A partition of connected graphs

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Abstract

We define an algorithm k which takes a connected graph G on a totally ordered vertex set and returns an increasing tree R (which is not necessarily a subtree of G). We characterize the set of graphs G such that $k(G) = R$. Because this set has a simple structure (it is isomorphic to a product of non-empty power sets), it is easy to evaluate certain graph invariants in terms of increasing trees. In particular, we prove that, up to sign, the coefficient of x^q in the chromatic polynomial $\chi_G(x)$ is the number of increasing forests with q components that satisfy a condition that we call G-connectedness. We also find a bijection between increasing G-connected trees and broken circuit free subtrees of G.

We will work with finite labeled simple graphs. Usually we will identify a graph G with its edge set; this should not cause any serious ambiguities. If the vertex set is V then we say that G is a graph on V. A (spanning) subgraph Q of G is a graph with the same vertex set as G and a subset of the edges of G. The notation $Q \subseteq G$ means Q is a subgraph of G. A rooted graph is a graph with a distinguished vertex called the root.

Define $\mathrm{link}(v, S)$ to be the set of all possible edges joining v to an element of S (so if $v \notin S$, $\mathrm{link}(v, S)$ has $|S|$ elements). If G is a graph on V and $S \subseteq V$, we define the restriction of G to S, $G|_S$, to be the graph on S whose edge set consists of all edges of G with both ends in S.

We will use the symbols π and σ to denote set partitions. The notation $\pi \vdash S$ means π is a set partition of the set S. The length (number of blocks) of π is denoted by $\ell(\pi)$. A set partition σ is called a refinement of a set partition π if every block of σ is contained in some block of π.

To each graph G on V there corresponds a set partition $s(G)$ such that two vertices $v, w \in V$ are in the same block of $s(G)$ if and only if there is a path in G from v to w. Equivalently, $s(G)$ is the maximal set partition of V whose blocks are connected. The restriction of G to a block of $s(G)$ is called a component of G.

If G is a rooted connected graph on V with root r, we will call the set partition $\pi = s(G|_{V-\{r\}})$ of $V - \{r\}$ the depth-first partition of G. To obtain a connected subgraph of a rooted connected graph G on V, we can choose, for each block π_i of π, a connected subgraph of $G|_{\pi_i}$ and a nonempty set of edges (in G) connecting r to π_i. In fact, every connected subgraph of G can be obtained in this way. Our Theorem 1 may be regarded as an iteration of this correspondence. The depth-first partition and this correspondence have been studied by Gessel [3].

A forest is a graph with no circuits. A tree is a connected forest. A basic property of trees is that there is a unique path (a sequence of distinct, adjacent vertices) between any two vertices. The distance between two vertices is defined to be the length of this path. In a rooted tree, the height of a vertex is defined to be its distance from the root. A vertex w is called a descendant of a vertex v (or v is called an ancestor of w) if the heights of the vertices on the unique path from v to w are increasing (so in particular v is always a descendant of itself). We define the join of v and w to be their unique common ancestor on the unique path between them.

Let R be a rooted tree on the vertex set V, and let $v \in V$. We define $\mathrm{des}(v, R) \subseteq V$ to be the set of descendants of v (including v). If v is not the root of R, we define $\mathrm{parent}(v, R) \in V$ to be the closest vertex to v in R which is not a descendant of v. A rooted tree is increasing (according to a total order on V) if for each $v \in V$ and $w \in \mathrm{des}(v, R)$ we have $v \leq w$. Consequently, the root of an increasing tree must be the smallest element of V.

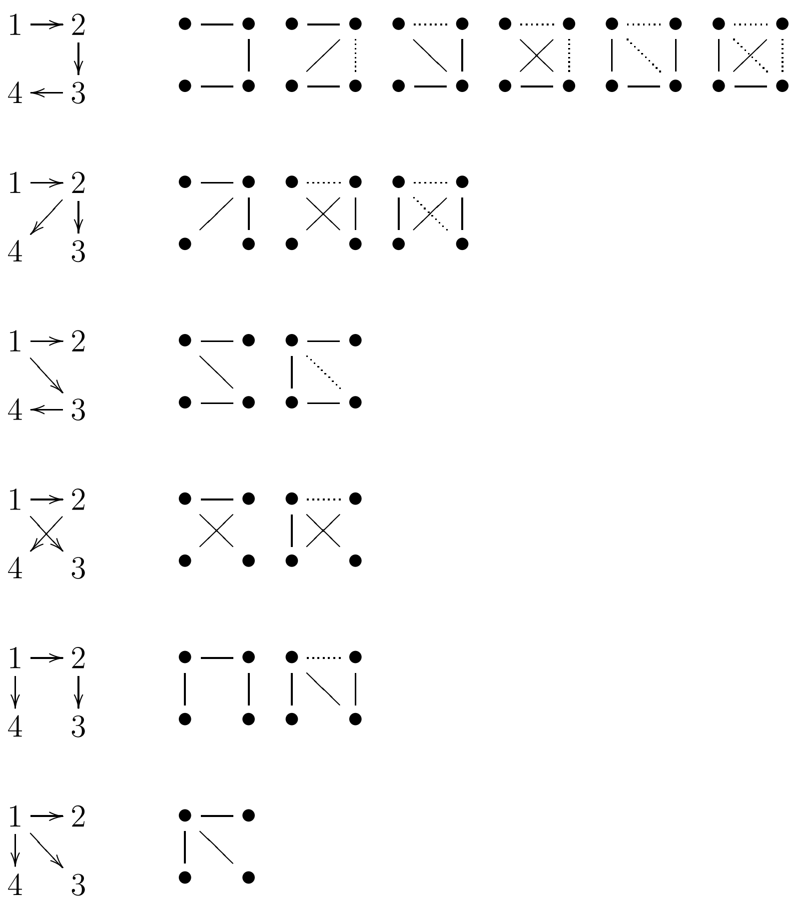
Definition 1 *Let R be a rooted tree on the totally ordered vertex set V with root r, and let $v \in V - \{r\}$. Define $J(v, R) = \mathrm{link}(\mathrm{parent}(v, R), \mathrm{des}(v, R))$. If G is a graph on V and if for each $v \in V - \{r\}$ we have $J(v, R) \cap G \neq \emptyset$ then we say that R is G-connected.*

Note that the sets $J(v, R)$ (as v ranges over $V - \{r\}$) are disjoint. Also note that a G-connected tree need not be a subgraph of G and that G must be connected for any rooted tree to be G-connected.

Definition 2 *For each connected graph G on a totally ordered vertex set V, define an increasing G-connected tree $k(G)$ by the following algorithm:*

1. *Let H be an empty graph on V, and set $S = V$.*
2. *Let π be the depth-first partition of $G|_S$ rooted at r=the smallest vertex in S. Add edges to H connecting r to the smallest vertex in each block of π.*
3. *For each block π_i of π with more than one element, return to step 2 with $S = \pi_i$.*
4. *Return $k(G) = H$.*

Example 1 *The 6 increasing trees on $V = \{1, 2, 3, 4\}$ are listed vertically. To the right of each increasing tree R are listed the subtrees T of the complete graph on V such that $k(T) = R$ (we have omitted the 22 connected subgraphs which are not trees). The breaks are indicated by dotted lines (see Theorem 3).*



There is a different algorithm, called depth-first search, which produces subforests of G. Some enumerative applications of this algorithm have been studied by Gessel and Sagan [4]. A distinguishing difference between depth-first search and our algorithm is that depth-first search only follows the edges of G, whereas here we add edges connecting to the smallest vertex in each block of π regardless of whether these are edges of G. The algorithms are related in that if G is a connected graph and R is a depth-first search subtree of G then parts 2 and 3 of the next theorem hold (although the converse is not true).

Theorem 1 *Let G be a connected graph on a totally ordered vertex set V, and let R be an increasing G-connected tree on V. Then the following are equivalent:*

1. $k(G) = R$
2. *For each vertex $v \in V$, $G|_{\mathrm{des}(v,R)}$ rooted at v is connected and has the same depth-first partition as $R|_{\mathrm{des}(v,R)}$ rooted at v.*
3. *For each non-root vertex $v \in V - \{r\}$ there is a nonempty set $E(v) \subseteq J(v,R)$ such that $G = \bigcup_{v \in V-\{r\}} E(v)$.*

Proof. 1 $\Leftrightarrow$ 2 This follows easily from Definition 2.

2 $\Rightarrow$ 3 Let $E(v) = J(v,R) \cap G$. We need to show that every edge of G lies in some $E(v)$. Let $e \in G$ and let $v < w$ be the vertices of e. We will show that w is a descendant of v. Suppose this is false, and let u be their join. Then $e \in G|_{\mathrm{des}(u,R)}$, so v and w are in the same block of the depth first partition of $G|_{\mathrm{des}(u,R)}$. This is a contradiction because they are in different blocks of the depth first partition of $R|_{\mathrm{des}(u,R)}$. Now, since w is a descendant of v, there is a unique vertex $z \in V$ (possibly equal to w) such that $\mathrm{parent}(z) = v$ and $w \in \mathrm{des}(z)$. Hence $e \in J(z,R) \cap G$.

3 $\Rightarrow$ 2 This is certainly true if v (in part 2) is a leaf of R (its only descendant is itself). Let $v \in V$ and suppose it is true for all $w \in \mathrm{des}(v,R) - \{v\}$. Let π be the depth-first partition of $R|_{\mathrm{des}(v,R)}$. Then $G|_{\pi_i}$ is connected by the inductive hypothesis. Furthermore, G contains an edge connecting v to π_i because π_i contains a vertex w whose parent in R is v and $J_G(w,R)$ consists of edges connecting v to π_i. Hence $G|_{\mathrm{des}(v,R)}$ is connected. Clearly π is a refinement of the depth-first partition of $G|_{\mathrm{des}(v,R)}$ (because $G|_{\pi_i}$ is connected), so to show that they are equal we have only to show that if x and y are in different blocks of π then they are in different blocks of the depth-first partition of G. Let $x < y \in V$ be in different blocks of π, and suppose G has an edge between x and y. Then y is a descendant of x in R because every edge of $J_G(w,R)$ (for any $w \in V$) connects a vertex to one of its descendants. This contradicts the fact that they are in different blocks of the depth-first partition of $R|_{\mathrm{des}(v,R)}$. $\square$

Remark 1 *Actually the condition in Theorem 1 that R be G-connected is not necessary because if R is not G-connected then parts 1, 2 and 3 will be false.*

Some algebraic invariants of graphs can be simply expressed in terms of connected subgraphs. We can use the algorithm k to express such invariants in terms increasing trees. Moreover, Theorem 1 shows that the set $k^{-1}(R)$ has a simple structure, as illustrated by the next theorem.

Definition 3 *Let G be a connected graph on V. Define*

$$\eta^G(t) = \sum_{\substack{Q \subseteq G \\ \text{connected}}} t^{|Q|}$$

where $|Q|$ denotes the number of edges in Q.

Theorem 2

$$\eta^G(t) = \sum_{\substack{R \\ \text{increasing} \\ G-\text{connected}}} \prod_{v \in V - \{r\}} [(1+t)^{|J(v,R)\cap G|} - 1]$$

Proof. We have

$$\eta^G(t) = \sum_{\substack{R \\ \text{increasing} \\ G-\text{connected}}} \sum_{\substack{Q \subseteq G \\ k(Q)=R}} t^{|Q|}$$

Now, the generating function for the cardinality of nonempty subsets of a set S is

$$f_S(x) = \sum_{\emptyset \neq T \subseteq S} x^{|T|} = (1+x)^{|S|} - 1$$

Hence from Theorem 1 part 3,

$$\sum_{\substack{Q\subseteq G \\ k(Q)=R}} t^{|Q|} = \sum_{\substack{Q=\bigcup_{v\in V-\{r\}} E(v) \\ \emptyset\neq E(v)\subseteq J(v,R)\cap G}} t^{|Q|} = \prod_{v\in V-\{r\}} f_{J(v,R)\cap G}(t)$$

from which the result follows. □

The chromatic polynomial $\chi_G(x)$ of a graph G is a polynomial which evaluates to the number of proper colorings of G with x colors. The subgraph expansion of $\chi_G(x)$ is

$$\chi_G(x) = \sum_{Q\subseteq G} (-1)^{|Q|} x^{c(Q)}$$

where $c(Q)$ is the number of components of Q. See [1] for background on the chromatic polynomial.

We define an increasing G-connected forest R to be a forest where each component $R|_{s(R)_i}$ is an increasing $G|_{s(R)_i}$-connected tree. For a graph G, let $t(G)$ be the (integer) partition whose parts are the sizes of the blocks of $s(G)$. For background on the chromatic symmetric function $X_G = X_G(x_1, x_2, \ldots)$ of a graph G, see [5] and [6]. For background on the chromatic symmetric function in non-commuting variables $Y_G = Y_G(x_1, x_2, \ldots)$, see [2].

Corollary 1 *Let G be a graph on a totally ordered vertex set V with $|V| = n$.*

1. *The coefficient of $(-1)^{n-1}x$ in the chromatic polynomial $\chi_G(x)$ is the number of increasing G-connected trees.*

2. *The coefficient of $(-1)^{n-q}x^q$ in the chromatic polynomial $\chi_G(x)$ is the number of increasing G-connected forests with q components (or, equivalently, with $n-q$ edges).*

3. *The coefficient of $(-1)^{n-\ell(\lambda)}p_\lambda$ in the chromatic symmetric function X_G is the number of increasing G-connected forests R such that $t(R) = \lambda$.*

4. *The coefficient of $(-1)^{n-\ell(\pi)}p_\pi$ in the chromatic symmetric function in non-commuting variables Y_G is the number of increasing G-connected forests R such that $s(R) = \pi$.*

Proof. 1. Let a^G be the coefficient of x in $\chi_G(x)$. From the subgraph expansion we have

$$a^G = \sum_{\substack{Q\subseteq G \\ \text{connected}}} (-1)^{|Q|} = \eta^G(-1) = \sum_{\substack{R \\ \text{increasing} \\ G-\text{connected}}} \prod_{v\in V-\{r\}} (-1)$$

We don't need to worry about 0^0 because the G-connectedness of R implies that $J(v, R) \cap G$ is never empty.

4. We will prove part 4, the others being simple specializations. Let H^G_π be the number of increasing G-connected forests R such that $s(R) = \pi$, and let H^G be the number of increasing G-connected trees. Then using part 1 we have

$$H^G_\pi = \prod_{i=1}^{\ell(\pi)} H^{G|_{\pi_i}} = (-1)^{n-\ell(\pi)} \prod_{i=1}^{\ell(\pi)} \sum_{\substack{Q \subseteq G|_{\pi_i} \\ connected}} (-1)^{|Q|} \tag{1}$$

The subgraph expansion of Y_G is

$$Y_G = \sum_{Q \subseteq G} (-1)^{|Q|} p_{s(Q)}$$

Hence

$$Y_G = \sum_{\pi \vdash V} p_\pi \sum_{\substack{Q \subseteq G \\ s(Q)=\pi}} (-1)^{|Q|} = \sum_{\pi \vdash V} p_\pi \prod_{i=1}^{\ell(\pi)} \sum_{\substack{Q \subseteq G|_{\pi_i} \\ connected}} (-1)^{|Q|}$$

Substituting (1), we obtain the desired result. $\square$

If G is a graph on a totally ordered vertex set V, we extend the ordering of the vertices to an ordering of the edges lexicographically. A broken circuit of $H \subseteq G$ is a set of edges $B \subseteq H$ such that there is some edge $e \in G$, smaller than every edge of B, such that $B \cup e$ is a circuit. Note that B being a broken circuit of H depends both on H and G. If $H \subseteq G$ contains no broken circuits then it is called broken circuit free. Note that if H contains a circuit then it also contains a broken circuit. Consequently, a broken circuit free subgraph is always a forest. If $T \subseteq G$ is a subtree of G and the edge $e \in G$, $e \notin T$ is the smallest edge in the unique circuit in $T \cup \{e\}$ then we will call e a break in T. Hence the set of breaks in a subtree T is in bijection with the set of broken circuits of T.

Whitney's Broken Circuit Theorem [7] shows that if G is a connected graph with n vertices, the coefficient of $(-1)^{n-1}x$ in $\chi_G(x)$ is the number of broken circuit free subtrees of G. Hence there should be a bijection between broken circuit free subtrees and increasing G-connected trees.

Theorem 3 *Let V be a totally ordered vertex set with smallest element r, and let G be a connected graph on V. Let $T \subseteq G$ be a subtree of G, and let $R = k(T)$. Let $E(v)$ for $v \in V - \{r\}$ be as in Theorem 1 part 3. Then $E(v)$ contains only one element $e(v)$ (otherwise T would have more than $|V|-1$ edges so it could not be a tree). For $v \in V - \{r\}$, let $d(v)$ be the set of elements of $J(v, R) \cap G$ which are smaller than $e(v)$. Then the set of breaks in T is*

$$\bigcup_{v \in V - \{r\}} d(v)$$

Proof. Let $J = \bigcup_{v \in V-\{r\}} J(v,R) \cap G$. Since $k(G)$ may be different from R, J may be different from G. We will first show that if $e \in G$ but $e \notin J$ then e is not a break. Let $v < w \in V$ be the vertices of e. Then w is not a descendant of v because otherwise we would have $e \in J$. Let $u \in V$ be the join of v and w in R. Then Theorem 1 part 2 implies that u is also the join of v and w in $T|_{\mathrm{des}(u,R)}$ (rooted at u). Therefore, the cycle created by adding e to T contains an edge connected to u. Since $u < v < w$, e cannot be a break.

Now suppose $e \in J(v,R) \cap G$ is smaller than $e(v)$. We will show that e is a break. Let $H = T|_{\mathrm{des}(v,R) \cup \mathrm{parent}(v,R)}$. Then parent$(v,R)$ is the smallest vertex in the vertex set of H. Therefore, e is smaller than any other edge in H. Since H is a tree, adding e would create a unique circuit in H. Hence e is a break.

Now suppose $e \in J(v,R) \cap G$ is larger than $e(v)$. Then, letting H be as before, we see that $e(v)$ must belong to the circuit which e creates. But $e(v)$ is smaller than e, so e cannot be a break. $\square$

Corollary 2 *The function*

$$f(R) = \bigcup_{v \in V-\{r\}} \min(J(v,R) \cap G)$$

is a bijection between increasing G-connected trees and broken circuit free subtrees, and $f^{-1}(T) = k(T)$.

Of course, this bijection generalizes to a bijection between increasing G-connected forests with q components and broken circuit free subforests of G with q components.

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