sequences, which is reflected by the asymptotic behaviour at singularities of their generating functions, is rather strongly constrained. For instance, iterated logarithms or negative powers of logarithms are "forbidden" and estimates like

$$a_n \underset{n\to+\infty}{\sim} \log\log n, \qquad b(z) \underset{z\to 1-}{\sim} \frac{1}{\log(1-z)}, \tag{3}$$

are sufficient to conclude that the sequence (a_n) and the function $b(z)$ are non-holonomic. (See details below.) A conjunction of the previous two ideas then perfectly describes the strategy of this note: In order to prove that a sequence is non-holonomic, it suffices to locate a "derived sequence" (produced by holonomicity-preserving transformations) that exhibits a suitable combination of kosher asymptotic terms with a foreign non-holonomic element, like in (3).

We choose here to operate directly with generating functions. Under this scenario, one can rely on the well established classification of singularities elaborated at the end of the nineteenth century by Fuchs [22], Fabry [14] and others. A summary of what is known is found in standard treatises, for instance the ones by Wasow [44] and Henrici [28]. What we need of this theory is summarised by Theorem 2 of the next section.

The relation between asymptotic behaviour of sequences and local behaviour of the generating functions is provided by a classical Abelian theorem, stated as Theorem 3 below.

Note 1. A methodological remark is in order at this stage. A glance at (3) suggests two possible paths for proving non-holonomicity: one may *a priori* operate equally well with sequences or with generating functions. The latter is what we have opted to do here. The former approach with sequences seems workable, but it requires a strong structure theorem analogous to Theorem 2 below for recurrences, i.e., difference equations. An ambitious programme towards such a goal was undertaken by Birkhoff and Trjitzinsky [5, 6] in the 1930's, their works being later followed by Wimp and Zeilberger in [45]. However, what is available in the classical literature is largely a set of *formal* solutions to difference equations and recurrences, and the relation of these to actual (analytic) solutions represents a difficult problem evoked in [45, p. 168] and [35, p. 1138]; see also [8] for recent results relying on multisummability. $\lhd$

Note 2. In this short paper, we do nothing but assemble some rather well-known facts of complex analysis, and logically organise them towards the goal of proving certain sequences to be non-holonomic. Our purpose is thus essentially pedagogical. As it should become transparent soon, a rough heuristic in this range of problem is the following: *Almost anything is non-holonomic unless it is holonomic by design.* (This naïve remark cannot of course be universally true and there are surprises, e.g., some sequences may eventually admit algebraic or holonomic descriptions for rather deep reasons. Amongst such cases, we count the enumeration of k-regular graphs and various types of maps [24, 25], the enumeration of permutations with bounded-length increasing subsequences, the Apéry sequence [42] related to a continued fraction expansion of $\zeta(3)$, as well as the appearance of holonomic functions in the theory of modular forms, for which we refer to the beautiful exposition of Kontsevich-Zagier [30].) $\lhd$

1 Methods

From the most basic theorems regarding the existence of analytic solutions to differential equations (e.g., [28, Th. 9.1]), any function $f(z)$ analytic at 0 that is holonomic can be continued analytically along any path that avoids the finite set Σ of points defined as roots of the equation $p_0(z) = 0$, where p_0 is the leading coefficient in (1). Figuratively:

Theorem 1 (Finiteness of singularities). *A holonomic function has only finitely many singularities.*

This theorem gives immediately as non-holonomic a number of sequences enumerating classical combinatorial structures.

— Integer partitions, whose generating function is $P(z) = \prod(1-z^n)^{-1}$, as the function admits the unit circle as a natural boundary. The same argument applies to integer partitions with summands restricted to any infinite set (e.g., primes), partitions into distict summands, plane partitions, and so on. More generally, combinatorial classes defined by an unlabelled set or multiset construction [20] are non-holonomic, unless a rather drastic combinatorial simplification occurs.

— Alternating (also known as zig-zag, up-and-down, cf[2] *EIS* A000111) permutations with exponential generating function $\tan z + \sec z$, as they have the odd multiples of $\frac{\pi}{2}$ as set of poles[3]. A similar argument applies to preferential arrangements (also

[2] In order to keep this note finite, we refer to some of the combinatorial problems by means of their number in Sloane's *Encyclopedia of Integer Sequences (EIS)*, see [39].

[3] Stanley [40] describes an algebraic proof dependent on the fact that $\exp(z)$ is nonalgebraic (his Example 4.5), then goes on to observe in his §4.a that $\sec z$ "*has infinitely many poles*".

known as ordered set partitions or surjections, cf *EIS* A000670), Bernoulli numbers, and the like.

— Necklaces (equivalently Lyndon words, irreducible polynomials over finite fields), whose generating function admits the unit circle as a natural boundary. More generally, "most" unlabelled cycles are non-holonomic.

— Unlabelled plane trees (*EIS* A000081), whose implicit specification involves an unlabelled multiset construction.

— Knight's walks in the quarter plane (*EIS* A057790). This is described by a simple two-dimensional linear recurrence with constant coefficients. Bousquet-Mélou and Petkovšek [7] have established that an associated generating function has infinitely many singularities.

In many cases, the criterion above is too brutal. For instance it does not preclude holonomicity for the Cayley tree function,

$$T(z) = \sum_{n\geq 1} n^{n-1}\frac{z^n}{n!}. \tag{4}$$

Indeed, the (multivalued) function $T(z)$ has singularities at $0, \infty, e^{-1}$ only.

A major theorem constrains the possible growth of a holonomic function near any of its singularities. Paraphrasing Theorem 19.1 of [44, p. 111], we can state:

Theorem 2 (Structure Theorem for singularities). *Let there be given a differential equation of the form* (2)*, a singular point z_0, and a sector S with vertex at z_0. Then, for z in a sufficiently narrow subsector S' of S and for $|z-z_0|$ sufficiently small, there exists a basis of d linearly independent solutions to* (2)*, such that any solution Y in that basis admits as $z \to z_0$ in the subsector an asymptotic expansion of the form*

$$Y \sim \exp\left(P(Z^{-1/r})\right) z^{\alpha} \sum_{j=0}^{\infty} Q_j(\log Z) Z^{js}, \qquad Z := (z-z_0), \tag{5}$$

where P is a polynomial, r an integer of $\mathbb{Z}_{\geq 0}$, α a complex number, s a rational of $\mathbb{Q}_{>0}$, and the Q_j are a family of polynomials of uniformly bounded degree. The quantities r, P, α, s, Q_j depend on the particular solution and the formal asymptotic expansions of (5) *are $\mathbb{C}$-linearly independent.*

(The argument is based on first constructing a *formal* basis of independent solutions, each of the form (5), and then applying to the possibly divergent expansions a summation mechanism that converts such formal solutions into actual analytic solutions. The restriction of the statement to a subsector is related to the Stokes phenomena associated to so-called "irregular" singularities.)

This theorem implies that the sequence $(n^{n-1}/n!)$ (hence (n^n)) is non-holonomic. Indeed the Cayley tree function satisfies the functional equation

$$T(z) = ze^{T(z)},$$

corresponding to the fact that it enumerates labelled nonplane trees. Set $W(z) = -T(-z)$, which is otherwise known as the "Lambert W-function". One has

$$W(x) \underset{x \to +\infty}{=} \log x - \log\log x + O(1),$$

as verified by bootstrapping (see De Bruijn's monograph [10, p. 26]). This is enough to conclude that W, hence T, is non-holonomic as the $\log\log(\cdot)$ term is incompatible with Eq. (5). Observe that, conceptually, the proof involves considering the *analytic continuation* of $T(z)$ and then extracting a clearly non-holonomic term in the expansion near a singularity. More of this in the next sections.

Amongst other applications, we may cite:

— Stanley's children rounds [*EIS* A066166], with exponential generating function $(1-z)^{-z}$. The expansion as $z \to 1$,

$$(1-z)^{-z} \underset{z \to 1}{\sim} \frac{1}{1-z}\left(1 + (1-z)\log(1-z) + \frac{(1-z)^2\log^2(1-z)}{2!} + \cdots\right),$$

contradicts the fact that logarithms can only appear with bounded degrees in holonomic functions.

— Bell numbers have OGF e^{e^z-1}. In this case, the double exponential behaviour as $z \to +\infty$ excludes them from the holonomic ring.

Finally, what is given is often a sequence rather than a function. Under such circumstances, it proves handy to be able to relate the asymptotic behaviour of f_n as $n \to +\infty$ to the asymptotic form of its generating function $f(z)$, near a singularity. Such transfers exist and are widely known in the literature as *Abelian theorems*. We make use here of well-established principles in this theory, as found, e.g., in the reference book by Bingham, Goldie, and Teugels [4]. For convenience of exposition, we state explicitly one version used repeatedly here:

Theorem 3 (Basic Abelian theorem). *Let $\phi(x)$ be any of the functions*

$$x^\alpha(\log x)^\beta(\log\log x)^\gamma, \qquad \alpha \geq 0, \quad \beta, \gamma \in \mathbb{C}. \tag{6}$$

Let (u_n) be a sequence that satisfies the asymptotic estimate

$$u_n \underset{n \to \infty}{\sim} \phi(n).$$

Then, the generating function,

$$u(z) := \sum_{n \geq 0} u_n z^n,$$

satisfies the asymptotic estimate

$$u(z) \underset{z\to 1-}{\sim} \Gamma(\alpha+1)\frac{1}{(1-z)}\phi\left(\frac{1}{1-z}\right). \tag{7}$$

This estimate remains valid when z tends to 1 in any sector with vertex at 1, symmetric about the horizontal axis, and with opening angle $< \pi$.

Proof (sketch). We shall content ourselves here with brief indications since Corollary 1.7.3 p. 40 of [4] provides simultaneously the needed Abelian property and its real-analysis Tauberian converse[4], at least in the case when z tends to 1^- along the real axis.

For simplicity, consider first the representative case where $\phi(x) = \log\log x$ and one has exactly $u_n = \phi(n)$ for $n \geq 2$, with $u_0 = u_1 = 0$. Assume at this stage that z is real positive and set $z = e^{-t}$, where $t \to 0$ as $z \to 1$. We have

$$u(z) = \sum_{n\geq 2} \phi(n)e^{-nt}.$$

Take $n_1 = \lfloor t^{-1}/\log t^{-1}\rfloor$. Basic majorizations imply that the sum of the terms corresponding to $n < n_1$ is bounded from above by $n_1 \log\log n_1$, which is smaller than the right hand side of (7). Similarly, define $n_2 = \lfloor t^{-1}\log t^{-1}\rfloor$. The sum of terms with $n > n_2$ is easily checked to be $O(1)$. The remaining "central" terms $n_1 \leq n \leq n_2$ are such that $\phi(n)$ varies slowly over the interval and one has $\phi(n_1) \sim \phi(n_2) \sim \phi(1/t)$. One can thus take out a factor of $\phi(1/t)$ and conclude, upon approximating the sum by an integral, that

$$\sum_{n=n_1}^{n_2} \phi(n)e^{-nt} \sim \frac{\phi(1/t)}{t}\int_{1/\log t^{-1}}^{\log t^{-1}} e^{-x}\,dx \sim \frac{\log\left(\log(1-z)^{-1}\right)}{1-z}. \tag{8}$$

(Use the Euler-Maclaurin summation formula, then complete the tails.)

The proof above applies when

$$z = e^{-t+i\theta}, \qquad \text{with} \quad |\theta| < \theta_0,$$

for some $\theta_0 < \frac{\pi}{2}$. Once more only the central terms matter asymptotically; the integral is then to be taken along a line of angle θ, but it reduces to the corresponding integral along the positive real line, by virtue of the residue theorem. This suffices to justify the extension of the estimate to sectors. The case $\phi(x) = \log\log x$ is then settled.

The extension to $u_n = \phi(n)$ when ϕ only involves powers of logarithms and of iterated logarithms follows similar lines, as $\log n$ is also of slow variation. The inclusion of a power of n, in the form n^α implies that the integral in (8) should be modified to include a factor t^α leading to a real line integral that evaluates to the Gamma function, $\Gamma(\alpha+1)$.

Finally, simple modifications of the previous arguments show that if a sequence (v_n) satisfies $v_n = o(\phi(n))$, for $\phi(n)$ any of the sequences of (6), then its generating function $v(z)$ is a little-oh of the right hand side of (7). Decomposing $u_n = \phi(n) + v_n$ completes the proof of the statement. □

[4]The singularity analysis technology of Flajolet and Odlyzko [18, 35] provides sufficient conditions for the converse complex-Tauberian implication.

Note 3. We have chosen to state the Abelian Theorem (Theorem 3) for z varying in a cone of the complex plane, rather than the more customary real line. In this way we can avail ourselves of the comparatively simple Structure Theorem (as stated above in Theorem 2) and avoid some of the possible hardships due to the Stokes phenomenon. $\triangleleft$

Here is a direct application of Theorem 3. Let $\pi(x)$ be the number of primes less than or equal to x. By the Prime Number Theorem, one knows that

$$\pi(n) \sim \frac{n}{\log n}.$$

The Abelian Theorem permits us to conclude about the non-holonomic character of the sequence $(\pi(n))$, since

$$\sum_{n\geq 1} \pi(n) z^n \underset{z\to 1^-}{\sim} \frac{1}{(1-z)^2 \log(1-z)^{-1}},$$

which contradicts what the Structure Theorem permits.

Note 4. In this article, we concentrate on proofs of non-holonomicity based on analysis, that is, eventually, *asymptotic approximations*. On a different register, powerful algebraic tools can be put to use in a number of situations. Considerations on power series have been used by Harris and Sibuya [26] to show: *The reciprocal* $(1/f)$ *of a holonomic function* f *is holonomic if and only if* f'/f *is algebraic.* For instance, this proves the non-holonomicity of the reciprocal of Gauss' ${}_2F_1$ hypergeometric, except in degenerate cases. Using differential Galois theory, Singer generalized this result in [38]. He characterized the possible polynomial relations between holonomic functions and also showed the following: *A holonomic function* f *has to be algebraic if any of* $\exp\int f$ *or* $\phi(f)$ *is holonomic, with* ϕ *an algebraic function of genus* ≥ 1. An analogous result for sequences is given in [43, Chap. 4]: *If both* f_n *and* $1/f_n$ *are holonomic, then* f_n *is an interlacing of hypergeometric sequences.* $\triangleleft$

2 The logarithmic sequence

Let $f_n = \log n$ (with $\log 0 \equiv 0$) and let $f(z)$ be its generating function,

$$f(z) = \sum_{n\geq 1} (\log n) z^n.$$

We propose to show that a sequence derived from f_n by means of holonomicity preserving transformations is non-holonomic. Consider a variant of the nth difference of the sequence f_n, namely

$$\widehat{f}_n := \sum_{k=1}^{n} \binom{n}{k} (-1)^k \log k,$$

whose ordinary generating function $\widehat{f}(z) := \sum_{n\geq 1} \widehat{f}_n z^n$ has positive radius of convergence and satisfies

$$\widehat{f}(z) = \frac{1}{1-z} f\left(-\frac{z}{1-z}\right). \tag{9}$$

It is known that holonomic functions are closed under product and algebraic (hence also, rational) substitutions. Thus, f and $\widehat{f}$ are such that either both of them are holonomic or none of them is holonomic. (See also Stanley's paper [40, p. 181] for a discussion of the fact that differencing preserves holonomicity.)

Next, Flajolet and Sedgewick proved in [19] that the sequence $\widehat{f}_n$ satisfies the asymptotic estimate

$$\widehat{f}_n = \log \log n + O(1). \tag{10}$$

As a matter of fact, a full expansion is derived in [19, Th. 4], based on the Nörlund-Rice integral representation [34]

$$\widehat{f}_n = \frac{(-1)^n}{2i\pi} \int_{\mathcal{H}} (\log s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds, \tag{11}$$

the use of a Hankel contour $\mathcal{H}$, and estimates akin to those used for the determination of inverse Mellin transforms affected with an algebraic-logarithmic singularity [11].

The proof of Proposition 1 can now be easily completed. By the Abelian estimate of Theorem 3 applied to the asymptotic form (10) of $\widehat{f}_n$, we have

$$\widehat{f}(z) \sim \frac{\log\left(\log(1-z)^{-1}\right)}{1-z} \qquad (z \to 1,\ z \in S),$$

this in the whole of a sector S with vertex at 1 and of opening angle $< \pi$ extending towards the negative real axis symmetrically about the real axis. Assume *a contrario* that $f(z)$ is holonomic. Then $\widehat{f}(z)$, which is associated to differences, is also holonomic. But then, given the Structure Theorem, a log-log asymptotic element valid in a subsector S' can never result from a $\mathbb{C}$-linear combination of elements, each having the form (5). A contradiction has thus been reached, and Proposition 1 is established.

Note 5. We have presented our proof in a way that seems to depend on the imported estimate (10) of the logarithmic differences. In this way, we could save a few analytic steps. A conceptually equivalent and self-contained proof would proceed from the asymptotic behaviour of the analytic continuation of $f(z)$ as $z \to -\infty$. (See our earlier discussion of $T(z)$ for a similar situation.) This can be achieved directly by means of a Lindelöf integral representation,

$$f(-z) = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} (\log s)\, z^s \frac{\pi}{\sin \pi s}\, ds.$$

(See Lindelöf's monograph [31] for explanations from the mouth of the master, Ford's book [21] and Flajolet's paper [16] for related developments.) It can then be verified, by deforming the line of integration into a Hankel contour, that non-holonomic elements crop up in the asymptotic expansion of $f(z)$ at $-\infty$. $\triangleleft$

Note 6. (Added in proof). Two weeks after the present article was submitted, Klazar [29] provided an alternative proof of Prop. 1 that only requires elementary real analysis. $\lhd$

3 The sequence of powers

The proof of Proposition 2 relies once more on the consideration of diagonal differences. It has been established in [19] that

$$w_n := \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^\alpha,$$

satisfies, for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$,

$$w_n = \frac{(\log n)^{-\alpha}}{\Gamma(1-\alpha)} \left(1 + O\left(\frac{1}{\log n}\right)\right).$$

For instance,

$$\sum_{k=1}^{\infty} \binom{n}{k} (-1)^k \sqrt{k} = \frac{1}{\sqrt{\pi} \log n} + O\left((\log n)^{-3/2}\right).$$

By the Abelian theorem, this implies for instance that the generating function $w(z)$ of w_n satisfies, for $\alpha = \frac{1}{2}$ as $z \to 1^-$,

$$w(z) \sim \frac{1}{\pi} \frac{1}{\sqrt{\log(1-z)^{-1}}} \frac{1}{1-z},$$

while, for general $\alpha \notin \mathbb{Z}$, the dominant asymptotics involve a factor of the form

$$\left(\log(1-z)^{-1}\right)^{-\alpha}.$$

Such a factor is foreign to what the Structure Theorem provides as legal holonomic asymptotics, as soon as α is nonintegral. This completes the proof of Proposition 2.

Proposition 2 implies in particular that the power sequences,

$$n^{\sqrt{17}}, \quad n^i = \cos\log n + i \sin\log n, \quad n^\pi, \ldots,$$

are non-holonomic. Note that some of these sequences *do* occur as valid *asymptotic* approximations of holonomic sequences, some of which even appear in natural combinatorial problems. For instance, it is proved in [17] that the expected cost of a partial match in a quadtree is holonomic and has the asymptotic form $n^{(\sqrt{17}-3)/2}$.

4 The nth prime function

Our proof of Proposition 3 will similarly involve detecting, in the generating function $g(z)$ associated to primes, some elements that are incompatible with holonomy and contradict the conclusions of the Structure Theorem.

The nth prime function $n \mapsto g_n$ (often also written $p(n)$) is a much researched function. In a way, this function is an inverse of the function $\pi(x)$ that gives the number of primes in the interval $[1, x]$. By the Prime Number Theorem, the function $\pi(x)$ satisfies

$$\pi(x) = \mathrm{Li}(x) + R(x),$$

where $R(x)$ is of an order smaller than the main term and $\mathrm{Li}(x)$ is the logarithmic integral,

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}\left(1 + \frac{1!}{\log x} + \frac{2!}{(\log x)^2} + \cdots\right).$$

The precise description of the remainder term $R(x)$ depends upon the Riemann hypothesis. However, it is known *unconditionally* that R is small enough that the relation $\pi(x) = y$ can be inverted asymptotically by just inverting the main term $\mathrm{Li}(x)$. In this way, one obtains (see [37] and references therein) an estimate due to Cipolla [9],

$$g_n = n \log n + n \log\log n + O(n). \tag{12}$$

The log-log term is once more a barrier to holonomicity.

To see this, we introduce now the function,

$$\ell(z) = \frac{z}{(1-z)^2} \log \frac{1}{1-z} + \frac{z}{(1-z)^2},$$

which is clearly holonomic. It satisfies, with $H_n = 1 + \frac{1}{2} + \cdots \frac{1}{n}$ denoting the harmonic number,

$$\ell_n = nH_n = n \log n + O(n).$$

Then, by taking a difference, one gets

$$g_n - \ell_n = n \log\log n + O(n).$$

Thus, the difference $g(z) - \ell(z)$ satisfies

$$g(z) - h(z) \underset{z \to 1^-}{\sim} \frac{\log\left(\log(1-z)^{-1}\right)}{(1-z)^2}.$$

This last fact is incompatible with holonomicity, by the very same argument as in the previous section.

5 Conclusion

As we have striven to illustrate and as is otherwise seen in many areas of mathematics (see the opening quotation), singularities are central to the understanding of properties of numeric sequences. It should be clear by now that a large number of sequences can be proved to be non-holonomic. Here is a brief recapitulation of methods and possible extensions.

1. ***Asymptotic discrepancies***. In order to conclude that a sequence (u_n) is non-holonomic, the following conditions are sufficient: (i) *the generating function of the sequence* (u_n) *(or one of its cognates) admits, near a singularity, an asymptotic expansion in a scale that involves logarithms and iterated logarithms;* (ii) *at least one term in that expansion is an iterated logarithm or a power of a logarithm with an exponent not in* $\mathbb{Z}_{\geq 0}$. It is then apparent from our earlier developments that sequences like

$$\sqrt{n^7+1}, \qquad \frac{1}{H_n}, \quad \sqrt{\frac{\log n}{H_n}}, \quad \log \frac{p(2n)}{p(n)}, \quad \frac{n}{\sqrt{n}+\log n}, \quad p(n^2),$$

(H_n the harmonic number and $p(n)$ the nth prime function) fail to be holonomic. Also, techniques of this note extend easily to other slowly varying sequences, like

$$e^{\sqrt{\log n}}, \quad \log\log\log n, \dots,$$

as well as to any sequence that involves any such term somewhere in its asymptotic expansion.

The singularity-based technology has otherwise served to establish the non-algebraic character of sequences arising from combinatorics and the theory of formal languages in [15]. Once more, such transcendence results imply that, structurally, the corresponding objects cannot be (unambiguously) encoded by words of a context-free language. For instance, two-dimensional walks on a regular lattice that are constrained to the first quadrant cannot be described (via a length-preserving encoding) by means of an unambiguous context-free grammar. This property is neatly visible from a logarithmic component in the generating function of walks (see e.g., [20]), which contradicts the Structure Theorem for algebraic functions (also known as Newton-Puiseux!).

2. ***Infinitude of singularities***. A celebrated theorem of Pólya and Carlson [3] implies the following[5]: *A function analytic at the origin having integer coefficients and assumed to converge in the open unit disc is either a rational function or else it admits the unit circle as a natural boundary.* Consider then the generating function $g(z)$ of $g_n \equiv p(n)$, the nth prime function, which can be subjected to Pólya-Carlson. Either it has a natural boundary, in which case it cannot be holonomic, since holonomic functions have isolated singularities. Else, it is rational; but this would be a clear contradiction, since no rational function can have coefficients of the asymptotic form $n \log n$ (by virtue of the Abelian

[5]Stefan Gerhold is grateful to Richard Stanley for pointing out the Pólya-Carlson connection.

Theorem, say). This proof[6] seems to require the Prime Number Theorem but many weaker bounds that are elementary (e.g., the ones due to Chebyshev) are sufficient for this purpose. It is pleasant to note that the Pólya-Carlson Theorem was earlier employed in a similar fashion [2] in order to establish a structural lower bound in the theory of formal languages.

3. *Arithmetic discrepancies*[7]. In [15, p. 294], a sketch was given of the solution to a conjecture of Stanley [40] to the effect that the generating function

$$S_k(z) = \sum_{n\geq 0} \binom{2n}{n}^k z^n,$$

is transcendental for odd values of k, $k = 3, 5, \ldots$. (As already noted by Stanley, S_k is clearly transcendental when k is even, because of the presence of logarithmic factors.) Longer algebraic proofs have since been published, see [46] and [1] for a discussion. It suffices to remark, as shown by an Abelian argument, that the local expansion at the finite singularity of S_k for $k = 2\ell + 1$ odd, satisfies

$$\frac{d^\ell}{dz^\ell} S_{2\ell+1}\left(\frac{z}{4^{2\ell+1}}\right) \underset{z\to 1^-}{\sim} c_\ell \pi^{-\ell} \frac{1}{\sqrt{1-z}} \qquad c_\ell \in \mathbb{Q}. \tag{13}$$

This asymptotic form is incompatible with algebraicity. Indeed, if S_k were algebraic over $\mathbb{C}(z)$, it would be algebraic over $\mathbb{Q}(z)$, as it has rational Taylor coefficients (this, by a famous lemma of Eisenstein [3]). But in that case, its Puiseux expansion could only involve algebraic numbers. Equation (13) contradicts this, by virtue of the transcendence of π. Et voila! In this last case, we have to play not only with the *shape* of an asymptotic expansion, but also with the *arithmetic nature* of its coefficients.

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[6] Alternatively, Erdős, Maxsein, and Smith [12] have shown that an integer recurrent sequence [i.e., one with a rational generating function] which consists only of primes is necessarily a periodic sequence, therefore involving only finitely many different values. We are indebted to Dr Alin Bostan for this remark.

[7] See the recent monograph of Everest *al.* [13] for a compendium of arithmetic properties relative to recurrence sequences.

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