Ramsey $(K_{1,2}, K_3)$-minimal graphs

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Submitted: Jul 21, 2004; Accepted: Apr 14, 2005; Published: May 6, 2005
Mathematics Subject Classifications: 05C55, 05C70

Abstract

For graphs G, F and H we write $G \to (F, H)$ to mean that if the edges of G are coloured with two colours, say red and blue, then the red subgraph contains a copy of F or the blue subgraph contains a copy of H. The graph G is (F, H)-*minimal* (*Ramsey-minimal*) if $G \to (F, H)$ but $G' \not\to (F, H)$ for any proper subgraph $G' \subseteq G$. The class of all (F, H)-minimal graphs shall be denoted by $R(F, H)$. In this paper we will determine the graphs in $R(K_{1,2}, K_3)$.

1 Introduction and Notation

We consider finite undirected graphs without loops or multiple edges. A graph G has a vertex set $V(G)$ and an edge set $E(G)$. We say that G *contains* H whenever G contains a subgraph isomorphic to H. The subgraph of G isomorphic to K_3 we will call a *triangle* of G and sometimes denoted by its vertices.

Let G_1, G_2 be subgraphs of G. We write $G_1 \cup G_2$ ($G_1 \cap G_2$) for a subgraph of G with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ ($V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$).

Let x and y be two nonadjacent vertices of G. Then $G + xy$ is the graph obtained from G by adding to G the edge xy.

Let G, F and H be graphs. We write $G \to (F, H)$ if whenever each edge of G is coloured either red or blue, then the red subgraph of G contains a copy of F or the blue subgraph of G contains a copy of H.

A graph G is (F, H)-*minimal* (*Ramsey-minimal*) if $G \to (F, H)$ but $G' \not\to (F, H)$ for any proper subgraph $G' \subseteq G$.

The class of all (F, H)-minimal graphs will be denoted by $R(F, H)$.

A (F, H)-*decomposition* of G is a partition (E_1, E_2) of $E(G)$, such that the graph $G[E_1]$ does not contain the graph F and the graph $G[E_2]$ does not contain the graph H. Obviously, if there is no (F, H)-decomposition of G then $G \to (F, H)$ holds.

In general, we follow the terminology of [4].

There are several papers dealing with the problem of determining the set $R(F, H)$. For example, Burr, Erdős and Lovász [1] proved that $R(2K_2, 2K_2) = \{3K_2, C_5\}$ and $R(K_{1,2}, K_{1,2}) = \{K_{1,3}, C_{2n+1}\}$ for $n \geq 1$. Burr et al. [3] determined the set $R(2K_2, K_3)$. In [6] the graphs belonging to $R(2K_2, K_{1,n})$ were characterized. It is shown in [2] that if m, n are odd then $R(K_{1,m}, K_{1,n}) = \{K_{m+n+1}\}$. Also the problem of characterizing pairs of graphs (F, H), for which the set $R(F, H)$ is finite or infinite has been investigated in numerous papers. In particular, all pairs of two forest for which the set $R(F, H)$ is finite are specified in a theorem of Faudree [5]. Łuczak [7] states that for each pair which consists of a non-trivial forest and non-forest the set of Ramsey-minimal graphs is infinite. From Łuczak's results it follows that the set $R(K_{1,2}, K_3)$ is infinite. In the paper we shall describe all graphs belonging to $R(K_{1,2}, K_3)$.

2 Definitions of some classes of graphs

To prove the main result we need some classes of graphs.

Let k be an integer such that $k \geq 2$. A graph G with $V(G) = \{v_1, v_2, ..., v_k, w_1, w_2, ..., w_{k-1}\}$ and $E(G) = \{v_i v_{i+1} : i = 1, 2, ..., k-1\} \cup \{v_i w_i : i = 1, 2, ..., k-1\} \cup \{w_i v_{i+1} : i = 1, 2, ..., k-1\}$ is called the K_3-*path*. The edges of $\{v_i v_{i+1} : i = 1, 2, ..., k-1\}$ are *internal* edges of the K_3-path and $\{v_i w_i : i = 1, 2, ..., k\} \cup \{w_i v_{i+1} : i = 1, 2, ..., k-1\}$ is the set of *external* edges of the K_3-path. The vertex v_1 or w_1 is called the *first vertex* of K_3-path. The vertex v_k or w_{k-1} is called the *last vertex* of K_3-path.

Let k be an integer such that $k \geq 4$. A graph G with $V(G) = \{v_1, v_2, ..., v_k, w_1, w_2, ..., w_k\}$ and $E(G) = \{v_i v_j : i = 1, 2, ..., k,\ j \equiv i + 1 \pmod k\} \cup \{w_i v_i : i = 1, 2, ..., k\} \cup \{w_i v_j : i = 1, 2, ..., k\ j \equiv i + 1 \pmod k\}$ is called the K_3-*cycle*. We will say that $\{v_i v_j : i = 1, 2, ..., k,\ j \equiv i + 1 \pmod k\}$ is the set of *internal* edges of the K_3-cycle and $\{w_i v_i : i = 1, 2, ..., k\} \cup \{w_i v_j : i = 1, 2, ..., k\ j \equiv i + 1 \pmod k\}$ is the set of *external* edges of the K_3-cycle.

A *length of* K_3-path (K_3^2-path, K_3-cycle) is the number of triangles in K_3-path (K_3^2-path, K_3-cycle).

If we add to a K_3-path the edges $w_i w_{i+1}$ ($i = 1, ..., k-2$) then we obtain the graph, which we call the K_3^2-*path of odd length*. If we add to a K_3^2-path of odd length a new vertex w_k and edges $w_{k-1}w_k, v_kw_k$ then we obtain the *K_3^2-path of even length*.

By R we will denote the graph with the root r, which is presented in Figure 1.

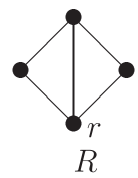

Figure 1.

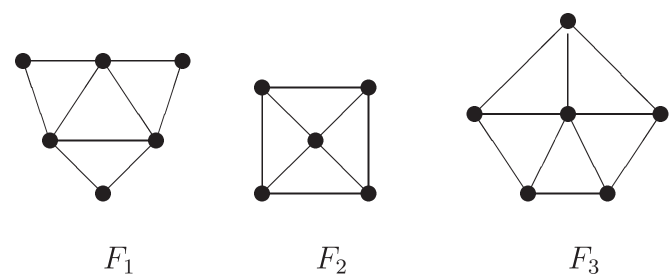

Figure 2.

Let $\mathcal{T}$ be the family of graphs, which contains:

(1) F_1, F_2, F_3 (see Fig. 2.);

(2) $F_4(k),\ k \geq 0$ — two vertex-disjoint copies of R with a K_3-path of length k, joining two roots (if $k = 0$ we have two copies of R, which are stuck together by the roots);

(3) $F_5(t_1, t_2, k),\ t_1 \geq 4,\ t_2 \geq 4,\ k \geq 0$ — two vertex disjoint copies of K_3-cycles of lengths t_1 and t_2 with a K_3-path of length k joining the two arbitrary vertices of K_3-cycles (if $k = 0$ we have two copies of K_3-cycles, which are stuck together by an arbitrary vertex);

(4) $F_6(t, k),\ t \geq 4, k \geq 0$ — a copy of R and a copy of a K_3-cycle of length t with a K_3-path of length k joining the root of R and an arbitrary vertex of the K_3-path;

(5) $F_7(t, k),\ t \geq 4, k \geq 1$ — a K_3-cycle H of length t with a K_3-path of length k joining two arbitrary vertices x, y of the K_3-cycle, such that $k + d_H(x, y) \geq 4$;

(6) $F_8(t),\ F_9(t)$,..., $F_{15}(t),\ t \geq 4$ — graphs, which are obtained from a K_3-cycle of length t by adding some new triangles as in Fig. 3;

(7) $F_{16}(t),\ t \geq 5$ — the graph, which is obtained from a K_3^2-path of odd length t in the following way: Let xyz and $x'y'z'$ be the last triangles of the K_3^2-path such that z and z' are degree 2, y, y' are degree 3, x, x' are degree 4. Then we add new edges zy', yz' and zz'.

For short we omit the parameters t, t_1, t_2, k if it does not lead to a misunderstanding.

It is easy to see that $\kappa(G) \leq 3$ for any graph $G \in \mathcal{T}$. Let us denote denote by

$$\mathcal{T}_i = \{G \in \mathcal{T} : \kappa(G) = i\},\ i = 1, 2, 3.$$

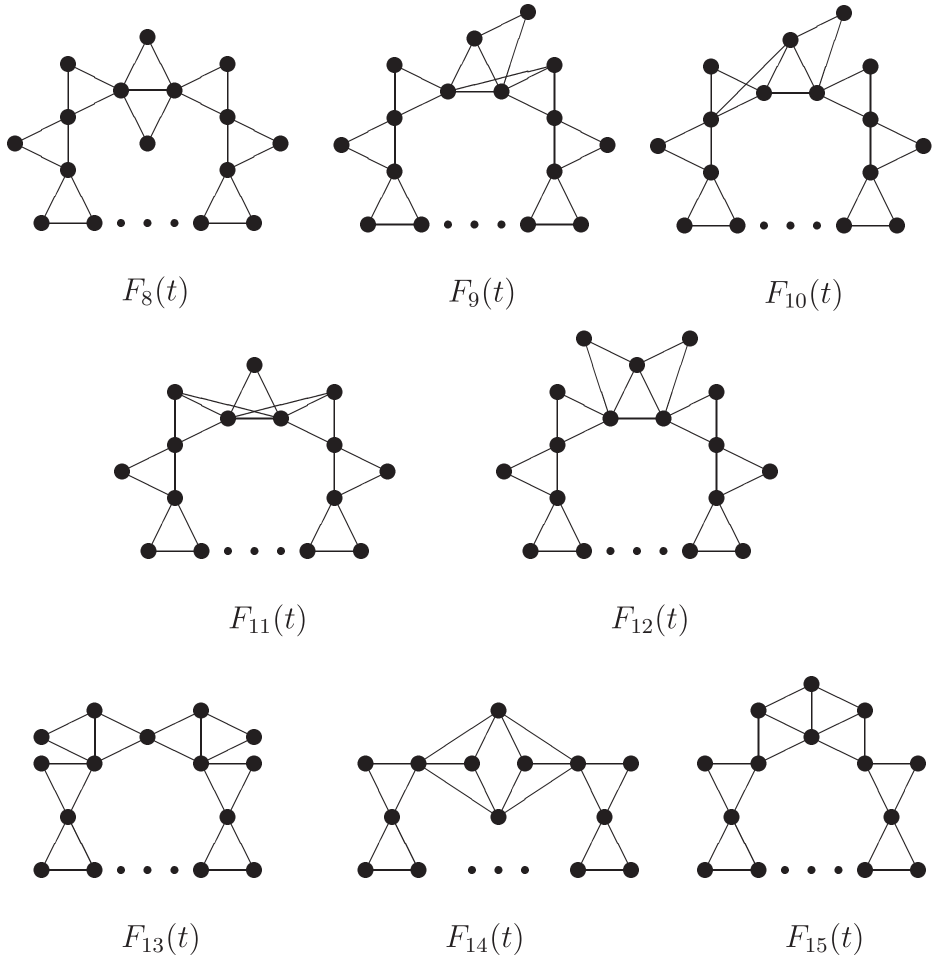

Figure 3.

Let $\mathcal{A}$ be the family of graphs each with a root denoted by x. To the family $\mathcal{A}$ belong:

(1) $L_1(k),\ k \geq 0$ — a copy of R and a copy of a K_3-path of length k, which are stuck together by the root of R and the first vertex of the K_3-path. The last vertex of the K_3-path is the root x of $L_1(k)$; if $k = 0$ then $L_1(0)$ is isomorphic to R;

(2) $L_2(t,k),\ t \geq 4, k \geq 0$ — a copy of a K_3-cycle of length t and a copy of a K_3-path of length k, which are stuck together by an arbitrary vertex of degree two of the K_3-cycle and the first vertex of the K_3-path. The last vertex of the K_3-path is the root x of $L_2(k)$; if $k = 0$ then $L_2(0)$ is isomorphic to a K_3-cycle and an arbitrary vertex of degree two is the root;

(3) $L_3(t,k),\ t \geq 4, k \geq 0$ — a copy of a K_3-cycle of length t and a copy of a K_3-path of length k, which are stuck together by an arbitrary vertex of degree four of the K_3-cycle and the first vertex of the K_3-path. The root x of $L_3(k)$ is the last vertex of the K_3-path;

if $k = 0$ then $L_3(0)$ is isomorphic to a K_3-cycle and an arbitrary vertex of degree 4 is the root.

The graphs of the family $\mathcal{A}$ will be also denoted briefly by L_1, L_2, L_3, if the parameters t, k are clear.

Let P be a subgraph of G isomorphic to a K_3-path such that $V(P) = \{v_1, v_2, ..., v_k, w_1, w_2, ..., w_{k-1}\}$ and $d_G(v_1) \geq 2$ (the first vertex of P), $d_G(v_k) \geq 2$ (the last vertex of P), $d_G(v_i) = 4$ $(i = 2, .., k-1), d_G(w_j) = 2$ $(j = 1, ..., k-1)$ then P we will call a *diagonal* K_3*-path*. If $k = 2$ (P is a triangle) and each edge of P is only in one triangle then we will say that P is a *diagonal* triangle in G.

Let $\mathcal{B}$ be the family of graphs with two roots denoted by x, y, constructed in the following way. Let G be a graph of $\mathcal{T}_2$ which has a diagonal K_3-path P (i.e., $G \in \{F_7, F_8, ..., F_{15}\}$). Let x, y be the first and the last vertex of P, respectively. We delete from G vertices $V(P) \setminus \{x, y\}$. The vertices x and y are the roots in the new graph. We denote such graphs in the following way:

(1) $B_1(t, k_1, k_2), B_2(t, k_1, k_2), B_3(t, k_1, k_2)$ $t \geq 4, k_1, k_2 \geq 0$ — a graph constructed from $F_7(t, k)$, which we also can obtain in the following way:

$B_1(t, k_1, k_2)$ — we stick together a K_3-cycle of length t and two K_3-paths of lengths k_1 and k_2 with the first vertex of each K_3-path and an arbitrary vertex of degree 4 of the K_3-cycle (the K_3-path are stuck on different vertices of the K_3-cycle);

$B_2(t, k_1, k_2)$ — we stick together a K_3-cycle of length t and two K_3-paths of lengths k_1 and k_2, we stick the first vertex of the first K_3-path on an arbitrary vertex of degree 4 and the first vertex of the second K_3-path on an arbitrary vertex of degree 2;

$B_3(t, k_1, k_2)$ — we stick together a K_3-cycle of length t and two K_3-paths of lengths k_1 and k_2 with the first vertex of each K_3-path and an arbitrary vertex of degree 2 of the K_3-cycle (the K_3-path are stuck on different vertices of K_3-cycle);

(2) $B_8(k_1, k_2), B_9(k_1, k_2), ..., B_{15}(k_1, k_2),\ k_1, k_2 \geq 0$ — the graphs obtained from $F_8(t)$, $F_9(t)$,..., $F_{15}(t)$, respectively (k_1, k_2 are the lengths of the diagonal K_3-paths).

Sometimes the graphs of the family $\mathcal{B}$ will be denoted by $B_1, B_2, B_3, B_8, ..., B_{15}$ for short.

$Z_1(k)$ $(k \geq 2)$ is a graph, which is obtained in the following way: A copy of R and a copy of a K_3-path of length k we stick together by the root of R and the first vertex of the K_3-path. Then we add a new edge, which joins two vertices of degree 2 of the neighbouring triangles of the K_3-path.

$Z_2(t)$ $(t \geq 4)$ is a graph obtained from a K_3-cycle H of length t by adding two new edges. Each new edge joins two vertices of degree 2 in H of the neighbouring triangles.

$Z_3(t, k)$ $(t \geq 4, k \geq 4)$ is a graph obtained in the following way: A copy of a K_3-cycle of length t and a copy of a K_3-path of length k we stick together by an arbitrary vertex of the K_3-cycle and the first vertex of the K_3-path. Then we add an edge, joining two vertices of degree 2 of the neighbouring triangles of the K_3-path.

3 Preliminary Results

Let G be a graph, which has a $(K_{1,2}, K_3)$-decomposition and x, y be vertices of G. If for any $(K_{1,2}, K_3)$-decomposition (E_1, E_2) of $E(G)$ at least one of the vertices x, y is incident with an edge of E_1 then we say that the pair (x, y) is *stable* in G. If $x = y$, then we say that x is a *stable vertex* in G.

First we prove some lemmas characterizing the graphs, which have a $(K_{1,2}, K_3)$-decomposition.

Lemma 1 *Let $H \not\to (K_{1,2}, K_3)$ and x be a stable vertex in H. Then H contains a subgraph H' such that $H' \in \mathcal{A}$ and x is the root of H'.*

Proof. Assume that in H there is no subgraph with the root x, which is isomorphic to a member of $\mathcal{A}$. Let (E_1, E_2) be any $(K_{1,2}, K_3)$-decomposition of H. Let v_0 be the vertex such that $xv_0 \in E_1$. Let x_1 be the third vertex of the triangle which contains the edge xv_0. If such the triangle does not exist then $(E_1/\{xv_0\}, E_2 \cup \{xv_0\})$ is a $(K_{1,2}, K_3)$-decomposition such that the vertex x is not incident with any edge of the set inducing the $K_{1,2}$-free graph, a contradiction. If there is a second triangle containing xv_0 then x is the root of $L_1(0) \subseteq H$. The vertex x_1 must be incident with an edge of E_1, otherwise $((E_1/\{xv_0\}) \cup v_0x_1, (E_2/\{v_0x_1) \cup \{xv_0\})$ is a $(K_{1,2}, K_3)$-decomposition, which contradicts that x is stable. Let $x_1v_1 \in E_1$ and x_2 be the third vertex of the triangle which contains the edge x_1v_1. Note, that the vertex x_2 such that $x_2 \neq x$ and $x_2 \neq v_0$ must exist. If $x_2 = x$ then x is the root of $L_1(0)$. If $x_2 = v_0$ then $((E_1/\{xv_0, x_1v_1\}) \cup x_1v_0, (E_2/\{x_1v_0\}) \cup \{xv_0, x_1v_1\})$ is a $(K_{1,2}, K_3)$-decomposition, which contradicts that x is stable. Since x is not the root of L_1, it follows that $x_1x_2v_1$ and $x_1v_1v_0$ are the only triangles which contain x_1v_1 (the second triangle need not exist). If x_2 is not incident with any edge of E_1 then similarly as above we can show that there exists a $(K_{1,2}, K_3)$-decomposition, which contradicts that x is stable.

In a similar manner we can obtain the next triangle and then we obtain a K_3-path starting in x. Let P be the longest K_3-path, which is obtained in such way and let $x_{k-1}x_kv_{k-1}$ be the last triangle in P. The edges $xv_0, x_1v_1, ..., x_{k-1}v_{k-1}$ of P are in E_1 and in H the edge x_iv_i is contained in at most two triangles $x_ix_{i+1}v_i$ and $x_iv_iv_{i-1}$ for $i = 1, 2, ..., k-1$ (the second triangle need not exist). If x_k is not incident with any edge of E_1 then similarly as above we can show that there exists a $(K_{1,2}, K_3)$-decomposition of H, which contradicts that x is stable. Let $x_kv_k \in E_1$. Since all vertices of P are incident with any edge of E_1, we have that $v_k \notin V(P)$. If $x_{k-1}x_kv_k$ is the triangle then x is the root of $L_1 \subseteq H$. If $x_kv_{k-1}v_k$ is the triangle then we can show that there exists a $(K_{1,2}, K_3)$-decomposition, which contradicts the stability of x. Then the triangle which contains x_kv_k is edge disjoint with P and the third vertex x_{k+1} of this triangle is in P (otherwise we obtain a longer K_3-path). If $x_{k+1} = v_{k-2}$ or $x_{k+1} = x_{k-2}$ then H contains F_1, otherwise x is the root of L_2 or L_3, a contradiction. ■

Lemma 2 *Let $H \not\to (K_{1,2}, K_3)$ and (x, y) be a stable pair in H $(x \neq y)$. Then H contains a graph of the family $\mathcal{A}$ with the root in one of the vertices x, y or there is a K_3-path joining x and y.*

Proof. Assume that H does not contain a subgraph with the root x or y isomorphic to a member of $\mathcal{A}$ and there is no K_3-path joining x and y. Since x and y are not stable in H, it follows that there is a $(K_{1,2}, K_3)$-decomposition (E_1, E_2) of $E(H)$ such that x is incident with an edge of E_1 and y is not incident with any edge of E_1. Let v_0 be a neighbour of x such that $xv_0 \in E_1$ and xv_0x_1 is the triangle, which contains xv_0. If $x_1 = y$ then there is a K_3-path joining x and y, a contradiction. Suppose that the vertex x_1 is not incident with any edge of E_1. Then $((E_1/\{xv_0\}) \cup v_0x_1, (E_2/\{v_0x_1) \cup \{xv_0\})$ is a $(K_{1,2}, K_3)$-decomposition, in which neither x nor y is incident with any edge of the set, which induces a $K_{1,2}$-free graph, a contradiction. Hence x_1 is incident with an edge of E_1. Let v_1 be the second vertex of this edge (i.e., $x_1v_1 \in E_1$) . Note, that there is no second triangle containing xv_0, otherwise x is the root of $L_1 \subseteq H$. Similarly if $v_1x \in E(H)$ then x is the root of $L_1 \subseteq H$. We show that there is a triangle disjoint with the triangle xx_1v_0, containing x_1v_1. If $x_1v_1v_0$ is the only triangle which contains x_1v_1 then $((E_1/\{xv_0, x_1v_1\}) \cup x_1v_0, (E_2/\{x_1v_0\}) \cup \{xv_0, x_1v_1\})$ is the $(K_{1,2}, K_3)$-decomposition, which contradicts that the pair (x, y) is stable. Then there is a triangle vertex-disjoint with the triangle xx_1v_0 containing x_1v_1. Let x_2 be the third vertex of this triangle. Since x is not the root of L_1, it follows that $x_1v_1x_2$ and $x_1v_1v_0$ are the only triangles containing x_1v_1 (the second triangle need not exist). If $x_2 = y$ then there is a K_3-path joining x and y, a contradiction. If $x_2 \neq y$ then x_2 is incident with the edge of E_1, otherwise there exists a $(K_{1,2}, K_3)$-decomposition, contradicting the stability of the pair (x, y).

In a similar manner we can obtain the next triangle and then we obtain a K_3-path starting in x. Let P be the longest K_3-path obtained in such way and let $x_{k-1}x_kv_{k-1}$ be the last triangle in P. The edge x_iv_i is contained in at most two triangles $x_ix_{i+1}v_i$ and $x_iv_iv_{i-1}$ for $i = 1, 2, ..., k-1$. Since there is no K_3-path joining x and y, we have $x_k \neq y$ and $v_{k-1} \neq y$. The vertex x_k must be incident with an edge of E_1, otherwise there exists a $(K_{1,2}, K_3)$-decomposition, which contradicts that the pair (x, y) is stable. Let v_k be the second vertex of this edge and $x_{k+1} \in V(P)$ be the third vertex of the triangle containing the edge x_kv_k. Similarly as in the proof of Lemma 2 we can show that $F_1 \subseteq H$ or x is the root of L_2 or L_3. ■

Lemma 3 *Let $H \not\to (K_{1,2}, K_3)$ and let x, y be two different, nonadjacent vertices such that x and y are not isolated in H and the pair (x, y) is stable in H. If the following condition holds:*
(*) in any proper subgraph of H, containing vertices x and y, the pair (x, y) is not stable; then H is a K_3-path.

Proof. If there is a K_3-path joining x and y then for any $(K_{1,2}, K_3)$-decomposition (E_1, E_2) of the K_3-path the vertex x or the vertex y is incident with an edge of E_1. Then by (*) H is the K_3-path. Suppose that there is no K_3-path in H, which joins x and y. By Lemma 2 one of vertices x, y is stable in H, say x is stable in H. Hence x is the root of a graph $L \in \mathcal{A}$ in H. The condition (*) implies that $E(H) = E(L)$. Since y is not isolated, we have $y \in V(L)$. Then H contains a K_3-path in H, which joins x and y, a contradiction. ■

The next lemmas provide necessary conditions for graphs belonging to $R(K_{1,2}, K_3)$.

Lemma 4 *If $G \in R(K_{1,2}, K_3)$ then it does not contain $Z_1(k)$.*

Proof. Suppose that G contains $Z_1(k)$. Let us denote by $v_1, v_2, ..., v_p, x_1, x_2, ..., x_p, x_{p+1}$ vertices of the K_3-path in $Z_1(k)$ such that v_i is the vertex of degree 2 and x_i is the vertex of degree 4 ($k = 1, 2, ..., p$) in the K_3-path and vertices $x_i x_{i+1} v_i$ form a triangle, x_{p+1} is the common vertex of the K_3-path and R in $Z_1(k)$. Let $e = v_i v_{i+1}$. Let (E_1, E_2) be the $(K_{1,2}, K_3)$-decomposition of $G - e$. The set E_1 must contain edges $x_i v_i$, $(i = 1, 2, .., p)$. If $v_i v_{i+1} x_{i+1}$ is the only triangle which contains e then $(E_1, E_2 \cup e)$ is a $(K_{1,2}, K_3)$-decomposition of G, a contradiction. Suppose that $v_i v_{i+1} w$ is the second triangle containing e. If $w \neq x_{i+1}$ and $w \neq x_{i+2}$ then G contains F_1. If $w = x_{i+2}$ then $F_4(k) \subseteq G$. If $w = x_{i+1}$ then $(E_1, E_2 \cup e)$ is a $(K_{1,2}, K_3)$-decomposition of G. ■

Lemma 5 *If $G \in R(K_{1,2}, K_3)$ then it does not contain $Z_2(t)$.*

Proof. Suppose that G contains $Z_2(t)$. Let us denote by $v_1, v_2, ..., v_k, x_1, x_2, ..., x_k$ the vertices of the K_3-cycle in $Z_2(t)$ such that v_i is the vertex of degree 2 and x_i is the vertex of degree 4 ($k = 1, 2, ..., k$) in the K_3-cycle and $v_i x_i v_j$, $j \equiv i + 1 \pmod k$ form a triangle. Assume that one edge of e_1, e_2 is only in one triangle in G, say e_1. Let (E_1, E_2) be a $(K_{1,2}, K_3)$-decomposition of $G - e_1$. Then $(E_1, E_2 \cup e_1)$ is a $(K_{1,2}, K_3)$-decomposition of G, a contradiction. Hence each edge e_1 and e_2 is contained in at least two triangles.

Case 1. The edges e_1, e_2 are not incident.

W.l.o.g assume that $e_1 = v_1 v_2$. Let $T = v_1 v_2 y$ be the triangle which contains e_1 such that $y \neq x_2$. Since G does not contain F_1, it follows that $y = x_1$ or $y = x_3$. In both cases we obtain a subgraph $Z_1(k)$ contained in G, a contradiction.

Case 2. The edges e_1, e_2 are incident.

W.l.o.g assume that $e_1 = v_1 v_2$ and $e_2 = v_2 v_3$. Let $T_1 = v_1 v_2 y$ be the triangle which contains e_1 such that $y \neq x_2$ and $T_2 = v_2 v_3 z$ be the triangle which contains e_2 such that $z \neq x_3$. We may assume that ($y = x_1$ or $y = x_3$) and ($z = x_2$ or $z = x_4$), otherwise G contains F_1. Suppose that $y = x_1$ and $z = x_2$. Let (E_1, E_2) be a $(K_{1,2}, K_3)$-decomposition of $G - e_1$. Since E_1 must contain $x_i v_i$ $(i = 1, ..., k)$, it follows that $(E_1, E_2 \cup e_1)$ is a $(K_{1,2}, K_3)$-decomposition of G. Using the same arguments we can obtain a $(K_{1,2}, K_3)$-decomposition of G if $y = x_3$ and $z = x_4$. If $y = x_1$ and $z = x_4$ then G contains F_4. If $y = x_3$ and $z = x_2$ then G contains F_{11}. ■

Similarly as Lemma 4 we can prove the next lemma.

Lemma 6 *If $G \in R(K_{1,2}, K_3)$ then it does not contain $Z_3(t, k)$.*

4 Main result

Theorem 1 *$G \in R(K_{1,2}, K_3)$ if and only if $G \in \mathcal{T}$.*

To prove the sufficient condition for a graph to be in $R(K_{1,2}, K_3)$ it is enough to check that each graph $G \in \mathcal{T}$ has no $(K_{1,2}, K_3)$-decomposition, but if we delete an edge from G then we obtain a graph which has a $(K_{1,2}, K_3)$-decomposition. The proof of the necessary condition is partitioned into three cases depending on the connectivity of the graph. The conclusion follows by Lemmas 7, 13, 20.

4.1 $\kappa(G) = 1$

Lemma 7 *Let $G \in R(K_{1,2}, K_3)$ and $\kappa = 1$. Then $G \in \mathcal{T}_1$.*

Proof. Let x be a cut vertex of G. Let $H_1, H_2, ..., H_p$ be components of $G - x$. Let $G_i = G[H_i \cup \{x\}]$, $i = 1, ..., p$. Since G is minimal, the graph G_i $(i = 1, 2, ..., p)$ has a $(K_{1,2}, K_3)$-decomposition. Suppose that there is a graph G_i and there is a $(K_{1,2}, K_3)$-decomposition (E_1, E_2) of G_i such that x is not incident with any edge of E_1. Then the $(K_{1,2}, K_3)$-decomposition of $G - H_i$ can be extended to a $(K_{1,2}, K_3)$-decomposition of G, a contradiction. Therefore in each $(K_{1,2}, K_3)$-decomposition of G_i $(i = 1, 2, ..., p)$ the vertex x is incident with an edge of the set inducing the $K_{1,2}$-free graph. Hence the vertex x is stable in G_i, $i = 1, 2, ..., p$. Moreover $G - x$ has only two components (i.e., $p = 2$). By Lemma 1 x is the root of the graph of the family $\mathcal{A}$ in G_1 and x is the root of a graph of the family $\mathcal{A}$ in G_2. Since G is minimal, it follows that for any proper subgraph G'_i of G_i containing x, the vertex x is not stable. Then G_i $(i = 1, 2)$ is isomorphic to a graph of $\mathcal{A}$. Hence $G \in \mathcal{T}_1$. ■

4.2 $\kappa(G) = 2$

Lemma 8 *Let $H \not\to (K_{1,2}, K_3)$ and let x, y be two nonadjacent stable vertices in H. If for any proper subgraph H' of H containing x and y at least one of vertices x or y is not stable in H' then H does not contain $Z_1(k), Z_2(t)$ and $Z_3(t, k)$.*

Proof. Let G be the graph obtained from H by adding a K_3-path joining vertices x and y. It is easy to see that $G \in R(K_{1,2}, K_3)$. Then by Lemmas 4, 5, 6 the graph G does not contain $Z_1(k), Z_2(t)$ and $Z_3(t, k)$. Hence any subgraph of G does not contain such graphs, too and the lemma follows. ■

Lemma 9 *Let $H \not\to (K_{1,2}, K_3)$ and let x, y be two nonadjacent stable vertices in H. If the following conditions hold*

(1) $\kappa(H + xy) \geq 2$,

(2) for any proper subgraph H' of H containing x and y at least one of the vertices x or y is not stable in H',

then the vertices x, y are the pair of roots of any graph of the family $\mathcal{B}$ in H.

Proof. (Sketch of proof. A complete proof of Lemma 9 can be found at: http://www.wmie.uz.zgora.pl/badania/raporty/) By Lemma 1 vertices x and y are roots of subgraphs isomorphic to some graphs of $\mathcal{A}$. Let L and L' be subgraphs with roots x and y, respectively. By the condition (2) we have $E(H) = E(L) \cup E(L')$. Since $\kappa(H + xy) \geq 2$, the subgraphs L and L' are not vertex-disjoint. Then H is obtained by sticking together L and L'. We stick together L and L' in such way that we obtain a graph, which has a $(K_{1,2}, K_3)$-decomposition (does not contain graphs $F_1, F_2, ..., F_{16}$) and is minimal (by Lemma 8 does not contain $Z_1(k), Z_2(t)$ and $Z_3(t, k)$). ■

Lemma 10 *Let $H \not\to (K_{1,2}, K_3)$ and let x, y be two adjacent stable vertices in H. If the following conditions hold*

1) $\kappa(H) \geq 2$,

2) for any proper subgraph H' of H, containing x and y, at least one of the vertices x or y is not stable in H',

then H is isomorphic to the graph $B_{12}(0,0)$ or H contains a diagonal triangle.

Proof. Similarly as in Lemma 9 vertices x and y are the roots of subgraphs isomorphic to some graphs of the family $\mathcal{A}$. Let us denote by L and L' these subgraphs with roots x and y, respectively. By the condition (2) we have $E(H) = E(L) \cup E(L')$. Since $\kappa(H) \geq 2$, the subgraphs L and L' are not vertex-disjoint. Then H is obtained by sticking together L and L'. If L and L' are isomorphic to $L_1(0)$ then we obtain the graph $B_{12}(0,0)$. Otherwise H contains a diagonal triangle. ■

To prove the main lemma of this part we need the next two lemmas.

Lemma 11 *Let $H \not\to (K_{1,2}, K_3)$ and x and y be two nonadjacent vertices of H such that x is stable in H. If the following conditions hold*

1) $\kappa(H + xy) \geq 2$,

2) for any proper subgraph H' of H the vertex x is not stable in H',

then H contains a diagonal triangle.

Proof. By Lemma 1 the vertex x is a root of a graph $L \in \mathcal{A}$. By the condition (2) we have $E(H) = E(L)$. Because $\kappa(H) \geq 2$, we have $y \in V(L)$. Since the vertices x and y are not adjacent, it follows that L is not isomorphic to $L_1(0)$. Then L contains a diagonal triangle and the lemma follows. ■

The next lemma can be proved similarly as Lemma 11.

Lemma 12 *Let $H \not\to (K_{1,2}, K_3)$ and let $xy \in E(G)$ and x is stable in H. If the following conditions hold*

1) $\kappa(H) \geq 2$,

2) for any proper subgraph H' of H the vertex x is not stable in H',

then H is isomorphic to the graph $L_1(0)$ and x is the root or H contains a diagonal triangle.

Lemma 13 *If $G \in R(K_{1,2}, K_3)$ and $\kappa(G) = 2$, then $G \in \mathcal{T}_2$.*

Proof. First assume that G contains a diagonal triangle $T = xyz$. Let z be a vertex of degree 2. Since G has no $(K_{1,2}, K_3)$-decomposition, it follows that in the graph $(G-z)-\{xy\}$ the vertices x and y are stable. Because of the minimality of G and Lemma 9 we have that the graph $(G - z) - \{xy\} \in \mathcal{B}$. Hence $G \in \mathcal{T}_2$.

Now, assume that G has no diagonal triangle. Let $S \subseteq V(G)$ be a cut set of G such that $|S| = 2$. Let H_1 be a component of $G-S$. Let us denote by $G_1 = G[V(H_1) \cup S], G_2 = G-H_1$. By the minimality of G we have that G_i $(i = 1, 2)$ has a $(K_{1,2}, K_3)$-decomposition.

Let $S = \{x, y\}$. If there is i $(i = 1, 2)$ such that G_i has a $(K_{1,2}, K_3)$-decomposition (E_1, E_2), in which x and y are not incident with any edge of E_1 (in G_i the pair (x, y) is not stable) then we can extend the $(K_{1,2}, K_3)$-decomposition of $G - H_i$ to a $(K_{1,2}, K_3)$-decomposition of G, a contradiction. Then the pair (x, y) is stable in G_i for $i = 1, 2$. Moreover, since $\kappa(G) = 2$, it follows that $\kappa(G_i + xy) = 2$.

Case 1. $xy \notin E(G)$

Suppose that x and y are stable in G_1. Because of the minimality of G we have that G_1 does not contain a proper subgraph in which x and y are stable and for any proper subgraph G'_2 of G_2 containing x and y the pair (x, y) is not stable in G'_2. Then by Lemma 3 G_2 is isomorphic to the K_3-path (the length of G_2 is at least 2 because $xy \notin E(G)$). Hence G contains a diagonal triangle.

If only one vertex of x, y is stable in G_1 then by Lemma 11 the graph G contains a diagonal triangle.

If neither x nor y is stable in G_1 then G_1 is a K_3-path of length at least 2 because the pair (x, y) is stable in G_1. Then again G contains a diagonal triangle.

Case 2. $xy \in E(G)$

If x and y are stable in one graph of G_1, G_2, say x and y are stable in G_1, then by Lemma 10 $G_1 = B_{12}(0, 0)$. Since there is no $(K_{1,2}, K_3)$-decomposition of $B_{12}(0, 0)$ in which xy is in the set inducing the $K_{1,2}$-free graph, we have $G_2 = K_3$. Hence $G = F_1$.

If only one vertex of x, y is stable in G_1 then the same vertex is stable in G_2, say that x is stable in G_1 and G_2. Moreover there is no $(K_{1,2}, K_3)$-decomposition of G_1 or G_2 in which xy is in the set inducing the $K_{1,2}$-free graph. Assume that G_2 has no such $(K_{1,2}, K_3)$-decomposition. Since for each proper subgraph of G_2 containing x and y, the vertex x is not stable and by Lemma 12, it follows that $G_2 = L_1(0)$. But G_2 contains the $(K_{1,2}, K_3)$-decomposition in which xy is in the set inducing the $K_{1,2}$-free graph, a contradiction.

If no vertex of x, y is stable in G_1 then x and y are stable in G_2 because G has no $(K_{1,2}, K_3)$-decomposition. As above we obtain $G = F_1$. ■

4.3 $\kappa(G) \geq 3$

Lemma 14 *If $G \in R(K_{1,2}, K_3)$ and $\kappa(G) \geq 3$ then G does not contain $L_1(k)$, $k \geq 2$ and $L_2(t, k), L_3(t, k)$, $t \geq 4, k \geq 1$.*

Proof. Suppose that G contains one of these graphs. Let us denote it by L. Let $T = xyz$ be the last triangle in L, x and y be the vertices of degree 2 in L and z be the vertex of degree 4. Let (E_1, E_2) be a $(K_{1,2}, K_3)$-decomposition of $G - xy$. If x and y are not incident with any edge of E_1 then $(E_1 \cup \{xy\}, E_2)$ is the $(K_{1,2}, K_3)$-decomposition of G. Hence the pair (x, y) is stable in $G - xy$.

Let L' be the minimal subgraph of $G - xy$, in which the pair (x, y) is stable (i.e., the pair (x, y) is not stable in any proper subgraph of L'). Because of the minimality of G we can deduce that G is obtained by sticking together L and L'. Since $\kappa(G) \geq 3$, it follows that there is no vertex of degree less than 3 in G. Then x and y are not isolated in L'. Hence by Lemma 3 L' is isomorphic to the K_3-path, which joins x and y.

Let x_1, x_2 be the neighbours of x such that xx_1x_2 is the triangle in L' and x_1 is the vertex of degree 2 and x_2 is the vertex of degree 4 in L'. Similarly let y_1, y_2 be the neighbours of y such that yy_1y_2 is the triangle in L' and y_1 is the vertex of degree 2 and y_2 is the vertex of degree 4 in L'. Note that z is the root of a graph of $\mathcal{A}$ in $L - xy$. Hence $xz \notin E_1$ and $yz \notin E_1$.

Knowing all $(K_{1,2}, K_3)$-decompositions of L' we can see that at least one of the vertices x_1, y_1 is incident with an edge of E_1, say x_1 is incident with an edge of E_1 ($xx_1 \in E_1$). Because G does not contain any vertex of degree less than 3, we have that x_1 is also in $V(L)$. Since in each decomposition of $L_2(k,t)$ and $L_3(k,t)$ each vertex is incident with an edge of the set inducing the $K_{1,2}$-free graph, it follows that L is isomorphic to $L_1(k)$ and x_1 is the vertex of degree 2 of subgraph R of L_1. Then $G[V(L) \cup \{x, x_1, x_2\}]$ contains the K_3-cycle. Let w be the vertex of degree 2 of K_3-path of L other than x and y. Then w is also in $V(L')$. Hence G contains $F_7(t,k)$ or F_1. ■

Similarly as Lemma 14 we can prove the next lemma.

Lemma 15 *Let $G \in R(K_{1,2}, K_3)$ and $\kappa(G) \geq 3$. If G contains $L_1(1)$ then G is isomorphic to F_3.*

Lemma 16 *If $G \in R(K_{1,2}, K_3)$ and $\kappa(G) \geq 3$ then G does not contain any K_3-cycle.*

Proof. Suppose that G contains a K_3-cycle. Let v be a vertex of degree two in the K_3-cycle. Let w be the third neighbour of v (such vertex exists because $\kappa(G) \geq 3$). Since G does not contain $L_2(t,k)$ ($k \geq 1$), the triangle containing the edge vw is not vertex-disjoint with the K_3-cycle. Since G does not contain $L_1(1)$ and F_1, it follows that w is a vertex of degree 2 of the K_3-cycle such that the triangle containing vw consist of two external edges of the K_3-cycle. The same property has each vertex of degree 2 of the K_3-cycle. From the definition we have that a K_3-cycle has at least 4 vertices of degree 2, hence we obtain that G contains $Z_2(t)$, a contradiction. ■

Lemma 17 *Let $G \in R(K_{1,2}, K_3)$ and $\kappa(G) \geq 3$. Then each triangle of G contains at most one edge, which is contained in one triangle.*

Proof. Let $T = xyz$ be the triangle of G containing two edges, which are only in T. Let $e_1, e_2 \in E(T)$ be the edges which are only in T and $e_1 = xy,\ e_2 = xz$. Let e_3 be the third edge of T. In each $(K_{1,2}, K_3)$-decomposition (E_1, E_2) of $G - \{e_1, e_2\}$ at least two vertices of T are incident with an edge of E_1. Then $(x,y), (x,z)$ and (y,z) are stable pairs in $G - \{e_1, e_2\}$. Moreover there is no $(K_{1,2}, K_3)$-decomposition of $G - \{e_1, e_2\}$ in which the edge e_3 is in the set inducing the $K_{1,2}$-free graph. From Lemma 14 and Lemma 15 it follows that G does not contain $L_1(k), L_2(t,k)$ and $L_3(t,k)$ ($t \geq 4, k \geq 1$). Then x is not stable in $G - \{e_1, e_2\}$.

Suppose that y and z are stable in $G - \{e_1, e_2\}$. Lemma 1 implies that y and z are the roots of graphs of $\mathcal{A}$ in $G - \{e_1, e_2\}$. Let L and L' be the graph with root in y and z, respectively. By Lemma 14 and Lemma 15 we have that L and L' are isomorphic to $L_1(0), L_2(t,0)$ or $L_3(t,0)$ and they contain the edge e_3. If one graph of

L, L', say L, is isomorphic to $L_2(t,0)$ or $L_3(t,0)$ then the triangle containing e_3 and T form the subgraph R of $L_1(1)$ in G, a contradiction. Hence L and L' are isomorphic to $L_1(0)$. If $V(L) \cap V(L') = \{y, z\}$ then G also contains $L_1(1)$. Since there is no $(K_{1,2}, K_3)$-decomposition of $G - \{e_1, e_2\}$, in which the edge e_3 is in the subgraph inducing the $K_{1,2}$-free graph, we may assume that there are vertices y_1, y_2, which are the neighbours of y in L, which have degree 2 and 3 in L, respectively (others than z). Similarly there are the neighbours z_1 and z_2 of z, which have degree 2 and 3 in L'. If $y_2 \neq z_2$ and ($y_1 \neq z_2$ or $z_1 \neq y_2$) then G contains $L_1(1)$. If $y_2 = z_2$ and $y_1 \neq z_1$ then G contains F_1. If $y_2 = z_2$ and $y_1 = z_1$ then G contains K_4, which has a common edge (the edge e_3) with T. If $y_2 \neq z_2$ and $y_1 = z_2$ and $z_1 = y_2$ then G also contains K_4, which has a common edge with T. Then assume that G contains a subgraph K isomorphic to K_4 such that $y, z \in V(K)$. Let y', z' be the remaining vertices of K. Note that the edges $y'y$ and $y'z$ are contained only in two triangles in G (two triangles of K) and the edges $z'y$ and $z'z$ are contained in only two triangles in G. Since $e_3 \notin E_1$, it follows that $yz', zy' \in E_1$ or $yy', zz' \in E_1$. W.l.o.g suppose that $yy', zz' \in E_1$. Then $(E_1/\{yy', zz'\}) \cup \{yz, y'z'\}, (E_2/\{yz, y'z'\}) \cup \{yy', zz'\})$ is a $(K_{1,2}, K_3)$-decomposition of G, a contradiction.

Suppose that at most one vertex of y, z is stable in $G - \{e_1, e_2\}$, say z is not stable in $G - \{e_1, e_2\}$. Then there is a K_3-path joining x and z. If this K_3-path does not contain the edge e_3 then G contains a K_3-cycle, which contradicts Lemma 16. Then assume the K_3-path consists of the edge e_3. Let w be the third vertex of the triangle of the K_3-path containing e_3 (the vertex w has degree 4 in the K_3-path). Let ww_1w_2 be the second triangle of the K_3-path. Suppose that w_1 is a vertex of degree 2 in the K_3-path and w_2 is a vertex of degree 4 in the K_3-path. Since z is not stable and G does not contain $L_1(1)$, it follows that z is contained in one triangle ywz and y is contained in at most two triangles ywz and yww_1. Since G does not contain $L_1(1)$, we have that two edges of the K_3-path, which are incident with x, are contained in only one triangle. Moreover, each edge of the K_3-path, which is contained in E_1 is in at most two triangles: the triangle of the K_3-path and the triangle containing two external edges of the K_3-path. Then we can change the edges of E_1, which are in the K_3-path, in such a way that we obtain a $(K_{1,2}, K_3)$-decomposition of $G - \{e_1, e_2\}$ containing e_3, a contradiction. ■

Lemma 18 *Let $G \in R(K_{1,2}, K_3)$ and $\kappa(G) \geq 3$. Then G does not contain K_4.*

Proof. Suppose that G contains a subgraph K isomorphic to K_4. Since K has a $(K_{1,2}, K_3)$-decomposition, it follows that there is a triangle T in G such that $T \not\subseteq K$. By Lemma 15 and the connectivity of G we have that T is not edge-disjoint with K. Let x be the vertex of T, which is not in K and let y, z be vertices of $V(K) \cap V(T)$. Because of Lemma 17 we have that xy or xz is in the second triangle T'. Let w be the third vertex of T'. The triangle T' is not edge-disjoint with K (a contradiction to Lemma 15), then $w \in V(K)$. Hence G contains F_2, a contradiction. ■

Let xyz and $x'y'z'$ be the last triangles of the K_3^2-path of odd length t ($t \geq 5$) such that z and z' have degree 2, y, y' have degree 3, x, x' have degree 4. We denote by $W(t)$ the graph which is obtained from a K_3^2-path of odd length by adding new edges zy' and zz'.

Lemma 19 *Let $t \geq 5$ and $G \in R(K_{1,2}, K_3)$, $\kappa(G) \geq 3$. If G contains $W(t)$, then $G = F_{16}(t)$.*

Proof. Suppose that G contains a subgraph W isomorphic to $W(t)$. Let us denote the vertices of W as in the definition of $W(t)$. By Lemma 17 we have that in G there is a second triangle containing zy' or there is a second triangle containing zz'. If this triangle contains neither y nor x then G contains $L_1(1)$ (a contradiction to Lemma 15). If xz' or yz' or yy' is in G then G contains Z_2 (a contradiction to Lemma 5). If $yz' \in E(G)$ then $G = F_{16}(t)$. ■

Lemma 20 *Let $G \in R(K_{1,2}, K_3)$ and $\kappa(G) \geq 3$. Then $G \in \mathcal{T}_3$.*

Proof. Let P be the longest K_3^2-path in G. From Lemma 17 and Lemma 18 it follows that the length of P is at least 3.

Case 1. The length of P is equal to 3.

Let x be a vertex of degree 4 in P and y, z, z', y' be the neighbours of x such that $yz, zz', z'y' \in E(P)$. By Lemma 17 we have that xy or yz is contained in at least two triangles in G and xy' or $y'z'$ is contained in at least two triangles in G.

First we show that there is no second triangle containing yz (similarly there is no second triangle containing $y'z'$). Suppose that w is the third vertex of such triangle. If $w \notin V(P)$ then there is a K_3^2-path of length 4 in G, a contradiction. If $w = z'$ or $w = y'$ then G contains K_4, which contradicts Lemma 18.

Now we show that if xy and xy' are in the second triangle then $G = F_2$ or $G = F_3$. Let w be the third vertex of the second triangle containing xy. If $w \notin V(P)$ then by Lemma 15 we have $G = F_3$. If $w = z'$ then G contains K_4 (a contradiction to Lemma 18). If $w = y'$ then $G = F_2$.

Case 2. The length of P is equal to 4.

Let x, x' be vertices of degree 4, y, y' vertices of degree 3, z, z' vertices of degree 2 in P such that xyz and $x'y'z'$ are triangles. By Lemma 17 we have that xz or yz is contained in at least two triangles in G and $x'z'$ or $y'z'$ is contained in at least two triangles in G.

Similarly as above we show that there is no second triangle containing yz (and there is no second triangle containing $y'z'$). Suppose that w is the third vertex of such triangle. If $w \notin V(P)$ then there is a K_3^2-path of length 5 in G, a contradiction. If $w = x'$ or $w = y'$ then G contains K_4, which contradicts Lemma 18. If $w = z'$ then G contains F_2.
Now suppose that xz is in the second triangle in G. Let w be the third vertex of this triangle. If $w \neq x'$ and $w \neq y'$ then G contains $L_1(1)$. Hence by Lemma 15 we have $G = F_3$. If $w = x'$ then G contains K_4, which contradicts Lemma 18. If $w = y'$ then G contains F_1.

Case 3. The length of P is at least 5.

Let us denote $V(P) = \{x_1, x_2, ..., x_k, y_1, y_2, ..., y_k\}$ ($V(P) = \{x_1, ..., x_k, y_1, y_2, ..., y_{k-1}\}$ if P is odd length) such that $x_i x_{i+1} y_i$ ($i = 1, 2, ..., k-1$) and $y_i y_{i+1} x_{i+1}$ ($i = 1, 2, ..., k-1$ for P of even length and $i = 1, 2, ..., k-2$ for P of the odd length) form the triangle. By Lemma 17 we have that $x_1 x_2$ or $x_1 y_1$ is contained in at least two triangles in G.

Suppose that x_1x_2 is in the second triangle in G. Let w be the third vertex of this triangle. If $w \neq x_3$ and $w \neq y_2$ then G contains $L_1(1)$ then by Lemma 15 G contains F_3. If $w = x_3$ or $w = y_2$ then G contains K_4, which contradicts Lemma 18.

Suppose that x_1y_1 is in the second triangle in G. Let w be the third vertex of this triangle. Since P is the longest K_3^2-path, we have $w \in V(P)$. Similarly as in Case 2 we can show that w is not any vertex of $\{x_2, x_3, y_2, y_3\}$. If $w = y_i$ ($i \geq 4$) then G contains $Z_2(t)$ (a contradiction to Lemma 5). If $w = x_i$ ($i \geq 4$) then G contains $W(t)$. Hence by Lemma 18 $G = F_{16}(t)$ or G contains $F_{16}(t)$. ■

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