Chain polynomials of distributive lattices are 75 % unimodal

Anders Björner
Department of Mathematics
Royal Institute of Technology
S-100 44 Stockholm, Sweden
bjorner@math.kth.se

Jonathan David Farley
Department of Applied Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, USA

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Abstract

It is shown that the numbers c_i of chains of length i in the proper part $L \setminus \{0,1\}$ of a distributive lattice L of length $\ell+2$ satisfy the inequalities

$$c_0 < \ldots < c_{\lfloor \ell/2 \rfloor} \quad \text{and} \quad c_{\lfloor 3\ell/4 \rfloor} > \ldots > c_\ell.$$

This proves 75 % of the inequalities implied by the Neggers unimodality conjecture.

1 Introduction

The *chain polynomial* of a finite poset P is defined as

$$C(P,t) = \sum_i c_i t^i,$$

where c_i is the number of chains (totally ordered subsets) in P of length i (i.e., cardinality $i+1$). One of the equivalent forms of a well-known poset conjecture due to Neggers [14] implies that the chain polynomial of the proper part $L \setminus \{0,1\}$ of a distributive lattice L of length $d+1$ is *unimodal*, meaning that for some k the coefficients of $C(L \setminus \{0,1\}, t)$ satisfy the inequalities

$$c_0 \leq \ldots \leq c_k \geq \ldots \geq c_{d-1}.$$

See [8] and [20] for background, references and more details concerning this unimodality conjecture, and see the Appendix for pointers to recent progress on related problems.

The purpose of this note is to show that the unimodality conjecture for chain polynomials of distributive lattices is 75% correct, in the sense that violations of unimodality can occur only for indices (roughly) between $d/2$ and $3d/4$. More precisely, we prove the following.

Theorem 1 *The numbers c_i of chains of length i in the proper part of a distributive lattice L of length $d+1$ satisfy the inequalities*

$$c_0 < \ldots < c_{\lfloor (d-1)/2 \rfloor} \quad \text{and} \quad c_{\lfloor 3(d-1)/4 \rfloor} > \ldots > c_{d-1}.$$

The proof consists in observing that the order complex of $L \setminus \{0,1\}$ is a nicely behaved ball, and then gathering and combining some known facts from f-vector theory. The pieces of the argument are stated as Propositions 2, 3, 4 and 5. Of these, only Proposition 3 seems to be new.

2 Some f-vector inequalities

For standard notions concerning simplicial complexes we refer to the literature, see e.g. the books [7, 22].

Let Δ be a $(d-1)$-dimensional simplicial complex, and let f_i be the number of i-dimensional faces of Δ. The sequence $(f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ. We put $f_{-1} = 1$. The *h-vector* $(h_0, \ldots, h_d)$ of Δ is defined by the equation

$$\sum_{i=0}^{d} f_{i-1} x^{d-i} = \sum_{i=0}^{d} h_i (x+1)^{d-i}. \tag{1}$$

In the following two results we assume that $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of a $(d-1)$-dimensional simplicial complex Δ, and that $f_0 > d$. From now on, let $d \geq 3$ and $\delta \stackrel{\text{def}}{=} \lfloor \frac{d}{2} \rfloor$, $\varepsilon \stackrel{\text{def}}{=} \lfloor \frac{d-1}{2} \rfloor$.

Proposition 2 *Suppose that $h_i \geq 0$, for all $0 \leq i \leq d$. Then*

$$f_i < f_j, \text{ for all } i < j \text{ such that } i + j \leq d - 2.$$

In particular, $f_0 < f_1 < \ldots < f_\varepsilon$.

Proof. This implication is well known. See e.g. [6, Proposition 7.2.5 (i)]. □

Proposition 3 *Suppose that $h_i \geq h_{d-i} \geq 0$, for all $0 \leq i \leq \delta$. Then*

$$f_{\lfloor 3(d-1)/4 \rfloor} > \ldots > f_{d-2} > f_{d-1}.$$

Proof. By (1), the f-vector $\mathbf{f} = (f_0, f_1, \ldots, f_{d-1})$ and the h-vector $\mathbf{h} = (h_0, h_1, \ldots, h_d)$ satisfy

$$f_k = \sum_{i=0}^{d} h_i \binom{d-i}{d-1-k} \quad , \qquad k = -1, \ldots, d-1. \tag{2}$$

Define integer vectors $\mathbf{b}^i$ as follows:

$$\mathbf{b}^i = \left(b^i_0, b^i_1, \ldots, b^i_{d-1}\right) \qquad , \qquad \text{where } b^i_k = \binom{i}{d-1-k}.$$

Then, by (2), $\mathbf{f} = \sum_{i=0}^{d} h_i \mathbf{b}^{d-i}$, which we rewrite

$$\mathbf{f} = \sum_{i=0}^{\varepsilon} (h_i - h_{d-i}) \mathbf{b}^{d-i} + \sum_{i=0}^{\delta} h_{d-i} \tilde{\mathbf{b}}^i, \tag{3}$$

where

$$\tilde{\mathbf{b}}^i \stackrel{\text{def}}{=} \begin{cases} \mathbf{b}^i + \mathbf{b}^{d-i} & , \quad \text{if } 2i \neq d \\ \mathbf{b}^{d/2} & , \quad \text{if } 2i = d. \end{cases}$$

Let us say that a unimodal sequence

$$a_0 \leq a_1 \leq \ldots \leq a_k \geq a_{k-1} \geq \ldots \geq a_n$$

peaks at k (note that this does not necessarily determine k uniquely).

It is shown in [5, Proof of Thm. 5, p. 50] that the vector $\tilde{\mathbf{b}}^i$ is unimodal and peaks at $d-1-\lfloor\frac{(d-i)}{2}\rfloor$. The vector $\mathbf{b}^{d-i}$ is a segment of a row in Pascal's triangle, so it is easy to see that it is unimodal and, in fact, also peaks at $d-1-\lfloor\frac{(d-i)}{2}\rfloor$. One easily checks that

$$d-1-\lfloor\frac{(d-i)}{2}\rfloor = \begin{cases} \lfloor\frac{d}{2}\rfloor + \lfloor\frac{i}{2}\rfloor - 1 & , \quad \text{if } d \text{ and } i \text{ are even} \\ \lfloor\frac{d}{2}\rfloor + \lfloor\frac{i}{2}\rfloor & , \quad \text{otherwise.} \end{cases}$$

Hence, both the vectors $\mathbf{b}^{d-i}$ $(0 \leq i \leq \varepsilon)$ and the vectors $\tilde{\mathbf{b}}^i$ $(0 \leq i \leq \delta)$ are unimodal and peak between δ and $\delta + \lfloor\delta/2\rfloor$.

By equation (3), $\mathbf{f}$ is a nonnegative linear combination of the vectors $\mathbf{b}^{d-i}$ and $\tilde{\mathbf{b}}^i$. It follows from the previous paragraph that the inequalities hold for each of these vectors separately, strictly for $\mathbf{b}^d$, and non-strictly otherwise. For the computation of the index $\lfloor 3(d-1)/4\rfloor$, see again [5, pp. 50–51]. Hence, if $h_d = 0$ the result follows. The case when $h_d = 1$ requires a small extra argument to see that the inequalities are in fact strict. For this case one can proceed as in [5, Proof of Thm. 5]. □

3 On the h-vectors of balls

We say that a simplicial complex is a *polytopal* $(d-1)$*-sphere* if it is combinatorially isomorphic to the boundary complex of some convex d-polytope. See Ziegler [22] for notions relating to polytopes and convex geometry.

We now review some definitions and results from the general theory of face numbers. For more about this topic, see e.g. [22] or the survey [2].

It follows from (1) that $h_0 = 1$, $h_1 = f_0 - d$, and $h_d = (-1)^{d-1}\tilde{\chi}(\Delta)$, where $\tilde{\chi}(\Delta)$ is the reduced Euler characteristic of Δ. In particular,

$$h_d = \begin{cases} 1, & \text{if } \Delta \text{ is a sphere,} \\ 0, & \text{if } \Delta \text{ is a ball,} \end{cases}$$

where the conditions are shorthand for saying that Δ's geometric realization is homeomorphic to a sphere, resp. a ball.

The following are the *Dehn-Sommerville relations*:

$$\text{If } \Delta \text{ is a sphere then} \quad h_i = h_{d-i}, \quad \text{for all } 0 \leq i \leq d. \tag{4}$$

Therefore, for spheres all f-vector information is encoded in the shorter *g-vector* $g = (g_0, \ldots, g_{\lfloor \frac{d}{2} \rfloor})$, defined by $g_i = h_i - h_{i-1}$. The relevance of the g-vector for this paper is the following result, due to Stanley [17]:

$$\text{If } \Delta \text{ is a polytopal sphere, then} \quad g_i \geq 0 \text{ for all } i \geq 0. \tag{5}$$

If Δ is a $(d-1)$-ball, its boundary complex $\partial\Delta$ is a $(d-2)$-sphere. Furthermore, $\partial\Delta$'s f-vector is determined by that of Δ, as shown by the following consequence of the Dehn-Sommerville relations, due to McMullen and Walkup [13], see also [3, Coroll. 3.9]:

$$\text{If } \Delta \text{ is a ball with boundary } \partial\Delta, \text{ then} \quad h_i^{\Delta} - h_{d-i}^{\Delta} = g_i^{\partial\Delta}. \tag{6}$$

Say that a $(d-1)$-ball Δ admits a *polytopal embedding* if Δ is isomorphic to a subcomplex of the boundary complex of some simplicial d-polytope. The following was shown by Kalai [12, §8] and Stanley [19, Coroll. 2.4].

$$\text{If } \Delta \text{ admits a polytopal embedding, then} \quad g_i^{\partial\Delta} \geq 0 \text{ for all } i \geq 0. \tag{7}$$

Combining (5), (6) and (7), we deduce the following result.

Proposition 4 *If Δ is a $(d-1)$-ball, such that either the boundary sphere $\partial\Delta$ is polytopal or Δ admits a polytopal embedding, then*

$$h_i \geq h_{d-i} \geq 0, \text{ for all } 0 \leq i \leq \delta.$$

□

4 Proof of Theorem 1

We refer to [18, Ch. 3] for basic facts and notation concerning distributive lattices.

Let L be a distributive lattice of length $d+1$, and let $\Delta_L = \Delta(L \setminus \{0, 1\})$ be the order complex of its proper part. Thus, Δ_L is a pure simplicial complex of dimension $d-1$.

Proposition 5 *Suppose that L is not Boolean. Then the complex Δ_L is a $(d-1)$-ball satisfying*

(i) Δ_L admits a polytopal embedding,

(ii) $\partial\Delta_L$ is polytopal.

Proof. By Birkhoff's representation theorem (see [18, Ch. 3]) we have that $L = J(P)$, where $J(P)$ is the family of order ideals of some poset P ordered by inclusion. Let B denote the Boolean lattice of *all* subsets of P. Then $\Delta_B = \Delta(B \setminus \{0, 1\})$ is a polytope boundary (the barycentric subdivision of the boundary of a d-simplex). Furthermore, Δ_L is embedded in Δ_B as a full-dimensional subcomplex. Finally, Δ_L is a shellable ball [4, 15]. Thus, part (i) is proved.

Part (ii) requires a small convexity argument. Alternatively, it follows from Provan's result [15] that Δ_L can be obtained from a simplex via repeated stellar subdivisions. Since this part is not needed for the main result of this paper, details of the proof are left out. □

We now have all the pieces needed to prove Theorem 1. We may assume that L is not Boolean, since in that case Δ_L is a sphere and Theorem 1 is a special case of [5, Thm. 5]. Then, by Propositions 4 and 5 we have that

$$h_i \geq h_{d-i} \geq 0, \text{ for all } 0 \leq i \leq \delta.$$

Furthermore, by Propositions 2 and 3 it follows that the f-vector of Δ_L satisfies

$$f_0 < \ldots < f_{\lfloor (d-1)/2 \rfloor} \quad \text{and} \quad f_{\lfloor 3(d-1)/4 \rfloor} > \ldots > f_{d-1}.$$

Since $f_i = c_i$ for all i, the proof of Theorem 1 is complete.

5 Appendix (added in proof)

By equation (1), the f-polynomial $f(x) = \sum_{i=0}^{d} f_{i-1} x^{d-i}$ and the h-polynomial $h(x) = \sum_{i=0}^{d} h_i x^{d-i}$ are related by $f(x) = h(x+1)$. The conjecture of Neggers [14] is that all roots of the h-polynomial of a distributive lattice are real. Equivalently, by equation (1), that all roots of its f-polynomial are real. It was recently shown by Brändén [10] that an extension of Neggers conjecture proposed by Stanley is false. Soon after, Stembridge [21] showed that the Neggers real-rootedness conjecture itself is false.

Real-rootedness of a polynomial implies unimodality. Furthermore, the counterexamples to real-rootedness given by Brändén and Stembridge are unimodal. Thus there remain *two unimodality conjectures*, one for the f-polynomial (the one referred to in this paper), and one for the h-polynomial. Recent progress on the latter appears in [1], [9], [11] and [16].

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