New lower bound for multicolor Ramsey numbers for even cycles

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Submitted: Oct 5, 2004; Accepted: Jun 3, 2005; Published: Aug 30, 2005
Mathematics Subject Classifications: Primary 05C55; Secondary 05C15, 05C38

Abstract

For given finite family of graphs $G_1, G_2, \ldots, G_k, k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph on n vertices with k colors then there is always a monochromatic copy of G_i colored with i, for some $1 \leq i \leq k$. We give a lower bound for $k-$color Ramsey number $R(C_m, C_m, \ldots, C_m)$, where $m \geq 4$ is even and C_m is the cycle on m vertices.

1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. By K_m we denote the complete graph on m vertices, and by C_m we denote the cycle of length m. For given graphs $G_1, G_2, \ldots, G_k, k \geq 2$, the *multicolor*

*The first author was partially supported by KBN grant 4 T11C 047 25.
†The second author was partially supported by grant BW/5100-5-0145-4.
‡The third author was partially supported by grant BW/5100-5-0095-5.

Ramsey number $R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with k colors, then it always contains a monochromatic copy of G_i colored with i, for some $1 \leq i \leq k$. We denote such a number by $R_k(G)$ if $G = G_1 = G_2 = \cdots = G_k$. Here in, we consider only 3-color Ramsey number $R_3(G)$ (i.e. we color the edges of the complete graph K_n with color red, blue and green.) A 3-coloring of K_n is called a $(G; n)_3$-coloring if it contains neither a red G nor a blue G nor a green G, $(G; n)_k$-coloring is defined analogously. We refer the reader to [6] for a survey.

2 The Ramsey numbers for even cycles

Up to now, there have been known only two exact values for 3-color Ramsey numbers for even cycles. More precisely, in [2] it was proved that $R_3(C_4) = 11$, and $R_3(C_6) = 12$ was showed in [7] with a help of the computer support. When talking about lower bounds, let us recall that Graham *et al.* [5] proved that for any k and m, $R_k(C_{2m}) \geq (k-1)(m-1)+1$. This bound was improved to $(k+1)m - k + 1$ in [3]. Finally, recall that Figaj and Łuczak proved the following theorem.

Theorem 1 ([4]). *For any constants $\alpha_1, \alpha_2, \alpha_3 > 0$,*

$$R\left(C_{2\lfloor \alpha_1 n \rfloor}, C_{2\lfloor \alpha_2 n \rfloor}, C_{2\lfloor \alpha_3 n \rfloor}\right) = (\alpha_1 + \alpha_2 + \alpha_3 + max\{\alpha_1, \alpha_2, \alpha_3\} + o(1))\, n$$

while $n \to \infty$.

Consequently, notice that if $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $n = m$ we obtain that

$$R_3(C_{2m}) = (4 + o(1))\, m.$$

In this paper, our main result is the following theorem.

Theorem 2. *For all integers $m \geq 2$ and an odd integer $k \geq 1$,*

$$R_k(C_{2m}) \geq (k+1)m.$$

Proof. We shall give a k-coloring of all edges of a complete graph $G'' = K_n$ on $n = (k+1)m - 1$ vertices which is a $(C_{2m}; n)_k$-coloring. The situation for $k = 1$ is obvious, so we may assume that $k \geq 3$.

Let $k \geq 3$ be an odd integer. Using a fact that $\chi'(K_{k+1}) = k$ when k is odd (see e.g. [8]), color properly edges of the complete graph K_{k+1} with k colors. "Blow-up" the coloring $m-1$ times, i.e. replace each vertex of K_{k+1} by the set G_i $(1 \leq i \leq k+1)$ of $m-1$ vertices and each colored edge by a complete monochromatic bipartite graph $K_{m-1,m-1}$ of an appropriate color (See Fig. 1 for illustration.) Formally, consider the complete graph G on $k+1$ vertices. Let $c\colon V \to \{1, \ldots, k+1\}$ be a proper edge-coloring of graph G. For a vertex $i \in V$, where $i \in \{1, \ldots, k+1\}$, let G_i denote a complete graph on $m-1$ vertices.

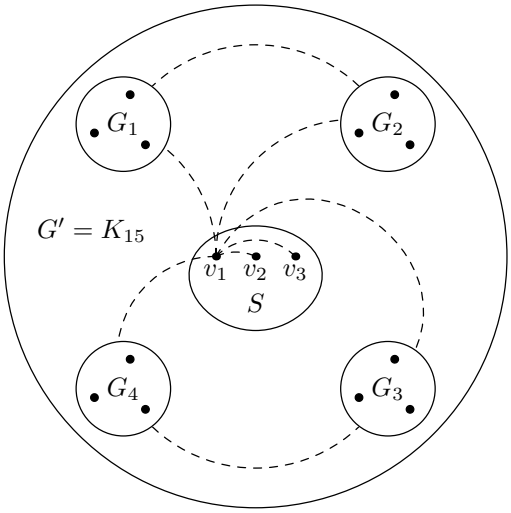

Figure 1: An illustration of coloring from the proof of Theorem 2 for the case $m = 4$ and $k = 3$. To make the picture readable there are shown only edges colored with color 1: ones which join v_1 with each G_i, and edges joining G_1 with G_2, and G_3 with G_4. Edges which join v_2 with each G_i, G_2 with G_3, and G_1 with G_4 are colored with color 2. Edges which join v_3 with each G_i, G_1 with G_3 and G_2 with G_4 are colored with color 3. Subgraphs G_1, G_2, G_3 and G_4 replace "blown-up" vertices of the complete graph K_4.

Let $G'(V', E')$ be a complete graph with the set of vertices

$$V' = \bigcup_{i=1}^{k+1} V\left(G_i\right).$$

The coloring c' of the graph G' is as follows:

$$c'\left(\{p,q\}\right) = \begin{cases} c\left(x,y\right) & \text{if } p \in V(G_x), \text{ and } q \in V(G_y), \text{ and } x \neq y; \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, such a graph G' contains no monochromatic path of more than $2m-2$ vertices.

Next, extend graph G' to the graph G'' by adding k new vertices $S = \{v_1, \ldots, v_k\}$. Now, color all edges between v_i and V' with the color i, and for any pair $\{i, j\}$ such that $j > i > 0$, color edge $\{v_i, v_j\}$ with the color i (thus there are no monochromatic cycles in the subgraph induced by S.)

More formally, let $G''(V'', E'')$ be a complete graph with the set of vertices $V'' = V' \cup S$. The coloring c'' of G'' is as follows:

$$c''\left(\{p,q\}\right) = \begin{cases} i & \text{if } p = v_i \in S, \text{ and } q \in V\left(G_j\right), \text{ and } 1 \leq j \leq k+1; \\ i & \text{if } p = v_i \in S, \text{ and } q = v_j \in S, \text{ and } 1 \leq i < j \leq k; \\ c'\left(\{p,q\}\right) & \text{otherwise.} \end{cases}$$

It remains to show that in G'' there are no monochromatic cycles of length at least $2m$. Suppose, contrary to our claim, that G'' contains a cycle C of color d longer than

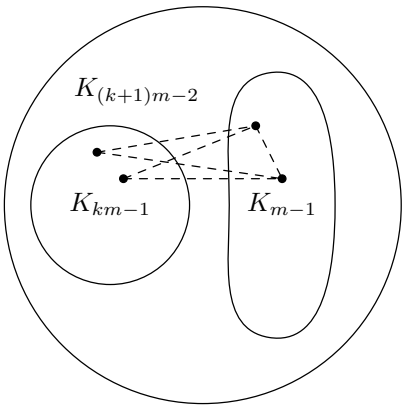

Figure 2: An illustration of coloring from the proof of Corollary 1. Edges of K_{km-1} are colored with $k-1$ colors without monochromatic cycle of length $2m$. All edges from the graph K_{m-1} and a bipartite graph $K_{km-1,m-1}$ are colored with color k (dashed line denote edges assigned with color k.)

$2m-1$. Since in G' there is no monochromatic path of length greater than $2m-2$, we have

$$C \cap S \neq \emptyset.$$

Next, the only vertex from S which is adjacent by an edge of color d with G is v_d, hence

$$C \cap S = \{v_d\}.$$

Since v_d can be contained only once in the cycle C, this implies that the cardinality of the set

$$\{i \ : \ C \cap G_i \neq \emptyset\}$$

is at most 2. Thus the length of C is less than $2m$, a contradiction. □

Corollary 1. *For all integers $m \geq 2$ and an even integer $k \geq 2$,*

$$R_k(C_{2m}) \geq (k+1)\,m - 1.$$

Proof. Let $n = (k+1)m - 2$. By Theorem 2, there exists $(C_{2m}; km-1)_{k-1}$-coloring of a complete subgraph K_{km-1} of K_n. A $(C_{2m}; n)_k$-coloring of K_n is obtained by assigning the last color k to all remaining edges (See Fig. 2.) Indeed, on the contrary suppose that there exists a monochromatic cycle of length $2m$. This cycle has the last k-th color. The number of vertices from K_{km-1} which belong to cycle is at least $m+1$ and is greater than the number of such vertices from K_{m-1}. The maximal possible number of edges between K_{km-1} and K_{m-1} is $2m-2$. Thus there exists an edge contained in the graph K_{km-1}, what is impossible. □

The following corollary is straightforward:

Corollary 2. *For all integers* $m \geq 2$,

$$R_3(C_{2m}) \geq 4m.$$

In particular, notice that we obtain $R_3(C_8) \geq 16$. Moreover, by using upper bound for Ramsey number for even cycles ([5, Section 5.7, Theorem 10]), we have $R_3(C_8) \leq 2412$, and by using known upper bound for Ramsey number for symmetric bipartite graph $K_{4,4}$ ([1]), we have $R_3(C_8) \leq 648$.

3 Acknowledgment

We would like to thank the anonymous referees for many helpful suggestions, especially for the simplified proof of Theorem 2. Also we would like to express our gratitude to Professor Paweł Żyliński for his critical reading of the manuscript and for pointing out a number of corrections.

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