On a Partition Function of Richard Stanley

George E. Andrews*

Department of Mathematics
The Pennsylvania State University
University Park, PA 16802

andrews@math.psu.edu

Submitted: Sep 5, 2003; Accepted: Nov 19, 2003; Published: Jun 3, 2004
MR Subject Classifications: 05A17

In honor of my friend Richard Stanley

Abstract

In this paper, we examine partitions π classified according to the number $r(\pi)$ of odd parts in π and $s(\pi)$ the number of odd parts in π', the conjugate of π. The generating function for such partitions is obtained when the parts of π are all $\leqq N$. From this a variety of corollaries follow including a Ramanujan type congruence for Stanley's partition function $t(n)$.

1 Introduction

Let π denote a partition of some integer and π' its conjugate. For definitions of these concepts, see [1; Ch.1]. Let $\mathcal{O}(\pi)$ denote the number of odd parts of π. For example, if π is $6+5+4+2+2+1$, then the Ferrers graph of π is

$$\begin{array}{llllll}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & & & & \\
\cdot & \cdot & & & & \\
\cdot & & & & &
\end{array}$$

Reading columns we see that π' is $6+5+3+3+2+1$. Hence $\mathcal{O}(\pi)=2$ and $\mathcal{O}(\pi')=4$.

Richard Stanley ([4] and [5]) has shown that if $t(n)$ denotes the number of partitions π of n for which $\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod 4$, then

*Partially supported by National Science Foundation Grant DMS-0200047

$$t(n) = \frac{1}{2}\Big(p(n) + f(n)\Big), \tag{1}$$

where $p(n)$ is the total number of partitions of n [1, p. 1], and

$$\sum_{n=0}^{\infty} f(n)q^n = \prod_{i \geqq 1} \frac{(1+q^{2i-1})}{(1-q^{4i})(1+q^{4i-2})^2}. \tag{2}$$

Note that $t(n)$ is Stanley's partition function referred to in the title of this paper. Stanley's result for $t(n)$ is related nicely to a general study of sign-balanced, labeled posets [5]. In this paper, we shall restrict our attention to $S_N(n,r,s)$, the number of partition π of n where each part of π is $\leqq N$, $\mathcal{O}(\pi) = r$, $\mathcal{O}(\pi') = s$. In Section 2, we shall prove our main result:

Theorem 1.

$$\sum_{n,r,s \geqq 0} S_{2N}(n,r,s)q^n z^r y^s = \frac{\sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}; q^4\Big] (-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}}{(q^4;q^4)_N(z^2q^2;q^4)_N}, \tag{3}$$

and

$$\sum_{n,r,s \geqq 0} S_{2N+1}(n,r,s)q^n z^r y^s = \frac{\sum_{j=0}^{N} \Big[\begin{matrix} N \\ j \end{matrix}; q^4\Big] (-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}}{(q^4;q^4)_N(z^2q^2;q^4)_{N+1}}, \tag{4}$$

where

$$\left[\begin{matrix} N \\ j \end{matrix}; q\right] = \begin{cases} \frac{(1-q^N)(1-q^{N-1})\dots(1-q^{N-j+1})}{(1-q^j)(1-q^{j-1})\dots(1-q)}, & \textit{for } 0 \leqq j \leqq N, \\ 0\ , & \textit{if } j < 0 \textit{ or } j > N, \end{cases} \tag{5}$$

and

$$(A;q)_M = (1-A)(1-Aq)\dots(1-Aq^{M-1}). \tag{6}$$

From Theorem 1 follows an immediate lovely corollary:

Corollary 1.1.

$$\sum_{n,r,s \geqq 0} S_{\infty}(n,r,s)q^n z^r y^s = \prod_{j=1}^{\infty} \frac{(1+yzq^{2j-1})}{(1-q^{4j})(1-z^2q^{4j-2})(1-y^2q^{4j-2})}. \tag{7}$$

From Corollary 1.1, we shall see in Section 3 that

Corollary 1.2.

$$t(5n+4) \equiv 0 \pmod 5. \tag{8}$$

Also,

Corollary 1.3.

$$\sum_{n=0}^{\infty} t(n)q^n = \frac{Q(q^2)^2 Q(q^{16})^5}{Q(q)Q(q^4)^5 Q(q^{32})^2}, \tag{9}$$

where

$$Q(q) = (q;q)_\infty = \prod_{j=1}^{\infty}(1-q^j). \tag{10}$$

We conclude with some open questions.

2 The Main Theorem

We begin with some preliminaries about partitions and their conjugates. For a given partition π with parts each $\leqq N$, we denote by $f_i(\pi)$ the number of appearances of i as a part of π. The parts of π' in non-increasing order are thus

$$\sum_{i=1}^{N} f_i(\pi),\ \sum_{i=2}^{N} f_i(\pi),\ \sum_{i=3}^{N} f_i(\pi), \dots, \sum_{i=N}^{N} f_i(\pi). \tag{11}$$

Note that some of the entries in this sequence may well be zero; the non-zero entries make up the parts of π'. However in light of the fact that 0 is even, we see that $\mathcal{O}(\pi')$ is the number of odd entries in the sequence (11) while

$$\mathcal{O}(\pi) = f_1(\pi) + f_3(\pi) + f_5(\pi) + \dots. \tag{12}$$

We now define

$$\sigma_N(q,z,y) = \left(\sum_{n,r,s\geqq 0} S_N(n,r,s)q^n z^r y^s\right)(q^4;q^4)_{\lfloor \frac{N}{2}\rfloor}(z^2q^2;q^4)_{\lfloor \frac{N+1}{2}\rfloor}. \tag{13}$$

Lemma 2.1. $\sigma_0(q,z,y) = 1$*, and for* $N \geqq 1$*,*

$$\sigma_{2N}(q,z,y) = \sigma_{2N-1}(q,z,y) + y^{2N}q^{2N}\sigma_{2N-1}(q,z,y^{-1}) \tag{14}$$

$$\sigma_{2N-1}(q,z,y) = \sigma_{2N-2}(q,z,y) + zy^{2N-1}q^{2N-1}\sigma_{2N-2}(q,z,y^{-1}). \tag{15}$$

Proof. We shall in the following be dealing with partitions whose parts are all $\leqq$ some given N. We let $\bar{\pi}$ be that partition made up of the parts of π that are $< N$. In light of (11) we see that if N is a part of π an even number of times, then $\mathcal{O}(\pi') = \mathcal{O}(\bar{\pi}')$ and if N appears an odd number of times in π, then $\mathcal{O}(\bar{\pi}') = N - \mathcal{O}(\pi)$ (because the removal of $f_N(\pi)$ from each sum in (11) reverses parity). Initially we note that the only partition with at most zero parts is the empty partition of 0; hence $\sigma_0(q,z,y) = 1$.

Next, for $N \geqq 1$,

$$\begin{aligned}
&\frac{\sigma_{2N}(q,z,y)}{(q^4;q^4)_N(z^2q^2;q^4)_N} \\
= &\sum_{\pi,\text{parts}\leqq 2N} q^{\sum if_i(\pi)} z^{f_1(\pi)+f_3(\pi)+\ldots+f_{2N-1}(\pi)} y^{\mathcal{O}(\pi')} \\
= &\sum_{\substack{\pi,\text{parts}\leqq 2N\\ f_{2N}(\pi)\text{even}}} q^{\sum if_i(\bar\pi)+2Nf_{2N}(\pi)} z^{f_1(\pi)+f_3(\pi)+\ldots+f_{2N-1}(\pi)} y^{\mathcal{O}(\bar\pi')} \\
&+ \sum_{\substack{\pi,\text{parts}\leqq 2N\\ f_{2N}(\pi)\text{odd}}} q^{\sum if_i(\bar\pi)+2Nf_{2N}(\pi)} z^{f_1(\pi)+f_3(\pi)+\ldots+f_{2N-1}(\pi)} y^{2N-\mathcal{O}(\bar\pi')} \\
= &\frac{1}{(1-q^{4N})}\frac{\sigma_{2N-1}(q,z,y)}{(q^4;q^4)_{N-1}(z^2q^2;q^4)_N} + \frac{y^{2N}q^{2N}}{(1-q^{4N})}\frac{\sigma_{2N-1}(q,z,y^{-1})}{(q^4;q^4)_{N-1}(z^2q^2;q^4)_N},
\end{aligned}$$

which is equivalent to (14).

Finally,

$$\begin{aligned}
&\frac{\sigma_{2N+1}(q,z,y)}{(q^4;q^4)_N(z^2q^2;q^4)_{N+1}} \\
= &\sum_{\pi,\text{parts}\leqq 2N+1} q^{\sum if_i(\pi)} z^{f_1(\pi)+f_3(\pi)+\ldots+f_{2N+1}(\pi)} y^{\mathcal{O}(\pi')} \\
= &\sum_{\substack{\pi,\text{parts}\leqq 2N+1\\ f_{2N+1}(\pi)\text{even}}} q^{\sum if_i(\bar\pi)+(2N+1)f_{2N+1}(\pi)} z^{f_1(\pi)+\ldots+f_{2N+1}(\pi)} y^{\mathcal{O}(\bar\pi')} \\
&+ \sum_{\substack{\pi,\text{parts}\leqq 2N+1\\ f_{2N+1}(\pi)\text{odd}}} q^{\sum if_i(\bar\pi)+(2N+1)f_{2N+1}(\pi)} z^{f_1(\pi)+\ldots+f_{2N-1}(\pi)+f_{2N+1}(\pi)} y^{2N+1-\mathcal{O}(\bar\pi')} \\
= &\frac{1}{(1-z^2q^{4N+2})}\frac{\sigma_{2N}(q,z,y)}{(q^4;q^4)_N(z^2q^2;q^4)_N} + \frac{y^{2N+1}q^{2N+1}z}{(1-z^2q^{4N+2})}\frac{\sigma_{2N}(q,z,y)}{(q^4;q^4)_N(z^2q^2;q^4)_N},
\end{aligned}$$

which is equivalent to (15) with N replaced by $N+1$. □

Proof of Theorem 1. We let $\tau_{2N}(q,z,y)$ denote the numerator on the right-hand side of (3) and $\tau_{2N+1}(q,z,y)$ denote the numerator on the right-hand side of (4). If we can show that $\tau_N(q,z,y)$ satisfies (14) and (15), then noting immediately that $\tau_0(q,z,y)=1$, we will have proved that $\sigma_N(q,z,y)=\tau_N(q,z,y)$ for each $N \geqq 0$ (by mathematical induction) and will then prove Theorem 1 once we recall (13).

First,

$$
\begin{aligned}
&\tau_{2N-1}(q,z,y) + y^{2N}q^{2N}\tau_{2N-1}(q,z,y^{-1})\\
&= \sum_{j\geqq 0} \begin{bmatrix} N-1 \\ j \end{bmatrix}; q^4 \Big] (-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j-1}(yq)^{2(N-1-j)}\\
&\qquad + y^{2N}q^{2N}\sum_{j\geqq 0} \left[\begin{matrix} N-1 \\ j \end{matrix}; q^4\right] (-zy^{-1}q;q^4)_{N-j}(-zyq;q^4)_j(y^{-1}q)^{2j}\\
&\qquad\qquad\qquad \text{(where } j\to N-1-j \text{ in the second sum)}\\
&= \sum_{j\geqq 0} \left[\begin{matrix} N-1 \\ j-1 \end{matrix}; q^4\right] (-zyq;q^4)_{j}(-zy^{-1}q;q^4)_{N-j}(yq)^{2(N-j)}\\
&\qquad + y^{2N}q^{2N}\sum_{j\geqq 0} \left[\begin{matrix} N-1 \\ j \end{matrix}; q^4\right] (-zy^{-1}q;q^4)_{N-j}(-zyq;q^4)_j(y^{-1}q)^{2j}\\
&\qquad\qquad\qquad \text{(where } j\to j-1 \text{ in the first sum)}\\
&= \sum_{j\geqq 0}(-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2(N-j)}\left\{ \left[\begin{matrix} N-1 \\ j-1 \end{matrix}; q^4\right] + q^{4j}\left[\begin{matrix} N-1 \\ j \end{matrix}; q^4\right] \right\}\\
&= \sum_{j\geqq 0}(-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2(N-j)}\left[\begin{matrix} N \\ j \end{matrix}; q^4\right]\\
&\qquad\qquad\qquad \text{(by [1, p.35, eq.(3.3.4)])}\\
&= \tau_{2N}(q,z,y).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\tau_{2N}(q,z,y) + zy^{2N+1}q^{2N+1}\tau(q,z,y^{-1})\\
&= \sum_{j=0} \left[\begin{matrix} N \\ j \end{matrix}; q^4\right] (-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}\\
&\qquad + zq^{2N+1}y^{2N+1}\sum_{j=0}^{N} \left[\begin{matrix} N \\ j \end{matrix}; q^4\right] (-zy^{-1}q;q^4)_{N-j}(-zyq;q^4)_j(qy^{-1})^{2j}\\
&\qquad\qquad\qquad \text{(where } j\to N-j \text{ in the second sum)}\\
&= \sum_{j=0}^{N} \left[\begin{matrix} N \\ j \end{matrix}; q^4\right] (-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}(1+zyq^{4j+1})\\
&= \sum_{j=0}^{N} \left[\begin{matrix} N \\ j \end{matrix}; q^4\right] (-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}\\
&= \tau_{2N+1}(q,z,y).
\end{aligned}
$$

□

Proof of Corollary 1.1. From Theorem 1 (either (3) or (4) with $j \to N - j$),

$$\begin{aligned}
&\sum_{n,r,s\geqq 0} S_\infty(n,r,s)q^n z^r y^s \qquad (16)\\
&= \frac{1}{(q^4;q^4)_\infty (z^2q^2;q^4)_\infty} \sum_{j=0}^{\infty} \frac{1}{(q^4;q^4)_j} (-zyq;q^4)_\infty (-zy^{-1}q;q^4)_j (yq)^{2j}\\
&= \frac{(-zyq;q^4)_\infty}{(q^4;q^4)_\infty (z^2q^2;q^4)_\infty} \frac{(-zyq^3;q^4)_\infty}{(y^2q^2;q^4)_\infty}\\
&\qquad \text{(by [1, p.17, eq.(2.2.1)])}\\
&= \frac{(-zyq;q^2)_\infty}{(q^4;q^4)_\infty (z^2q^2;q^4)_\infty (y^2q^2;q^4)_\infty},
\end{aligned}$$

which is Corollary 1.1. ☐

Corollary 2.1. *Identity* (1) *is valid.*

Proof. We note that $\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod 2$ because each is clearly congruent $\pmod 2$ to the number being partitioned. Hence,

$$\begin{aligned}
\sum_{n\geqq 0} t(n)q^n &= \sum_{\substack{n,r,s\geqq 0\\ \frac{r-s}{2}\,\text{even}}} S_\infty(n,r,s)q^n \qquad (17)\\
&= \frac{1}{2} \sum_{n,r,s\geqq 0} S_\infty(n,r,s)q^n(1+i^{r-s})\\
&= \frac{1}{2}\left(\frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty (q^2;q^4)_\infty^2} + \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty (-q^2;q^4)_\infty^2} \right)\\
&= \frac{1}{2}\left(\frac{1}{(q;q)_\infty} + \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty (-q^2;q^4)_\infty^2} \right)\\
&= \frac{1}{2} \sum_{n=0}^{\infty} (p(n)+f(n))q^n,
\end{aligned}$$

and comparing coefficients of q^n in the extremes of this identity we deduce (1). ☐

3 Further Properties of $t(n)$

Corollary 1.2. $t(5n+4) \equiv 0 \pmod 5$.

Proof. Ramanujan proved [3, p.287, Th. 359] that

$$p(5n+4) \equiv 0 \pmod 5.$$

So it follows from (1) that to prove $5|t(5n+4)$ we need only prove that $5|f(5n+4)$.

By (2),

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n)q^n &= \frac{(-q;q^2)}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2} \qquad (18)\\
&= \frac{(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty}{(q^4;q^4)_\infty^2(-q^2;q^4)_\infty^2}\\
&= \frac{1}{(-q^2;-q^2)_\infty^2}\sum_{n=-\infty}^{\infty} q^{2n^2-n} \qquad \text{(by [1, p.21, eq.(2.2.10)])}\\
&= \frac{(-q^2;-q^2)_\infty^3}{(-q^2;-q^2)_\infty^5}\sum_{n=-\infty}^{\infty} q^{2n^2-n}\\
&= \frac{1}{(-q^{10};-q^{10})_\infty}\sum_{n=\infty}^{\infty} q^{2n^2-n}\sum_{j=0}^{\infty}(-1)^{j+(j+1)/2}(2j+1)q^{j^2+j} \pmod 5\\
&\qquad \text{(by [3, p.285, Thm. 357]).}
\end{aligned}$$

Now the only time an exponent of q in the numerator is congruent to 4 (mod 5) is when $n \equiv 4 \pmod 5$ and $j \equiv 2 \pmod 5$. But then $(2j+1) \equiv 0 \pmod 5$, i.e. the coefficient of q^{5m+4} in the numerator must be divisible by 5. Given that the denominator is a function of q^5, it cannot possibly affect the residue class of any term when it is divided into the numerator. So,

$$f(5n+4) \equiv 0 \pmod 5.$$

Therefore,

$$t(5n+4) \equiv 0 \pmod 5.$$

□

Corollary 1.3.

$$\sum_{n\geqq 0} t(n)q^n = \frac{Q(q)^2Q(q^{16})^5}{Q(q)Q(q^4)^5Q(q^{32})^2}, \qquad (19)$$

where

$$Q(q) = (q;q)_\infty. \qquad (20)$$

Proof. By (17),

$$\begin{aligned}
\sum_{n\geqq 0} t(n)q^n &= \frac{1}{2}\left(\frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(q^2;q^4)_\infty^2} + \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2}\right) \\
&= \frac{(-q;q^2)_\infty}{2(q^4;q^4)_\infty^2(q^2;q^4)_\infty^2(-q^2;q^4)_\infty^2}\left((q^4;q^4)_\infty(-q^2;q^4)_\infty^2 + (q^4;q^4)_\infty(q^2;q^4)_\infty^2\right) \\
&= \frac{(-q;q^2)_\infty}{2(q^4;q^4)_\infty^2(q^4;q^8)_\infty^2}\left(\sum_{n=-\infty}^{\infty} q^{2n^2} + \sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\right) \\
&\qquad \text{(by [1, p.21, eq.(2.2.10)])} \\
&= \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty^2(q^4;q^8)_\infty^2}\sum_{n=-\infty}^{\infty} q^{8n^2} \\
&= \frac{(-q;q^2)_\infty(q^{16};q^{16})_\infty(-q^8;q^{16})_\infty^2}{(q^4;q^4)_\infty^2(q^4;q^8)_\infty^2} \\
&= \frac{Q(q^2)^2Q(q^{16})^5}{Q(q)Q(q^4)^5Q(q^{32})^2},
\end{aligned}$$

where the last line follows from several applications of the two identities

$$(q;q^2)_\infty = \frac{Q(q)}{Q(q^2)}$$

and

$$(-q;q^2)_\infty = \frac{Q(q^2)^2}{Q(q)Q(q^4)}.$$

□

Corollary 1.3 allows us to multisect the generating function for $t(n)$ modulo 4.

Corollary 3.1.

$$\sum_{n\geqq 0} t(4n)q^n = (q^{16};q^{16})_\infty(-q^7;q^{16})_\infty(-q^9;q^{16})_\infty W(q), \tag{21}$$

$$\sum_{n\geqq 0} t(4n+1)q^n = (q^{16};q^{16})_\infty(-q^5;q^{16})_\infty(-q^{11};q^{16})_\infty W(q), \tag{22}$$

$$\sum_{n\geqq 0} t(4n+2)q^n = q(q^{16};q^{16})_\infty(-q;q^{16})_\infty(-q^{15};q^{16})_\infty W(q), \tag{23}$$

$$\sum_{n\geqq 0} t(4n+3)q^n = (q^{16};q^{16})_\infty(-q^3;q^{16})_\infty(-q^{13};q^{16})_\infty W(q), \tag{24}$$

where

$$W(q) = \frac{Q(q^4)^5}{Q(q)^5Q(q^8)^2}. \tag{25}$$

Proof. We begin with Gauss's special case of the Jacobi Triple Product Identity [1, p.23, eq.(2.2.13)]

$$\sum_{n=-\infty}^{\infty} q^{2n^2-n} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{Q(q^2)^2}{Q(q)} \tag{26}$$

Therefore by Corollary 1.3, we see that

$$\sum_{n\geqq 0} t(n)q^n = W(q^4)\sum_{n=-\infty}^{\infty} q^{2n^2-n}. \tag{27}$$

Now $2n^2-n\equiv n \pmod 4$. So to obtain (3.4)–(3.7) we multisect the right-hand series in (27) by setting $n=4m+j$ $(0\leqq j\leqq 3)$, so

$$\sum_{n\geqq 0} t(n)q^n = W(q^4)\sum_{j=0}^{3}\sum_{m=-\infty}^{\infty} q^{2(4m+j)^2-(4m+j)}.$$

One then obtains four identities arising from the four residue classes mod 4. We carry out the full calculations in the case $j=0$:

$$\begin{aligned}\sum_{n\geqq 0} t(4n)q^{4n} &= W(q^4)\sum_{m=-\infty}^{\infty} q^{32m^2-4m}\\ &= W(q^4)(q^{64};q^{64})_\infty(-q^{28};q^{64})_\infty(-q^{36};q^{64})_\infty,\end{aligned}$$

a result equivalent to (3.4) once q is replaced by $q^{1/4}$. The remaining results are proved similarly. □

4 Conclusion

As is obvious, Theorem 1 is easily proved once it is stated, but the sums appearing in (3) and (4) seem to arise from nowhere.

I note that by considering the cases $N=1,2,3,4$, I discovered empirically that

$$\sum_{n,r,s\geqq 0} S_{2N}(n,r,s)q^n z^r y^s = \frac{1}{(q^4;q^4)_N}\sum_{j=0}^{N}\frac{(-zyq;q^2)_{2j}}{(z^2q^2;q^4)_j}\begin{bmatrix}N\\ j\end{bmatrix}_{q^4}(y^2q^2)^{N-j} \tag{28}$$

and

$$\sum_{n,r,s\geqq 0} S_{2N+1}(n,r,s)q^n z^r y^s = \frac{1}{(q^4;q^4)_N}\sum_{j=0}^{N}\frac{(-zyq;q^2)_{2j+1}}{(z^2q^2;q^4)_{j+1}}\begin{bmatrix}N\\ j\end{bmatrix}_{q^4}(y^2q^2)^{N-j}. \tag{29}$$

One can then pass to (3) and (4) by means of a ${}_3\phi_2$ transformation [2, p.242, eq.(III.13)], and the proof of Theorem 1 is easiest using (3) and (4).

The referee notes that both (1.3) and (1.4) can be written as a ${}_2\phi_1$. These ${}_2\phi_1$ series can both be transformed into ${}_3\phi_1$ series, equivalent to (1.4) and (4.2) by (III.8) of [2].

There are many mysteries surrounding many of the identities in this paper.

Problem 1. Is there a partition statistic that will divide the partitions enumerated by $t(5n+4)$ into five equinumerous classes? Dyson's rank (largest part minus number of parts) provides such a division at least for $n=0$ and 1 (cf. [1, p.175]).

Problem 2. Identity (7) cries out for combinatorial proof.

I have been informed that A. Sills, A. J. Yee, and C. Boulet have independently found such proofs in addition to further results.

References

[1] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, 1976 (Reissued: Cambridge University Press, Cambridge, 1998).

[2] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[3] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 4 ed., Oxford University Press, Oxford, 1960.

[4] R.P. Stanley, Problem 10969, *Amer. Math. Monthly*, **109** (2002), 760.

[5] R.P. Stanley, Some remarks on sign-balanced and maj-balanced posets (to appear).