Construction of Codes Identifying Sets of Vertices

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Abstract

In this paper the problem of constructing graphs having a $(1, \leq \ell)$-identifying code of small cardinality is addressed. It is known that the cardinality of such a code is bounded by $\Omega\left(\frac{\ell^2}{\log \ell}\log n\right)$. Here we construct graphs on n vertices having a $(1, \leq \ell)$-identifying code of cardinality $O\left(\ell^4 \log n\right)$ for all $\ell \geq 2$. We derive our construction from a connection between identifying codes and superimposed codes, which we describe in this paper.

1 Codes identifying sets of vertices

Let $G = (V, E)$ be a simple, non-oriented graph. For a vertex $v \in V$, let us denote by $N[v]$ the closed neighborhood of $v :$ $N[v] = N(v) \cup \{v\}$. Let $C \subseteq V$ be a subset of vertices of G, and for all nonempty subset of at most ℓ vertices $X \subseteq V$, let us denote

$$I(X) = I(X, C) := \bigcup_{x \in X} N[x] \cap C.$$

If all the $I(X, C)$'s are distinct, then we say that C *separates* the sets of at most ℓ vertices of G, and if all the $I(X, C)$'s are nonempty then we say that C *covers* the sets of at most ℓ vertices of G. We say that C is a code identifying sets of at most ℓ vertices of G if and only if C covers and separates all the sets of at most ℓ vertices of G. The dedicated terminology [12] for such codes is $(1, \leq \ell)$-*identifying codes.* The sets $I(X)$ are said to be the *identifying sets* of the corresponding X's.

Whereas $C = V$ is trivially always a code covering the sets of at most ℓ vertices of any graph $G = (V, E)$, not every graph has a $(1, \leq \ell)$-identifying code. For example, if G contains two vertices u and v such that $N[u] = N[v]$, then G has no $(1, \leq \ell)$-identifying code, since for any subset of vertices C we have $N[u] \cap C = N[v] \cap C$. Actually, a graph admits a $(1, \leq \ell)$-identifying code if and only if for every pair of subsets $X \neq Y$, $|X|, |Y| \leq \ell$, we have $N[X] \neq N[Y]$, where $N[X]$ denotes $\bigcup_{x \in X} N[x]$. In the case where G admits a $(1, \leq \ell)$-identifying code, then $C = V$ is always a $(1, \leq \ell)$-identifying code of G, hence we are usually interested in finding a $(1, \leq \ell)$-identifying code of minimum cardinality.

These codes are used for fault diagnosis in multiprocessor systems, and were first defined in [9]. The problem of constructing such codes has already been addressed in [1, 2, 12, 9, 10, 7]. In these papers the authors used covering codes, that are quite well known [3]. We refer the reader to [14] for an online up-to date bibliography about identifying codes.

In the general case $\ell \geq 1$, another good framework to construct such codes is to use *ℓ-superimposed codes*, as suggested in [6]. Indeed, given a graph $G = (V, E)$ together with a $(1, \leq \ell)$-identifying code C of G, the characteristic vectors of the subsets $I(X, C)$, for $|X| \leq \ell$, satisfy the following property :

> The boolean sum (`OR`) of any set of at most ℓ vectors is distinct from the boolean sum of any other set of at most ℓ vectors. (1)

A set of vectors satisfying (1) is a *UD_ℓ-code*, or *ℓ-superimposed code*. These codes were defined by Kautz and Singleton in [11], and about such codes we know the following :

Theorem 1 *Let K be a maximum ℓ-superimposed code of $\{0,1\}^N$. Then there exist two constants c_1 and c_2, not depending on N or ℓ, such that*

$$2^{c_1 N/\ell^2} \leq |K| \leq 2^{c_2 N \log \ell / \ell^2}.$$

Moreover the lower bound is constructive : there exists an algorithm which, given N and ℓ, constructs an ℓ-superimposed code of $\{0,1\}^N$ of cardinality $2^{c_1 N/\ell^2}$.

The lower bound comes from [11], and a combinatorial proof of the upper bound, originally established in [4], can be found, for example, in [13]. A greedy algorithm constructing an ℓ-superimposed code of cardinality $2^{c_1 N/\ell^2}$ can be found in [8].

It was already explained in [6] that it was easy to get an ℓ-superimposed code from a $(1, \leq \ell)$-identifying code. In this paper we show that we can also get a $(1, \leq \ell)$-identifying code from an ℓ-superimposed code, which answers to a question of [6]. We give such a construction and prove the following :

Theorem 2 *For all $\ell \geq 1$, there exists a function $c(n) = O(\ell^4 \log n)$ and an infinite family of graphs $(G_i)_{i \in \mathbb{N}}$, such that, for all $i \in \mathbb{N}$, G_i has n_i vertices and admits a $(1, \leq \ell)$-identifying code of cardinality $c(n_i)$, with $n_i \to \infty$ when $i \to \infty$. Moreover we can explicitly construct such a family of graphs $(G_i)_{i \in \mathbb{N}}$.*

In the next section we describe our construction. In section 3 we show the validity of our construction, which proves Theorem 2. In the last section, we give an open problem connected to our construction.

2 Construction of Identifying Codes

Let $\ell \geq 2$. In this section we describe the construction of a graph $\mathcal{G}$ together with a $(1, \leq \ell)$-identifying code C of $\mathcal{G}$. Its validity is proved in the next section.

1. Let $N = \lceil \ell^2 \log n \rceil$ and let K be a maximal ℓ-superimposed code of $\{0,1\}^N$, that is to say there is no $K' \supset K$, $K' \neq K$, such that K' is an ℓ-superimposed code. Let k denote the cardinality of K : $K = V_1, \ldots, V_k$.

2. Consider the $N \times k$ matrix M whose columns are the vectors of K. Let M' be a $N \times N$ submatrix of M such that there is a 1 on every row of M'.

3. Let H be a connected graph admitting a $(1, \leq \ell)$-identifying code. From M and M', let us construct a graph $\mathcal{G} = G(M, M')$ together with $C = C(M, M')$ a $(1, \leq \ell)$-identifying code of $\mathcal{G}$ as follows. The subgraph induced by the code $\mathcal{G}[C]$ consists in the disjoint union of N copies of H. In each copy H_i of H we specify one vertex h_i, $i = 1, \ldots, N$. These vertices $h_1, \ldots, h_N$ will be such that

   $$N(V(\mathcal{G}) \setminus C) = \{h_1, \ldots, h_N\}.$$

   Now, to each column V_j of $M \setminus M'$ we associate a vertex $v_j = \phi(V_j)$ of $\mathcal{G}$, whose neighbors are the h_i's for each i such that the i-th coordinate of V_j is equal to 1 (see Figure 1). There are no edges between the v_j's, hence V_j is the characteristic vector of the identifying set of v_j, which is also the neighborhood of v_j.

3 Proof of the validity of the construction

We show the validity of the construction described in the previous section and we prove Theorem 2. In Step 2 of the construction, we needed the following:

Lemma 1 *Let M be an $n \times m$ $(n \leq m)$ $0-1$-matrix which has no row consisting only of* 0*'s. Then there exists an $n \times n'$ $(n' \leq n)$ submatrix M' of M such that there is a* 1 *on every row of M'.*

Proof : Let M be a matrix satisfying the requirements of the lemma. Let $M_1, \ldots, M_m$ be the columns of M.

The proof works by induction on n. Without loss of generality, we may assume that there exists $p \leq n$ such that $M_{i,1} = 1$ for all $i \leq p$ and $M_{j,1} = 0$ for all $j > p$. If $p = n$ then the lemma holds. Otherwise, let P be the matrix consisting in the restriction of the

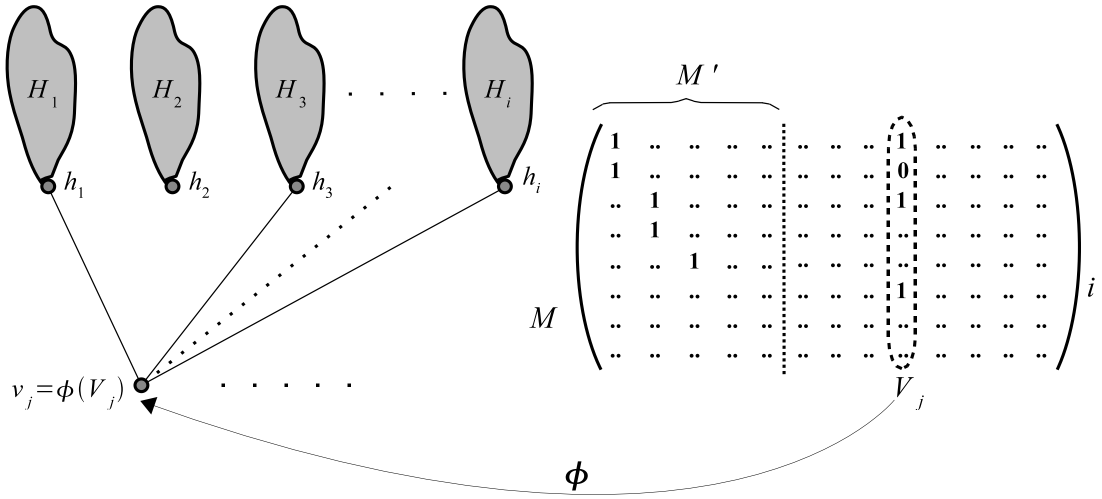

Figure 1: Construction of a graph $\mathcal{G} = \mathcal{G}(M, M')$ together with a $(1, \leq \ell)$- identifying code $C = C(M, M')$ of $\mathcal{G}$ from M and M'.

columns $M_2, \ldots, M_m$ to the rows indexed by $p+1, \ldots, n$. By induction, there exists a submatrix P' of P such that there is a 1 on every row of P'. Now, the submatrix M' of M defined by the columns of P' plus M_1 satisfies the requirement. □

Since a matrix of a maximal ℓ-superimposed code of $\{0,1\}^N$ is a $0-1$-matrix with no row consisting only of 0's, we get, by the previous lemma :

Lemma 2 *Let M be an $N \times k$ matrix whose columns are the vectors of a maximal ℓ-superimposed code of $\{0,1\}^N$. Then there exists an $N \times N'$ $(N' \leq N)$ submatrix M' of M such that there is a 1 on every row of M'.*

Later we will also need the following :

Lemma 3 *Let M be an $N \times k$ matrix whose columns are the vectors of K, a maximal ℓ-superimposed code of $\{0,1\}^N$, and let M' be an $N \times N'$ $(N' \leq N)$ submatrix of M such that there is a 1 on every row of M' (by the previous Lemma such a submatrix exists). Then every column of $M \setminus M'$ has at least ℓ nonzero coordinates.*

Proof : Let V be a column of $M \setminus M'$ having less than ℓ nonzero coordinates. Since there is a 1 on every row of M' then we can find $\{V_1, \ldots, V_m\}$, $m \leq \ell - 1$, a set of at most $\ell - 1$ columns of M', such that

$$V \leq \sum_{i=1}^{m} V_i$$

where $\sum$ stands for the boolean sum. This implies $\sum_{i=1}^{m} V_i + V = \sum_{i=1}^{m} V_i$, which contradicts the fact that K is an ℓ-superimposed code. □

With the use of projective planes, we can prove that, in the case where ℓ is a prime power, there exist connected graphs admitting $(1,\leq\ell)$-identifying codes of cardinality $\Theta(\ell^2)$. We recall that a *projective plane* of order n is an hypergraph on n^2+n+1 vertices such that :

- Any pair of vertices lie in a unique hyperedge,
- Any two hyperedges have a unique common vertex,
- Every vertex is contained in $n+1$ hyperedges, and
- Every hyperedge contains $n+1$ vertices.

Note that some of these properties are redundant. We denote $\mathbb{P}_n$ the projective plane of order n. It is known that $\mathbb{P}_n$ exists if n is the power of a prime number. Projective planes of order n are also known as 2-$(n^2+n+1, n+1, 1)$ designs, or $S(2, n+1, n^2+n+1)$ Steiner systems.

Lemma 4 *If q is a prime power, then there exists a connected graph G_q on $2(q^2+q+1)$ vertices admitting a $(1,\leq q)$-identifying code. Moreover, G_q is $(q+1)$-regular.*

Proof : Assume that q is a prime power, and consider a finite projective plane $\mathbb{P}_q$ of order q. In other words, we have a (q^2+q+1)-element set S and $\mathbb{P}_q$ consists of q^2+q+1 hyperedges, each hyperedge being a $(q+1)$-element subset of S. $\mathbb{P}_q$ has the property that every pair of elements of S is contained in a unique hyperedge. The number of hyperedges is q^2+q+1; each element of S is contained in exactly $q+1$ hyperedges; and, finally, every two hyperedges have exactly one element in common.

Denote by A the adjacency matrix of $\mathbb{P}_q$, where the rows are labelled by the elements of S and the columns by the hyperedges, and the entry A_{ij} is 1 if the i-th element is in the j-th hyperedge, and 0, otherwise. (By labelling the elements and hyperedges suitably, we could make A symmetric, but we do not need it here.) Now, every row (resp. column) of A has exactly $q+1$ ones; and every two rows (resp. every two columns) of A have exactly one 1 in common.

We now use A to construct a graph G_q as follows. Let

$$B = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

and let G_q be the simple, non-oriented graph whose adjacency matrix is B, *i.e.* vertices i and j are adjacent in G_q if and only if $B_{ij}=1$. The graph G_q is well-defined since B is a symmetric matrix having only 0's on its diagonal.

Obviously, the graph G_q has $2(q^2+q+1)$ vertices and is $(q+1)$-regular. Moreover, G_q is bipartite, as all the edges go between the first q^2+q+1 and the last q^2+q+1 vertices. Clearly, G_q is connected: Given any two of the first q^2+q+1 vertices, there is a unique vertex among the last q^2+q+1 vertices which is connected to both of them, and the connectivity easily follows.

Moreover, we can prove that the whole vertex set is a $(1, \leq q)$-identifying code of G_q. Assume that X is a subset of the vertex set having at most q elements. Assume further that we do not know X, but that we know $I(X)$. Let v be an arbitrary vertex. Clearly $|I(v)| = q + 2$, and

$$\text{For every vertex } u \neq v, \text{ the set } I(u) \text{ contains at most one element of } I(v) \setminus \{v\}. \quad (2)$$

(Remark that we can obtain the identifying sets of individual vertices by changing all the diagonal elements of B into 1's: We get a matrix B' where the i-th row gives the identifying set of the i-th vertex.) For the vertices u in the same part of the bipartition as v, (2) follows from the properties of projective planes; and for the other vertices (2) is trivial by construction. Consequently, if $v \in X$, then all the $q + 2$ elements of $I(v)$ are in $I(X)$; but if $v \notin X$, then at most $q + 1$ elements of $I(v)$ are in $I(X)$. So, we can immediately tell by looking at $I(X)$, whether v is in X or not; and this is true for all $v \in X$, completing the proof. □

Finally, we need the following :

Lemma 5 *Let C be a $(1, \leq \ell)$-identifying code of a graph G, and let X and Y be distinct subsets of at most ℓ vertices of G. Then we have either*

$$|X| + |I(X)\Delta I(Y)| > \ell \qquad \textit{or} \qquad |Y| + |I(X)\Delta I(Y)| > \ell.$$

Proof : Let $X' := X \cup I(X)\Delta I(Y)$ and $Y' := Y \cup I(X)\Delta I(Y)$. It is easy to see that $I(X')\Delta I(Y') = \emptyset$. Since C is a $(1, \leq \ell)$-identifying code, this implies $|X'| > \ell$ or $|Y'| > \ell$. □

Now we are ready to prove the validity of the construction described in the previous section.

Proof of Theorem 2 : The case $\ell = 1$ is already known [9], and derive from the case $\ell = 2$. Now let $\ell \geq 2$. Let $N = \lceil \ell^2 \log n \rceil$ and let K be a maximal ℓ-superimposed code of $\{0, 1\}^N$. By Theorem 1 we know that there exists such a K satisfying $|K| \geq \Omega(n)$. Let M be the matrix whose columns are the vectors of K. In Step 2 of the construction we need to find an $N \times N$ submatrix M' of M having a 1 on each one of its rows : since K is maximal, then by Lemma 2 such a submatrix exists. In Step 3 of the construction we need a graph H having a $(1, \leq \ell)$-identifying code. If ℓ is a prime power then we take $H = G_\ell$ as constructed in Lemma 4. If ℓ is not a prime power, then by Bertrand's Conjecture – proved in 1850 by Chebyshev and later by Erdős in his first paper [5] – we know that there exists a prime number p in the interval $[\ell, 2\ell]$, and we take $H = G_p$ as constructed in Lemma 4. Since $p \geq \ell$, then G_p admits a $(1, \leq p)$-identifying code implies that G_p admits a $(1, \leq \ell)$-identifying code. Both $H = G_\ell$ and $H = G_p$ have $\Theta(\ell^2)$ vertices.

Now let $\mathcal{G}$ and C be as constructed in Step 3 of the construction. We prove that C is a $(1, \leq \ell)$-identifying code of $\mathcal{G}$. Let X and Y be two subsets of vertices of $\mathcal{G}$ of cardinality less or equal to ℓ. We show that $I(X) = I(Y)$ if and only if $X = Y$. We proceed in

two steps: first we prove that $I(X) = I(Y) \Rightarrow X \cap C = Y \cap C$, and then we prove that $I(X) = I(Y) \Rightarrow X \setminus C = Y \setminus C$. In the rest of the proof, we assume that $I(X) = I(Y)$.
(a) By way of contradiction, let us assume that $X \cap C \neq Y \cap C$, and let H_i be a connected component of $\mathcal{G}[C]$ on which X and Y differ. Denoting $X_i = X \cap H_i$ and $Y_i = Y \cap H_i$, we have $X_i \neq Y_i$. Since $H_i \subset C$ and $V(H_i)$ is a $(1, \leq \ell)$-identifying code of H_i, then we have $I(X_i) \neq I(Y_i)$. If there is an $h \in H_i$, $h \neq h_i$, such that $h \in I(X_i)\Delta I(Y_i)$, then we obtain a contradiction since $h \notin N(X \setminus X_i) \cup N(Y \setminus Y_i)$: the neighborhood of $h \neq h_i$ is contained in H_i, and consequently $h \in I(X_i)\Delta I(Y_i) \Rightarrow h \in I(X)\Delta I(Y)$. Hence $I(X_i)\Delta I(Y_i) = \{h_i\}$. By Lemma 5 we may assume that $|X_i| = \ell$, that is to say $X = X_i \subseteq H_i$ and $h_i \in I(X) \setminus I(Y_i)$. Since our assumption is that $I(X) = I(Y)$, it means that there exists a neighbor y of h_i belonging to $Y \setminus C$. By Lemma 3, y is neighbor of at least ℓ vertices of C (remember that to each column vector W of $M - M'$ we associated a vertex $\phi(W)$ which is neighbor to h_i for all i such that the i-th coordinate of W is 1). Since $\ell \geq 2$, then there exists $h_j \in C$, $h_j \neq h_i$, such that $h_j \in I(Y) \setminus I(X)$: this contradicts $I(X) = I(Y)$.
(b) Set $X' = X \setminus C$ and $Y' = Y \setminus C$. Assume that $X' \neq Y'$. Now, to each $h_i \in I(X')\Delta I(Y')$, we can associate a unique $h_i' \in X \cap C = Y \cap C$. Indeed, since $I(X) = I(Y)$, then for each h_i in, say, $I(X') \setminus I(Y')$, there exists an $h_i' \in Y \cap H_i = X \cap H_i$ such that $h_i \in N(h_i')$. Hence there exists an injection $I(X')\Delta I(Y') \hookrightarrow X \cap C = Y \cap C$. This shows that :

$$|X| \geq |X'| + |I(X')\Delta I(Y')| \quad \text{and} \quad |Y| \geq |Y'| + |I(X')\Delta I(Y')| \tag{3}$$

Now, remind that $X' = \{v_p\}_{p\in P}$ and $Y' = \{v_q\}_{q\in P}$ correspond to two different sets $\phi^{-1}(X) = \{V_p\}_{p\in P}$ and $\phi^{-1}(Y) = \{V_q\}_{q\in Q}$ of column vectors of the matrix $M \setminus M'$. Note that $|I(X')\Delta I(Y')|$ is the number of coordinates on which $\sum_{p\in P} V_p$ and $\sum_{q\in Q} V_q$ differ, where $\sum$ stands for the boolean sum. Let $\mathcal{I}$ denote the set of coordinates on which $\sum_{p\in P} V_p$ and $\sum_{q\in Q} V_q$ differ: $|\mathcal{I}| = |I(X')\Delta I(Y')|$. Now, for each coordinate $i \in \mathcal{I}$, let $W_{\tau(i)}$ be a column vector of M' having its i-th coordinate equal to 1. By definition of the $W_{\tau(i)}$'s, we have :

$$\sum_{p\in P} V_p + \sum_{i\in\mathcal{I}} W_{\tau(i)} \;=\; \sum_{q\in Q} V_q + \sum_{i\in\mathcal{I}} W_{\tau(i)}.$$

Since M is the matrix of an ℓ-superimposed code, this implies that :

$$|P| + |\mathcal{I}| > \ell \quad \text{or} \quad |Q| + |\mathcal{I}| > \ell.$$

Recalling (3), since $|P| = |X'|$, $|Q| = |Y'|$, and $|\mathcal{I}| = |I(X')\Delta I(Y')|$, we obtain:

$$|X| > \ell \quad \text{or} \quad |Y| > \ell$$

which is a contradiction.

Hence C is a $(1, \leq \ell)$-identifying code of $\mathcal{G}$. C has cardinality $N \times |H|$, and $\mathcal{G}$ has $N \times |H| + (|K| - N)$ vertices. Since $N = \lceil \ell^2 \log n \rceil$, $|K| \geq \Omega(n)$ and $|H| = \Theta(\ell^2)$, then we have

$$|C| = \Theta(\ell^2)\lceil \ell^2 \log n \rceil \qquad \text{and} \qquad |\mathcal{G}| = \Omega(n)$$

hence

$$|C| = O\left(\ell^4 \log |\mathcal{G}|\right). \qquad \square$$

4 Conclusion

In this paper we showed a correspondence between $(1, \leq \ell)$-identifying codes and ℓ-superimposed codes, which enabled us to construct a $(1, \leq \ell)$-identifying code of cardinality $O\left(\ell^4 \log n\right)$ in a graph on n vertices from a maximal ℓ-superimposed code of length $\lceil \ell^2 \log n \rceil$. This answers a question of [6].

Our method can be used to answer another interesting question. In [12] it is shown that a graph admitting a $(1, \leq \ell)$-identifying code has its minimum degree greater or equal to ℓ. We wondered if there existed graphs admitting a $(1, \leq \ell)$-identifying code with minimum degree equal to ℓ. The idea of the construction of Section 2 can be used to answer this question : take ℓ copies $H_1, \ldots, H_\ell$ of a connected graph H admitting a $(1, \leq \ell)$-identifying code (from Lemma 4 we know that such an H exists), specify ℓ vertices $h_i \in H_i$ for $i = 1, \ldots, \ell$ and then construct a graph $\mathcal{G}'$ by joining the H_i's with a new vertex u such that uh_i is an edge of $\mathcal{G}'$ for all $i = 1, \ldots, \ell$. It is easy to see that $\mathcal{G}'$ is a graph admitting a $(1, \leq \ell)$-identifying code. Indeed, let X and Y be two distinct subsets of at most ℓ vertices of $\mathcal{G}'$. If $u \notin X \cup Y$, then clearly $N[X] \neq N[Y]$ since H admits a $(1, \leq \ell)$-identifying code. If $u \in X \cap Y$, then let i be such that $X \cap H_i =: X_i \neq Y_i := Y \cap H_i$. As $|Xi| \leq \ell - 1$ and $|Yi| \leq \ell - 1$, then by Lemma 5 we know that $|N[Xi] \Delta N[Yi]| \geq 2$. Since u has only one neighbor h_i in H_i, then $N[X] \neq N[Y]$. Finally, if, say, $u \in X \setminus Y$, then Y has to have a nontrivial intersection with each copy $H_1, \ldots, H_\ell$. Hence $|Y| = \ell$ and for all $i = 1, \ldots, \ell$ we have $|Y \cap H_i| = 1$. Since H admits a $(1, \leq \ell)$-identifying code then $\delta(H) \geq \ell \geq 1$ and then $|N[Y] \cap H_i| \geq 2$ for all $i = 1, \ldots, \ell$. This implies that for all $i = 1, \ldots, \ell$ there exists an $x_i \in X \cap H_i$. Since X contains also u, this contradicts $|X| \leq \ell$.

Thus, we proved the following :

Proposition 1 *For all $\ell \geq 1$ there exists a graph G_ℓ admitting a $(1, \leq \ell)$- identifying code with minimum degree equal to ℓ.*

We wonder if there exists ℓ-regular graphs admitting $(1, \leq \ell)$-identifying codes. Remind that Lemma 4 says that, if ℓ is a prime power, then there exists $(\ell + 1)$-regular graphs admitting a $(1, \leq \ell)$-identifying code.

We recall from [6] that a $(1, \leq \ell)$-identifying code of a graph on n vertices has a cardinality greater or equal to $\Omega\left(\frac{\ell^2}{\log \ell} \log n\right)$. This is a direct consequence of Theorem 1. Here we showed how to construct graphs having a $(1, \leq \ell)$-identifying code of cardinality $O\left(\ell^4 \log n\right)$. Our construction is based on the existence of connected graphs on $\Theta(\ell^2)$ vertices admitting a $(1, \leq \ell)$- identifying code (Lemma 4). If we could improve Lemma 4 by constructing graphs on less than $\Theta(\ell^2)$ vertices admitting a $(1, \leq \ell)$-identifying code, then this would directly result in an improvement of Theorem 2.

Hence the minimum number of vertices of a connected graph admitting a $(1, \leq \ell)$-identifying code is an interesting question, that we pose here as an open problem.

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