On plethysm conjectures of Stanley and Foulkes: the $2 \times n$ case

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Abstract

We prove Stanley's plethysm conjecture for the $2 \times n$ case, which composed with the work of Black and List provides another proof of Foulkes conjecture for the $2 \times n$ case. We also show that the way Stanley formulated his conjecture, it is false in general, and suggest an alternative formulation.

1 Introduction

Denote by V a finite-dimensional complex vector space, and by $S^m V$ its m-th symmetric power. Foulkes in [4] conjectured that the $GL(V)$-module $S^n(S^m V)$ contains the $GL(V)$-module $S^m(S^n V)$ for $n \geq m$. For $m = 2, 3$ and 4 the conjecture was proved; see [7], [3], [1]. An extensive list of references can be found in [8].

In [2] Black and List showed that Foulkes conjecture follows from the following combinatorial statement. Denote $I_{m,n}$ to be the set of dissections of $\{1, \ldots, mn\}$ into sets of cardinality m. Let $s = \bigsqcup_{i=1}^{n} S_i$ and $t = \bigsqcup_{i=1}^{m} T_i$ be elements of $I_{m,n}$ and $I_{n,m}$ respectively. Define matrix $M^{m,n} = (M^{m,n}_{t,s})$ by

$$M^{m,n}_{t,s} = \begin{cases} 1 & \text{if } |S_i \cap T_j| = 1 \text{ for any } 1 \leq i \leq n, 1 \leq j \leq m; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.1 (Black, List 89). *If the rank of $M^{m,n}$ is equal to $|I_{n,m}|$ for $n \geq m > 1$, then Foulkes conjecture holds for all pairs of integers (n, r) such that $1 \leq r \leq m$.*

Let λ be a partition of N. A *tableau* is a filling of a Young diagram of shape λ with numbers from 1 to N, and let T_λ to be the set of such tableaux. Define two tableaux to be h-equivalent, denoted $\equiv_h$, if they can be obtained one from the other by permuting

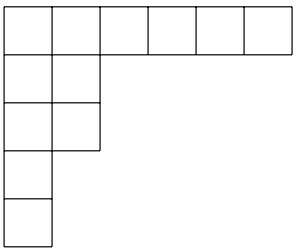

Figure 1: A counterexample for Stanley's conjecture.

elements in rows and permuting rows of equal length. Define a *horizontal tableau* to be an element of $H_\lambda := T_\lambda / \equiv_h$. In other words, rows of a horizontal tableau form a partition of the set $\{1, \ldots, N\}$. Similarly, define v-equivalence $\equiv_v$ and the set $V_\lambda := T_\lambda / \equiv_v$ of *vertical tableaux* of shape λ. Consider a horizontal tableau Γ with rows $r_1, \ldots, r_p$ and a vertical tableau Δ with columns $c_1, \ldots, c_q$. Call Γ and Δ *orthogonal*, denoted $\Gamma \perp \Delta$, if the inequality $|r_i \cap c_j| \leq 1$ holds for all i, j . Equivalently, Γ and Δ are orthogonal if and only if there exists a tableau ρ consistent with both Γ and Δ.

Define the matrix $K_\lambda = (K_\lambda^{\Gamma,\Delta})$ by

$$K_\lambda^{\Gamma,\Delta} = \begin{cases} 1 & \text{if } \Gamma \perp \Delta; \\ 0 & \text{otherwise.} \end{cases}$$

The rows of K_λ are naturally labelled by horizontal tableaux, while the columns are labelled by vertical tableaux. Let λ' be the conjugate partition. In [6], Stanley formulated a conjecture, which can be equivalently stated as follows.

Conjecture 1.2. *If $\lambda \geq \lambda'$ in dominance order, i.e. $\lambda_1 + \cdots + \lambda_i \geq \lambda'_1 + \cdots + \lambda'_i$ for all i, then the rows of K_λ are linearly independent.*

This conjecture is false. For the shape λ shown in Figure 1, the inequality $\lambda \geq \lambda'$ holds. However, the matrix K_λ has more rows than columns, thus the rows cannot be linearly independent. Indeed, $|H_\lambda| = \frac{12!}{6!2!2!1!1!2!2!}$, which is greater than $|V_\lambda| = \frac{12!}{5!3!1!1!1!1!4!}$. This counterexample was suggested by Richard Stanley as the smallest possible one. The following conjecture seems to be a reasonable alternative formulation, although Max Neunhöffer has recently shown that in general approach of Black and List does not work, see [5].

Conjecture 1.3. *K_λ has full rank for all λ.*

Let $\boldsymbol{m} \times \boldsymbol{n}$ denote the rectangular shape with m rows and n columns. For rectangular shapes, Stanley's conjecture implies Foulkes conjecture since $K_{\boldsymbol{m} \times \boldsymbol{n}} = M^{m,n}$. For hook shaped λ, the conjecture is known to be true; see [6]. In Section 2 we present a proof of Stanley's conjecture for $\lambda = \boldsymbol{2} \times \boldsymbol{n}$.

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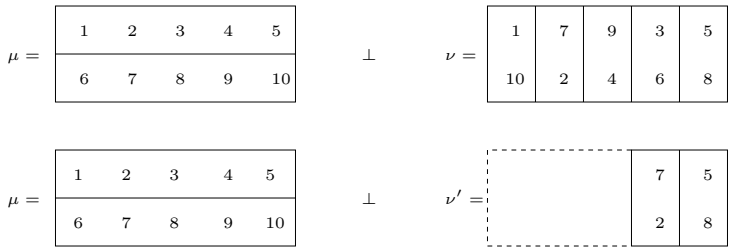

Figure 2: Partial tableau Δ' is a subtableau of Δ. Since $\Gamma \perp \Delta$, also $\Gamma \perp \Delta'$.

2 The Main Result

Our aim is to prove the following theorem.

Theorem 2.1. *Conjecture 1.3 is true for* $\lambda = \mathbf{2} \times \boldsymbol{n}$.

Note that for rectangular shapes, Conjectures 1.2 and 1.3 are equivalent, because for $m \leq n$, the inequality $|H_{\boldsymbol{m} \times \boldsymbol{n}}| \leq |V_{\boldsymbol{m} \times \boldsymbol{n}}|$ holds. Therefore, proving that $K_{\mathbf{2} \times \boldsymbol{n}}$ has full rank is equivalent to proving that its rows are linearly independent. Suppose for contradiction that there is a nontrivial linear combination of rows of $K_{\mathbf{2} \times \boldsymbol{n}}$ equal to 0. Let τ_Γ be the coefficient of the row corresponding to a horizontal tableau Γ in this combination. Then for a column of $K_{\mathbf{2} \times \boldsymbol{n}}$ labelled by a vertical tableau Δ, the linear combination $\sum_\Gamma K^{\Gamma,\Delta}_{\mathbf{2} \times \boldsymbol{n}} \tau_\Gamma$ equals 0. Alternatively, this sum can be written as $\sum_{\Gamma \perp \Delta} \tau_\Gamma = 0$. Call a 0-*filter* a condition on horizontal tableax such that sum of τ_Γ over all Γ satisfying this condition is 0. Thus, orthogonality to Δ is a 0-filter. Our aim is to show that being Γ is a 0-filter for every horizontal tableau Γ. Indeed, this is just saying that all τ_Γ are equal to 0, which contradicts the assumption above.

Definition 2.2. For $k < n$, a *subtableau* of shape $\mathbf{2} \times \boldsymbol{k}$ of a vertical tableau Δ of shape $\mathbf{2} \times \boldsymbol{n}$ is a subset of k columns of Δ. A *partial tableau* is a collection of k columns which is a subtableau of at least one vertical tableau Δ.

In other words, a partial tableau is a vertical tableau of shape $\mathbf{2} \times \boldsymbol{k}$, filled with numbers from $\{1, \ldots, 2n\}$. We can now generalize the concept of orthogonality as follows. Call a horizontal tableau Γ of shape $\mathbf{2} \times \boldsymbol{n}$ orthogonal to a partial tableau Δ' of shape $\mathbf{2} \times \boldsymbol{k}$, where $k < n$, if there exists vertical tableau Δ of shape $\mathbf{2} \times \boldsymbol{n}$ such that $\Gamma \perp \Delta$, and Δ' is a subtableau of Δ. An example is presented in Figure 2. The reason for considering such a generalization is evident from the following theorem.

Theorem 2.3. *Orthogonality to a certain partial tableau* Δ' *is a* 0*-filter.*

Proof. For a given partial tableau Δ' of shape $\mathbf{2} \times \boldsymbol{k}$, denote

$$F(\Delta') = \{\Delta \in V_{\mathbf{2} \times \boldsymbol{n}} \mid \ \Delta' \text{ is a subtableau of } \Delta\}.$$

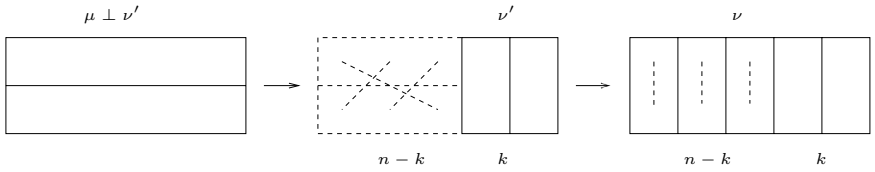

Figure 3: Each matching corresponds to exactly one possible tableau $\Delta \perp \Gamma$ containing Δ' as a subtableau.

Consider the sum $\sum_{\Delta \in F(\Delta')} \sum_{\Gamma \perp \Delta} \tau_\Gamma$. We claim that for each horizontal tableau Γ, τ_Γ enters this sum with the same coefficient. Indeed, the coefficient of a particular τ_Γ is the number of tableaux Δ containing Δ' and orthogonal to Γ. Such Δ's are in one-to-one correspondance with matchings between two sets of size $n-k$, as can be seen from the Figure 3. The number of such matchings is $(n-k)!$, which obviously does not depend on particular Γ. Therefore, $\frac{1}{(n-k)!} \sum_{\Delta \in F(\Delta')} \sum_{\Gamma \perp \Delta} \tau_\Gamma = \sum_{\Gamma \perp \Delta'} \tau_\Gamma$. Since each $\sum_{\Gamma \perp \Delta} \tau_\Gamma$ is zero by the assumption above, the sum $\sum_{\Gamma \perp \Delta'} \tau_\Gamma$ is also 0, which means that orthogonality to Δ' is a 0-filter.

□

We now continue the proof of Theorem 2.1. Choose a particular horizontal tableau Γ_0, for example with one row filled with numbers $1, \ldots, n$ and the other row filled with the rest of the numbers. If we show that $\tau_{\Gamma_0} = 0$, then in a similar fashion (just by relabelling numbers) we can show that all τ_Γ's are 0, which would be a contradiction with the assumption that the combination of rows of $K_{\mathbf{2} \times \boldsymbol{n}}$ is nontrivial. For a given horizontal tableau Γ, let a_Γ and b_Γ be the numbers of elements of $\{1, \ldots, n\}$ in the rows of Γ, so that $a_\Gamma + b_\Gamma = n$. We do not distinguish between the rows of Γ, therefore we can assume that $a_\Gamma \geq b_\Gamma$. Observe that Γ_0 is the only horizontal tableau such that $(a_{\Gamma_0}, b_{\Gamma_0}) = (n, 0)$. Let T_a be the collection of horizontal tableaux Γ with $a_\Gamma = a$, and call elements of T_a *horizontal tableaux of type* a. Then Γ_0 is the only horizontal tableau of type n.

Theorem 2.4. *For $a \geq n/2$, being a horizontal tableau of type a is a 0-filter.*

Proof. For $k \leq [n/2]$, consider the set P_k of all possible partial tableaux of shape $\mathbf{2} \times \boldsymbol{k}$, filled with numbers from $\{1, \ldots, n\}$. Consider the sum $\sum_{\Delta' \in P_k} \sum_{\Gamma \perp \Delta'} \tau_\Gamma$. We claim that only τ_Γ's for Γ of type at most $n-k$ appear in this sum. We also claim that the coefficient of τ_Γ in the sum depends only on the type of Γ.

The first statement is easy to verify. Let Γ be orthogonal to some $\Delta' \in P_k$. Then each of the two rows of Γ contains at least k numbers from $\{1, \ldots, n\}$, which means it cannot have type larger than $n-k$. As for the second statement, we can calculate exactly the number of different $\Delta' \in P_k$ that are orthogonal to a given Γ of type a. Indeed, first choose an unordered k-tuple among the $n-a$ elements of $\{1, \ldots, n\}$ in one row of Γ. Then match them with a ordered k-tuple taken from the a elements of $\{1, \ldots, n\}$ in the other row. Obviously, such a procedure gives all possible Δ', each exactly once. Therefore, the coefficient of τ_Γ which we are looking for is $c_a^k = \frac{(n-a)!a!}{k!(n-a-k)!(a-k)!}$.

We now proceed by induction. For the base case, take $k = [n/2]$. The only horizontal tableaux in the sum $\sum_{\Delta' \in P_k} \sum_{\Gamma \perp \Delta'} \tau_\Gamma$ are those of type $n - [n/2]$. Since they all have the same coefficient, and $\sum_{\Delta' \in P_k} \sum_{\Gamma \perp \Delta'} \tau_\Gamma = 0$ because each $\sum_{\Gamma \perp \Delta'} \tau_\Gamma = 0$, we conclude that $\sum_{\Gamma \in T_{n-[n/2]}} \tau_\Gamma = 0$.

Given that being a type a tableau is a 0-filter for $n - [n/2] \leq a \leq a' < n$, let us show that being a type $a' + 1$ tableau is a 0-filter. Indeed, $\sum_{\Delta' \in P_{n-a'-1}} \sum_{\Gamma \perp \Delta'} \tau_\Gamma = 0$ as before. This equality can be written as $\sum_{n-[n/2] \leq a \leq a'+1} c_a^{n-a'-1} \sum_{\Gamma \in T_a} \tau_\Gamma = 0$, where $c_a^{n-a'-1}$ is the coefficient calculated above. By the induction assumption, we know that for $n - [n/2] \leq a \leq a'$, the sum $\sum_{\Gamma \in T_a} \tau_\Gamma$ is 0. Since $c_{a'+1}^{n-a'-1} \neq 0$, we conclude that $\sum_{\Gamma \in T_{a'+1}} \tau_\Gamma = 0$. □

A trivial observation to make is that for $a = n$ this theorem implies that $\tau_{\Gamma_0} = 0$, which leads to the desired contradiction. Therefore, rows of $K_{\mathbf{2} \times \boldsymbol{n}}$ are linearly independent, which proves Theorem 2.1.

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