Pan-factorial Property in Regular Graphs*

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Abstract

Among other results, we show that if for any given edge e of an r-regular graph G of even order, G has a 1-factor containing e, then G has a k-factor containing e and another one avoiding e for all k, $1 \le k \le r-1$.

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For a function $f : V(G) \to \{0, 1, 2, 3, \ldots\}$, a spanning subgraph F of G with $\deg_F(x) = f(x)$ for all $x \in V(G)$ is called an *f-factor* of G, where $\deg_F(x)$ denotes the degree of x in F. If $f(x) = k$ for all vertices $x \in V(G)$, then an f-factor is also called a *k-regular factor* or a *k-factor*. An *$[a, b]$-factor* is a spanning subgraph F of G such that $a \le \deg_F(x) \le b$ for all $x \in V(G)$.

A graph G is *pan-factorial* if G contains all k-factors for $1 \le k \le \delta(G)$. In this note, we investigate the pan-factor property in regular graphs. Moreover, we proved that the existence of 1-factor containing any given edge implies the existence of k-factors containing or avoiding any given edge.

The first of our main results is the following.

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Theorem 1 *Let G be a connected r-regular graph of even order. If for every edge e of G, G has a 1-factor containing e, then G has a k-factor containing e and another k-factor avoiding e for all integers k, $1 \leq k \leq r-1$.*

The next theorem is also one of our main results.

Theorem 2 *Let G be a connected graph of even order, e be an edge of G, and a, b, c be odd integers such that $1 \leq a < c < b$. If G has both an a-factor and a b-factor containing e, then G has a c-factor containing e. Similarly, if G has both an a-factor and a b-factor avoiding e, then G has a c-factor avoiding e.*

The above theorem shows that there exists a kind of continuity relation among regular factors, which is an improvement of the following theorem obtained by Katerinis [1].

Theorem 3 (*Katerinis* [1]) *Let G be a connected graph of even order, and a, b and c be odd integers such that $1 \leq a < c < b$. If G has both an a-factor and a b-factor, then G has a c-factor.*

We need a few known results as lemmas for the proof of our theorems. Firstly, we quote Petersen's classic decomposition theorem about regular graphs of even degree.

Lemma 1 (*Petersen* [2]) *Every $2r$-regular graph can be decomposed into r disjoint 2-factors.*

For the introduction of Tutte's f-factors theorem, we require the following notation.

For a graph G and $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, define

$$\delta_G(S,T) = \sum_{x \in S} f(x) + \sum_{x \in T} (d_{G-S}(x) - f(x)) - h_G(S,T),$$

where $h_G(S,T)$ is the number of components C of $G - (S \cup T)$ such that $\sum_{x \in V(C)} f(x) + e_G(V(C), T) \equiv 1 \pmod 2$ and such a component C is called an *f-odd component* of $G - (S \cup T)$.

Lemma 2 (*Tutte's f-factor Theorem* [3]) *Let G be a graph and $f : V(G) \to \{0, 1, 2, 3, \ldots\}$ be a function. Then*

(a) G has an f-factor if and only if $\delta_G(S,T) \geq 0$ for all S, $T \subseteq V(G)$ with $S \cap T = \emptyset$;

(b) $\delta_G(S,T) \equiv \sum_{x \in V(G)} f(x) \pmod 2$ for all S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.

Lemma 3 *Let G be a connected graph. If for any edge e there exists a 1-factor containing e, then there exists another 1-factor avoiding e.*

Proof. For any edge $e \in E(G)$, we will show that there exists a 1-factor avoiding e. Choose an edge e' incident to the given edge e, then there exists a 1-factor F containing e' and thus F is the 1-factor avoiding e. □

Now we are ready to show the main results. We start with the proof of Theorem 2 and then derive the proof of Theorem 1 from it.

Proof of Theorem 2. Let e be an edge of G. Assume that G has both a-factor and b-factor avoiding e. By applying Theorem 3 to $G-e$, we see that $G-e$ has a c-factor, which implies that G has a c-factor avoiding e.

We now prove that if G has both a-factor and b-factor containing e, then G has a c-factor containing e.

We define a new graph G^* by inserting a new vertex w on the edge e, and define an integer-value function $f_k : V(G^*) \to \{k, 2\}$ such that

$$f_k(x) = \begin{cases} k & \text{if } x \in V(G); \\ 2 & \text{if } x = w. \end{cases}$$

Then G has a k-factor containing e if and only if G^* has a f_k-factor. It is obvious that $\sum_{x \in V(G^*)} f_k(x) = k|V(G)| + 2 \equiv 0 \pmod 2$ since G is of even order.

Assume that G^* has no f_c-factor. Then, by Tutte's f-factor Theorem, there exist two disjoint subsets $S, T \subseteq V(G^*)$ such that

$$\delta(S,T;f_c) = \sum_{x \in S} f_c(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_c(x)) - h(S,T;f_c) \leq -2. \tag{1}$$

On the other hand, since G^* has both f_a-factor and f_b-factor, we have

$$\delta(S,T;f_a) = \sum_{x \in S} f_a(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_a(x)) - h(S,T;f_a) \geq 0, \tag{2}$$

$$\delta(S,T;f_b) = \sum_{x \in S} f_b(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_b(x)) - h(S,T;f_b) \geq 0. \tag{3}$$

Now depending on the location of w, we consider three cases:

Case 1. $w \notin S \cup T$.

(1), (2) and (3) can be rewritten as

$$c|S| + \sum_{x \in T} \deg_{G^*-S}(x) - c|T| - h(S,T;f_c) \quad \leq \quad -2, \tag{4}$$

$$a|S| + \sum_{x \in T} \deg_{G^*-S}(x) - a|T| - h(S,T;f_a) \quad \geq \quad 0, \tag{5}$$

$$b|S| + \sum_{x \in T} \deg_{G^*-S}(x) - b|T| - h(S,T;f_b) \quad \geq \quad 0. \tag{6}$$

Subtracting (5) from (4), we have

$$(c-a)(|S| - |T|) + h(S,T;f_a) - h(S,T;f_c) \leq -2. \tag{7}$$

Similarly, from (6) and (4), we have

$$(c-b)(|S| - |T|) + h(S,T;f_b) - h(S,T;f_c) \leq -2. \tag{8}$$

Recall that $h(S,T;f_k)$ is the number of f_k-odd components C of $G^* - (S \cup T)$, which satisfies $\sum_{x \in V(C)} f_k(x) + e_{G^*}(C,T) \equiv 1 \pmod 2$. Since all a, b and c are odd integers, it follows that if $w \notin V(C)$, then

$$\begin{aligned} & \sum_{x \in V(C)} f_a(x) + e_{G^*}(C,T) = a|C| + e_{G^*}(C,T) \\ \equiv\ & b|C| + e_{G^*}(C,T) = \sum_{x \in V(C)} f_b(x) + e_{G^*}(C,T) \pmod 2 \\ \equiv\ & c|C| + e_{G^*}(C,T) = \sum_{x \in V(C)} f_c(x) + e_{G^*}(C,T) \pmod 2. \end{aligned}$$

Therefore we obtain

$$h(S,T;f_c) - h(S,T;f_a) \le 1 \qquad \text{and} \qquad h(S,T;f_c) - h(S,T;f_b) \le 1.$$

If $|S| \ge |T|$, then (7) implies

$$-1 \le (c-a)(|S| - |T|) + h(S,T;f_a) - h(S,T;f_c) \le -2,$$

a contradiction. If $|S| < |T|$, then (8) implies

$$-1 \le (c-b)(|S| - |T|) + h(S,T;f_b) - h(S,T;f_c) \le -2,$$

a contradiction again.

Case 2. $w \in S$.

In this case, (1), (2) and (3) become

$$\begin{aligned} 2 + c(|S|-1) + \sum_{x \in T} \deg_{G^*-S}(x) - c|T| - h(S,T;f_c) &\le -2 \\ 2 + a(|S|-1) + \sum_{x \in T} \deg_{G^*-S}(x) - a|T| - h(S,T;f_a) &\ge 0 \\ 2 + b(|S|-1) + \sum_{x \in T} \deg_{G^*-S}(x) - b|T| - h(S,T;f_b) &\ge 0. \end{aligned}$$

It is clear that $h(S,T;f_c) = h(S,T;f_a) = h(S,T;f_b)$. If $|S| \ge |T| + 1$, we have $0 \le (c-a)(|S|-1-|T|) \le -2$, a contradiction; if $|S| < |T|+1$, then $0 \le (c-b)(|S|-1-|T|) \le -2$, a contradiction as well.

Case 3. $w \in T$.

In this case, (1), (2) and (3) become

$$\begin{aligned} c|S| + \sum_{x \in T} \deg_{G^*-S}(x) - 2 - c(|T|-1) - h(S,T;f_c) &\le -2 \\ a|S| + \sum_{x \in T} \deg_{G^*-S}(x) - 2 - a(|T|-1) - h(S,T;f_a) &\ge 0 \\ b|S| + \sum_{x \in T} \deg_{G^*-S}(x) - 2 - b(|T|-1) - h(S,T;f_b) &\ge 0. \end{aligned}$$

Discussing similarly as in Case 2, we yield contradictions. Consequently the theorem is proved. □

With help of Theorem 2 and Petersen's Theorem (Lemma 1), we can provide a clean proof for Theorem 1.

Proof of Theorem 1. For any edge e of G, let F_1 be a 1-factor containing e. From Lemma 3, there exists another 1-factor F_2 avoiding e. According to the parity of r we consider two cases.

Case 1. r is odd.

Since $G - F_1$ is an even regular graph, by Lemma 1, $G - F_1$ can be decomposed into 2-factors $T_1, T_2, \ldots, T_m$, where $m = (r-1)/2$. For an integer k $(1 \leq k \leq m-1)$, $F_1 \cup T_1 \cup \cdots \cup T_k$ is a $(2k+1)$-factor containing e. In the mean time, $T_1 \cup \cdots \cup T_k$ is a $2k$-factor avoiding e. Moreover, $G - F_1$ is a $2m$-factor avoiding e.

Similarly, $G - F_2$ has disjoint 2-factors $T_1, T_2, \ldots, T_m$. Without loss of generality, we may assume $e \in T_1$. Then $F_2 \cup T_2 \cup \cdots \cup T_{k+1}$ is a $(2k+1)$-factor avoiding e, and $T_1 \cup \cdots \cup T_k$ is a $2k$-factor containing e. Furthermore, $G - F_2$ is a $2m$-factor containing e. Therefore the theorem holds in this case.

Case 2. r is even.

For even k, similar to Case 1, G can be decomposed into 2-factors $T_1, T_2, \ldots, T_m$, where $m = r/2$. Without loss of generality, assume $e \in T_1$. Then T_1, $T_1 \cup T_2$, …, $T_1 \cup \ldots \cup T_m$ are 2-factor, 4-factor, …, r-factor containing e, respectively. Moreover, T_2, $T_2 \cup T_3$, …, $T_2 \cup T_3 \cup \ldots \cup T_m$ are 2-factor, 4-factor, …, $(r-2)$-factor avoiding e, respectively.

For odd k, it is clear that $G - F_2$ is a $(r-1)$-factor containing e and $G - F_1$ is an $(r-1)$-factor avoiding e. By Theorem 2, the odd-factors F_1 and $G - F_2$ containing e, respectively, imply the existence of k-factors containing e, $1 \leq k \leq r-1$. Similarly, we obtain k-factors avoiding e, $1 \leq k \leq r-1$.

So the desired statement holds and consequently the theorem is proved. □

Next we consider the existence of factors containing or avoiding a given edge in a regular graph of *odd* order and prove a similar but slightly weaker result than Theorem 1.

Theorem 4 *Let G be a connected $2r$-regular graph of odd order. For any given edge e and any vertex $v \in V(G) - V(e)$, if $G - v$ has a 1-factor containing e, then $G - v$ has a $[k, k+1]$-factor containing or avoiding e for $1 \leq k \leq 2r-2$.*

Proof. For any edge e of G and any vertex $u \in V(G) - V(e)$, let the neighbor vertices of u be $x_1, x_2, \ldots, x_{2r}$. We construct a new graph G^* by using two copies of $G - u$ and joining two sets of vertices $\{x_1, x_2, \ldots, x_{2r}\}$ by a matching M. Then the resulting graph G^* is a $2m$-regular graph with $2(|V(G)| - 1)$ vertices. Since $G - u$ has a 1-factor containing e, so does G^*. By Theorem 1, G^* has a k-factor containing e and another k-factor avoiding e for all k, $1 \leq k \leq 2r-1$. Deleting the matching M from G^*, we obtain a $[k, k+1]$-factor containing or avoiding e for $1 \leq k \leq 2r-2$. $\square$

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