On the Kronecker Product $s_{(n-p,p)} * s_\lambda$

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Abstract

The Kronecker product of two Schur functions s_λ and s_μ, denoted $s_\lambda * s_\mu$, is defined as the Frobenius characteristic of the tensor product of the irreducible representations of the symmetric group indexed by partitions of n, λ and μ, respectively. The coefficient, $g_{\lambda,\mu,\nu}$, of s_ν in $s_\lambda * s_\mu$ is equal to the multiplicity of the irreducible representation indexed by ν in the tensor product. In this paper we give an algorithm for expanding the Kronecker product $s_{(n-p,p)} * s_\lambda$ if $\lambda_1 - \lambda_2 \geq 2p$. As a consequence of this algorithm we obtain a formula for $g_{(n-p,p),\lambda,\nu}$ in terms of the Littlewood-Richardson coefficients which does not involve cancellations. Another consequence of our algorithm is that if $\lambda_1 - \lambda_2 \geq 2p$ then every Kronecker coefficient in $s_{(n-p,p)} * s_\lambda$ is independent of n, in other words, $g_{(n-p,p),\lambda,\nu}$ is stable for all ν.

Introduction

Let χ^λ and χ^μ be the irreducible characters of S_n (the symmetric group on n letters) indexed by the partitions λ and μ of n. The *Kronecker product* $\chi^\lambda\chi^\mu$ is defined by $(\chi^\lambda\chi^\mu)(w) = \chi^\lambda(w)\chi^\mu(w)$ for all $w \in S_n$. Hence, $\chi^\lambda\chi^\mu$ is the character that corresponds to the diagonal action of S_n on the tensor product of the irreducible representations indexed by λ and μ. Then we have

$$\chi^\lambda\chi^\mu = \sum_{\nu \vdash n} g_{\lambda,\mu,\nu}\chi^\nu,$$

where $g_{\lambda,\mu,\nu}$ is the multiplicity of χ^ν in $\chi^\lambda\chi^\mu$. Hence the $g_{\lambda,\mu,\nu}$ are non-negative integers.

By means of the Frobenius map one can define the Kronecker (internal) product on the Schur symmetric functions by

$$s_\lambda * s_\mu = \sum_{\nu\vdash n} g_{\lambda,\mu,\nu} s_\nu.$$

A reasonable formula for decomposing the Kronecker product is unavailable, although the problem has been studied since the early twentieth century. In recent years Lascoux [La], Remmel [R-1], Remmel and Whitehead [RWd] and Rosas [Ro] derived closed formulas for Kronecker products of Schur functions indexed by two row shapes or hook shapes. Gessel [Ge] obtained a combinatorial interpretation for zigzag partitions.

More general results include a formula of Garsia and Remmel [GR-1] which decomposes the product of homogeneous symmetric functions with a Schur function. Dvir [D] and Clausen and Meier [CM] have given for any λ and μ a simple and precise description for the maximum length of ν and the maximum size of ν_1 whenever $g_{\lambda,\mu,\nu}$ is nonzero. Bessenrodt and Kleshchev [BK] have looked at the problem of determining when the decomposition of the Kronecker product has one or two constituents.

The main result of this paper is an algorithm for decomposing the Kronecker product $s_{(n-p,p)} * s_\lambda$ whenever $\lambda_1 - \lambda_2 \geq 2p$. We use this algorithm to obtain a closed formula for $g_{(n-p,p),\lambda,\nu}$ in terms of Littlewood-Richardson coefficients that does not involve cancellations. Our algorithm is a generalization of the following simple algorithm for the decomposition of $s_{(n-1,1)} * s_\lambda$ whenever $\lambda_1 - \lambda_2 \geq 2$. Let $\bar\lambda = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ denote the Young diagram obtained by removing the first part from λ.

First Step: Everywhere possible delete zero or one box from $\bar\lambda$ such that the resulting diagram corresponds to a partition.

Second step: To each diagram $\beta \neq \bar\lambda$ obtained in the first step, everywhere possible add zero or one box so that the resulting diagram corresponds to a partition. And to $\beta = \bar\lambda$ add everywhere possible one box.

Finally, we complete the resulting diagrams $\bar\nu$ obtained in the second step such that $\nu = (n - |\bar\nu|, \bar\nu)$ is a partition of n. Then $s_{(n-1,1)} * s_\lambda$ is equal to the sum of the Schur functions corresponding to all diagrams ν obtained via the remove/add steps above.

In 1937 Murnaghan [M] noticed that for large n the Kronecker product did not depend on the first part of the partitions λ and μ. That is, if λ is a partition of n and $\bar\lambda = (\lambda_2, \ldots, \lambda_{\ell(\lambda)})$ denotes the partition obtained by removing the first part of λ, then there exists an n such that $g_{(n-|\bar\lambda|,\bar\lambda),(n-|\bar\mu|,\bar\mu),(n-|\bar\nu|,\bar\nu)} = g_{(m-|\bar\lambda|,\bar\lambda),(m-|\bar\mu|,\bar\mu),(m-|\bar\nu|,\bar\nu)}$ for all $m \geq n$. In this case we say that $g_{\lambda,\mu,\nu}$ is *stable*. Vallejo [V1] has recently found a bound for n for the stability of $g_{\lambda,\mu,\nu}$. As a consequence of our algorithm we have that $g_{(n-p,p),\lambda,\nu}$ is stable for all ν if $\lambda_1 - \lambda_2 \geq 2p$. This improves Vallejo's bound in some cases.

Other consequences of our algorithm are bounds for the size of ν_1 and ν_2 whenever $g_{(n-p,p),\lambda,\nu} \neq 0$.

Our main tools for establishing the algorithm are the Garsia-Remmel identity [GR-1, Lemma 6.3] and the Remmel-Whitney algorithm for multiplying Schur functions [RWy]. The main strength of the algorithm relies in the fact that it does not involve cancellations.

The paper is organized as follows. In Section 1 we review basic terminology and establish notation. We also give a variation of the Remmel-Whitney algorithm for multiplying Schur functions. In Section 2 we state our algorithm for the product $s_{(n-p,p)} * s_\lambda$ and give an example of the algorithm. In Section 3 we prove the main theorem which states that the result of the algorithm in Section 2 yields the decomposition of $s_{(n-p,p)} * s_\lambda$. In Section 4 we give a closed formula for the coefficient $g_{(n-p,p),\lambda,\nu}$ in terms of Littlewood-Richardson coefficients when $\lambda_1 - \lambda_2 \geq 2p-1$. We also give bounds for ν_1 and ν_2 so that $g_{(n-p,p),\lambda,\nu} \neq 0$. In Section 5 we discuss the stability of the coefficients $g_{(n-p,p),\lambda,\nu}$.

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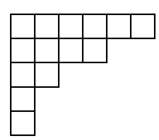
1 Notation and Basic Algorithms

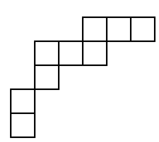
For details and proofs of the contents of this section see [Ma] or [S, Chapter 7]. Let n be a non-negative integer. A *partition* of n is a weakly decreasing sequence of non-negative integers, $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$, such that $|\lambda| = \sum \lambda_i = n$. We write $\lambda \vdash n$ to mean λ is a partition of n. The nonzero integers λ_i are called the *parts* of λ. We identify a partition with its *Young diagram*, i.e. the array of left-justified squares (boxes) with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. The rows are arranged in matrix form from top to bottom. By the box in position (i, j) we mean the box in the i-th row and j-th column of λ. The *length* of λ, $\ell(\lambda)$, is the number of rows in the Young diagram.



$\lambda = (6, 4, 2, 1, 1), \quad \ell(\lambda) = 5, \quad |\lambda| = 14$

Fig. 1

Given two partitions λ and μ, we write $\mu \subseteq \lambda$ if and only if $\ell(\mu) \leq \ell(\lambda)$ and $\lambda_i \geq \mu_i$ for $1 \leq i \leq \ell(\mu)$. If $\mu \subseteq \lambda$, we denote by λ/μ the skew shape obtained by removing the boxes corresponding to μ from λ.



λ/μ where $\lambda = (6, 4, 2, 1, 1)$ and $\mu = (3, 1, 1)$

Fig.2

A *horizontal strip* is a skew shape λ/μ with no two squares in the same column.

Let $D = \lambda/\mu$ be a skew shape and let $a = (a_1, a_2, \cdots, a_k)$ be a sequence of positive integers such that $\sum a_i = |D| = |\lambda| - |\mu|$. A *decomposition* of D of type a, denoted

$D_1 + \cdots + D_k = D$, is given by a sequence of shapes $\mu = \lambda^{(0)} \subseteq \lambda^{(1)} \ldots \subseteq \lambda^{(k)} = \lambda$, where $D_i = \lambda^{(i)}/\lambda^{(i-1)}$ and $|D_i| = a_i$.

For example, if $\lambda = (4,4,4,3,1)$, $\mu = \emptyset$ and $a = (3,6,7)$ the sequence

$$\emptyset \subseteq (2,1) \subseteq (4,2,1,1,1) \subseteq (4,4,4,3,1)$$

gives the decomposition

$$(2,1) + (4,2,1,1,1)/(2,1) + (4,4,4,3,1)/(4,2,1,1,1) = \lambda$$

of λ of type $(3,6,7)$.

A *semi-standard Young tableau* (SSYT) *of shape* λ/μ is a filling of the boxes of the skew shape λ/μ with positive integers so that the numbers weakly increase in each row from left to right and strictly increase in each column from top to bottom. The *type* of a SSYT T is the sequence of non-negative integers $(t_1, t_2, \ldots)$, where t_i is the number of i's in T.

$$\begin{array}{ccccccc} & & & 2 & 2 & 3 & 4 \\ & & 1 & 4 & 4 & 6 & \\ & 1 & 3 & 6 & 6 & & \\ 2 & 2 & 4 & & & & \end{array}$$

is a SSYT of shape $\lambda/\mu = (7,6,5,3)/(3,2,1)$ and type $(2,4,2,4,0,3)$.
Fig. 3

Given a SSYT T of shape λ/μ and type $(t_1, t_2, \ldots)$, we define its *weight*, $w(T)$, to be the monomial obtained by replacing each i in T by x_i and taking the product over all boxes, i.e. $w(T) = x_1^{t_1} x_2^{t_2} \cdots$. For example, the weight of the SSYT in Fig. 3 is $x_1^2 x_2^4 x_3^2 x_4^4 x_6^3$. The skew Schur function $s_{\lambda/\mu}$ is defined combinatorially by the formal power series

$$s_{\lambda/\mu} = \sum_T w(T),$$

where the sum runs over all SSYTs of shape λ/μ. To obtain the usual Schur function one sets $\mu = \emptyset$.

The space of homogeneous symmetric functions of degree n is denoted by Λ^n. A basis for this space is given by the Schur functions $\{s_\lambda \,|\, \lambda \vdash n\}$. The Hall inner product on Λ^n is denoted by $\langle\ ,\ \rangle_{\Lambda^n}$ and it is defined by

$$\langle s_\lambda, s_\mu \rangle_{\Lambda^n} = \delta_{\lambda\mu},$$

where $\delta_{\lambda\mu}$ denotes the Kronecker delta.

For a positive integer r, let $p_r = x_1^r + x_2^r + \cdots$. Then $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{\ell(\mu)}}$ is the power symmetric function corresponding to the partition μ of n. If CS_n denotes the space of class functions of S_n, then the *Frobenius characteristic map* $F : CS_n \to \Lambda^n$ is defined by

$$F(\sigma) = \sum_{\mu \vdash n} z_\mu^{-1} \sigma(\mu) p_\mu,$$

where $z_\mu = 1^{m_1}\, m_1!\, 2^{m_2}\, m_2! \cdots n^{m_n}\, m_n!$ if $\mu = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$, i.e. k is repeated m_k times in μ, and $\sigma(\mu) = \sigma(\omega)$ for an $\omega \in S_n$ of cycle type μ. Note that F is an isometry. If χ^λ is an irreducible character of S_n then, by the Murnaghan-Nakayama rule [S, 7.17.5], $F(\chi^\lambda) = s_\lambda$.

For a positive integer r, let $h_r = s_{(r)}$. Then $h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_{\ell(\mu)}}$ is the *homogeneous symmetric function* corresponding to the partition μ of n. The Jacobi-Trudi identity allows us to express a Schur function in terms of homogeneous symmetric functions:

$$s_\lambda = \det \| h_{\lambda_i - i + j} \|_{1 \le i,j \le \ell(\lambda)},$$

where we set $h_0 = 1$ and $h_k = 0$ for $k < 0$.

The *Littlewood-Richardson coefficients* are defined via the Hall inner product on symmetric functions as follows:

$$c^\lambda_{\mu\,\nu} := \langle s_\lambda, s_\mu s_\nu \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle.$$

That is, skewing is the adjoint operator of multiplication with respect to this inner product. The Littlewood-Richardson coefficients are best described combinatorially by the Littlewood-Richardson rule. Before presenting the rule we need to recall two additional notions. A *lattice permutation* is a sequence $a_1 a_2 \cdots a_n$ such that in any initial factor $a_1 a_2 \cdots a_j$, the number of i's is at least as great as the number of $(i+1)$'s for all i. For example 11122321 is a lattice permutation. The *reverse reading word* of a tableau is the sequence of entries of T obtained by reading the entries from right to left and top to bottom, starting with the first row.

Example: The reverse reading word of the tableau $\begin{array}{cccc} & & 1 & 2 \\ 3 & 5 & 6 & 8 \\ 4 & 7 & 9 & \end{array}$ is 218653974.

The *Littlewood-Richardson rule* states that the Littlewood-Richardson coefficient $c^\lambda_{\mu\,\nu}$ is equal to the number of SSYTs of shape λ/μ and type ν whose reverse reading word is a lattice permutation.

We now recall an algorithm given by Remmel-Whitney [RWy] for expanding the product of Schur functions $s_\lambda s_\mu$. In this paper we give two slight variations of the Remmel-Whitney algorithm: one for multiplication and the other for skewing. This will allow us to give a nicer presentation of our main result. The algorithm for expanding the skew Schur function $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\,\nu} s_\nu$ is a special case of the algorithm for the product of Schur functions. We will refer to the algorithm for multiplying $s_\lambda s_\mu$ as *Add[μ] to* λ, and we will refer to skewing algorithm as *Delete[μ] from* λ.

The *reverse lexicographic filling* of μ, $rl(\mu)$, is a filling of the Young diagram μ with the numbers $1, 2, \ldots, |\mu|$ so that the numbers are entered in order from right to left and top to bottom. For example, the reverse lexicographic filling of (5,3,1) is $\begin{array}{ccccc} 5 & 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & & \\ 9 & & & & \end{array}$.

Definition: A tableau T is (λ, μ)-*compatible* if it contains $|\lambda|$ unlabelled boxes and $|\mu|$ labelled boxes (with labels $1, 2 \ldots, |\mu|$) and all of the following conditions are satisfied:

(a) T contains $|\lambda|$ unlabelled boxes in the shape λ. They are positioned in the upper-left corner of T.

(b) The labelled boxes in T are in increasing order in each row from left to right and in each column from top to bottom. If one box of T is labelled, so are all the boxes in the same row that are to the right of it.

(c) If a box labelled $i+1$ occurs immediately to the left of the box labelled i in $rl(\mu)$, then in T the label $i+1$ occurs weakly above and strictly to the right of i.

(d) If the box labelled y occurs immediately below the box labelled x in $rl(\mu)$, then in T the label y occurs strictly below and weakly to the left of x.

Remmel and Whitney showed that $c^{\nu}_{\lambda\mu}$ is the number of (λ,μ)-compatible tableaux of shape ν [RWy].

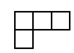
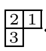
Multiplication: $s_\lambda s_\mu$ - Add[μ] to λ

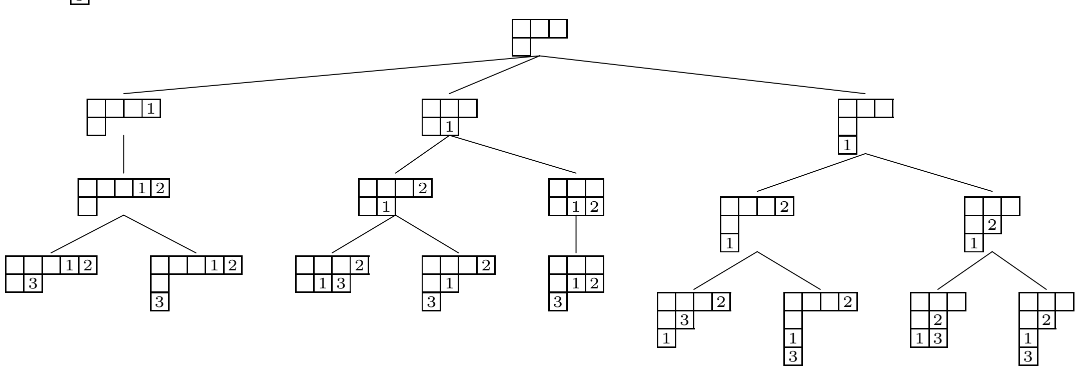
The *Add[μ] to* λ algorithm for computing $s_\lambda s_\mu = \sum_{|\nu|=|\lambda|+|\mu|} c^{\nu}_{\lambda\mu} s_\nu$ is as follows:

(1) To the Young diagram λ add a box labelled 1 everywhere possible so that the rows are weakly increasing in size.

(2) We add each subsequent number so that, at each step, the conditions of the definition of (λ,μ)-compatible tableau are satisfied.

In this way we obtain a tree. The leaves of this tree are the elements of the multi-set Add[μ] to λ. They are the summands in the decomposition of $s_\lambda s_\mu$.

Example: The decomposition of $s_\lambda s_\mu$, where $\lambda = (3,1)$, $\mu = (2,1)$: $\lambda =$  and $rl(\mu) =$ .



Add[μ] to $\lambda = \{(5,2),(5,1,1),(4,3),2(4,2,1),(3,3,1),(4,1,1,1),(3,2,2),(3,2,1,1)\}$.
Hence $s_\lambda s_\mu = s_{(5,2)} + s_{(5,1,1)} + s_{(4,3)} + 2s_{(4,2,1)} + s_{(3,3,1)} + s_{(4,1,1,1)} + s_{(3,2,2)} + s_{(3,2,1,1)}$.

Remark: The *Add[μ] to* λ algorithm is the same as the Remmel-Whitney algorithm. We do not label the boxes of λ since, by Remark 1 of [RWy], they will always be placed in the shape of λ in the upper left corner.

The Remmel-Whitney algorithm for multiplying Schur functions is a special case of a skew Schur function expansion rule [RWy][Remark 3]. See also [R-2]. The Remmel-Whitney algorithm for the decomposition of the skew Schur function $s_{\eta/\nu}$ requires forming the reverse lexicographic filling of η/ν and placing the labels in increasing order such that (c) and (d) in the definition of compatible tableau are satisfied at each step. Consider now the skew shape $(\mu/\rho) \times \lambda$ given by

$$(\mu_1+\lambda_1, \mu_2+\lambda_1, \ldots, \mu_{\ell(\mu)}+\lambda_1, \lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})/(\lambda_1+\rho_1, \lambda_2+\rho_2, \ldots, \lambda_{\ell(\rho)}+\rho_{\ell(\rho)}, \lambda_1^{\ell(\mu)-\ell(\rho)}).$$

To obtain the expansion of $s_{(\mu/\rho)\times\lambda}$, the Remmel-Whitney algorithm first decomposes the skew Schur function $s_{\mu/\rho} = \sum s_{\gamma_i}$. Continuing the algorithm, we place the labels of λ thus obtaining the decomposition for each $s_{\gamma_i} s_\lambda$. The leaves of the obtained tree are the diagrams indexing the Schur functions in the decomposition of $s_{\mu/\rho} s_\lambda$. In performing the algorithm, the labels themselves are irrelevant; only their relative position to each other is important. Thus, expanding $s_{(\mu/\rho)\times\lambda}$ gives the same decomposition as expanding $s_{\lambda\times(\mu/\rho)}$, where $\lambda \times (\mu/\rho)$ is the skew shape

$$(\lambda_1 + \mu_1, \lambda_2 + \mu_1, \ldots, \lambda_{\ell(\lambda)} + \mu_1, \mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})/(\mu_1^{\ell(\lambda)}, \rho).$$

We have the following lemma.

Lemma 1.1. *The* Add *algorithm can be applied to compute the product of a skew Schur function and a straight Schur function. To perform* Add $[\mu/\rho]$ to λ *form the reverse lexicographic filling of* μ/ρ *and add the labels of* μ/ρ *to* λ *according to the* Add *algorithm above. The leaves of the obtained tree correspond to the summands in the decomposition of* $s_{\mu/\rho} s_\lambda$.

Skew: $s_{\lambda/\mu}$ - Delete[μ] from λ

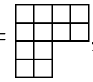
The *Delete[μ] from* λ for computing $s_{\lambda/\mu} = \sum_{|\nu|=|\lambda|-|\mu|} c^{\lambda}_{\mu\,\nu} s_\nu$ is as follows:

(1) Form the reverse lexicographic filling of μ.

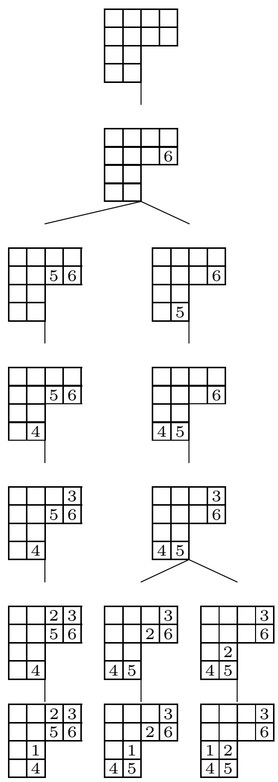
(2) Starting with the Young diagram λ we will label its outermost boxes with the numbers $1, 2, \ldots, |\mu|$ in decreasing order, starting with $|\mu|$, in the following way. At every step, the diagram obtained from λ by deleting the labelled boxes must be a Young diagram. Suppose the position (i, j) in $rl(\mu)$ is labelled x. If $j > 1$, let x^- be the label in position $(i, j-1)$ in $rl(\mu)$. If $i < \ell(\mu)$, let x^+ be the label in position $(i+1, j)$ in $rl(\mu)$. In λ, x will be placed to the left and weakly below (to the SW) of x^- and above and weakly to the right (to the NE) of x^+.

    From each of the diagrams obtained (with $|\mu|$ labelled boxes) we remove all labelled boxes. The resulting diagrams are the elements in the multi-set Delete[μ] from λ. They are the summands in the decomposition of $s_{\lambda/\mu}$.

Remark: Suppose (i, j) is the position of the label x in $rl(\mu)$ and (l, m) is the new position of x in λ. Because of the above rules, there will be constraints on l and m. It can be easily verified that we must have $l \geq i$ and $m \geq \mu_i - j + 1$, where μ_i is the number of boxes in the i-th row of μ.

Example: The decomposition of $s_{\lambda/\mu}$, $\lambda = (4,4,2,2)$, $\mu = (3,3)$: $\lambda =$ , $rl(\mu) = \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 6&5&4\\\hline\end{array}$.
First we establish the constraints on the position of each label in λ.

label	position (i,j) in $rl(\mu)$	position (l,m) in λ	position relative to x^- and x^+
6	$(2,1)$	$l \geq 2$ and $m \geq 3-1+1=3$	
5	$(2,2)$	$l \geq 2$ and $m \geq 3-2+1=2$	SW of 6
4	$(2,3)$	$l \geq 2$ and $m \geq 3-3+1=1$	SW of 5
3	$(1,1)$	$l \geq 1$ and $m \geq 3-1+1=3$	NE of 6
2	$(1,2)$	$l \geq 1$ and $m \geq 3-2+1=2$	SW of 3 and NE of 5
1	$(1,3)$	$l \geq 1$ and $m \geq 3-3+1=1$	SW of 2 and NE of 4



Thus Delete$[\mu]$ from $\lambda = \{(2,2,1,1),(3,2,1),(3,3)\}$. Hence $s_{\lambda/\mu} = s_{(2,2,1,1)} + s_{(3,2,1)} + s_{(3,3)}$.
Remark: The *Delete[μ] from λ* algorithm follows from the *Add[μ] to ν* algorithm and the fact that skewing is the adjoint operation of multiplication, i.e. $< s_{\lambda/\mu}, s_\nu > = < s_\lambda, s_\mu s_\nu >$.

1.1 Kronecker Product

The *Kronecker product* of homogenous symmetric polynomials is defined in terms of the Frobenius characteristic map F. Let χ_1, χ_2 be two class functions in the center of the group algebra of S_n. Then $\chi_1\chi_2$, defined by $\chi_1\chi_2(\sigma) = \chi_1(\sigma)\chi_2(\sigma)$ for all $\sigma \in S_n$, is also a class function. If $P_1 = F(\chi_1)$ and $P_2 = F(\chi_2)$, we define the Kronecker product of P_1

and P_2 by:

$$P_1 * P_2 = F(\chi_1 \chi_2).$$

The following are well-known rules for the Kronecker product:

(1) $s_{(n)} * s_\lambda = s_\lambda$

(2) $s_{(1^n)} * s_\lambda = s_{\lambda'}$, where λ' denotes the conjugate of λ.

(3) $s_\lambda * s_\mu = s_\mu * s_\lambda = s_{\lambda'} * s_{\mu'} = s_{\mu'} * s_{\lambda'}$

(4) $(P+Q) * R = P * R + Q * R$, for any symmetric homogenous polynomials P, Q, R.

(5) $(s_\lambda s_\mu) * s_\nu = \sum_{\substack{\tau \vdash |\lambda| \\ \eta \vdash |\mu|}} c^{\nu}_{\tau\,\eta} (s_\tau * s_\lambda)(s_\eta * s_\mu)$, where λ, μ, τ, η are straight shapes.

Formula (5) was proved by Littlewood [Li]. Garsia and Remmel [GR-2] used this formula to prove the following more general result:

$$(s_A s_B) * s_D = \sum_{\substack{D_1 + D_2 = D \\ |D_1| = |A|, |D_2| = |B|}} (s_A * s_{D_1})(s_B * s_{D_2}),$$

where A, B and D are skew shapes and the sum runs over all decompositions of the skew shape D. In particular, an inductive argument establishes that

$$\left(s_{(n_1)} s_{(n_2)} \cdots s_{(n_k)}\right) * s_D = \sum_{\substack{D_1 + D_2 + \cdots D_k = D \\ |D_i| = n_i}} s_{D_1} \cdots s_{D_k},$$

where the sum runs over all decompositions of D of length k such that $|D_i| = n_i$ for all i. This in turn helps in the computation of arbitrary Kronecker products using the Jacobi-Trudy identity.

Kronecker products of Schur functions, as well as Kronecker products of skew Schur functions, are homogenous symmetric functions. Thus they can be written as linear combinations of Schur functions. Since Schur functions are images of characters of symmetric group representations under the Frobenius characteristic map, it is known that the coefficients in their expansion are non-negative integers. More specifically, the coefficients are multiplicities of irreducible representations.

2 Algorithm for computing $s_{(n-p,p)} * s_\lambda$

If $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, we denote by $\bar{\mu}$ the partition $\bar{\mu} = (\mu_2, \ldots, \mu_k)$. We will follow the philosophy of [M] and work with the partition $\bar{\mu}$ instead of μ whenever possible. Knowing that $\mu \vdash n$, μ_1 is completely determined by $\bar{\mu}$.

Let p be a positive integer and λ a partition of n such that $\lambda_1 - \lambda_2 \geq 2p$. We consider the subset of partitions of p contained in λ: $S_\lambda = \{\alpha \vdash p \mid \alpha \subseteq \lambda\}$.

Algorithm: For every $\alpha \in S_\lambda$ form the following set of Young diagrams:

$$Q(\alpha) = \bigcup_{j=0}^{\alpha_1} \{\nu |\, \nu \text{ is obtained by removing a horizontal strip with } j \text{ boxes from } \alpha\}$$

$$= \bigcup_{j=0}^{\alpha_1} \text{ Delete } [(j)] \text{ from } \alpha$$

For each $\alpha \in S_\lambda$ perform the following two steps:

(1) Remove$[\alpha]$**:** For each $\delta \in Q(\alpha)$ perform *Delete[δ] from* $\bar{\lambda}$. Record all diagrams obtained from *Delete[δ] from* $\bar{\lambda}$, with multiplicity, in the multi-set $D(\alpha)$. Denote by $d_{\alpha\lambda\beta}$ the multiplicity of β in $D(\alpha)$. If $\alpha_1 > \alpha_2$, let $D'(\alpha)$ be the submulti-set of $D(\alpha)$ of diagrams obtained by performing *Delete[δ] from* $\bar{\lambda}$ whenever $\delta_1 = \alpha_1$. Denote the multiplicity of $\beta \in D'(\alpha)$ by $d'_{\alpha\lambda\beta}$. If $\alpha_1 = \alpha_2$, set $d'_{\alpha\lambda\beta} = 0$.

(2) Add$[\alpha]$**:** For each (distinct) $\beta \in D(\alpha)$,

(a) If $d'_{\alpha\lambda\beta} = 0$, then for each $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ perform *Add[γ] to* β. The multiplicity of each resulting diagram is multiplied by $d_{\alpha\lambda\beta}$.

(b) If $0 < d'_{\alpha\lambda\beta} = d_{\alpha\lambda\beta}$, then for each $\gamma \in Q(\alpha)$ perform *Add[γ] to* β. The multiplicity of each resulting diagram is multiplied by $d_{\alpha\lambda\beta}$.

(c) If $0 < d'_{\alpha\lambda\beta} < d_{\alpha\lambda\beta}$, then for each $\gamma \in Q(\alpha)$ perform *Add[γ] to* β. For each $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ the multiplicity of each resulting diagram is multiplied by $d_{\alpha\lambda\beta}$. And for each γ such that $\gamma_1 < \alpha_1$ the multiplicity of each resulting diagram is multiplied by $d'_{\alpha\lambda\beta}$.

Finally, we record all diagrams obtained in step (2), for every β, in a multi-set R_α.
Note: Whenever we perform *Delete[η] from* η, the empty diagram, denoted ϵ, will be recorded. Thus, if $\alpha = (p)$, then $\epsilon \in Q(\alpha)$. Similarly, in the **Remove**$[\alpha]$ step, if $\delta = \bar{\lambda} \in Q(\alpha)$, then $\epsilon \in D(\alpha)$.

If $\eta = (\eta_1, \ldots, \eta_{\ell(\eta)}) \in R_\alpha$, let $\tilde{\eta} = (\eta_0, \eta_1, \ldots, \eta_{\ell(\eta)})$, where $\eta_0 = n - |\eta|$. Thus $\tilde{\eta} \vdash n$.

Theorem 2.1. *Let p be a positive integer and λ a partition of n such that $\lambda_1 - \lambda_2 \geq 2p$. Then*

$$s_{(n-p,p)} * s_\lambda = \sum_{\alpha \in S_\lambda} \sum_{\eta \in R_\alpha} s_{\tilde{\eta}}.$$

We prove this theorem in the next section.

Remark: The multiplicity of each $\beta \in D(\alpha)$ is

$$d_{\alpha\lambda\beta} = \sum_{\substack{\delta \in Q(\alpha) \\ |\delta| = |\bar{\lambda}| - |\beta|}} c^{\bar{\lambda}}_{\delta\,\beta},$$

where $c^{\bar{\lambda}}_{\delta\,\beta}$ are Littlewood-Richardson coefficients.

Corollary 2.2. *The coefficient of s_ν in $s_{(n-p,p)} * s_\lambda$ is $g_{(n-p,p),\lambda,\nu} = \sum_{\alpha \in S_\lambda} c(\alpha, \lambda, \nu)$ where $c(\alpha, \lambda, \nu)$ is the multiplicity of $\bar{\nu} \in R_\alpha$.*

Example: We will perform the algorithm for $s_{(n-p,p)} * s_\lambda$ in the case when $n = 12$, $p = 3$ and $\lambda = (8, 2, 1, 1)$. Since $\lambda_1 - \lambda_2 = 8 - 2 = 6 \geq 2p$, the condition of the algorithm is satisfied. The Young diagrams for λ and $\bar{\lambda}$ are

$$\lambda = \ydiagram{8,2,1,1} \quad \text{and} \quad \bar{\lambda} = \ydiagram{2,1,1}.$$

We have $S_\lambda = \{\alpha \vdash 3 \mid \alpha \leq \lambda\} = \left\{ \ydiagram{3}\, , \ydiagram{2,1}\, , \ydiagram{1,1,1} \right\}$

$\boldsymbol{\alpha =}$ $\ydiagram{3}$: From α remove j boxes, $0 \leq j \leq 3$, no two in the same column.

$$Q(\alpha) = \{ \ydiagram{3}\, , \ydiagram{2}\, , \ydiagram{1}\, , \epsilon \}$$

(1) Remove$[\alpha]$**:** For each $\delta \in Q(\alpha)$ perform *Delete*[δ] *from* $\bar{\lambda}$.

Delete$[\begin{ytableau}3&2&1\end{ytableau}]$, *Delete*$[\begin{ytableau}2&1\end{ytableau}]$, *Delete*$[\begin{ytableau}1\end{ytableau}]$, and *Delete*$[\epsilon]$ *from* $\ydiagram{2,1,1}$. Then we have

$$D(\alpha) = \left\{ \ydiagram{1,1}\, , \ydiagram{1,1,1}\, , \ydiagram{2,1}\, , \ydiagram{2,1,1} \right\} \qquad \text{and} \qquad D'(\alpha) = \emptyset.$$

(2) Add$[\alpha]$**:** Since $D'(\alpha) = \emptyset$, we have $d'_{\alpha\lambda\beta} = 0$ for all $\beta \in D(\alpha)$. We are in case (a). The only $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ is $\gamma = \ydiagram{3}$. For every $\beta \in D(\alpha)$ we perform *Add*$[\ydiagram{3}]$ *to* β.

Add$[\begin{ytableau}3&2&1\end{ytableau}]$ *to* $\ydiagram{1,1} = \{(4, 1), (3, 1, 1)\}$;

Add$[\begin{ytableau}3&2&1\end{ytableau}]$ *to* $\ydiagram{1,1,1} = \{(4, 1, 1), (3, 1, 1, 1)\}$;

Add$[\begin{ytableau}3&2&1\end{ytableau}]$ *to* $\ydiagram{2,1} = \{(5, 1), (4, 2), (4, 1, 1), (3, 2, 1)\}$;

Add$[\begin{ytableau}3&2&1\end{ytableau}]$ *to* $\ydiagram{2,1,1} = \{(5, 1, 1), (4, 2, 1), (4, 1, 1, 1), (3, 2, 1, 1)\}$.

We take the union of these four multi-sets to get:

$R_{\ydiagram{3}} = \{(4, 1), (3, 1, 1), 2(4, 1, 1), (3, 1, 1, 1), (5, 1), (4, 2), (3, 2, 1), (5, 1, 1), (4, 2, 1),$
$(4, 1, 1, 1), (3, 2, 1, 1)\}$

$\boldsymbol{\alpha =}$ $\ydiagram{2,1}$: From α remove j boxes, $0 \leq j \leq 2$, no two in the same column.

$$Q(\alpha) = \left\{ \ydiagram{2,1}\, , \ydiagram{1,1}\, , \ydiagram{2}\, , \ydiagram{1} \right\}$$

(1) Remove$[\alpha]$**:** For each $\delta \in Q(\alpha)$ perform *Delete*[δ] *from* $\bar{\lambda}$.

Delete$[\begin{ytableau}2&1\\3\end{ytableau}]$ *from* $\ydiagram{2,1,1} = \{(1)\}$; *Delete*$[\begin{ytableau}1\\2\end{ytableau}]$ *from* $\ydiagram{2,1,1} = \{(1, 1), (2)\}$;

Delete$[\begin{ytableau}2&1\end{ytableau}]$ *from* $\ydiagram{2,1,1} = \{(1, 1)\}$; *Delete*$[\begin{ytableau}1\end{ytableau}]$ *from* $\ydiagram{2,1,1} = \{(2, 1), (1, 1, 1)\}$.

This yields:

$$D(\alpha) = \left\{ \ydiagram{1}\,, 2\,\ydiagram{1,1}\,, \ydiagram{2}\,, \ydiagram{2,1}\,, \ydiagram{1,1,1} \right\} \quad \text{and} \quad D'(\alpha) = \left\{ \ydiagram{1}\,, \ydiagram{1,1} \right\}.$$

(2) Add$[\alpha]$: If $\beta = \ydiagram{1}$, then we have $d'_{\alpha\lambda\beta} = 1 = d_{\alpha\lambda\beta}$ and we are in case (b). For each $\gamma \in Q(\alpha)$ we perform *Add[γ] to* $\ydiagram{1}$.

$Add[\begin{ytableau}2&1\\3\end{ytableau}]\ to\ \ydiagram{1} = \{(3,1),(2,2),(2,1,1)\}$; $\qquad$ $Add[\begin{ytableau}1\\2\end{ytableau}]\ to\ \ydiagram{1} = \{(2,1),(1,1,1)\}$;

$Add[\begin{ytableau}2&1\end{ytableau}]\ to\ \ydiagram{1} = \{(3),(2,1)\}$; $\qquad$ $Add[\begin{ytableau}1\end{ytableau}]\ to\ \ydiagram{1} = \{(2),(1,1)\}$.

If $\beta = \ydiagram{1,1}$, then $d'_{\alpha\lambda\beta} = 1$ and $d_{\alpha\lambda\beta} = 2$. Thus we are in case (c).

For each $\gamma \in Q(\alpha)$ we perform *Add[γ] to* $\ydiagram{1,1}$ and if $\gamma_1 = \alpha_1$ count the resulting diagrams with multiplicity $d_{\alpha\lambda\beta} = 2$.

$2 \times Add[\begin{ytableau}2&1\\3\end{ytableau}]\ to\ \ydiagram{1,1} = \{2(3,2),2(3,1,1),2(2,2,1),2(2,1,1,1)\}$;

$Add[\begin{ytableau}1\\2\end{ytableau}]\ to\ \ydiagram{1,1} = \{(2,2),(2,1,1),(1,1,1,1)\}$;

$2 \times Add[\begin{ytableau}2&1\end{ytableau}]\ to\ \ydiagram{1,1} = \{2(3,1),2(2,1,1)\}$;

$Add[\begin{ytableau}1\end{ytableau}]\ to\ \ydiagram{1,1} = \{(2,1),(1,1,1)\}$.

If $\beta = \ydiagram{2}$, then $d'_{\alpha\lambda\beta} = 0$. We are in case (a). The only $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ are $\gamma = \ydiagram{2,1}$ and $\gamma = \ydiagram{2}$.

$Add[\begin{ytableau}2&1\\3\end{ytableau}]\ to\ \ydiagram{2} = \{(4,1),(3,2),(3,1,1),(2,2,1)\}$;

$Add[\begin{ytableau}2&1\end{ytableau}]\ to\ \ydiagram{2} = \{(4),(3,1),(2,2)\}$;

If $\beta = \ydiagram{2,1}$, then $d'_{\alpha\lambda\beta} = 0$. We are in case (a). As before, the only $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ are $\gamma = \ydiagram{2,1}$ and $\gamma = \ydiagram{2}$.

$Add[\begin{ytableau}2&1\\3\end{ytableau}]\ to\ \ydiagram{2,1} = \{(4,2),(4,1,1),(3,3),2(3,2,1),(3,1,1,1),(2,2,2),(2,2,1,1)\}$;

$Add[\begin{ytableau}2&1\end{ytableau}]\ to\ \ydiagram{2,1} = \{(4,1),(3,2),(3,1,1),(2,2,1)\}$.

If $\beta = \ydiagram{1,1,1}$, then $d'_{\alpha\lambda\beta} = 0$. We are in case (a). As before, the only $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ are $\gamma = \ydiagram{2,1}$ and $\gamma = \ydiagram{2}$.

$Add[\begin{ytableau}2&1\\3\end{ytableau}]\ to\ \ydiagram{1,1,1} = \{(3,2,1),(3,1,1,1),(2,2,1,1),(2,1,1,1,1)\}$;

$Add[\begin{ytableau}2&1\end{ytableau}]\ to\ \ydiagram{1,1,1} = \{(3,1,1),(2,1,1,1)\}$.

We take the union of all the multi-sets above (from the Add step):

$$\begin{aligned} R_{\ydiagram{2,1}} = \{&4(3,1),3(2,2),4(2,1,1),3(2,1),2(1,1,1),(3),(2),(1,1),4(3,2),\\ &5(3,1,1),4(2,2,1),3(2,1,1,1),(1,1,1,1),2(4,1),(4),(4,2),(4,1,1),\\ &(3,3),3(3,2,1),2(3,1,1,1),(2,2,2),2(2,2,1,1),(2,1,1,1,1)\} \end{aligned}$$

$\boldsymbol{\alpha} = \ydiagram{1,1,1}$: From α remove j boxes, $0 \le j \le 1$, no two in the same column.

$$Q(\alpha) = \left\{ \ydiagram{1,1,1}\, , \ydiagram{1,1} \right\}$$

(1) Remove$[\alpha]$**:** For each $\delta \in Q(\alpha)$ perform *Delete[δ] from* $\bar{\lambda}$.

$Delete[\young(1,2,3)]\ from\ \ydiagram{2,1} = \{(1)\};$ $\qquad$ $Delete[\young(1,2)]\ from\ \ydiagram{2,1} = \{(2), (1,1)\}.$

This yields:

$$D(\alpha) = \left\{ \ydiagram{1}\, , \ydiagram{1,1}\, , \ydiagram{2} \right\}.$$

(2) Add$[\alpha]$**:** Since $\alpha_1 = \alpha_2$, $d'_{\alpha\lambda\beta} = 0$ for all $\beta \in D(\alpha)$. We are in case (a). For $\alpha = (1,1,1)$, all $\gamma \in Q(\alpha)$ satisfy $\gamma_1 = \alpha_1$. We perform *Add[γ] to β* for all $\gamma \in Q(\alpha)$ and all $\beta \in D(\alpha)$.

$Add[\young(1,2,3)]\ to\ \ydiagram{1} = \{(2,1,1), (1,1,1,1)\}$

$Add[\young(1,2,3)]\ to\ \ydiagram{1,1} = \{(2,2,1), (2,1,1,1), (1,1,1,1,1)\}$

$Add[\young(1,2,3)]\ to\ \ydiagram{2} = \{(3,1,1), (2,1,1,1)\}$

$Add[\young(1,2)]\ to\ \ydiagram{1} = \{(2,1), (1,1,1)\}$

$Add[\young(1,2)]\ to\ \ydiagram{1,1} = \{(2,2), (2,1,1), (1,1,1,1)\}$

$Add[\young(1,2)]\ to\ \ydiagram{2} = \{(3,1), (2,1,1)\}$

We take the union of all the multi-sets above:

$$\begin{aligned} R_{\ydiagram{1,1,1}} = \{&3(2,1,1), 2(1,1,1,1), (2,1), (1,1,1), (2,2,1), \\ &2(2,1,1,1), (1,1,1,1,1), (2,2), (3,1,1), (3,1)\} \end{aligned}$$

Finally, we use Theorem 2.1 to obtain the decomposition of $s_{(9,3)} * s_{(8,2,1,1)}$. Consider the union of the multi-sets R_α, for all $\alpha \in S_{(8,2,1,1)}$, and "complete" each shape to size 12.

Thus

$s_{(9,3)} * s_{(8,2,1,1)} = 3s_{(7,4,1)} + 7s_{(7,3,1,1)} + 3s_{(6,4,1,1)} + 3s_{(6,3,1,1,1)} + s_{(6,5,1)} + 2s_{(6,4,2)} + 4s_{(6,3,2,1)} + s_{(5,5,1,1)} + s_{(5,4,2,1)} + s_{(5,4,1,1,1)} + s_{(5,3,2,1,1)} + 5s_{(8,3,1)} + 4s_{(8,2,2)} + 7s_{(8,2,1,1)} + 4s_{(9,2,1)} + 3s_{(9,1,1,1)} + s_{(9,3)} + s_{(10,2)} + s_{(10,1,1)} + 4s_{(7,3,2)} + 5s_{(7,2,2,1)} + 5s_{(7,2,1,1,1)} + 3s_{(8,1,1,1,1)} + s_{(8,4)} + s_{(6,3,3)} + s_{(6,2,2,2)} + 2s_{(6,2,2,1,1)} + s_{(6,2,1,1,1,1)} + s_{(7,1,1,1,1,1)}$.

3 Proof of Theorem 2.1

In this section we prove Theorem 2.1, but first we establish a few facts about the multiplicities $d_{\alpha\lambda\beta}$ and $d'_{\alpha\lambda\beta}$ of the elements β in $D(\alpha)$ and $D'(\alpha)$ respectively.

As in Section 1, we denote by $c^{\mu}_{\nu\,\eta}$ the Littlewood-Richardson coefficient. If we denote by $T^{\eta}_{\mu/\nu}$ the set of semistandard Young tableaux of shape μ/ν and type η whose reverse reading word is a lattice permutation, then the cardinality of $T^{\eta}_{\mu/\nu}$ is equal to $c^{\mu}_{\nu\,\eta}$. Let $T^{\eta}_{\mu/\nu}(i,j)$ be the subset of $T^{\eta}_{\mu/\nu}$ of SSYTs of shape μ/ν and type η with label 1 in position (i,j). Note that this multi-subset could be empty. Define

$$a^{\mu}_{\nu\,\eta} := \begin{cases} |T^{\eta}_{\mu/\nu}(2,\nu_1)|, & \text{if } \mu_2 \geq \nu_1 \text{ and } \nu_1 > \nu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Remarks: (1) Since $a^{\mu}_{\nu\,\eta}$ counts SSYTs whose reverse reading word is a lattice permutation, the first row of each such tableau must be filled with 1's. Thus, if $\eta_1 \leq \mu_1 - \nu_1$, we have $a^{\eta}_{\mu\,\nu} = 0$. Moreover, if position $(2,\nu_1)$ is to be filled with 1, then all boxes in the second row of μ/ν that are to the left of position $(2,\nu_1)$ must be filled with 1's. Thus, if $\eta_1 < \mu_1 - \nu_2$, we have $a^{\mu}_{\nu\,\eta} = 0$.

(2) In general, if $\nu_1 > \nu_2$, $a^{\mu}_{\nu\,\eta}$ equals the number of SSYTs of shape μ/ν and type η whose reverse reading word is a lattice permutation and having exactly $\nu_1 - \nu_2$ boxes in the second row of μ/ν labelled 1.

Theorem 3.1. *Let p and n be positive integers such that $n \geq p$. Let $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ be a partition of n such that $\lambda_1 - \lambda_2 \geq p$ and let $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$ be a partition of p. Let $\hat{\beta} = (\beta_0, \beta_1, \ldots, \beta_{\ell(\beta)})$ be a partition of $n-p$. Denote by β the partition $\beta = (\beta_1, \beta_2, \ldots, \beta_{\ell(\beta)})$, i.e. $\beta = \bar{\hat{\beta}}$.*

(a) If $|\beta| > |\bar{\lambda}| - |\bar{\alpha}|$ or $|\beta| < |\bar{\lambda}| - p$, then

$$c^{\lambda}_{\alpha\,\hat{\beta}} = 0.$$

(b) If $|\bar{\lambda}| - p \leq |\beta| \leq |\bar{\lambda}| - |\bar{\alpha}|$, then

$$c^{\lambda}_{\alpha\,\hat{\beta}} = \sum_{\substack{\delta \in Q(\alpha);\ \delta \subseteq \bar{\lambda} \\ |\delta| = |\bar{\lambda}| - |\beta|}} c^{\bar{\lambda}}_{\delta\,\beta}.$$

Proof: (a) If $|\beta| > |\bar{\lambda}| - |\bar{\alpha}|$, then $\beta_0 = n - p - |\beta| < n - p - |\bar{\lambda}| + |\bar{\alpha}| = n - |\bar{\lambda}| - (p - |\bar{\alpha}|) = \lambda_1 - \alpha_1$. Since $\beta_0 < \lambda_1 - \alpha_1$, there are not enough 1's to create any SSYT of shape λ/α and type $\hat{\beta}$ (the first row of λ/α has length $\lambda_1 - \alpha_1$ and needs to be filled with 1's).

If $|\beta| < |\bar{\lambda}| - p$, then $\beta_0 = n - p - |\beta| > n - p - |\bar{\lambda}| + p = n - |\bar{\lambda}| = \lambda_1$. Since $\beta_0 > \lambda_1$, there are too many 1's. The maximum number of 1's, in creating a SSYT of shape λ/α and type $\hat{\beta}$ is λ_1 (since the numbers in the filling of the SSYT must increase strictly in the columns).

In both cases we obtain $c^{\lambda}_{\alpha\,\hat{\beta}} = 0$.

(b) Assume $|\bar{\lambda}| - p \leq |\beta| \leq |\bar{\lambda}| - |\bar{\alpha}|$. Consider the skew diagram λ/α. To determine $c^{\lambda}_{\alpha\,\hat{\beta}}$ we need to find the number of SSYTs of shape λ/α and type $\hat{\beta}$ whose reverse reading word is a lattice permutation. Thus, in the skew-shape λ/α we need to fill β_0 places with 1's, β_1 places with 2's, ..., and $\beta_{\ell(\beta)}$ places with $(\ell(\beta)+1)$'s.

In filling β_0 places with 1's one is forced to place $\lambda_1 - \alpha_1$ of them in the first row of λ/α. There are now $\beta_0 - \lambda_1 + \alpha_1 = n - p - |\beta| - \lambda_1 + \alpha_1 = |\bar{\lambda}| - |\bar{\alpha}| - |\beta| \geq 0$ remaining 1's. Since we are to obtain SSYTs, the remaining 1's need to be placed at the beginning (left end) of the remaining rows of λ/α (i.e. at the beginning of the rows of $\bar{\lambda}/\bar{\alpha}$) such that no two 1's are placed directly under each other. Thus, there is a one-to-one correspondence between the possible ways of filling the 1's in λ/α and the Young diagrams $\eta \subseteq \bar{\lambda}$ obtained from $\bar{\alpha}$ by adding $|\bar{\lambda}| - |\bar{\alpha}| - |\beta|$ boxes, no two in the same column and no more that $\alpha_1 - \alpha_2$ boxes in the first row of $\bar{\alpha}$.

After all 1's have been placed, completing the SSYT of shape λ/α and type $\hat{\beta}$ such that the reverse reading word is a lattice permutation is equivalent to determining the SSYT of shape $\bar{\lambda}/\eta$ and type β whose reverse reading word is a lattice permutation for the corresponding η: each label i ($i = 1, 2, \ldots, \ell(\beta)$) in the SSYT of shape $\bar{\lambda}/\eta$ and type β corresponds to a label $i+1$ in the SSYT of shape λ/α and type $\hat{\beta}$. The maximum number of 2's that can be placed in λ/α equals λ_2. Since $\lambda_1 - \lambda_2 \geq p$, we have $\lambda_1 - \alpha_1 \geq \lambda_2$ and therefore the total number of 2's in λ/α is less or equal than the number of 1's in the first row. If the SSYT of shape $\bar{\lambda}/\eta$ and type β is such that the reverse reading word is a lattice permutation, then the same will be true of the SSYT of shape λ/α and type $\hat{\beta}$.

Claim: The diagrams $\eta \subseteq \bar{\lambda}$ obtained above by adding $|\bar{\lambda}| - |\bar{\alpha}| - |\beta|$ boxes to $\bar{\alpha}$, no two in the same column and no more that $\alpha_1 - \alpha_2$ boxes in the first row of $\bar{\alpha}$, are precisely the elements of $\delta \in Q(\alpha)$ of size $|\delta| = |\bar{\lambda}| - |\beta|$, such that $\delta \subseteq \bar{\lambda}$.

Proof of the claim: Let $\eta \subseteq \bar{\lambda}$ be a Young diagram obtained from $\bar{\alpha}$ by adding $|\bar{\lambda}| - |\bar{\alpha}| - |\beta|$ boxes, no two in the same column and no more than $\alpha_1 - \alpha_2$ boxes in the first row of $\bar{\alpha}$. Then $\eta \vdash |\bar{\lambda}| - |\beta|$, $\bar{\alpha} = (\alpha_2, \alpha_3, \ldots, \alpha_{\ell(\alpha)}) \subseteq \eta = (\eta_1, \eta_2, \ldots, \eta_{\ell(\eta)})$ and $\ell(\alpha) - 1 = \ell(\bar{\alpha}) \leq \ell(\eta) \leq \ell(\bar{\alpha}) + 1 = \ell(\alpha)$.

Since $\bar{\alpha} \subseteq \eta$, we have $\alpha_{j+1} \leq \eta_j$ for all $j = 1, 2, \ldots, \ell(\alpha) - 1$. Since no two boxes are added to the same column, we have $\eta_j \leq \alpha_j$ for all $j = 2, 3, \ldots, \ell(\alpha)$. Since no more than $\alpha_1 - \alpha_2$ boxes are added to the first row of $\bar{\alpha}$, we have $\eta_1 \leq \alpha_2 + \alpha_1 - \alpha_2 = \alpha_1$. Thus, $\alpha_{j+1} \leq \eta_j \leq \alpha_j$ for all $j = 1, 2, \ldots, \ell(\alpha) - 1$ and $\eta_{\ell(\alpha)} \leq \alpha_{\ell(\alpha)}$ ($\eta_{\ell(\alpha)}$ might be zero).

The diagram η is exactly the Young diagram δ obtained from α by removing, for each $j = 1, 2, \ldots, \ell(\alpha)$, $\alpha_j - \eta_j \geq 0$ boxes from the j-th row of α. Since, for each $j = 1, 2, \ldots, \ell(\alpha) - 1$, $\alpha_{j+1} \leq \eta_j = \alpha_j - (\alpha_j - \eta_j)$, no two boxes have been removed from the same column and thus $\delta \in Q(\alpha)$.

Conversely, if $\delta \in Q(\alpha)$, $|\delta| = |\bar{\lambda}| - |\beta|$, $\delta \subseteq \bar{\lambda}$, then $\delta_j \leq \alpha_j$ for all $j = 1, 2 \ldots, \ell(\alpha)$ and, since no two boxes are removed from the same column of α, $\delta_j \geq \alpha_{j+1}$ for all $j = 1, 2 \ldots, \ell(\alpha) - 1$. Moreover, $\ell(\alpha) - 1 \leq \ell(\delta) \leq \ell(\alpha)$.

The diagram δ is exactly the Young diagram η obtained from $\bar{\alpha}$ by adding, for each $j = 1, 2 \ldots, \ell(\alpha) - 1$, $\delta_j - \alpha_{j+1} \geq 0$ boxes to the j-th row of $\bar{\alpha}$ (i.e. the $j+1$-st row of α) and $\delta_{\ell(\alpha)}$ boxes below the last row of $\bar{\alpha}$. Note that $\delta_{\ell(\alpha)}$ might be zero.

Thus δ is in fact a Young diagram η obtained from $\bar{\alpha}$ by adding $\sum_{j=1}^{\ell(\alpha)-1}(\delta_j - \alpha_{j+1}) + \delta_{\ell(\alpha)} = |\bar{\lambda}| - |\beta| - |\bar{\alpha}|$ boxes. Since $\alpha_{j+1} + (\delta_j - \alpha_{j+1}) = \delta_j \leq \alpha_j$, no two boxes have been added to the same column. Since $\delta_1 = \alpha_2 + (\delta_1 - \alpha_2)$ and $\delta_1 \leq \alpha_1$, no more than $\alpha_1 - \alpha_2$

boxes have been added to the first row of $\bar{\alpha}$. $\square$

Corollary 3.2. *The Littlewood-Richardson coefficient* $c^{\lambda}_{\alpha\,\hat{\beta}}$ *is equal to* $d_{\alpha\lambda\beta}$*, the multiplicity of* β *in* $D(\alpha)$*.*

Proof: Since $\delta \in Q(\alpha)$ is such that $|\bar{\alpha}| = p - \alpha_1 \leq |\delta| \leq p$ and $\beta \in D(\alpha)$ is obtained by performing *Delete[δ] from* $\bar{\lambda}$, the size of β must satisfy $|\bar{\lambda}| - p \leq |\beta| \leq |\bar{\lambda}| - |\bar{\alpha}|$. Thus, if $|\beta| > |\bar{\lambda}| - |\bar{\alpha}|$ or $|\beta| < |\bar{\lambda}| - p$, then β does not occur in $D(\alpha)$.

If $|\bar{\lambda}| - p \leq |\beta| \leq |\bar{\lambda}| - |\bar{\alpha}|$, the corollary follows from Theorem 3.1 and the Remark after Theorem 2.1. $\square$

Theorem 3.3. *With the notation of Theorem 3.1,*

(a) If $|\beta| > |\bar{\lambda}| - p + \alpha_2$ *or* $|\beta| < |\bar{\lambda}| - p$*, then*

$$a^{\lambda}_{\alpha\,\hat{\beta}} = 0.$$

(b) If $|\bar{\lambda}| - p \leq |\beta| \leq |\bar{\lambda}| - p + \alpha_2$*, then*

$$a^{\lambda}_{\alpha\,\hat{\beta}} = \sum_{\substack{\delta \in Q(\alpha);\ \delta \subseteq \bar{\lambda} \\ |\delta| = |\bar{\lambda}| - |\beta| \\ \delta_1 = \alpha_1}} c^{\bar{\lambda}}_{\delta\,\beta}$$

Proof: (a) Since position $(2, \alpha_1)$ must be filled with 1, the entire first row and exactly $\alpha_1 - \alpha_2$ boxes in the second row of λ/α must be filled with 1's. Thus $\beta_0 = n - p - |\beta|$ must be at least $\lambda_1 - \alpha_1 + \alpha_1 - \alpha_2 = \lambda_1 - \alpha_2$. Therefore, we must have $|\beta| \leq |\bar{\lambda}| - p + \alpha_2$. If $|\beta| < |\bar{\lambda}| - p$, the result follows from Theorem 3.1.

(b) The proof of this part of the theorem is similar to the proof of Theorem 3.1, part (b). We are forced to place $\lambda_1 - \alpha_1$ 1's in the first row, and when placing the remaining $|\bar{\lambda}| - |\bar{\alpha}| - |\beta|$ 1's we are forced to place exactly $\alpha_1 - \alpha_2$ of them in the second row.

Thus, there is a one-to-one correspondence between the possible ways of filling the 1's in λ/α and the Young diagrams $\eta \subseteq \bar{\lambda}$ obtained from $\bar{\alpha}$ by adding $|\bar{\lambda}| - |\bar{\alpha}| - |\beta|$ boxes, no two in the same column and exactly $\alpha_1 - \alpha_2$ boxes in the first row of $\bar{\alpha}$.

The claim of Theorem 3.1 becomes:

Claim: The diagrams $\eta \subseteq \bar{\lambda}$ obtained above by adding $|\bar{\lambda}| - |\bar{\alpha}| - |\beta|$ boxes to $\bar{\alpha}$, no two in the same column and exactly $\alpha_1 - \alpha_2$ boxes in the first row of $\bar{\alpha}$, are precisely the elements of $\delta \in Q(\alpha)$ of size $|\delta| = |\bar{\lambda}| - |\beta|$, such that $\delta \subseteq \bar{\lambda}$ and $\delta_1 = \alpha_1$ (i.e. no boxes have been removed from the first row of α).

The proof of the claim in this case is similar to that of the claim of Theorem 3.1. $\square$

Corollary 3.4. *If* $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}) \vdash p$*, with* $\alpha_1 > \alpha_2$*, let* $D'(\alpha)$ *be defined as in the algorithm. Then* $a^{\lambda}_{\alpha\,\hat{\beta}}$ *is equal to* $d'_{\alpha\lambda\beta}$*, the multiplicity of* β *in* $D'(\alpha)$*.*

Lemma 3.5. *Let n, p be non-negative integers such that $n \geq 2p$. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_{\ell(\beta)})$ be a partition such that $\beta_1 \leq n - 2p - |\beta|$ and suppose that $\alpha \vdash p$. As before, let $\hat{\beta} = (\beta_0, \beta_1, \ldots, \beta_{\ell(\beta)}) \vdash n-p$ be such that $\beta = \bar{\hat{\beta}}$. Thus $\beta_0 - \beta_1 \geq p$. Then performing* Add$[\alpha]$ *to $\hat{\beta}$ is equivalent to performing* Add$[\gamma]$ *to β for each $\gamma \in Q(\alpha)$ and then adding a row of the correct size to the resulting diagrams.*

Proof: To perform the *Add[α] to $\hat{\beta}$* algorithm we form $rl(\alpha)$, the reverse lexicographic filling of α, and add the labelled boxes of $rl(\alpha)$ to the rows of $\hat{\beta}$ according to the *Add* algorithm. The only labels that could be added to the first row of $\hat{\beta}$ are the labels of the first row of α. If exactly j, $0 \leq j \leq \alpha_1$, labels of the first row of α are added to the first row of $\hat{\beta}$, then these labels are $\alpha_1 - j + 1, \alpha_1 - j + 2, \ldots, \alpha_1 - 1, \alpha_1$. Notice that since $\beta_0 - \beta_1 \geq p$, no more than $\beta_0 - \beta_1$ boxes are added to the first row of β. Adding the remaining labels of α to $\hat{\beta}$ is equivalent to adding the labels of the reverse lexicographic filling of $\alpha/(j)$ to β according to the rules of the *Add* algorithm. By Lemma 1.1 this gives the decomposition of $s_{\alpha/(j)} s_\beta$. One can first straighten $\alpha/(j)$ using the *Delete* algorithm. Letting j run from 0 to α_1, we have that performing *Add[α] to $\hat{\beta}$* is equivalent to performing *Add[γ] to β* for each $\gamma \in Q(\alpha)$ and adding a first row of the correct size to each diagram to obtain diagrams of size n. $\square$

Proof of Theorem 2.1

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a fixed partition of n such that $\lambda_1 - \lambda_2 \geq 2p$ (thus $n \geq 2p$). First, we expand $s_{(n-p,p)} * s_\lambda$ using the Jacobi-Trudi identity and the Garsia-Remmel formula from Section 1.

By the Jacobi-Trudi identity, we have

$$s_{(n-p,p)} = \begin{vmatrix} h_{n-p} & h_{n-p+1} \\ h_{p-1} & h_p \end{vmatrix} = h_{n-p}h_p - h_{n-p+1}h_{p-1}.$$

If $p = 1$, then $h_{p-1} = h_0 = 1$, by convention.

Thus, $s_{(n-p,p)} * s_\lambda = (h_{n-p}h_p - h_{n-p+1}h_{p-1}) * s_\lambda = (h_{n-p}h_p) * s_\lambda - (h_{n-p+1}h_{p-1}) * s_\lambda$ and, by Garsia-Remmel, we have

$$s_{(n-p,p)} * s_\lambda = \sum_{\substack{D_1 + D_2 = \lambda \\ |D_2| = n-p \\ |D_1| = p}} s_{D_1} s_{D_2} - \sum_{\substack{L_1 + L_2 = \lambda \\ |L_2| = n-p+1 \\ |L_1| = p-1}} s_{L_1} s_{L_2}.$$

Since λ is a shape (and not a skew-shape), D_1 and L_1 must be shapes. We write $D_1 = \alpha \vdash p$ and $L_1 = \beta \vdash p - 1$. Therefore

$$s_{(n-p,p)} * s_\lambda = \sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda}} s_\alpha s_{\lambda/\alpha} - \sum_{\substack{\beta \vdash p-1 \\ \beta \subseteq \lambda}} s_\beta s_{\lambda/\beta}.$$

There is a $1 - 1$ correspondence between the partitions $\alpha \vdash p$ with $\alpha_1 > \alpha_2$ and the partitions $\alpha^- \vdash p - 1$ given by

$$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_{\ell(\alpha)}) \longleftrightarrow \alpha^- = (\alpha_1 - 1, \alpha_2, \ldots \alpha_{\ell(\alpha)}).$$

The Young diagram of α^- is obtained from the Young diagram of α by removing the last box of the first row. Hence, we have

$$s_{(n-p,p)} * s_\lambda = \sum_{\substack{\alpha\vdash p,\alpha_1>\alpha_2\\ \alpha\subseteq\lambda}} (s_\alpha s_{\lambda/\alpha} - s_{\alpha^-}s_{\lambda/\alpha^-}) + \sum_{\substack{\alpha\vdash p,\alpha_1=\alpha_2\\ \alpha\subseteq\lambda}} s_\alpha s_{\lambda/\alpha}.$$

The symmetric function $s_\alpha s_{\lambda/\alpha} - s_{\alpha^-}s_{\lambda/\alpha^-}$ above is Schur positive (i.e. when expanded as a linear combination of Schur functions, all the coefficients are positive) [BO].

Now, one has to straighten each $s_{\lambda/\alpha}$ and s_{λ/α^-} using the *Delete[α] from* λ (respectively *Delete[α⁻] from* λ) algorithm and then use the *Add* algorithm to multiply each summand by the corresponding s_α (respectively s_{α^-}). Finally, in the first sum, one would cancel out all negative summands with the corresponding positive summands. However, we will figure out how the cancellations will take place and skip these shapes in the *Delete* and *Add* algorithms, thus eliminating the redundant work.

When we compute $s_{\lambda/\alpha}$ for $\alpha\vdash p$ and $\lambda_1-\lambda_2\geq 2p$ using the *Delete[α] from* λ *algorithm* we obtain a multi-set of Young diagrams with $n-p$ boxes. We will show that there is a one-to-one correspondence between the multi-sets:

$$\textit{Delete}[\alpha] \textit{ from } \lambda \longleftrightarrow D(\alpha),$$

where $D(\alpha)$ is the multi-set obtained in the **Remove** $[\alpha]$ step in our algorithm. By the proof of Theorem 3.1, to each $\beta\in D(\alpha)$ there corresponds a partition $\hat\beta$ of $n-p$, $\hat\beta=(\beta_0,\beta_1,\ldots,\beta_{\ell(\beta)})\in$ *Delete[α] from* λ. The bijection is given by adding a row β_0 to β so that $\hat\beta$ is a partition of $n-p$. By Corollary 3.2, the multiplicity of $\hat\beta$ in Delete$[\alpha]$ from λ is the same as the multiplicity of β in $D(\alpha)$. Hence, performing the *Delete [α] from* λ algorithm is equivalent to performing the **Remove** $[\alpha]$ step.

First we compare the results of straightening both λ/α and λ/α^-. If $\alpha_1>\alpha_2$ in α, then $Q(\alpha)-Q(\alpha^-)=\{\delta\in Q(\alpha)\,|\,\delta_1=\alpha_1\}$. To see this consider $\delta=(\alpha_1,\alpha_2-t_2,\ldots,\alpha_{\ell(\alpha)}-t_{\ell(\alpha)})\in Q(\alpha)$, $t_i\geq 0$, $i=2,3,\ldots,\ell(\alpha)$ (no box has been removed from the first row of α). Then $\delta\notin Q(\alpha^-)$. On the other hand, if $\delta=(\alpha_1-t_1,\alpha_2-t_2,\ldots,\alpha_{\ell(\alpha)}-t_{\ell(\alpha)})\in Q(\alpha)$, with $t_1\geq 1$ and $t_i\geq 0$, $i=2,3,\ldots,\ell(\alpha)$, then $\delta=(\alpha_1-1-(t_1-1),\alpha_2-t_2,\ldots,\alpha_{\ell(\alpha)}-t_{\ell(\alpha)})\in Q(\alpha^-)$. Conversely, if $\delta'\in Q(\alpha^-)$ is given by $\delta'=(\alpha_1-1-j_1,\alpha_2-j_2,\ldots,\alpha_{\ell(\alpha)}-j_{\ell(\alpha)})$ with $j_i\geq 0$ for all $i=1,2,\ldots,\ell(\alpha)$, then $\delta'\in Q(\alpha)$. If $\alpha_1=\alpha_2$ then all $\delta\in Q(\alpha)$ satisfy $\delta_1=\alpha_1$ since one cannot remove two boxes from the same column of α.

Since $Q(\alpha)-Q(\alpha^-)=\{\delta\in Q(\alpha)\;\;\delta_1=\alpha_1\}$, we have $D(\alpha)-D(\alpha^-)=D'(\alpha)$. Let $\hat\beta'\vdash n-p+1$ be a diagram obtained when performing *Delete[α⁻] from* λ. The Young diagram $\hat\beta'$ is obtained from a diagram $\beta\in D(\alpha^-)$ by adding a first row such that $|\hat\beta'|=n-p+1$. Since $D(\alpha^-)\subseteq D(\alpha)$, $\beta\in D(\alpha)$. If we add a first row to β to obtain $\hat\beta\vdash n-p$, then $\hat\beta$ appears when we perform *Delete[α] from* λ. The diagram $\hat\beta'=(\beta_0,\beta_1,\ldots,\beta_{\ell(\beta)})$, obtained when performing *Delete[α⁻] from* λ, corresponds to the diagram $\hat\beta=(\beta_0-1,\beta_1,\ldots,\beta_{\ell(\beta)})$ obtained when performing *Delete[α] from* λ. Note that $|\beta|\leq|\bar\lambda|=n-\lambda_1$. Therefore $\beta_0=n-p+1-|\beta|\geq n-p+1-n+\lambda_1=\lambda_1-p+1$. On the other hand, since $\lambda_1-\lambda_2\geq 2p$, we have $\beta_1\leq\lambda_2\leq\lambda_1-2p\leq\lambda_1-p+1\leq\beta_0$. Thus

$\beta \in D(\alpha^-)$ can always be completed to a Young diagram of size $n-p+1$ by adding a first row of the correct size.

Now we will address the *Add* algorithm. On the one hand we have to multiply

$$s_\alpha s_{\lambda/\alpha} = s_\alpha \sum_{\beta \in D(\alpha)} s_{\hat{\beta}} = \sum_{\beta \in D(\alpha)} s_{\hat{\beta}} s_\alpha.$$

On the other hand we will multiply

$$s_{\alpha^-} s_{\lambda/\alpha^-} = s_{\alpha^-} \sum_{\beta \in D(\alpha^-)} s_{\hat{\beta}'} = \sum_{\beta \in D(\alpha^-)} s_{\hat{\beta}'} s_{\alpha^-}.$$

Consider the product $s_{\hat{\beta}} s_\alpha$, where, as before, $\alpha \vdash p$ and $\hat{\beta} \vdash n-p$ such that $\beta = \bar{\hat{\beta}} \in D(\alpha)$. Since $\hat{\beta} = (\beta_0, \beta_1, \ldots, \beta_\ell)$ is the result of applying *Delete[α] from* λ and $\lambda_1 - \lambda_2 \geq 2p$, we have that $\beta_0 - \beta_1 \geq \lambda_1 - \alpha_1 - \lambda_2 \geq \lambda_1 - \lambda_2 - p \geq p$. Thus $\beta_1 \leq \beta_0 - p = n - p - |\beta| - p = n - 2p - |\beta|$. By Lemma 3.5, performing *Add[α] to* $\hat{\beta}$ is equivalent to performing *Add[γ] to* β for each $\gamma \in Q(\alpha)$ and adding a first row of the correct size to each diagram to obtain diagrams of size n.

We are finally able to decide which Schur functions remain after the cancellation. For every $\beta \in D(\alpha^-)$, since $D(\alpha^-) \subseteq D(\alpha)$, after performing *Add[$\alpha$] to* $\hat{\beta}$ and *Add[α^-] to* $\hat{\beta}'$, the only shapes that do not cancel are shapes obtained from *Add[γ] to* β for $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ (i.e $\gamma \in Q(\alpha) - Q(\alpha^-)$). For every $\beta \in D'(\alpha) = D(\alpha) - D(\alpha^-)$, after the *Add* algorithm is performed, all shapes remain, i.e. the shapes obtained from *Add[γ] to* β for each $\gamma \in Q(\alpha)$.

(1) Case $\alpha_1 > \alpha_2$:

(a) If $d'_{\alpha\lambda\beta} = 0$, β does not appear in $D'(\alpha)$ and thus we perform *Add[γ] to* β only for $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$. This is case (a) in part (2) of the algorithm.

(b) If $0 < d'_{\alpha\lambda\beta} = d_{\alpha\lambda\beta}$, β appears only in $D(\alpha)$ (and not in $D(\alpha^-)$). We perform *Add[γ] to* β for each $\gamma \in Q(\alpha)$. This is case (b) in part (2) of the algorithm.

(c) If $0 < d'_{\alpha\lambda\beta} < d_{\alpha\lambda\beta}$, β appears in both $D'(\alpha)$ and $D(\alpha^-)$. Since $\beta \in D'(\alpha)$, we perform *Add[γ] to* β for each $\gamma \in Q(\alpha)$. Since $\beta \in D(\alpha^-)$, we perform *Add[γ] to* β for all $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$. This is case (c) in part (2) of the algorithm.

(2) Case $\alpha_1 = \alpha_2$:

In this case, by definition $d'_{\alpha\lambda\beta} = 0$ and according to the algorithm we are in case (a) of part (2). On the other hand, if $\alpha_1 = \alpha_2$ all diagrams $\delta \in Q(\alpha)$ satisfy $\delta_1 = \alpha_1$ and thus $D'(\alpha) = D(\alpha)$. For each $\beta \in D(\alpha)$ we need to perform *Add[γ] to* β for each $\gamma \in Q(\alpha)$. Since all $\delta \in Q(\alpha)$ satisfy $\delta_1 = \alpha_1$, following case (a) in part (2) of the algorithm gives the correct answer. $\square$

Let us show that the condition $\lambda_1 - \lambda_2 \geq 2p$ is indeed necessary: Since $\lambda_1 - \lambda_2 \geq p$ (required in order for the **Remove**[α] step to give the diagrams occurring in λ/α), we have $\lambda_1 \geq p$. If $\lambda_1 = p$ and $\lambda_1 - \lambda_2 \geq p$, then $\lambda_2 = 0$ and $\lambda = (p)$. This would imply that $n = p$ which is impossible. Thus $\lambda_1 > p$. Then S_λ contains $\alpha = (p)$ and $Q((p))$ contains ϵ (the

empty diagram). When we perform *Delete[ϵ] from* $\bar{\lambda}$ we obtain $\beta = \bar{\lambda} \in D((p))$. For $\beta = \bar{\lambda}$ we have $\hat{\beta} = (\lambda_1 - p, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \vdash n - p$. This is indeed a partition since $\lambda_1 - p \geq \lambda_2$. When we perform *Add[(p)] to* $\bar{\lambda}$ one of the obtained diagrams is $\eta = (\lambda_2 + p, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$. To complete this diagram to a Young diagram $\tilde{\eta} = (\lambda_1 - p, \lambda_2 + p, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ of size n, we must have $\lambda_1 - p \geq \lambda_2 + p$ and thus $\lambda_1 - \lambda_2 \geq 2p$.
Note: The theorem holds even if $\lambda_1 - \lambda_2 = 2p - 1$. However, in this case one obtains the shape $\eta = (\lambda_2 + p, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ which does not complete to a Young diagram of size n and needs to be discarded.

4 Multiplicities in the Kronecker Product

In this section we give a closed formula for the coefficients $g_{(n-p,p),\lambda,\nu}$ in terms of Littlewood-Richardson coefficients.

Theorem 4.1. *Let n and p be positive integers such that $n \geq 2p$ and let λ be a partition of n with $\lambda_1 - \lambda_2 \geq 2p$. The multiplicity of s_ν in $s_{(n-p,p)} * s_\lambda$ is equal to*

$$\sum_{\substack{\beta \subseteq \bar{\lambda},\, \beta \subseteq \bar{\nu} \\ |\beta| \geq n - \lambda_1 - p}} \sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda}} \left(\sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 = \alpha_1,\, \gamma \subseteq \bar{\nu} \\ |\gamma| = |\bar{\nu}| - |\beta|}} c^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} + \sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 < \alpha_1,\, \gamma \subseteq \bar{\nu} \\ |\gamma| = |\bar{\nu}| - |\beta|}} a^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} \right),$$

where $\hat{\beta} = (n - p - |\beta|, \beta)$.

Proof: We use Corollaries 3.2 and 3.4 and the Remmel-Whitney result that the multiplicity of $\bar{\nu}$ obtained by performing *Add[γ] to β* equals $c^{\bar{\nu}}_{\beta\gamma}$. Then, Theorem 2.1 implies that

$$\begin{aligned} g_{(n-p,p),\lambda,\nu} &= \sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda}} \left(\sum_{\substack{\beta \in D(\alpha) \\ a^{\lambda}_{\alpha\hat{\beta}} = 0}} \left(\sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 = \alpha_1}} c^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} \right) + \sum_{\substack{\beta \in D(\alpha) \\ a^{\lambda}_{\alpha\hat{\beta}} \neq 0}} \left(\sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 = \alpha_1}} c^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} + \sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 < \alpha_1}} a^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} \right) \right) \\ &= \sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda}} \sum_{\substack{\beta \in D(\alpha) \\ \beta \subseteq \bar{\nu}}} \left(\sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 = \alpha_1}} c^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} + \sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 < \alpha_1}} a^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} \right) \end{aligned}$$

The second summation, where $\beta \in D(\alpha)$, runs over all *distinct* $\beta \in D(\alpha)$. The formula of the theorem is just a reformulation of the last formula. We use the statements of Theorems 3.1 and 3.3 that if $|\beta| < |\bar{\lambda}| - p = n - \lambda_1 - p$, then $c^{\lambda}_{\alpha\hat{\beta}} = a^{\lambda}_{\alpha\hat{\beta}} = 0$. $\square$

Example: We use the above theorem to determine the multiplicity of $s_{(13,4,2)}$ in the Kronecker product $s_{(15,4)} * s_{(11,3,2,2,1)}$.

We have $n = 19$, $p = 4$, $\bar{\lambda} = (3, 2, 2, 1)$ and $\bar{\nu} = (4, 2)$, i.e

$$\bar{\lambda} = \ydiagram{3,2,2,1}, \qquad \bar{\nu} = \ydiagram{4,2}.$$

Since $n - \lambda_1 - p = 19 - 11 - 4 = 4$, the first summation in the formula of Theorem 4.1 runs over all Young diagrams β such that $|\beta| \geq 4$, $\beta \subseteq \bar{\lambda}$ and $\beta \subseteq \bar{\nu}$. Thus β has at most two rows: $\beta = (\beta_1, \beta_2)$ with $\beta_1 \leq 3$ and $\beta_2 \leq 2$. The possible β's in the first summation are

$$\ydiagram{3,1}, \quad \ydiagram{3,2}, \quad \ydiagram{2,2}.$$

The second summation runs over all Young diagrams α of size $p = 4$ with $\alpha \subseteq \lambda$. They are the elements of

$$S_\lambda = \left\{ \ydiagram{4}, \ \ydiagram{3,1}, \ \ydiagram{2,2}, \ \ydiagram{2,1,1}, \ \ydiagram{1,1,1,1} \right\}.$$

(1) If $\beta = \ydiagram{3,1}$, then $\hat{\beta} = (11, 3, 1) \vdash n - p = 15$. For each α, the inner sums will run over all $\gamma \in Q(\alpha)$ with $|\gamma| = |\bar{\nu}| - |\beta| = 6 - 4 = 2$.

If $\alpha = \ydiagram{2,2}$, then the only SSYT of shape λ/α and type $\hat{\beta} = (11, 3, 1)$ is $\ytableaushort{\none\none111111111,\none\none2,11,22,3}$.

Thus $c^{\lambda}_{\alpha\,\hat{\beta}} = 1$ and, since $\alpha_1 = \alpha_2$, we have $a^{\lambda}_{\alpha\,\hat{\beta}} = 0$. The only $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ is $\gamma = \ydiagram{2}$. There is one SSYT of shape $\bar{\nu}/\beta$ and type $\gamma = (2)$: $\ytableaushort{\none\none\none1,\none1}$. Therefore $c^{\bar{\nu}}_{\beta\,\gamma} = 1$. Hence, $c^{\lambda}_{\alpha\,\hat{\beta}} c^{\bar{\nu}}_{\beta\,(2)} = 1$ *This contributes* 1 *to the multiplicity.*

If $\alpha = \ydiagram{2,1,1}$, then the only SSYT of shape λ/α and type $\hat{\beta} = (11, 3, 1)$ is $\ytableaushort{\none\none111111111,\none12,\none2,13,2}$.

Thus $c^{\lambda}_{\alpha\,\hat{\beta}} = a^{\lambda}_{\alpha\,\hat{\beta}} = 1$. The only $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ is $\gamma = \ydiagram{1,1}$. There is one SSYT of shape $\bar{\nu}/\beta$ and type $\gamma = (1, 1)$: $\ytableaushort{\none\none\none1,\none2}$. Therefore $c^{\bar{\nu}}_{\beta\,\gamma} = 1$. Hence, $c^{\lambda}_{\alpha\,\hat{\beta}} c^{\bar{\nu}}_{\beta\,(1,1)} = 1$. *This contributes* 1 *to the multiplicity.*
For all other $\alpha \in S_\lambda$ we have $c^{\lambda}_{\alpha\,\hat{\beta}} = a^{\lambda}_{\alpha\,\hat{\beta}} = 0$. Hence, they do not contribute to the multiplicity.

(2) If $\beta = \ydiagram{3,2}$, then $\hat{\beta} = (10, 3, 2) \vdash n - p = 15$. For each α, the inner sums will run over all $\gamma \in Q(\alpha)$ with $|\gamma| = |\bar{\nu}| - |\beta| = 6 - 5 = 1$.
If $\alpha = \ydiagram{4}$ then $c^{\lambda}_{\alpha\,\hat{\beta}} = a^{\lambda}_{\alpha\,\hat{\beta}} = 0$.
If $\alpha = \ydiagram{3,1}$, $\ytableaushort{\none\none\none11111111,\none12,12,23,3}$ is the only SSYT of shape λ/α and type $\hat{\beta} = (10, 3, 2)$. Thus $c^{\lambda}_{\alpha\,\hat{\beta}} = 1$ and $a^{\lambda}_{\alpha\,\hat{\beta}} = 0$. Since $\alpha_1 = 3$, there is no $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ and $|\gamma| = 1$.
If $\alpha = \ydiagram{2,2}$, $\alpha = \ydiagram{2,1,1}$ or $\alpha = \ydiagram{1,1,1,1}$, there is no $\gamma \in Q(\alpha)$ with $|\gamma| = 1$.

(3) Finally, if $\beta = \ydiagram{2,2}$, then $\hat{\beta} = (11, 2, 2) \vdash n - p = 15$. For each α, the inner sums will run over all $\gamma \in Q(\alpha)$ with $|\gamma| = |\bar{\nu}| - |\beta| = 6 - 4 = 2$.

If $\alpha = \ydiagram{3,1}$, $\ytableaushort{\none\none\none 11111111, \none 11, 12, 23, 3}$ is the only SSYT of shape λ/α and type $\hat\beta = (11, 2, 2)$. Thus $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 1$. The shapes $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ are $\gamma = \ydiagram{2}$ and $\gamma = \ydiagram{1,1}$. There is exactly one SSYT of shape $\bar\nu/\beta$ and type $\gamma = (2)$. Thus, for $\gamma = (2)$, $c^{\bar\nu}_{\beta\,\gamma} = 1$. Hence, $c^{\lambda}_{\alpha\,\hat\beta} c^{\bar\nu}_{\beta\,(2)} = 1$. *This contributes* 1 *to the multiplicity.* We also have $c^{\bar\nu}_{\beta\,(1,1)} = 0$.

If $\alpha = \ydiagram{2,2}$, then $\ytableaushort{\none\none 111111111, \none\none 2, 11, 23, 3}$ is the only SSYT of shape λ/α and type $\hat\beta = (11, 2, 2)$. Thus $c^{\lambda}_{\alpha\,\hat\beta} = 1$ and, since $\alpha_1 = \alpha_2$, $a^{\lambda}_{\alpha\,\hat\beta} = 0$. The only $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ (and $\gamma_1 = \alpha_1$) is $\gamma = \ydiagram{2}$. As before, there is one SSYT of shape $\bar\nu/\beta$ and type $\gamma = (2)$. Therefore $c^{\bar\nu}_{\beta\,(2)} = 1$. Hence, $c^{\lambda}_{\alpha\,\hat\beta} c^{\bar\nu}_{\beta\,(2)} = 1$. *This contributes* 1 *to the multiplicity.*

If $\alpha = \ydiagram{2,1,1}$, then $\ytableaushort{\none\none 111111111, \none 12, \none 2, 13, 3}$ is the only SSYT of shape λ/α and type $\hat\beta = (11, 2, 2)$. Thus $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 1$. The only $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ is $\gamma = \ydiagram{1,1}$. However, $c^{\bar\nu}_{\beta\,(1,1)} = 0$.
For all other $\alpha \in S_\lambda$ we have $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 0$.

Therefore the multiplicity of $s_{(13,4,2)}$ in $s_{(15,4)} * s_{(11,3,2,2,1)}$ equals 4.

Proposition 4.2. *Let n and p be positive integers, $n \geq 2p$, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ be a partition of n with $\lambda_1 - \lambda_2 \geq 2p$. Consider the partition $\nu = (\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)})$ of n. If the multiplicity $g_{(n-p,p),\lambda,\nu}$ of s_ν in $s_{(n-p,p)} * s_\lambda$ is non-zero, then $\lambda_1 - p \leq \nu_1 \leq \lambda_1 + p$. Moreover, if $\lambda_2 < p$ and $g_{(n-p,p),\lambda,\nu} \neq 0$, then $\lambda_1 - p \leq \nu_1 \leq \lambda_1 + \lambda_2$.*

Proof: First suppose all $\alpha \in S_\lambda$ are such that $\alpha_1 \leq \lambda_2$. For a fixed $\alpha \in S_\lambda$, each $\delta \in Q(\alpha)$ satisfies $p - \alpha_1 \leq |\delta| \leq p$. Thus β obtained in step (1) of the algorithm from *Delete[δ] from $\bar\lambda$* satisfies

$$n - \lambda_1 - p \leq |\beta| = |\bar\lambda| - |\delta| \leq n - \lambda_1 - p + \alpha_1.$$

Now, in step (2) of the algorithm, we perform *Add[γ] to β* for some $\gamma \in Q(\alpha)$. Thus $p - \alpha_1 \leq |\gamma| \leq p$ and after the *Add[γ] to β* algorithm we obtain diagrams $\bar\nu$ with

$$n - \lambda_1 - \alpha_1 \leq |\bar\nu| = |\beta| + |\gamma| \leq n - \lambda_1 + \alpha_1.$$

Thus, $\lambda_1 - \alpha_1 \leq \nu_1 = n - |\bar\nu| \leq \lambda_1 + \alpha_1$. Since $\alpha_1 \leq p$ (note that $\alpha = (p)$ always belongs to S_λ), we have $\lambda_1 - p \leq \nu_1 \leq \lambda_1 + p$.

If $\lambda_2 < p$ and α is such that $\alpha_1 > \lambda_2$ then, in step (1), the algorithm *Delete[δ] from $\bar\lambda$* will return no shapes β if $\delta_1 > \lambda_2$. Thus to obtain shapes β from *Delete[δ] from $\bar\lambda$*, the diagram $\delta \in Q(\alpha)$ must have been obtained from α by removing at least $\alpha_1 - \lambda_2$ boxes. Hence $p - \alpha_1 \leq |\delta| \leq p - \alpha_1 + \lambda_2$. Therefore

$$n - \lambda_1 - p + \alpha_1 - \lambda_2 \leq |\beta| \leq n - \lambda_1 - p + \alpha_1.$$

As before, $p - \alpha_1 \leq |\gamma| \leq p$ and thus $n - \lambda_1 - \lambda_2 \leq |\bar\nu| \leq n - \lambda_1 + \alpha_1$. Therefore $\lambda_1 - \alpha_1 \leq \nu_1 \leq \lambda_1 + \lambda_2$. Since $\alpha_1 \leq p$, we have $\lambda_1 - p \leq \nu_1 \leq \lambda_1 + \lambda_2$. $\square$.

Proposition 4.3. *Let n and p and $\lambda \vdash n$ be as in the previous proposition, i.e. $\lambda_1 - \lambda_2 \geq 2p$. Consider the partition $\nu = (\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)})$ of n. If $\nu_2 > \lambda_2 + p$, then the multiplicity $g_{\lambda,(n-p,p),\lambda,\nu}$ of s_ν in $s_{(n-p,p)} * s_\lambda$ is equal to zero. Moreover, if $\nu = (\lambda_1 - p, \lambda_2 + p, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$, then $g_{(n-p,p),\lambda,\nu} = 1$.*

Proof: As noted at the end of the proof of Theorem 2.1, since $\lambda_1 - \lambda_2 \geq 2p$, the partition $\alpha = (p)$ belongs to S_λ and $Q((p))$ contains the empty diagram ϵ. In Step (1), when we perform *Delete[ϵ] from* $\bar{\lambda}$, we obtain $\beta = \bar{\lambda} \in D((p))$. Thus $\hat{\beta} = (\lambda_1 - p, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$. We have $c^{\lambda}_{(p)\,\hat{\beta}} = 1$ and $a^{\lambda}_{(p)\,\hat{\beta}} = 0$. Moreover, $\alpha = (p)$ is the only partition in S_λ for which $\bar{\lambda}$ appears in $D(\alpha)$. Then, in step (2), when we perform *Add[(p)] to* $\bar{\lambda}$, we obtain the largest possible first row, of size $\lambda_2 + p$, when all p labels of (p) are added to the first row of $\bar{\lambda}$. Thus $\nu_2 \leq \lambda_2 + p$. Therefore, if $\nu_2 > \lambda_2 + p$, then $g_{(n-p,p),\lambda,\nu} = 0$.

Moreover, if ν was obtained as above, then $\nu = (\lambda_1 - p, \lambda_2 + p, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ and $g_{(n-p,p),\lambda,\nu} = 1$.

Note that it is possible to obtain diagrams $\beta \in D(\alpha)$ with $\beta_1 = \lambda_2$ other than $\bar{\lambda}$, and therefore the decomposition of $s_{(n-p,p)} * s_\lambda$ might contain s_ν for other ν with $\nu_2 = \lambda_2 + p$. These diagrams ν will have $\nu_1 > \lambda_1 - p$. $\square$

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}) \vdash m$, we say that λ is less than μ *in lexicographic order*, and write $\lambda <_l \mu$, if there is a non-negative integer k such that $\lambda_i = \mu_i$ for all $i = 1, 2, \ldots, k$ and $\lambda_{k+1} < \mu_{k+1}$. Note that the lexicographic order is a total order on the set of all partitions.

Corollary 4.4. *Let n and p be positive integers such that $n \geq 2p$ and let $\lambda \vdash n$ such that $\lambda_1 - \lambda_2 \geq 2p$. The smallest partition in lexicographic order $\nu \vdash n$ such that s_ν appears in the decomposition of $s_{(n-p,p)} * s_\lambda$ is the partition whose parts are $\lambda_1 - p, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, p$, reordered to form a partition. Moreover, this s_ν appears with multiplicity 1.*

Proof: If s_ν appears in the decomposition of $s_{(n-p.p)} * s_\lambda$, Proposition 4.2 implies that $\nu_1 \geq \lambda_1 - p$. If ν is such that $\nu_1 = \lambda_1 - p$, then $|\bar{\nu}| = n - \lambda_1 + p = |\bar{\lambda}| + p$.

Now, in the algorithm, $\bar{\nu}$ is obtained after performing the **Add**$[\alpha]$ step for some (possibly more than one) $\alpha \vdash p$, $\alpha \subseteq \lambda$. In the **Add**$[\alpha]$ step, one performs, for each $\beta \in D(\alpha)$, *Add[γ] to* β for certain (or all) $\gamma \in Q(\alpha)$. Thus, the diagrams obtained in this step have size equal to $|\beta| + |\gamma|$. This size can equal $|\bar{\lambda}| + p$ if and only if $|\beta| = |\bar{\lambda}|$ and $|\gamma| = p$. Therefore, as diagrams, we must have $\beta = \bar{\lambda}$ and $\gamma = \alpha$. The only α for which $\bar{\lambda} \in D(\alpha)$ is $\alpha = (p)$. Thus, if s_ν with $\nu_1 = \lambda_1 - p$ appears in the decomposition of $s_{(n-p,p)} * s_\lambda$, then $\bar{\nu}$ is obtained by performing *Add[(p)] to* $\bar{\lambda}$. By Corollary 2.2 of [BO], the smallest partition $\bar{\nu}$ in lexicographic order obtained by performing *Add[(p)] to* $\bar{\lambda}$ (i.e. smallest $s_{\bar{\nu}}$ in lexicographic order in the expansion of $s_{(p)} s_{\bar{\lambda}}$) is obtained by concatenating the parts of (p) and $\bar{\lambda}$ and reordering them to form a partition. Following Theorem 2.1 of [BO], one sees that this partition appears with multiplicity 1. This finishes the proof of the corollary. $\square$

Remark: The result of Corollary 4.4 is already known. It follows from Corollary 4.2.2 in [V2] and Theorem 6.2 in [V3]. The new contribution of the corollary is that the minimal component s_ν, where ν is as in Corollary 4.4, is the minimal component in lexicographic order.

5 Stability of Kronecker coefficients

As before, if n and p are positive integers such that $n \geq 2p$ and λ, $\nu \vdash n$, we denote by $g_{(n-p,p),\lambda,\nu}$ the coefficient of s_ν in the decomposition of the Kronecker product $s_{(n-p,p)} * s_\lambda$. We refer to $g_{(n-p,p),\lambda,\nu}$ as a Kronecker coefficient. Below we show that the Kronecker coefficients are stable if $\lambda_1 - \lambda_2 \geq 2p$, i.e. the Kronecker coefficients depend only on $\bar{\lambda}$, $\overline{(n-p,p)} = (p)$ and $\bar{\nu}$.

Theorem 5.1. *Given an arbitrary partition $\bar{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$, let n be a positive integer such that $n \geq 2p + |\bar{\lambda}| + \lambda_2$. Then $g_{(n-p,p),(n-|\bar{\lambda}|,\bar{\lambda}),(n-|\bar{\nu}|,\bar{\nu})} = g_{(m-p,p),(m-|\bar{\lambda}|,\bar{\lambda}),(m-|\bar{\nu}|,\bar{\nu})}$ for all $m \geq n$ and all partitions $\nu \vdash n$.*

Proof: The condition $n \geq 2p + |\bar{\lambda}| + \lambda_2$ is equivalent to $\lambda_1 - \lambda_2 \geq 2p$. In this case, Theorem 4.1 above gives a formula for the Kronecker coefficients. All indices of summation in the formula depend only on $\bar{\lambda}$, $\bar{\nu}$ and $p = |\overline{(n-p,p)}|$. To see this, note that

$$\{\alpha \vdash p | \alpha \subseteq \lambda\} = \bigcup_{m \leq p-1} \bigcup_{\{\eta \vdash m|\, \eta \subseteq \bar{\lambda},\, \eta_1 \leq p-m\}} \{\alpha = (p-m, \eta_1, \eta_2, \ldots, \eta_{\ell(\eta)})\}.$$

From Theorems 3.1(b) and 3.3(b), all Littlewood-Richardson type coefficients involved in the formula of Theorem 4.1 depend only of $\bar{\lambda}$, $\bar{\nu}$ and p. This proves the stability of the Kronecker coefficients. $\square$

Let us compare our stability result to Vallejo's result [V1] in the particular case in which one of the partitions is $\bar{\mu} = (p)$. The main difference is that Vallejo's lower bound on n (starting with which the Kronecker coefficients are stable) depends on $\bar{\lambda}$, $\bar{\nu}$ and p while our bound depends only on $\bar{\lambda}$ and p.

Consider partitions $\bar{\lambda}$, $\bar{\nu}$ and (p). If $n \geq \lambda_2 + |\bar{\lambda}| + 2p$ and s_ν appears in the decomposition of $s_\lambda * s_{(n-p,p)}$, by Proposition 4.2 we have $\lambda_1 - p \leq \nu_1 \leq \lambda_1 + p$ and thus $|\bar{\lambda}| - p \leq |\bar{\nu}| \leq |\bar{\lambda}| + p$. Also, by Proposition 4.3 we have $\nu_2 \leq \lambda_2 + p$. Thus, if $n \geq 2p + |\bar{\lambda}| + \lambda_2$ as in Theorem 5.1, then

$$n \geq |\bar{\lambda}| + \lambda_2, \quad n \geq |\bar{\nu}| + \nu_2, \quad n \geq 2p$$

and $(n - |\bar{\lambda}|, \bar{\lambda})$, $(n - |\bar{\nu}|, \bar{\nu})$ and $(n-p,p)$ are partitions of n.

If $\bar{\lambda} = \bar{\nu}$, Vallejo's lower bound for the stability of the Kronecker coefficients is given by

$$m = \max\{\lambda_2 + |\bar{\lambda}| + p, 2p\}.$$

In this case, our bound for stability, $\lambda_2 + |\bar{\lambda}| + 2p$, is worse than Vallejo's.

If $\bar{\lambda} \neq \bar{\nu}$, Vallejo's bound is given by

$$\begin{aligned} m = \min\{&\max\{\lambda_2 + |\bar{\lambda}| + p - 1, \nu_2 + |\bar{\nu}| + p - 1, 2p\},\\ &\max\{\lambda_2 + |\bar{\lambda}| + |\bar{\nu}| - 1, p + p + |\bar{\nu}| - 1, 2|\bar{\nu}|\},\\ &\max\{\nu_2 + |\bar{\nu}| + |\bar{\lambda}| - 1, p + p + |\bar{\lambda}| - 1, 2|\bar{\lambda}|\}\}. \end{aligned}$$

If Vallejo's bound m is such that $m > \lambda_2 + |\bar{\lambda}| + 2p$, then our bound is an improvement. This happens if, for example,

$$|\bar{\nu}| > \max\{2p + 1, \lambda_2 + |\bar{\lambda}| + 1, \frac{\lambda_2 + |\bar{\lambda}|}{2} + p\}$$

and

$$\nu_2 + |\bar{\nu}| > \max\{\lambda_2 + |\bar{\lambda}| + p + 1, \lambda_2 + 2p + 1\}.$$

Of course, depending on $\bar{\lambda}$, our bound could be an improvement even for "smaller" $\bar{\nu}$.

Example: Suppose $p = 3$, $\bar{\lambda} = (2, 1)$ and $\bar{\nu} = (4, 2, 1)$. In this case, Vallejo's bound for the stability of $g_{(n-|\bar{\lambda}|,|\bar{\lambda}|),(n-p,p),(n-|\bar{\nu}|,\bar{\nu})}$ is

$$m = \min\{\max\{7, 13, 6\}, \max\{11, 12, 14\}, \max\{13, 11, 6\}\} = 13,$$

while our bound is

$$\lambda_2 + |\bar{\lambda}| + 2p = 11.$$

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