A Note on Two Multicolor Ramsey Numbers

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Abstract

Two new bounds for multicolor Ramsey numbers are proved:
$R(K_3, K_3, C_4, C_4) \geq 27$ and $R_4(C_4) \leq 19$.

1 Introduction

We prove two new bounds for multicolor Ramsey numbers, a lower bound for $R(K_3, K_3, C_4, C_4)$ by coloring K_{26}, and an upper bound for $R_4(C_4)$ by a density argument.

2 The Ramsey number $R(K_3, K_3, C_4, C_4)$

From the survey of Ramsey numbers by Radziszowski [3] we know that $R(K_3, K_3, C_4, C_4) \geq 26$. We use C_5-decompositions to construct a four-coloring of the edges of K_{26}, which show that $R(K_3, K_3, C_4, C_4) \geq 27$. The technique used in this section was invented by Exoo and Reynolds [2].

Theorem 1 *$R(K_3, K_3, C_4, C_4) \geq 27$.*

Proof: Let $X, Y, I, \bar{0}$, and $\bar{1}$ be defined by

$$X = \begin{bmatrix} 0&1&0&0&1 \\ 1&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \end{bmatrix} \quad Y = \begin{bmatrix} 0&0&1&1&0 \\ 0&0&0&1&1 \\ 1&0&0&0&1 \\ 1&1&0&0&0 \\ 0&1&1&0&0 \end{bmatrix} \quad I = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} \quad \bar{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \bar{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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The critical colorings which show that $R(K_3, K_3) > 5$ and $R(C_4, C_4) > 5$ have the adjacency matrices X and Y. Observe that $X + Y + I$ is the all-ones 5×5 matrix.

We now construct four 26×26 adjacency matrices M_i, so that M_1 and M_2 contain no K_3, and M_3 and M_4 contain no C_4.

Given a triangle-free graph on n vertices, one can construct a triangle-free graph on nm vertices by replacing each vertex with m vertices and each edge with $K_{m,m}$. We construct the two first graphs, which are isomorphic, by beginning with C_5, replacing the edges with $K_{5,5} - e$, and then adding a vertex with five edges.

$$M_1 = \begin{bmatrix} 0 & X & X & X & X & \bar{1} \\ X & X & X & X & X & \bar{0} \\ X & X & X & X & X & \bar{0} \\ X & X & X & X & X & \bar{0} \\ X & X & X & X & X & \bar{0} \\ \bar{1}^T & \bar{0}^T & \bar{0}^T & \bar{0}^T & \bar{0}^T & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} Y & Y & Y & Y & Y & \bar{0} \\ Y & Y & Y & Y & Y & \bar{0} \\ Y & Y & Y & Y & Y & \bar{0} \\ Y & Y & Y & Y & Y & \bar{0} \\ Y & Y & Y & Y & 0 & \bar{1} \\ \bar{0}^T & \bar{0}^T & \bar{0}^T & \bar{0}^T & \bar{1}^T & 0 \end{bmatrix}$$

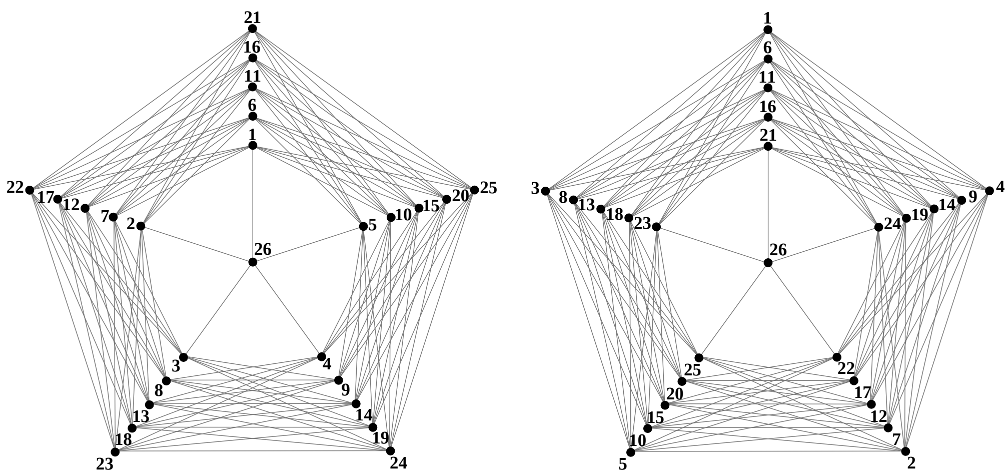

Figure 1: The graphs with adjacency matrices M_1 and M_2.

We denote the vertices from the top of the matrices as $1, 2, \ldots 26$. The graphs are shown in Figure 1. Vertices $1-25$ are the triangle-free constructions from C_5 and $K_{5,5}-e$. Vertex 26 is in no triangle, since its neighbors have no edges between them. Hence, the graphs are triangle-free.

The remaining edges are distributed as described by the adjacency matrices M_3 and M_4.

$$M_3 = \begin{bmatrix} X & I & 0 & 0 & I & \bar{0} \\ I & 0 & I & 0 & 0 & \bar{0} \\ 0 & I & 0 & I & 0 & \bar{1} \\ 0 & 0 & I & 0 & I & \bar{1} \\ I & 0 & 0 & I & 0 & \bar{0} \\ \bar{0}^T & \bar{0}^T & \bar{1}^T & \bar{1}^T & \bar{0}^T & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 0 & I & I & 0 & \bar{0} \\ 0 & 0 & 0 & I & I & \bar{1} \\ I & 0 & 0 & 0 & I & \bar{0} \\ I & I & 0 & 0 & 0 & \bar{0} \\ 0 & I & I & 0 & Y & \bar{0} \\ \bar{0}^T & \bar{1}^T & \bar{0}^T & \bar{0}^T & \bar{0}^T & 0 \end{bmatrix}$$

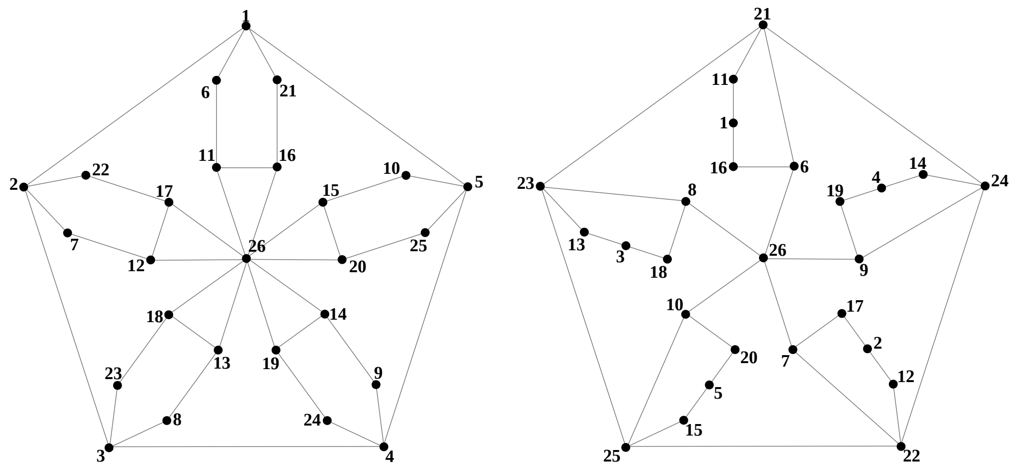

Figure 2: The graphs with adjacency matrices M_3 and M_4.

It is not hard to see that $M_1 + M_2 + M_3 + M_4$ is the adjacency matrix of K_{26}. It is clear from Figure 2 that there are no quadrilaterals. $\square$

3 The Ramsey number $R_4(C_4)$

From the Ramsey number survey [3] we also know that $18 \leq R_4(C_4) \leq 21$. It was shown by Clapham, Flockhart and Sheehan [1] that a C_4-free graph with 19 vertices has at most 42 edges. Since $4 \cdot 42 = 168$ and there are 171 edges in K_{19}, it is not possible to four-color the edges of K_{19} without a monochromatic quadrilateral.

Theorem 2 $R_4(C_4) \leq 19$.

References

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