A combinatorial proof of Postnikov's identity and a generalized enumeration of labeled trees

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Abstract

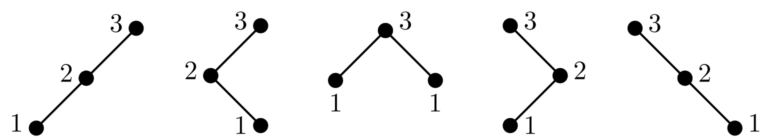
In this paper, we give a simple combinatorial explanation of a formula of A. Postnikov relating bicolored rooted trees to bicolored binary trees. We also present generalized formulas for the number of labeled k-ary trees, rooted labeled trees, and labeled plane trees.

1 Introduction

In Stanley's 60th Birthday Conference, Postnikov [3, p. 21] showed the following identity:

$$(n+1)^{n-1} = \sum_{\mathfrak{b}} \frac{n!}{2^n} \prod_{v \in V(\mathfrak{b})} \left(1 + \frac{1}{h(v)}\right), \tag{1}$$

where the sum is over unlabeled binary trees $\mathfrak{b}$ on n vertices and $h(v)$ denotes the number of descendants of v (including v). The figure below illustrates all five unlabeled binary trees on 3 vertices, with the value of $h(v)$ assigned to each vertex v. In this case, identity (1) says that $(3+1)^2 = 3+3+4+3+3$.



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Postnikov derived this identity from the study of a combinatorial interpretation for mixed Eulerian numbers, which are coefficients of certain reparametrized *volume polynomials* which introduced in [3]. For more information, see [2, 3].

In the same talk, he also asked for a combinatorial proof of identity (1). Multiplying both sides of (1) by 2^n and expanding the product in the right-hand side yields

$$2^n\,(n+1)^{n-1} = \sum_{\mathfrak{b}} n! \sum_{\alpha \subseteq V(\mathfrak{b})} \prod_{v \in \alpha} \frac{1}{h(v)}\,. \tag{2}$$

Let LHS_n (resp. RHS_n) denote the left-hand (resp. right-hand) side of (2).

The aim of this paper is to find a combinatorial proof of (2). In Section 2 we construct the sets $\mathcal{F}_n^{\mathrm{bi}}$ of labeled bicolored forests on $[n]$ and $\mathcal{D}_n$ of certain labeled bicolored binary trees, where the cardinalities equal LHS_n and RHS_n, respectively. In Section 3 we give a bijection between $\mathcal{F}_n^{\mathrm{bi}}$ and $\mathcal{D}_n$, which completes the bijective proof of (2). Finally, in Section 4, we present generalized formulas for the number of labeled k-ary trees, rooted labeled trees, and labeled plane trees.

2 Combinatorial objects for LHS_n and RHS_n

From now on, unless specified, we consider trees to be *labeled* and *rooted*.

A *tree* on $[n] := \{1, 2, \ldots, n\}$ is an acyclic connected graph on the vertex set $[n]$ such that one vertex, called the *root*, is distinguished. We denote by $\mathcal{T}_n$ the set of trees on $[n]$ and by $\mathcal{T}_{n,i}$ the set of trees on $[n]$ where vertex i is the root. A *forest* is a graph such that every connected component is a tree. Let $\mathcal{F}_n$ denote the set of forests on $[n]$. There is a canonical bijection $\gamma : \mathcal{T}_{n+1,n+1} \to \mathcal{F}_n$ such that $\gamma(T)$ is the forest obtained from T by removing the vertex $n+1$ and letting each neighbor of $n+1$ be a root. A graph is called *bicolored* if each vertex is colored with the color **b** (black) or **w** (white). We denote by $\mathcal{F}_n^{\mathrm{bi}}$ the set of bicolored forests on $[n]$. From Cayley's formula [1] and the bijection γ, we have

$$|\mathcal{F}_n| = |\mathcal{T}_{n+1,n+1}| = (n+1)^{n-1} \quad \text{and} \quad |\mathcal{F}_n^{\mathrm{bi}}| = 2^n \cdot (n+1)^{n-1}. \tag{3}$$

Thus LHS_n can be interpreted as the cardinality of $\mathcal{F}_n^{\mathrm{bi}}$.

Let F be a forest and let i and j be vertices of F. We say that j is a *descendant* of i if i is contained in the path from j to the root of the component containing j. In particular, if i and j are joined by an edge of F, then j is called a *child* of i. Note that i is also a descendant of i itself. Let $S(F, i)$ be the induced subtree of F on descendants of i, rooted at i. We call this tree the descendant subtree of F rooted at i. A vertex i is called *proper* if i is the smallest vertex in $S(F, i)$; otherwise i is called *improper*. Let $\mathrm{pv}(F)$ denote the the number of proper vertices in F.

A *plane tree* or *ordered tree* is a tree such that the children of each vertex are linearly ordered. We denote by $\mathcal{P}_n$ the set of plane trees on $[n]$ and by $\mathcal{P}_{n,i}$ the set of plane trees on $[n]$ where vertex i is the root. Define a *plane forest* on $[n]$ to be a finite ordered sequence of non-empty plane trees $(P_1, \ldots, P_m)$ such that $[n]$ is the disjoint union of the sets $V(P_r)$, $1 \leq r \leq m$. We denote by $\mathcal{PF}_n$ the set of plane forests on $[n]$ and by $\mathcal{PF}_n^{\mathrm{bi}}$ the set of bicolored plane forests on $[n]$. There is also a canonical bijection $\bar{\gamma} : \mathcal{P}_{n+1,n+1} \to \mathcal{PF}_n$ such that $\bar{\gamma}(P) = \big(S(P, j_1), \ldots, S(P, j_m)\big)$ where each vertex j_r is the rth child of $n+1$ in P. It is well-known that the number of unlabeled plane trees on $n+1$ vertices is given by the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see [4, ex. 6.19]). Thus we have

$$|\mathcal{PF}_n| = |\mathcal{P}_{n+1,n+1}| = n! \cdot C_n = 2n\,(2n-1)\cdots(n+2)\,. \tag{4}$$

A *binary tree* is a tree in which each vertex has at most two children and each child of a vertex is designated as its left or right child. We denote by $\mathcal{B}_n$ the set of binary trees on $[n]$ and by $\mathcal{B}_n^{\mathrm{bi}}$ the set of bicolored binary trees on $[n]$.

For $k \geq 2$, a *k-ary tree* is a tree where each vertex has at most k children and each child of a vertex is designated as its first, second, ..., or kth child. We denote by $\mathcal{A}_n^k$ the set of k-ary trees on $[n]$. Clearly, we have that $\mathcal{A}_n^2 = \mathcal{B}_n$. Since the number of unlabeled k-ary trees on n vertices is given by $\frac{1}{(k-1)n+1}\binom{kn}{n}$ (see [4, p. 172]), the cardinality of $\mathcal{A}_n^k$ is as follows:

$$|\mathcal{A}_n^k| = n! \cdot \frac{1}{(k-1)n+1}\binom{kn}{n} = kn\,(kn-1)\cdots(kn-n+2)\,.$$

Now we introduce a combinatorial interpretation of the number RHS_n. Let $\mathfrak{b}$ be an unlabeled binary tree on n vertices and $\omega : V(\mathfrak{b}) \to [n]$ be a bijection. Then the pair $(\mathfrak{b}, \omega)$ is identified with a (labeled) binary tree on $[n]$. Let $\Pi(\mathfrak{b}, \omega)$ be the set of vertices v in $\mathfrak{b}$ such that v has no descendant v' satisfying $\omega(v) > \omega(v')$, i.e., $\omega(v)$ is proper.

Let $\mathcal{D}_n$ be the set of bicolored binary trees on $[n]$ such that each proper vertex is colored with $\mathbf{b}$ or $\mathbf{w}$ and each improper vertex is colored with $\mathbf{b}$.

Lemma 1. *The cardinality of $\mathcal{D}_n$ is equal to* RHS_n.

Proof. Let $\mathcal{D}_n'$ be the set defined as follows:

$$\mathcal{D}_n' := \{\, (\mathfrak{b}, \omega, \alpha) \mid (\mathfrak{b}, \omega) \in \mathcal{B}_n \text{ and } \alpha \subseteq \Pi(\mathfrak{b}, \omega) \,\}\,.$$

There is a canonical bijection from $\mathcal{D}_n'$ to $\mathcal{D}_n$ as follows: Given $(\mathfrak{b}, \omega, \alpha) \in \mathcal{D}_n'$, if a vertex v of $\mathfrak{b}$ is contained in α then color v with $\mathbf{w}$; otherwise color v with $\mathbf{b}$. Thus it suffices to show that the cardinality of $\mathcal{D}_n'$ equals RHS_n.

Given an unlabeled binary tree $\mathfrak{b}$ and a subset α of $V(\mathfrak{b})$, let $l(\mathfrak{b}, \alpha)$ be the number of labelings ω satisfying $\alpha \subseteq \Pi(\mathfrak{b}, \omega)$. Then for each $v \in \alpha$, the label $\omega(v)$ of v should be the smallest among the labels of the descendants of v. If we pick a labeling ω uniformly at random, the probability that $\omega(v)$ is the smallest among the labels of the descendants of v is $1/h(v)$. So the number of possible labelings ω is $n!/\prod_{v\in\alpha} h(v)$. Thus we have

$$\begin{aligned} |\mathcal{D}_n'| &= \sum_{\mathfrak{b}} \sum_{\alpha \subseteq V(\mathfrak{b})} l(\,\mathfrak{b}, \alpha) \\ &= \sum_{\mathfrak{b}} \sum_{\alpha \subseteq V(\mathfrak{b})} n! \prod_{v \in \alpha} \frac{1}{h(v)}\,, \end{aligned}$$

which coincides with RHS_n. $\square$

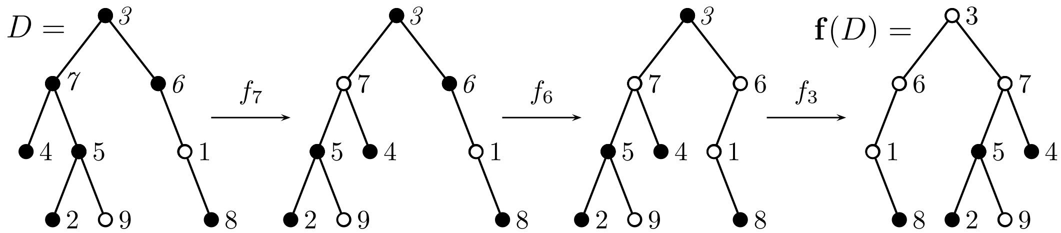

Figure 1: The map $\mathbf{f}$. (Right improper vertices are in italics.)

3 A bijection

In this section we construct a bijection between $\mathcal{F}_n^{\rm bi}$ and $\mathcal{D}_n$, which gives a bijective proof of (2).

Given a vertex v of a bicolored binary tree B, let $L(B,v)$ (resp. $R(B,v)$) be the descendant subtree of B, which is rooted at the left (resp. right) child of v. Note that $L(B,v)$ and $R(B,v)$ may be empty, but $L(B,v)$ or $R(B,v)$ is nonempty when v is improper. For any kind of tree T, let $m(T)$ be the smallest vertex in T. By convention, we put $m(\emptyset) = \infty$. For an improper vertex v of B, if $m\big(L(B,v)\big) > m\big(R(B,v)\big)$, then we say that v is *right improper*; otherwise *left improper*.

For a vertex v of B, define the *flip* on v, which will be denoted by f_v , by swapping $L(B,v)$ and $R(B,v)$ and changing the color of v. Note that the flip satisfies $f_v \circ f_v = id$ and $f_v \circ f_w = f_w \circ f_v$. For a bicolored binary tree D in $\mathcal{D}_n$, let $\mathbf{f}$ be the map defined by

$$\mathbf{f}(D) := (f_{v_1} \circ \cdots \circ f_{v_k})(D)\,,$$

where $\{v_1, \ldots, v_k\}$ is the set of right improper vertices in D. (See Figure 1.)

Let $\mathcal{E}_n$ be the set of bicolored binary trees E on $[n]$ such that every improper vertex v is left improper, i.e., $m\big(L(E,v)\big) < m\big(R(E,v)\big)$.

Lemma 2. *The map* $\mathbf{f}$ *is a bijection from* $\mathcal{D}_n$ *to* $\mathcal{E}_n$.

Proof. For a bicolored binary tree E in $\mathcal{E}_n$, let $\mathbf{f}'$ be the map defined by $\mathbf{f}'(E) := (f_{u_1} \circ \cdots \circ f_{u_j})(E)$, where $\{u_1, \ldots, u_j\}$ is the set of white-colored improper vertices in E. Then the map $\mathbf{f}'$ is the inverse of $\mathbf{f}$. ☐

Let $\mathcal{G}_n$ (resp. $\mathcal{Q}_n$) be the set of bicolored trees (resp. bicolored plane trees) on $[n+1]$ such that $n+1$ is the root colored with $\mathbf{b}$. Note that the map γ (resp. $\bar{\gamma}$) given at the beginning of Section 2 can be regarded as a bijection $\gamma : \mathcal{G}_n \to \mathcal{F}_n^{\rm bi}$ (resp. $\bar{\gamma} : \mathcal{Q}_n \to \mathcal{PF}_n^{\rm bi}$). For a vertex v of $Q \in \mathcal{Q}_n$, let $(w_1, \ldots, w_r)$ be the children of v, in order. Then for each $i = 1, \ldots, r-1$, we say that w_{i+1} is the *right sibling* of w_i. The set $\mathcal{G}_n$ can be viewed as a subset of $\mathcal{Q}_n$ satisfying the following condition: Suppose that v is the right sibling of u in $Q \in \mathcal{Q}_n$. Then $m\big(S(Q,u)\big) < m\big(\,S(Q,v)\big)$ holds.

Recall that $\mathcal{B}_n^{\rm bi}$ denotes the set of bicolored binary trees on $[n]$. Clearly we have $\mathcal{E}_n \subseteq \mathcal{B}_n^{\rm bi}$. Let Φ be a bijection from $\mathcal{B}_n^{\rm bi}$ to $\mathcal{Q}_n$, which maps B to Q as follows:

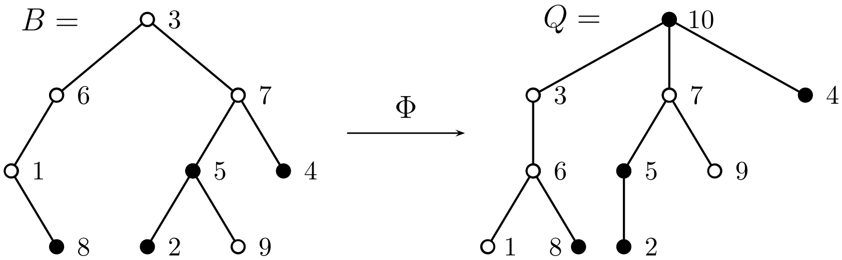

Figure 2: The bijection Φ.

1. The root of B is the first child of $n+1$ in Q.

2. v is the first child of u in Q iff v is a left child of u in B.

3. v is the right sibling of u in Q iff v is a right child of u in B.

4. The color of v in Q is the same as the color of v in B.

Note that here Φ is essentially an extension of a well-known bijection, which is described in [5, p. 60], from binary trees to plane trees.

Lemma 3. *The restriction ϕ of Φ to $\mathcal{E}_n$ is a bijection from $\mathcal{E}_n$ to $\mathcal{G}_n$.*

Proof. For any improper vertex v of $E \in \mathcal{E}_n$, we have $m(L(E,v)) < m(R(E,v))$. This guarantees that $m(S(G,v)) < m(S(G,w))$ in $G = \Phi(E)$, where w (if it exists) is the right sibling of v in G. Thus $\Phi(E) \in \mathcal{G}_n$, i.e., $\Phi(\mathcal{E}_n) \subseteq \mathcal{G}_n$. Similarly we can show that $\Phi^{-1}(\mathcal{G}_n) \subseteq \mathcal{E}_n$. So we have $\Phi(\mathcal{E}_n) = \mathcal{G}_n$, which implies ϕ is bijective. $\square$

From Lemma 3, we easily get that $\gamma \circ \phi$ is a bijection from $\mathcal{E}_n$ to $\mathcal{F}_n^{\mathrm{bi}}$. Combining this result with Lemma 2 yields the following consequence.

Theorem 4. *The map $\gamma \circ \phi \circ \mathbf{f}$ is a bijection from $\mathcal{D}_n$ to $\mathcal{F}_n^{\mathrm{bi}}$.*

Figure 3 shows how the bijection in Theorem 4 maps a bicolored binary tree D in $\mathcal{D}_{11}$ to a bicolored forest F on $[11]$. From equation (3) the cardinality of $\mathcal{F}_n^{\mathrm{bi}}$ equals LHS_n and from Lemma 1 the cardinality of $\mathcal{D}_n$ equals RHS_n. Thus Theorem 4 is a combinatorial explanation of identity (2).

4 Generalized formulas

Theorem 4 implies the set $\mathcal{D}_n$ of binary trees on $[n]$ such that each proper vertex is colored with the color $\mathbf{b}$ or $\mathbf{w}$ and each improper vertex is colored with the color $\mathbf{b}$ has cardinality $|\mathcal{D}_n| = 2^n\,(n+1)^{n-1}$. In this section we give a generalization of this result.

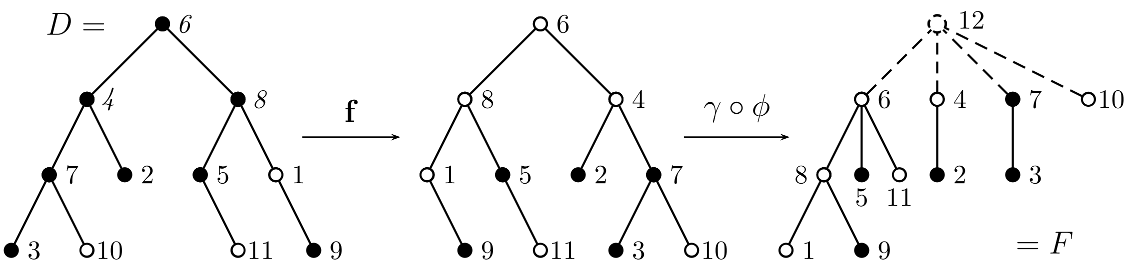

Figure 3: The bijection from $\mathcal{D}_n$ to $\mathcal{F}_n^{\mathrm{bi}}$.

For $n \geq 1$, let $a_{n,m}$ denote the number of k-ary trees on $[n]$ with m proper vertices. By convention, we put $a_{0,m} = \delta_{0,m}$. Let

$$a_n(t) = \sum_{m \geq 0} a_{n,m}\, t^m = \sum_{T \in \mathcal{A}_n^k} t^{\,\mathrm{pv}(T)}\,,$$

where $\mathrm{pv}(T)$ is the number of proper vertices of T. It is clear that for a positive integer t the number $a_n(t)$ is the number of k-ary trees on $[n]$ such that each proper vertex is colored with the color $\bar{1}, \bar{2}, \ldots,$ or $\bar{t}$ and each improper vertex has one color $\bar{1}$. Let $A(x)$ be denote the exponential generating function for $a_n(t)$, i.e.,

$$A(x) = \sum_{n \geq 0} a_n(t)\, \frac{x^n}{n!}\,.$$

Lemma 5. *The generating function $A = A(x)$ satisfies the following differential equation:*

$$A' = k\, x\, A^{k-1} A' + t\, A^k, \tag{5}$$

where the prime denotes the derivative with respect to x.

Proof. Let T be an k-ary tree on $[n] \cup \{0\}$. Delete all edges going from the root r of T. Then T is decomposed into $T' = (r; T_1, \ldots, T_k)$ where each T_i is a k-ary tree and $[n] \cup \{0\}$ is the disjoint union of $V(T_1), \ldots, V(T_k)$ and $\{r\}$. Consider two cases: (i) For some $1 \leq i \leq k$, T_i has the vertex 0; (ii) $r = 0$. Then we have

$$\begin{aligned} a_{n+1}(t) \;&=\; \sum_{i=1}^{k} \sum_{n_1+\cdots+n_k=n-1} \binom{n}{1, n_1, \ldots, n_k} a_{n_1}(t) \cdots a_{n_i+1}(t) \cdots a_{n_k}(t) \\ &+\; t \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, \ldots, n_k} a_{n_1}(t) \cdots a_{n_k}(t)\,. \end{aligned}$$

Multiplying both sides by $x^n/n!$ and summing over n yields (5). □

To compute $a_n(t)$ from (5) we need the following theorem.

Theorem 6. *Fix positive integers a and b. Let $u = 1 + \sum_{n=1}^{\infty} u_n\, x^n/n!$ be a formal power series in x satisfying*

$$u' = a\,x\,u^b u' + t\,u^{b+1}. \tag{6}$$

Then u_n is given by

$$u_n = t \prod_{i=1}^{n-1} \big((bi+1)\,t + a\,(n-i) \big)\,, \qquad n \geq 1\,.$$

Proof. Adding $(bt-a)\,x\,u^b u'$ to both sides of (6) yields

$$\big(1 + (bt-a)\,x\,u^b \big)\,u' = t\,\big(b\,x\,u^{b-1}u' + u^b \big)\,u\,.$$

Since $(1 + (bt-a)\,x\,u^b)' = (bt-a)\,(b\,x\,u^{b-1}u' + u^b)$, we have

$$(bt-a)\,\log u = t\,\log(\,1 + (bt-a)\,x\,u^b\,)\,.$$

Taking the exponential of both sides and the substitutions $x = y^b$ and $yu(y^b) = \hat{u}(y)$ yield

$$\hat{u}(y) = y\,\big(1 + (bt-a)\,\hat{u}(y)^b \big)^{t/(bt-a)}. \tag{7}$$

Applying the Lagrange Inversion Formula (see [4, p. 38]) to (7) yields that

$$\begin{aligned}
\big[y^{bn+1}\big]\,\hat{u}(y) &= \frac{1}{bn+1}\,\big[y^{bn}\big]\,\big(1 + (bt-a)\,y^b \big)^{\frac{t\,(bn+1)}{bt-a}} \\
&= \frac{1}{bn+1}\,(bt-a)^n \binom{\frac{t\,(bn+1)}{bt-a}}{n} \\
&= \frac{t}{n!} \prod_{i=1}^{n-1} \big(t\,(bn+1) - (bt-a)\,i \big)\,.
\end{aligned}$$

Since $u_n = n!\,[y^{bn+1}]\,\hat{u}(y)$, we obtain the desired result. $\square$

Since (5) is a special case of (6) ($a = k$, $b = k-1$), we can deduce a formula for $a_n(t)$.

Corollary 7 (k-ary trees). *For $n \geq 1$, $a_n(t)$ is given by*

$$a_n(t) = t \prod_{i=1}^{n-1} \big((ki - i + 1)\,t + k\,(n-i) \big)\,. \tag{8}$$

Clearly, substituting $t = 1$ in (8) yields the number of k-ary trees on $[n]$, i.e.,

$$a_n(1) = kn\,(kn-1)\cdots(kn-n+2) = |\mathcal{A}_n^k|\,.$$

For some values of k, we can get interesting results. In particular when $k = 2$ we have

$$a_n(t) = t \prod_{i=1}^{n-1} \big((i+1)t + 2(n-i) \big) \xrightarrow{t=2} 2^n (n+1)^{n-1},$$

so this is a generalization of $|\mathcal{D}_n| = 2^n \, (n+1)^{n-1}$, i.e., identity (2).

In fact Theorem 6 has more applications. For $n \geq 1$, let $f_{n,m}$ denote the number of forests on $[n]$ with m proper vertices and let $p_{n,m}$ denote the number of plane forests on $[n]$ with m proper vertices. Let

$$f_n(t) = \sum_{m \geq 1} f_{n,m} \, t^m \quad \text{and} \quad p_n(t) = \sum_{m \geq 1} p_{n,m} \, t^m .$$

Let $F(x)$ and $P(x)$ be the exponential generating function for $f_n(t)$ and $p_n(t)$, respectively, i.e.,

$$F(x) = 1 + \sum_{n \geq 1} f_n(t) \, \frac{x^n}{n!} \quad \text{and} \quad P(x) = 1 + \sum_{n \geq 1} p_n(t) \, \frac{x^n}{n!} .$$

Similarly to Lemma 5, we can get two differential equations:

$$F' = x \, F \, F' + t \, F^2 , \tag{9}$$

$$P' = x \, P^2 P' + t \, P^3 . \tag{10}$$

Since (9) and (10) are special cases of (6) ($a = b = 1$ and $a = 1$, $b = 2$, respectively), we have the following results.

Corollary 8. *Suppose $f_n(t)$ and $p_n(t)$ are defined as above. Then we have*

1. *For $n \geq 1$, $f_n(t)$ is given by*

$$f_n(t) = t \prod_{i=1}^{n-1} \big((i+1) \, t + (n-i) \big) . \tag{11}$$

2. *For $n \geq 1$, $p_n(t)$ is given by*

$$p_n(t) = t \prod_{i=1}^{n-1} \big((2i+1) \, t + (n-i) \big) . \tag{12}$$

Note that (11) and (12) are generalizations of (3) and (4), respectively. Moreover, from these formulas, we can easily get

$$\sum_{T \in \mathcal{T}_{n+1}} t^{\,\mathrm{pv}(T)} = t \prod_{i=0}^{n-1} \big((i+1) \, t + (n-i) \big),$$

$$\sum_{P \in \mathcal{P}_{n+1}} t^{\,\mathrm{pv}(P)} = t \prod_{i=0}^{n-1} \big((2i+1) \, t + (n-i) \big),$$

which are generalizations of $|\mathcal{T}_{n+1}| = (n+1)^n$ and $|\mathcal{P}_{n+1}| = (n+1)! \, C_n$.

Remark. In spite of the simple expressions, we have not proved (8), (11) and (12) in a bijective way. Also a direct combinatorial proof of Theorem 6 would be desirable.

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