The h-vector of a Gorenstein toric ring of a compressed polytope

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Dedicated to Richard P. Stanley on the occasion of his 60th birthday

Abstract

A compressed polytope is an integral convex polytope all of whose pulling triangulations are unimodular. A $(q-1)$-simplex Σ each of whose vertices is a vertex of a convex polytope $\mathcal{P}$ is said to be a special simplex in $\mathcal{P}$ if each facet of $\mathcal{P}$ contains exactly $q-1$ of the vertices of Σ. It will be proved that there is a special simplex in a compressed polytope $\mathcal{P}$ if (and only if) its toric ring $K[\mathcal{P}]$ is Gorenstein. In consequence it follows that the h-vector of a Gorenstein toric ring $K[\mathcal{P}]$ is unimodal if $\mathcal{P}$ is compressed.

A *compressed polytope* [10, p. 337] is an integral convex polytope all of whose "pulling triangulations" are unimodular. (Recall that an integral convex polytope is an convex polytope each of whose vertices has integer coordinates.) A typical example of compressed polytopes is the Birkhoff polytopes [10, Example 2.4 (b)]. Later, in [6], a large class of compressed polytopes including the Birkhoff polytopes is presented. Recently, Seth Sullivant [12] proved a surprising result that the class given in [6] does essentially contain *all* compressed polytopes.

Let $\mathcal{P} \subset \mathbb{R}^n$ be an integral convex polytope. Let K be a field and $K[\mathbf{x}, \mathbf{x}^{-1}, t] = K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, t]$ the Laurent polynomial ring in $n+1$ variables over K. The *toric ring* of $\mathcal{P}$ is the subalgebra $K[\mathcal{P}]$ of $K[\mathbf{x}, \mathbf{x}^{-1}, t]$ which is generated by those monomials $\mathbf{x}^{\mathbf{a}}t = x_1^{a_1} \cdots x_n^{a_n} t$ such that $\mathbf{a} = (a_1, \ldots, a_n)$ belongs to $\mathcal{P} \bigcap \mathbb{Z}^n$. We will regard $K[\mathcal{P}]$ as a homogeneous algebra [2, p. 147] by setting each $\deg \mathbf{x}^{\mathbf{a}}t = 1$ and write $F(K[\mathcal{P}], \lambda)$ for its Hilbert series. One has $F(K[\mathcal{P}], \lambda) = (h_0 + h_1\lambda + \cdots + h_s\lambda^s)/(1-\lambda)^{d+1}$, where each $h_i \in \mathbb{Z}$ with $h_s \neq 0$ and where d is the dimension of $\mathcal{P}$. The sequence $h(K[\mathcal{P}]) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of $K[\mathcal{P}]$. If the toric ring $K[\mathcal{P}]$ is normal, then $K[\mathcal{P}]$ is Cohen–Macaulay. If $K[\mathcal{P}]$ is Cohen–Macaulay, then the h-vector of $K[\mathcal{P}]$ is nonnegative, i.e., each $h_i \geq 0$. Moreover, if $K[\mathcal{P}]$ is Gorenstein, then the h-vector of $K[\mathcal{P}]$ is symmetric, i.e., $h_i = h_{s-i}$ for all i.

A well-known conjecture is that the h-vector $(h_0, h_1, \ldots, h_s)$ of a Gorenstein toric ring is *unimodal*, i.e., $h_0 \leq h_1 \leq \cdots \leq h_{[s/2]}$. One of the effective techniques to show that $(h_0, h_1, \ldots, h_s)$ is unimodal is to find a simplicial complex polytope of dimension $s-1$ whose h-vector [11, p. 75] coincides with $(h_0, h_1, \ldots, h_s)$ (Stanley [9]). In fact, Reiner and Welker [8] succeeded in showing that the h-vector of a Gorenstein toric ring arising from a finite distributive lattice (see, e.g., [4]) is equal to the h-vector of a simplicial convex polytope.

Christos Athanasiadis [1] introduced the concept of a "special simplex" in a convex polytope. Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex polytope. A $(q-1)$-simplex Σ each of whose vertices is a vertex of $\mathcal{P}$ is said to be a *special simplex* in $\mathcal{P}$ if each facet (maximal face) of $\mathcal{P}$ contains exactly $q-1$ of the vertices of Σ. It turns out [1, Theorem 3.5] that if $\mathcal{P}$ is compressed and if there is a special simplex in $\mathcal{P}$, then the h-vector of $K[\mathcal{P}]$ is equal to the h-vector of a simplicial convex polytope. In particular, if $\mathcal{P}$ is compressed and if there is a special simplex in $\mathcal{P}$, then $K[\mathcal{P}]$ is Gorenstein whose h-vector is unimodal. Examples for which [1, Theorem 3.5] can be applied include (i) toric rings of the Birkhoff polytopes ([1, Example 3.1]), (ii) Gorenstein toric rings arising from finite distributive lattices ([1, Example 3.2]), and (iii) Gorenstein toric rings arising from stable polytopes of perfect graphs ([7, Theorem 3.1 (b)]).

In the present paper we prove that there is a special simplex in a compressed polytope $\mathcal{P}$ if (and only if) its toric ring $K[\mathcal{P}]$ is Gorenstein.

Theorem 0.1 *Let $\mathcal{P}$ be a compressed polytope. Then there exists a special simplex in $\mathcal{P}$ if (and only if) its toric ring $K[\mathcal{P}]$ is Gorenstein.*

Proof. It follows from [12, Theorem 2.4] that every compressed polytope $\mathcal{P}$ is lattice isomorphic to an integral convex polytope of the form $C_n \bigcap L$, where $C_n \subset \mathbb{R}^n$ is the n-dimensional unit hypercube and where L is an affine subspace of $\mathbb{R}^n$. Without loss of generality, one can assume that $L \bigcap (C_n \setminus \partial C_n) \neq \emptyset$, where ∂C_n is the boundary of C_n. In other words, $\dim \mathcal{P} = \dim L$. Let $\mathcal{P} = C_n \bigcap L$ with $d = \dim \mathcal{P}$. Thus L is the intersection of $n-d$ hyperplanes in $\mathbb{R}^n$, say

$$
\begin{array}{rcl}
a_{11}x_1 + \cdots + a_{1d}x_d + x_{d+1} & = & b_1 \\
a_{21}x_1 + \cdots + a_{2d}x_d + x_{d+2} & = & b_2
\end{array}
$$

$$\cdots$$

$$a_{n-d,1}x_1 + \cdots + a_{n-d,d}x_d + x_n \;=\; b_{n-d},$$

where $a_{ij}, b_i \in \mathbb{Q}$ for all i and j. Since $\mathcal{P}$ possesses the integer decomposition property [6, p. 2544], its toric ring coincides with the Ehrhart ring [5, p. 97] of $\mathcal{P}$. Hence the criterion [3, Corollary 1.2] can be applied for $K[\mathcal{P}]$.

To state the criterion [3, Corollary 1.2], let $\delta > 0$ denote the smallest integer for which $\delta(\mathcal{P} \setminus \partial\mathcal{P}) \bigcap \mathbb{Z}^n \neq \emptyset$, where $\delta(\mathcal{P} \setminus \partial\mathcal{P}) = \{\delta\alpha \,:\, \alpha \in \mathcal{P} \setminus \partial\mathcal{P}\}$, and $(c_1, \ldots, c_n) \in \delta(\mathcal{P} \setminus \partial\mathcal{P}) \bigcap \mathbb{Z}^n$. Write $\mathcal{Q} \subset \mathbb{R}^d$ for the convex polytope defined by the inequalities

$$0 \leq x_i \leq 1, \qquad 1 \leq i \leq d$$

together with

$$\begin{aligned} 0 \leq b_1 - (a_{11}x_1 + \cdots + a_{1d}x_d) &\leq 1 \\ 0 \leq b_2 - (a_{21}x_1 + \cdots + a_{2d}x_d) &\leq 1 \\ &\cdots \\ 0 \leq b_{n-d} - (a_{n-d,1}x_1 + \cdots + a_{n-d,d}x_d) &\leq 1. \end{aligned}$$

Then $\mathcal{Q}$ is an integral convex polytope of dimension d with $K[\mathcal{Q}] \cong K[\mathcal{P}]$. Let $\mathcal{Q}^\sharp = \delta\mathcal{Q} - (c_1, \ldots, c_d)$. Then $\mathcal{Q}^\sharp$ is an integral convex polytope of dimension d and the origin of $\mathbb{R}^d$ belongs to the interior of $\mathcal{Q}^\sharp$. By using [3, Corollary 1.2] the toric ring $K[\mathcal{Q}]$ is Gorenstein if and only if the equation of the supporting hyperplane of each facet of $\mathcal{Q}^\sharp$ is of the form $q_1x_1 + \cdots + q_dx_d = 1$ with each $q_j \in \mathbb{Z}$.

Claim. *Suppose that $K[\mathcal{Q}]$ is Gorenstein. Then, for each $1 \leq i \leq n$, one has $c_i = \delta - 1$ (resp. $c_i = 1$) if the hyperplane in $\mathbb{R}^n$ defined by the equation $x_i = 1$ (resp. $x_i = 0$) is a supporting hyperplane of a facet of $\mathcal{P}$.*

Proof of Claim. Let $1 \leq i \leq d$. If the equation $x_i = 1$ (resp. $x_i = 0$) defines a facet of $\mathcal{P}$, then the equation $x_i + c_i = \delta$ (resp. $x_i + c_i = 0$) defines a facet of $\mathcal{Q}^\sharp$. Since $0 \leq c_i \leq \delta$, one has $c_i = \delta - 1$ (resp. $c_i = 1$), as desired.

Let $1 \leq i \leq n - d$. If the equation $x_{d+i} = 1$ defines a facet of $\mathcal{P}$, then the equation

$$a_{i1}(x_1 + c_1) + \cdots + a_{id}(x_d + c_d) = \delta(b_i - 1)$$

defines a facet of $\mathcal{Q}^\sharp$. Since $a_{i1}c_1 + \cdots + a_{id}c_d + c_{d+i} = \delta b_i$, the equation

$$a_{i1}x_1 + \cdots + a_{id}x_d = c_{d+i} - \delta \tag{1}$$

defines a facet of $\mathcal{Q}^\sharp$. We write the equation (1) of the form

$$(p/q)(a'_{i1}x_1 + \cdots + a'_{id}x_d) = c_{d+i} - \delta,$$

where $a'_{i1}, \ldots, a'_{id}$ are integers which are relatively prime, and where p and $q > 0$ are integers which are relatively prime. Then $q(c_{d+i} - \delta)/p = \pm 1$. Hence $q = 1$. Thus each

$a_{ij} \in \mathbb{Z}$ is divided by p. We write the equation $a_{i1}x_1 + \cdots + a_{id}x_d + x_{d+i} = b_i$ of the form $p(a'_{i1}x_1 + \cdots + a'_{id}x_d) + x_{d+i} = b_i$. Since $L \bigcap (C_n \setminus \partial C_n) \neq \emptyset$, there is a vertex $(v_1, \ldots, v_n)$ of $\mathcal{P} = C_n \bigcap L$ with $v_{d+i} = 0$. Thus $b_i \in \mathbb{Z}$ is divided by p, say, $b_i = pb'_i$ with $b'_i \in \mathbb{Z}$. Let $(v_1, \ldots, v_n)$ be a vertex of $\mathcal{P}$ with $v_{d+i} = 1$. However, unless $p = \pm 1$, such the vertex cannot lie on the hyperplane defined by the equation $p(a'_{i1}x_1 + \cdots + a'_{id}x_d) + x_{d+i} = pb'_i$. Thus $p = \pm 1$. Since $c_{d+i} - \delta = p$ and $c_{d+i} \leq \delta$, one has $p = -1$ and $c_{d+i} = \delta - 1$, as desired. On the other hand, modify the above technique slightly, and one has $c_{d+i} = 1$ if the hyperplane in $\mathbb{R}^n$ defined by the equation $x_{d+i} = 0$.

Now, we proceed to the final step of our proof of Theorem 0.1. Since $(c_1, \ldots, c_n)$ belongs to $\delta(\mathcal{P} \setminus \partial\mathcal{P}) \bigcap \mathbb{Z}^n$, there exists δ vertices $\mathbf{v}_1, \ldots, \mathbf{v}_\delta$ of $\mathcal{P}$ with $(c_1, \ldots, c_n) = \mathbf{v}_1 + \cdots + \mathbf{v}_\delta$. Write Σ for the convex hull of $\{\mathbf{v}_1, \ldots, \mathbf{v}_\delta\}$. Our work is to show that Σ is a special simplex in $\mathcal{P}$. Let a facet $\mathcal{F}$ of $\mathcal{P}$ be defined by the equation $x_i = 1$ (resp. $x_i = 0$). Then $c_i = \delta - 1$ (resp. $c_i = 1$). Since each vertex of $\mathcal{P}$ is a $(0,1)$-vector, exactly $\delta - 1$ vertices of $\mathbf{v}_1, \ldots, \mathbf{v}_\delta$ lie on $\mathcal{F}$. Finally, to see why Σ is a $(\delta - 1)$-simplex, suppose that, say, $\mathbf{v}_\delta$ belongs to the convex hull of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{\delta-1}\}$ and that $\mathbf{v}_\delta$ does not lie on a facet $\mathcal{G}$ of $\mathcal{P}$. Then all of $\mathbf{v}_1, \ldots, \mathbf{v}_{\delta-1}$ must belong to $\mathcal{G}$. Hence $\Sigma \subset \mathcal{G}$. Thus $\mathbf{v}_n \in \mathcal{G}$, which contradicts $\mathbf{v}_n \notin \mathcal{G}$. Q. E. D.

By virtue of [1, Theorem 3.5] together with Theorem 0.1 it follows that

Corollary 0.2 *Let $\mathcal{P}$ be a compressed polytope and suppose that the toric ring $K[\mathcal{P}]$ is Gorenstein. Then the h-vector of $K[\mathcal{P}]$ is unimodal.*

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