

# Counting segmented permutations using bicoloured Dyck paths

Anders Claesson

Division of Mathematics,  
Department of Chemistry and Biomedical Sciences,  
University of Kalmar, Sweden  
`anders.claesson@hik.se`

Submitted: Jun 14, 2005; Accepted: Aug 9, 2005; Published: Aug 17, 2005  
Mathematics Subject Classifications: 05A05, 05A15

## Abstract

A bicoloured Dyck path is a Dyck path in which each up-step is assigned one of two colours, say, red and green. We say that a permutation  $\pi$  is  $\sigma$ -segmented if every occurrence  $o$  of  $\sigma$  in  $\pi$  is a segment-occurrence (i.e.,  $o$  is a contiguous subword in  $\pi$ ).

We show combinatorially the following results: The 132-segmented permutations of length  $n$  with  $k$  occurrences of 132 are in one-to-one correspondence with bicoloured Dyck paths of length  $2n - 4k$  with  $k$  red up-steps. Similarly, the 123-segmented permutations of length  $n$  with  $k$  occurrences of 123 are in one-to-one correspondence with bicoloured Dyck paths of length  $2n - 4k$  with  $k$  red up-steps, each of height less than 2.

We enumerate the permutations above by enumerating the corresponding bicoloured Dyck paths. More generally, we present a bivariate generating function for the number of bicoloured Dyck paths of length  $2n$  with  $k$  red up-steps, each of height less than  $h$ . This generating function is expressed in terms of Chebyshev polynomials of the second kind.

## 1 Introduction

Let  $\mathcal{S}_n$  be the set of permutation of  $[n] = \{1, 2, \dots, n\}$ . Let  $\pi \in \mathcal{S}_n$  and  $\sigma \in \mathcal{S}_k$ , with  $k \leq n$ . An *occurrence* of  $\sigma$  in  $\pi$  is a subword  $o$  of length  $k$  in  $\pi$  such that  $o$  and  $\sigma$  are in same relative order. In this context  $\sigma$  is called a *pattern*. For example, an occurrence of the pattern 132 in  $\pi$  is a subword  $\pi(i)\pi(j)\pi(k)$  such that  $\pi(i) < \pi(k) < \pi(j)$ ; so 253 is an occurrence of 132 in 42513. A permutation  $\pi$  that does not contain any occurrence of  $\sigma$  is said to *avoid*  $\sigma$ .

It is relatively easy to show that number of permutations of  $[n]$  avoiding a pattern of length 3 is the Catalan number,  $C_n = \binom{2n}{n}/(n+1)$  (e.g., see [9] or [5]). In contrast, to count the permutations containing  $r$  occurrences of a fixed pattern of length 3, for a general  $r$ , is a very hard problem. The best result on this latter problem has been achieved by Mansour and Vainshtein [7]. They presented an algorithm that computes the generating function for the number of permutations with  $r$  occurrences of 132 for any  $r \geq 0$ . The algorithm has been implemented in C. It yields explicit results for  $1 \leq r \leq 6$ .

We say that an occurrence  $o$  of  $\sigma$  in  $\pi$  is a *segment-occurrence* if  $o$  is a segment (also called factor) of  $\pi$ , in other words, if  $o$  is a contiguous subword in  $\pi$ . Elizalde and Noy [2] presented exponential generating functions for the distribution of the number of segment-occurrences of any pattern of length 3. Related problems have also been studied by Kitaev [3] and by Kitaev and Mansour [4].

We say that  $\pi$  is  $\sigma$ -segmented if every occurrence of  $\sigma$  in  $\pi$  is a segment-occurrence. For instance, 4365172 contains 3 occurrences of 132, namely 465, 365, and 172. Of these occurrences, only 365 and 172 are segment-occurrences. Thus 4365172 is not 132-segmented. Note that if  $\pi$  is  $\sigma$ -avoiding then  $\pi$  is also  $\sigma$ -segmented. In this article we try to enumerate the  $\sigma$ -segmented permutations by length and by the number of occurrences of  $\sigma$ .

Krattenthaler [5] gave two bijections: one between 132-avoiding permutations and Dyck paths, and one between 123-avoiding permutations and Dyck paths. We obtain two new results by extending these bijections:

- The 132-segmented permutations of length  $n$  with  $k$  occurrences of 132 are in one-to-one correspondence with bicoloured Dyck paths of length  $2n - 4k$  with  $k$  red up-steps.
- The 123-segmented permutations of length  $n$  with  $k$  occurrences of 123 are in one-to-one correspondence with bicoloured Dyck paths of length  $2n - 4k$  with  $k$  red up-steps, each of height less than 2.

Here a bicoloured Dyck path is a Dyck path in which each up-step is assigned one of two colours, say, red and green. We enumerate the permutations above by enumerating the corresponding bicoloured Dyck paths. To be more precise, let  $\mathcal{B}_{n,k}$  be the set of bicoloured Dyck path of length  $2n$  with  $k$  red up-steps. Let  $\mathcal{B}_{n,k}^{[h]}$  be the subset of  $\mathcal{B}_{n,k}$  consisting of those paths where the height of each red up-step is less than  $h$ . It is plain that  $|\mathcal{B}_{n,k}| = \binom{n}{k} C_n$ . We show that

$$\sum_{n,k \geq 0} |\mathcal{B}_{n,k}^{[h]}| q^k t^n = \frac{C(t) - 2xqU_h(x)U_{h-1}(x)}{1 + q - qU_h^2(x)}, \quad x = \frac{1}{2\sqrt{(1+q)t}},$$

where  $C(t) = (1 - \sqrt{1 - 4t})/(2t)$  is the generating function for the Catalan numbers, and  $U_n(x)$  is the  $n$ th Chebyshev polynomial of the second kind. We also find formulas for  $|\mathcal{B}_{n,k}^{[1]}|$  and  $|\mathcal{B}_{n,k}^{[2]}|$ .

## 2 Bicoloured Dyck paths

By a *lattice path* we shall mean a path in  $\mathbb{Z}^2$  with steps  $(1, 1)$  and  $(1, -1)$ ; the steps  $(1, 1)$  and  $(1, -1)$  will be called *up-* and *down-steps*, respectively. Furthermore, a lattice path that never falls below the  $x$ -axis will be called *nonnegative*. A *Dyck path* of length  $2n$  is a nonnegative lattice path from  $(0, 0)$  to  $(2n, 0)$ . As an example, these are the 5 Dyck paths of length 6:



Letting  $u$  and  $d$  represent the steps  $(1, 1)$  and  $(1, -1)$ , we code a Dyck path with a word over  $\{u, d\}$ . For example, the paths above are coded by

$$ududud \quad udwudd \quad uuddud \quad uduudd \quad uuuddd$$

Let  $\mathcal{D}_n$  be the language over  $\{u, d\}$  obtained from Dyck paths of length  $2n$  via this coding, and let  $\mathcal{D} = \cup_{n \geq 0} \mathcal{D}_n$ . In general, if  $\mathcal{A}$  is a language over some alphabet  $X$ , then the *characteristic series* of  $\mathcal{A}$ , also (by slight abuse of notation) denoted  $\mathcal{A}$ , is the element of  $\mathbb{C}\langle\langle X \rangle\rangle$  defined by

$$\mathcal{A} = \sum_{w \in \mathcal{A}} w.$$

A nonempty Dyck path  $\beta$  can be written uniquely as  $u\beta_1d\beta_2$  where  $\beta_1$  and  $\beta_2$  are Dyck paths. This decomposition is called the *first return decomposition* of  $\beta$ , because the  $d$  in  $u\beta_1d\beta_2$  corresponds to the first place, after  $(0, 0)$ , where the path touches the  $x$ -axis. By this decomposition, the characteristic series of  $\mathcal{D}$  is uniquely determined by the functional equation

$$\mathcal{D} = \epsilon + u\mathcal{D}d\mathcal{D}, \tag{1}$$

where  $\epsilon$  denotes the empty word.

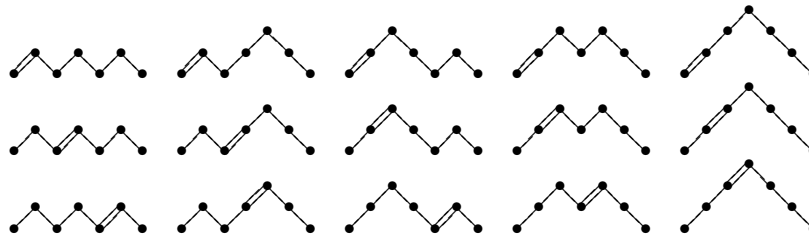
In a similar vein, we now consider the language  $\mathcal{B}$  over  $\{u, \bar{u}, d\}$  whose characteristic series is uniquely determined by the functional equation

$$\mathcal{B} = \epsilon + (u + \bar{u})\mathcal{B}d\mathcal{B}. \tag{2}$$

Let  $\mathcal{B}_n$  be the set of words in  $\mathcal{B}$  that are of length  $2n$ , and let  $\mathcal{B}_{n,k}$  be the set of words in  $\mathcal{B}_n$  with  $k$  occurrences of  $\bar{u}$ . As an example, when  $n = 3$  and  $k = 1$  there are 15 such words, namely

$$\begin{aligned} &\bar{u}dudud \quad \bar{u}dwudd \quad \bar{u}uddud \quad \bar{u}duudd \quad \bar{u}uwudd \\ &ud\bar{u}dud \quad ud\bar{u}udd \quad u\bar{u}ddud \quad u\bar{u}dudd \quad u\bar{u}wudd \\ &udud\bar{u}d \quad udw\bar{u}dd \quad uudd\bar{u}d \quad uud\bar{u}dd \quad uu\bar{u}ddd \end{aligned}$$

We may view the elements of  $\mathcal{B}$  as *bicoloured Dyck paths*. The words from the previous example are depicted below.



Here steps represented by double edges are, say, red, and steps represented by simple edges are, say, green.

**Proposition 1** *With  $C_n = |\mathcal{D}_n|$ , we have*

$$|\mathcal{B}_{n,k}| = \binom{n}{k} C_n \quad \text{and} \quad |\mathcal{B}_n| = 2^n C_n.$$

**Proof** A bicoloured Dyck path  $\beta$  of length  $2n$  naturally breaks up into two parts: (a) The Dyck path obtained from  $\beta$  by removing colours. (b) The subset of  $[n]$  consisting of those integers  $i$  for which the  $i$ th up-step is red.  $\square$

For  $h \geq 1$ , let  $\mathcal{B}^{[h]}$  be the subset of  $\mathcal{B}$  whose characteristic series is the solution to

$$\mathcal{B}^{[h]} = \epsilon + (u + \bar{u})\mathcal{B}^{[h-1]}d\mathcal{B}^{[h]}, \quad (3)$$

with the initial condition  $\mathcal{B}^{[0]} = \mathcal{D}$ , where  $\mathcal{D}$  is defined as above. Let

$\mathcal{B}_n^{[h]}$  be the set of words in  $\mathcal{B}^{[h]}$  that are of length  $2n$ , and let  
 $\mathcal{B}_{n,k}^{[h]}$  be the set of words in  $\mathcal{B}_n^{[h]}$  with  $k$  occurrences of  $\bar{u}$ .

To translate these definitions in terms of lattice paths we define the *height* of a step in a (bicoloured) lattice path as the height above the  $x$ -axis of its left point. Then  $\mathcal{B}^{[h]}$  is the set of bicoloured Dyck paths whose red up-steps all are of height less than  $h$ . As an example, there is exactly one element in  $\mathcal{B}_{3,1}$  that is not in  $\mathcal{B}^{[2]}$ , namely



To count words of given length in  $\mathcal{D}$ ,  $\mathcal{B}$  and  $\mathcal{B}^{[h]}$ , we will study the commutative counterparts of the functional equations (1), (2) and (3). Formally, we define the substitution  $\mu : \mathbb{C}\langle\langle u, \bar{u}, d \rangle\rangle \rightarrow \mathbb{C}[[q, t]]$  by

$$\mu = \{ u \mapsto 1, \bar{u} \mapsto q, d \mapsto t \}.$$

Let  $C = \mu(\mathcal{D})$ ,  $B = \mu(\mathcal{B})$ , and  $B^{[h]} = \mu(\mathcal{B}^{[h]})$ . We then get

$$C = 1 + tC^2, \quad (4)$$

$$B = 1 + (1 + q)tB^2, \quad (5)$$

$$B^{[h]} = 1 + (1 + q)tB^{[h-1]}B^{[h]}, \quad B^{[0]} = C. \quad (6)$$

By an easy application of the Lagrange inversion formula it follows from (4) that

$$[t^n](C(t))^i = \frac{i}{i+n} \binom{2n+i-1}{n}. \quad (7)$$

In particular, we obtain that  $C(t)$  is the familiar generating function of the Catalan numbers,  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . Thus we have derived the well known fact that the number of Dyck paths of length  $2n$  is the  $n$ th Catalan number. Furthermore, it follows from (5) that

$$B(q, t) = C((1+q)t), \quad (8)$$

and it follows from (6) that

$$B^{[h]}(q, t) = \frac{1}{1 - (1+q)tB^{[h-1]}}, \quad B^{[0]} = C. \quad (9)$$

From these series we generate the first few values of  $|\mathcal{B}_{n,k}|$ ,  $|\mathcal{B}_{n,k}^{[1]}|$  and  $|\mathcal{B}_{n,k}^{[2]}|$ ; tables with these values are given in Section 5.

Recall that the *Chebyshev polynomials of the second kind*, denoted  $U_n(x)$ , are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta},$$

where  $n$  is an integer,  $x = \cos \theta$ , and  $0 \leq \theta \leq \pi$ . Equivalently, these polynomials can be defined as the solution to the linear difference equation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

with  $U_{-1}(x) = 0$  and  $U_0(x) = 1$ .

In 1970 Kreweras [6] showed that

$$C^{[h]}(t) = \frac{U_h\left(\frac{1}{2\sqrt{t}}\right)}{\sqrt{t} \cdot U_{h+1}\left(\frac{1}{2\sqrt{t}}\right)} \quad (10)$$

is the generating function for Dyck paths that stay below height  $h$ . Note that, since  $C^{[0]} = 1$  and  $C^{[h]} = (1 - tC^{[h-1]})^{-1}$ , this result is also easy to prove by induction on  $h$ .

**Theorem 2** *With  $B^{[h]}$  being the generating function for the number of Dyck paths whose red up-steps all are of height less than  $h$ , and  $U_n$  being the  $n$ th Chebyshev polynomial of the second kind we have*

$$B^{[h]}(q, t) = \frac{4x^2U_{h-1}(x) - 2xU_{h-2}(x)C(t)}{2xU_h(x) - U_{h-1}(x)C(t)} = \frac{C(t) - 2xqU_h(x)U_{h-1}(x)}{1 + q - qU_h^2(x)},$$

where  $x = 1/(2\sqrt{(1+q)t})$ , and  $C(t) = (1 - \sqrt{1-4t})/(2t)$  is the generating function for the Catalan numbers.

**Proof** We shall prove the first equality by induction. To this end, we let

$$F^{[h]}(q, t) = \frac{4x^2U_{h-1}(x) - 2xU_{h-2}(x)C(t)}{2xU_h(x) - U_{h-1}(x)C(t)}.$$

From  $U_{-2}(x) = -1$ ,  $U_{-1}(x) = 0$ , and  $U_0(x) = 1$  it readily follows that  $F^{[0]}(q, t) = C(t) = B^{[0]}(q, t)$ . If  $B^{[h]} = F^{[h]}$ , for some fixed  $h \geq 0$ , then

$$\begin{aligned} B^{[h+1]} &= \frac{1}{1 - (1+q)tB^{[h]}} \\ &= \frac{1}{1 - (1+q)tF^{[h]}} \\ &= \frac{2xU_h - U_{h-1}C}{2xU_h - U_{h-1}C - (1+q)t(4x^2U_{h-1} - 2xU_{h-2}C)} \\ &= \frac{2xU_h - U_{h-1}C}{2xU_h - (1+q)t4x^2U_{h-1} - (U_{h-1} - (1+q)t2xU_{h-2})C} \\ &= \frac{4x^2U_h - 2xU_{h-1}C}{2x(2xU_h - (1+q)t4x^2U_{h-1}) - (2xU_{h-1} - (1+q)t4x^2U_{h-2})C} \\ &= \frac{4x^2U_h - 2xU_{h-1}C}{2x(2xU_h - U_{h-1}) - (2xU_{h-1} - U_{h-2})C} \\ &= \frac{4x^2U_h - 2xU_{h-1}C}{2xU_{h+1} - U_hC} \\ &= F^{[h+1]}, \end{aligned}$$

in which  $U_h = U_h(x)$  and  $C = C(t)$ . This completes the induction step, and thus the first equality holds for all  $h \geq 0$ . The second equality is plain algebra/trigonometry.  $\square$

**Proposition 3** For  $n, k \geq 0$  we have

$$\begin{aligned} |\mathcal{B}_{n,k}^{[1]}| &= b(n+k, n-k) = \frac{2k+1}{n+k+1} \binom{2n}{n-k}, \\ |\mathcal{B}_n^{[1]}| &= \binom{2n}{n}, \end{aligned}$$

where  $b(n, k) = \frac{n-k+1}{n+1} \binom{n+k}{n}$  is a ballot number.

**Proof** The ballot number  $b(n, k)$  is the number of nonnegative lattice paths from  $(0, 0)$  to  $(n+k, n-k)$ . Thus, the first claim of the proposition is that  $|\mathcal{B}_{n,k}^{[1]}|$  equals the number of nonnegative lattice paths from  $(0, 0)$  to  $(2n, 2k)$ . Let  $\mathcal{A}_{n,k}$  denote the language over  $\{u, d\}$  obtained from these paths via the usual coding. In addition, let  $\mathcal{A}_n = \cup_{k \geq 0} \mathcal{A}_{n,k}$  and  $\mathcal{A} = \cup_{n \geq 0} \mathcal{A}_n$ . The characteristic series of  $\mathcal{A}$  satisfies

$$\mathcal{A} = \epsilon + u\mathcal{D}(u+d)\mathcal{A}.$$

From (3) we also know that

$$\mathcal{B}^{[1]} = \epsilon + (u + \bar{u})\mathcal{D}d\mathcal{B}^{[1]}.$$

We exploit the obvious similarity between these two functional equations to define, by recursion, a length preserving bijection  $f$  from  $\mathcal{B}^{[1]}$  onto  $\mathcal{A}$  such that  $\beta \in \mathcal{B}^{[1]}$  has exactly  $k$  occurrences of  $\bar{u}$  precisely when  $f(\beta) \in \mathcal{A}$  ends at height  $2k$ :

$$f(\beta) = \begin{cases} \epsilon & \text{if } \beta = \epsilon, \\ u\beta_1df(\beta_2) & \text{if } \beta = u\beta_1d\beta_2, \beta_1 \in \mathcal{D}, \beta_2 \in \mathcal{B}^{[1]}, \\ u\beta_1uf(\beta_2) & \text{if } \beta = \bar{u}\beta_1d\beta_2, \beta_1 \in \mathcal{D}, \beta_2 \in \mathcal{B}^{[1]}. \end{cases}$$

For  $\beta \in \mathcal{B}^{[1]}$ , let  $|\beta|_{\bar{u}}$  denote the number of occurrences of  $\bar{u}$  in  $\beta$ , and for  $\alpha \in \mathcal{A}$  let  $h(\alpha)$  denote the height at which  $\alpha$  ends. To prove that  $f$  is length preserving, bijective, and that  $2|\cdot|_{\bar{u}} = h \circ f$ , we use induction on path-length:  $f$  trivially has these properties as a function from  $\mathcal{B}_0^{[1]}$  to  $\mathcal{A}_0$ . Let  $n$  be a positive integer and assume that  $f$  has the desired properties as a function from  $\cup_{k=0}^{n-1} \mathcal{B}_k^{[1]}$  to  $\cup_{k=0}^{n-1} \mathcal{A}_k$ . Any  $\beta$  in  $\mathcal{B}_n^{[1]}$  can be written as  $\beta = x\beta_1d\beta_2$  for some  $x \in \{u, \bar{u}\}$ ,  $\beta_1 \in \mathcal{D}$  and  $\beta_2 \in \mathcal{B}^{[1]}$ . Therefore, using induction,

$$|f(\beta)| = 2 + |\beta_1| + |f(\beta_2)| = 2 + |\beta_1| + |\beta_2| = |\beta|$$

and

$$(h \circ f)(\beta) = 2|x|_{\bar{u}} + (h \circ f)(\beta_2) = 2|x|_{\bar{u}} + 2|\beta_2|_{\bar{u}} = 2|\beta|_{\bar{u}}$$

To prove that  $f$  is injective, assume that  $f(\beta) = f(\beta')$ , where  $\beta' = x'\beta'_1d\beta'_2$  for some  $x' \in \{u, \bar{u}\}$ ,  $\beta'_1 \in \mathcal{D}$ , and  $\beta'_2 \in \mathcal{B}^{[1]}$ . Then

$$f(\beta) = u\beta_1yf(\beta_2) = u\beta'_1y'f(\beta'_2) = f(\beta'),$$

in which  $y, y' \in \{u, d\}$ . Thus  $\beta_1 = \beta'_1$ ,  $y = y'$ , and  $f(\beta_2) = f(\beta'_2)$ . By the induction hypothesis,  $f(\beta_2) = f(\beta'_2)$  implies that  $\beta_2 = \beta'_2$ , and hence  $\beta = \beta'$ .

To prove that  $f$  is surjective, take any  $\alpha = u\alpha'y\alpha''$  in  $\mathcal{A}_n$ , where  $y \in \{u, d\}$ ,  $\alpha' \in \mathcal{D}$ , and  $\alpha'' \in \mathcal{A}$ . By the induction hypothesis, there exists  $\beta''$  in  $\mathcal{B}^{[1]}$  such that  $f(\beta'') = \alpha''$ ; so  $f(u\alpha'y\beta'') = \alpha$ . This completes the proof of the first part of the proposition.

Given the first result, the second result may be formulated as saying that the central binomial coefficient  $\binom{2n}{n}$  is the sum of the ballot numbers  $b(n+k, n-k)$  for  $k = 0, 1, \dots, n$ . This is a known fact (see [8, p. 79]). Indeed,

$$\frac{2k+1}{n+k+1} \binom{2n}{n-k} = \binom{2n}{n-k} - \binom{2n}{n-k-1},$$

and hence the sum of these numbers is telescoping.

For a bijective proof of the second part we consider the set of all lattice paths from  $(0, 0)$  to  $(2n, 0)$ . Let  $\mathcal{P}_n$  be the language over  $\{u, d\}$  obtained from these  $\binom{2n}{n}$  paths via the usual coding, and let  $\mathcal{P} = \cup_{n \geq 0} \mathcal{P}_n$ . The characteristic series of  $\mathcal{P}$  satisfies

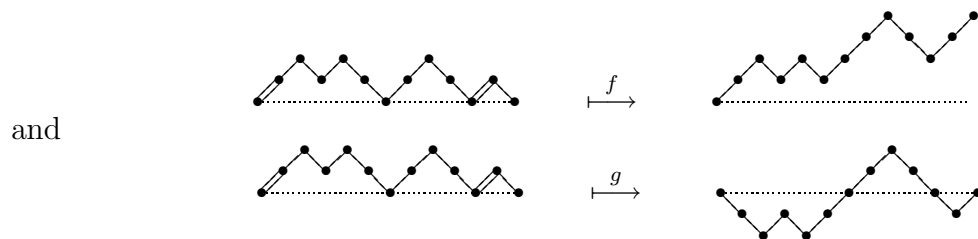
$$\mathcal{P} = \epsilon + u\mathcal{D}d\mathcal{P} + d\widehat{\mathcal{D}}u\mathcal{P},$$

where  $\widehat{\mathcal{D}}$  is the image of  $\mathcal{D}$  under the involution on  $\mathbb{C}\langle\langle u, d \rangle\rangle$  defined by  $u \mapsto d$  and  $d \mapsto u$ ; this involution has the effect of reflecting a Dyck path in the  $x$ -axis. A length preserving bijection  $g$  from  $\mathcal{B}^{[1]}$  onto  $\mathcal{P}$  is then recursively defined by

$$g(\beta) = \begin{cases} \epsilon & \text{if } \beta = \epsilon, \\ u\beta_1dg(\beta_2) & \text{if } \beta = u\beta_1d\beta_2, \beta_1 \in \mathcal{D}, \beta_2 \in \mathcal{B}^{[1]}, \\ d\widehat{\beta}_1ug(\beta_2) & \text{if } \beta = \bar{u}\beta_1d\beta_2, \beta_1 \in \mathcal{D}, \beta_2 \in \mathcal{B}^{[1]}. \end{cases}$$

Again, by induction on path-length it follows that  $g$  is a bijection. □

**Example** As an illustration of the bijections in the proof of Proposition 3, we have



**Proposition 4** For  $n, k \geq 0$  we have

$$|\mathcal{B}_{n,k}^{[2]}| = \sum_{i \geq 0} \frac{2k+i+1}{n+k+i+1} \binom{k-1}{k-i} \binom{2n+i}{n-k}.$$

**Proof** From (9) it follows that

$$B^{[2]}(q, t) = \frac{1 - t(1+q)C(t)}{1 - t(1+q)(1+C(t))}.$$

Using (4) we rewrite this as

$$B^{[2]}(q, t) = \frac{(1 - qt(C(t))^2)C(t)}{1 - (1+C(t))qt(C(t))^2}, \tag{11}$$

and on expanding the right hand side as a geometric series we get

$$[q^k]B^{[2]}(q, t) = t^k C(t)^{2k+1} (1+C(t))^{k-1} (\delta_{k,0} + C(t)), \tag{12}$$

where  $\delta_{k,0}$  is 1 if  $k = 0$ , and 0 otherwise. The result is easy to check for  $k = 0$ , so let us assume that  $k \geq 1$ . Then

$$[q^k]B^{[2]}(q, t) = t^k \sum_{i \geq 0} \binom{k-1}{i} (C(t))^{3k-i+1} = t^k \sum_{i \geq 0} \binom{k-1}{3k-i} (C(t))^{i+1}.$$

From (7) we get

$$\begin{aligned} [t^n q^k]B^{[2]}(q, t) &= \sum_{i \geq 0} \frac{i+1}{n-k+i+1} \binom{k-1}{3k-i} \binom{2n-2k+i}{n-k} \\ &= \sum_{i \geq 0} \frac{2k+i+1}{n+k+i+1} \binom{k-1}{i-1} \binom{2n+i}{n-k}, \end{aligned}$$

which completes the proof. □



### 3 Segmented permutations

Let  $v = v_1v_2 \cdots v_n$  be a word over  $\mathbb{N}$  without repeated letters. We define the *reduction* of  $v$ , denoted  $\text{red}(v)$ , by

$$\text{red}(v)(i) = |\{j : v_j \leq v_i\}|.$$

In other words,  $\text{red}(v)$  is the permutation of  $[n]$  obtained from  $v$  by replacing the smallest letter in  $v$  with 1, the second smallest with 2, etc. For instance,  $\text{red}(19453) = 15342$ . We will also need a map that is a kind of inverse to  $\text{red}$ . For a finite subset  $V$  of  $\mathbb{N}$ , with  $n = |V|$ , and a permutation  $\pi$  of  $[n]$ , we denote by  $\text{red}_V^{-1}(\pi)$  the word over  $V$  obtained from  $\pi$  by replacing  $i$  in  $\pi$  with the  $i$ th smallest element in  $V$ , for all  $i$ . Here is an example: If  $V = \{1, 3, 4, 5, 9\}$  then  $\text{red}_V^{-1}(15342) = 19453$ .

Given  $\pi$  in  $\mathcal{S}_n$  and  $\sigma$  in  $\mathcal{S}_k$  ( $\sigma$  is often referred to as a *pattern*), an *occurrence* of  $\sigma$  in  $\pi$  is a subword

$$o = \pi(i_1)\pi(i_2) \cdots \pi(i_k)$$

of  $\pi$  such that  $\text{red}(o) = \sigma$ . If, in addition,  $i_r + 1 = i_{r+1}$  for each  $r = 1, 2, \dots, k-1$ , then  $o$  is a *segment-occurrence* of  $\sigma$  in  $\pi$ . We say that  $\pi$  is  $(\sigma)^k$ -*segmented* if there are exactly  $k$  occurrences of  $\sigma$  in  $\pi$ , each of which is a segment-occurrence of  $\sigma$  in  $\pi$ . A  $(\sigma)^0$ -segmented permutation is usually called  $\sigma$ -*avoiding*, and the set of  $\sigma$ -avoiding permutations of  $[n]$  is denoted  $\mathcal{S}_n(\sigma)$ .

If  $\pi$  is  $(\sigma)^k$ -segmented for some  $k$ , then we say that  $\pi$  is  $\sigma$ -*segmented*. We also define

$$\mathcal{R}_n^k(\sigma) = \{ \pi \in \mathcal{S}_n : \pi \text{ is } (\sigma)^k\text{-segmented} \}$$

and  $\mathcal{R}_n(\sigma) = \cup_{k \geq 0} \mathcal{R}_n^k(\sigma)$ . In other words,  $\mathcal{R}_n(\sigma)$  is the set of  $\sigma$ -segmented permutations of length  $n$ . Let

$$R(\sigma; q, t) = \sum_{k, n \geq 0} |\mathcal{R}_n^k(\sigma)| q^k t^n.$$

The first nontrivial case is  $\sigma = 12$ . A permutation is 12-segmented if all its non-inversions are rises. For instance, the permutation 7653412 is 12-segmented while 7643512 is not (45 is a non-inversion, but not a rise).

Let  $\pi \in \mathcal{R}_n(12)$  with  $n \geq 1$ . If the letter 1 precedes the letter  $b$  in  $\pi$ , then  $1b$  is an occurrence of 12 in  $\pi$ . Thus, either 1 is the last letter in  $\pi$ , or 1 is the penultimate letter in  $\pi$  and 2 is the last letter in  $\pi$ . In terms of the generating function  $R = R(12; q, t)$  this amounts to

$$R = 1 + tR + qt^2R.$$

So  $R$  is a rational function in  $t$  and  $q$ . Extracting coefficients we get

$$|\mathcal{R}_n^k(12)| = \binom{n-k}{k} \quad \text{and} \quad |\mathcal{R}_n(12)| = F_n,$$

where  $F_n$  is the  $n$ th Fibonacci number (i.e.,  $F_{n+1} = F_n + F_{n-1}$  with  $F_0 = F_1 = 1$ ). This

is in fact an old result:

$$\begin{aligned}
 & \pi \text{ is 12-segmented} \\
 \iff & \text{ there is no subword } axb, \text{ with } a < b \text{ and } x \neq \epsilon, \text{ in } \pi \\
 \iff & \pi \text{ avoids all linear extensions of the poset } \begin{array}{c} 3 \\ | \\ 1 \quad 2 \end{array} \\
 \iff & \pi \text{ is } \{123, 132, 213\}\text{-avoiding}
 \end{aligned}$$

The  $\{123, 132, 213\}$ -avoiding permutations have been enumerated by Simion and Schmidt [9].

In general, to every pattern  $\sigma$  there is a set of patterns  $\Sigma(\sigma)$  such that a permutation is  $\sigma$ -segmented precisely when it is  $\Sigma(\sigma)$ -avoiding. For example,

$$\Sigma(123) = \{1243, 1234, 1324, 1423, 2134, 2314\};$$

these are the linear extensions of the two posets

$$\begin{array}{c} 4 \\ | \\ 2 \\ | \\ 1 \quad 3 \end{array} \quad \text{and} \quad \begin{array}{c} 4 \\ | \\ 3 \\ | \\ 1 \quad 2 \end{array} .$$

Similarly, we have

$$\Sigma(132) = \{1243, 1342, 1423, 1432, 2143, 2413\}.$$

To summarize,

$$\begin{aligned}
 \mathcal{R}_n(12) &= \mathcal{S}_n(123, 132, 213); \\
 \mathcal{R}_n(123) &= \mathcal{S}_n(1243, 1234, 1324, 1423, 2134, 2314); \\
 \mathcal{R}_n(132) &= \mathcal{S}_n(1243, 1342, 1423, 1432, 2143, 2413).
 \end{aligned}$$

**Theorem 5** *Let  $k \geq 0$  and  $n \geq 3k$ .*

*The 132-segmented permutations of length  $n$  with  $k$  occurrences of 132 are in one-to-one correspondence with bicoloured Dyck paths of length  $2n - 4k$  with  $k$  red up-steps. Thus*

$$|\mathcal{R}_n^k(132)| = |\mathcal{B}_{n-2k,k}| = \binom{n-2k}{k} C_{n-2k},$$

where the last equality is a consequence of Proposition 1.

*The 123-segmented permutations of length  $n$  with  $k$  occurrences of 123 are in one-to-one correspondence with bicoloured Dyck paths of length  $2n - 4k$  with  $k$  red up-steps, each of height less than 2. Thus*

$$|\mathcal{R}_n^k(123)| = |\mathcal{B}_{n-2k,k}^{[2]}| = \sum_{i \geq 0} \frac{2k+i+1}{n-k+i+1} \binom{k-1}{k-i} \binom{2n-4k+i}{n-3k},$$

where the last equality is a consequence of Proposition 4.

**First proof** Let  $n$  be a positive integer, and let  $\pi$  be a 132-segmented permutation of length  $n$ . If the letter  $n$  is not part of any occurrence of 132, then we can factor  $\pi$  as  $\pi = \pi_1 n \pi_2$ , where  $\pi_1$  and  $\pi_2$  are 132-segmented permutations, and  $\pi_2 < \pi_1$  (i.e., every letter in  $\pi_2$  is smaller than every letter in  $\pi_1$ ). On the other hand, if  $n$  is part of an occurrence of 132, then we can factor  $\pi$  as

$$\pi = \pi_1 a n b \pi_2, \quad \text{where } \pi_2 < a < b < \pi_1,$$

and  $\pi_1$  and  $\pi_2$  are 132-segmented permutations. In particular,  $a = |\pi_2| + 1$  and  $b = a + 1$ . Thus the generating function  $R = R(132; q, t)$  satisfies the functional equation

$$R = 1 + (t + qt^3)R^2. \tag{13}$$

It follows that  $R = C(t + qt^3)$ , where  $C(t)$  is the generating function for the Catalan numbers, and hence  $[t^n q^k]R = |\mathcal{B}_{n-2k, k}|$ , as claimed.

Let  $\pi \in \mathcal{R}_n^k(123)$  with  $n \geq 1$ . Then, either  $k = 0$  and  $\pi$  is 123-avoiding, or  $k \geq 1$  and  $\pi$  contains at least one occurrence of 123. Let us focus on the latter case, and let

$$\pi = \pi_1 a b c \pi_2,$$

where  $abc$  is the leftmost occurrence of 123 in  $\pi$ . Then  $a\pi_2$  is  $(123)^{k-1}$ -segmented and  $\pi_1 c$  is 123-avoiding, with the additional restriction that  $a\pi_2$  may not begin with an occurrence of 123. Moreover,

$$a\pi_2 < b < \pi_1 c,$$

or else a non segment-occurrence of 123 would be present. With regard to the generating function  $R = R(123; q, t)$  this decomposition of 123-segmented permutations amounts to the functional equation

$$R = C + qt(C - 1)(\tilde{R} - 1), \tag{14}$$

where  $C = C(t)$  is the generating function of the Catalan numbers, and the coefficient of  $q^k t^n$  in  $\tilde{R} = \tilde{R}(q, t)$  is the number of  $(123)^k$ -segmented permutations of length  $n$  that do not begin with an occurrence of 123. Considering the decomposition above in the special case when  $\pi_1$  is the empty word, we see that  $t^2 q(\tilde{R} - 1)$  is the generating function of the number of 123-segmented permutations that begin with an occurrence of 123; so

$$R = \tilde{R} + qt^2(\tilde{R} - 1). \tag{15}$$

Solving equations (14) and (15) for  $R$ , eliminating  $\tilde{R}$ , we get

$$R = \frac{(1 - qt^3 C^2)C}{1 - (1 + C)qt^3 C^2}. \tag{16}$$

It follows from (11) that  $R = B^{[2]}(qt^2, t)$ , as claimed. □

**Second proof** We shall define a bijection

$$f : \mathcal{R}(132) \rightarrow \mathcal{B}^{[2]},$$

such that  $|f(\pi)| = 2(n - 2k)$  and  $|f(\pi)|_{\bar{u}} = k$  whenever  $\pi \in \mathcal{R}_n^k(132)$ . Our definition of  $f$  will be recursive and we start by defining that  $f(\epsilon) = \epsilon$ . Now, assume that  $n$  is a positive integer, and let  $\pi$  be a 132-segmented permutation of length  $n$ . As in the first proof, if the letter  $n$  is not part of any occurrence of 132, then we can factor  $\pi$  as  $\pi = \pi_1 n \pi_2$ , where  $\pi_1$  and  $\pi_2$  are 132-segmented permutations, and  $\pi_2 < \pi_1$ ; in this case we define

$$f(\pi) = u(f \circ \text{red})(\pi_1)d(f \circ \text{red})(\pi_2).$$

If  $n$  is part of an occurrence of 132, then we can factor  $\pi$  as  $\pi = \pi_1 a n b \pi_2$  where  $\pi_2 < a < b < \pi_1$  and  $\pi_1$  and  $\pi_2$  are 132-segmented permutations; in this case we define

$$f(\pi) = \bar{u}(f \circ \text{red})(\pi_1)d(f \circ \text{red})(\pi_2).$$

For any  $\beta$  in  $\mathcal{B}$ , let

$$\lambda(\beta) = \frac{1}{2}|\beta| + 2|\beta|_{\bar{u}} = |\beta|_u + 3|\beta|_{\bar{u}}.$$

Using induction, it is plain to show that the inverse of  $f$  is given by

$$\begin{aligned} f^{-1}(\epsilon) &= \epsilon; \\ f^{-1}(u\beta_1 d\beta_2) &= (\text{red}_{V_1}^{-1} \circ f^{-1})(\beta_1) n (\text{red}_{V_2}^{-1} \circ f^{-1})(\beta_2), \end{aligned}$$

where  $n = \lambda(\beta_1) + \lambda(\beta_2) + 1$ ,  $V_1 = [\lambda(\beta_2) + 1, n - 1]$ , and  $V_2 = [1, \lambda(\beta_2)]$ ;

$$f^{-1}(\bar{u}\beta_1 d\beta_2) = (\text{red}_{V_1}^{-1} \circ f^{-1})(\beta_1) a n b (\text{red}_{V_2}^{-1} \circ f^{-1})(\beta_2),$$

where  $a = \lambda(\beta_2) + 1$ ,  $b = a + 1$ ,  $n = \lambda(\beta_1) + b + 1$ ,  $V_1 = [b + 1, n - 1]$ , and  $V_2 = [1, a - 1]$ .

To find a bijective proof of the second part of Theorem 5 we will first discuss a decomposition of paths in  $\mathcal{B}^{[2]}$  which is similar to the decomposition of permutations in  $\mathcal{R}(123)$  underlying (14). Let  $\beta \in \mathcal{B}^{[2]}$ . If there is a leftmost occurrence of  $\bar{u}$  in  $\beta$  then the height of that  $\bar{u}$  must be either 0 or 1. Thus we have

$$\mathcal{B}^{[2]} = \mathcal{D} + \mathcal{D}\bar{u}\mathcal{B}^{[1]}d\mathcal{B}^{[2]} + \mathcal{D}u\mathcal{D}\bar{u}\mathcal{D}d\mathcal{B}^{[1]}d\mathcal{B}^{[2]} \quad (17)$$

whose commutative counterpart is

$$\begin{aligned} B^{[2]} &= C + qtCB^{[1]}B^{[2]} + qt^2C^3B^{[1]}B^{[2]} \\ &= C + qt^{-1}(tC + t^2C^3)tB^{[1]}B^{[2]}. \end{aligned} \quad (18)$$

Since  $C = 1 + tC^2$ , the factor  $tC + t^2C^3$  simplifies to  $C - 1$ . Moreover, if we let  $\tilde{\mathcal{B}}^{[2]}$  denote the set of paths in  $\mathcal{B}^{[2]}$  whose first step is  $u$  (i.e., not  $\bar{u}$ ), then

$$\tilde{\mathcal{B}}^{[2]} = \epsilon + u\mathcal{B}^{[1]}d\mathcal{B}^{[2]}, \quad (19)$$

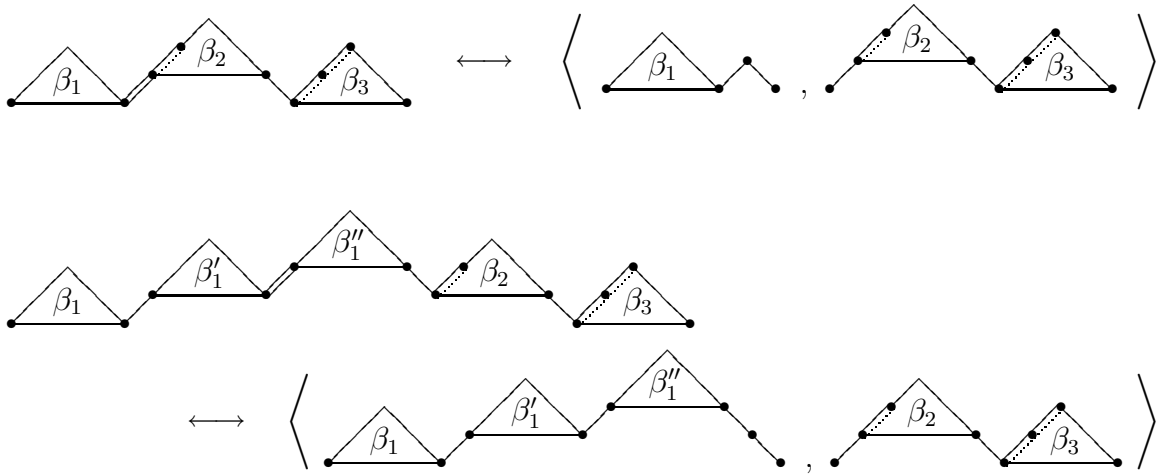


Figure 1: The bijection  $\mathcal{B}^{[2]} \setminus \mathcal{D} \cong (\mathcal{D} \setminus \{\epsilon\}) \times (\tilde{\mathcal{B}}^{[2]} \setminus \{\epsilon\})$ .

and, as a consequence,  $tB^{[1]}B^{[2]} = \tilde{B}^{[2]} - 1$ . Thus (18) can be rewritten as

$$B^{[2]} = C + qt^{-1}(C - 1)(\tilde{B}^{[2]} - 1),$$

which should be compared to (14). This suggests that we should be able to uniquely decompose any path  $\beta$  in  $\mathcal{B}^{[2]} \setminus \mathcal{D}$  into two nonempty paths  $\beta' \in \mathcal{D}$  and  $\beta'' \in \tilde{\mathcal{B}}^{[2]}$  such that  $|\beta| = |\beta'| + |\beta''| - 1$  and  $|\beta|_{\bar{u}} = |\beta''|_{\bar{u}} + 1$ . Indeed, using (17), (19) and

$$\mathcal{D} = \epsilon + \mathcal{D}ud + \mathcal{D}u\mathcal{D}u\mathcal{D}dd,$$

such a decomposition is defined by the map

$$\begin{aligned} \beta_1\bar{u}\beta_2d\beta_3 &\mapsto \langle \beta_1ud, u\beta_2d\beta_3 \rangle, \\ \beta_1u\beta'_1\bar{u}\beta''_1d\beta_2d\beta_3 &\mapsto \langle \beta_1u\beta'_1u\beta''_1dd, u\beta_2d\beta_3 \rangle, \end{aligned}$$

where  $\beta_1, \beta'_1, \beta''_1 \in \mathcal{D}$ ,  $\beta_2 \in \mathcal{B}^{[1]}$ , and  $\beta_3 \in \mathcal{B}^{[2]}$ . We denote by  $\Phi$  the inverse of this map; it is obtained by simply reversing the arrows. See Figure 3 for a schematic diagram of  $\Phi$ .

Let  $h$  be any bijection from  $\mathcal{S}_n(123)$  to  $\mathcal{D}_n$ . For definiteness, we can take  $h$  to be the bijection  $\Psi$  given by Krattenthaler in [5, p. 522]. (A description of  $\Psi$  can be found in the example following this proof.) We shall define a bijection

$$g : \mathcal{R}(123) \rightarrow \mathcal{B}^{[2]}$$

such that  $|g(\pi)| = 2(n - 2k)$  and  $|g(\pi)|_{\bar{u}} = k$ , whenever  $\pi \in \mathcal{R}_n^k(123)$ . If  $\pi$  avoids 123 then let  $g(\pi) = h(\pi)$ . If  $\pi$  does not avoid 123 then, as in the first proof, we can write  $\pi = \pi_1abc\pi_2$ , where  $abc$  is the leftmost occurrence of 123 in  $\pi$ ; in this case, we let

$$g(\pi) = \Phi \langle (g \circ \text{red})(\pi_1c), (g \circ \text{red})(a\pi_2) \rangle.$$

Proving that  $g$  is invertible is similar to proving that  $f$  is invertible. □

We remark that the bijection  $f$  from the first part of the preceding proof maps 132-avoiding permutations onto Dyck paths. In fact, the restriction of  $f$  to  $\mathcal{S}(132)$  is a bijection due to Krattenthaler [5, p. 512].

**Example** The permutation 846572931 is 132-segmented. It has two occurrences of 132, namely 465 and 293. We illustrate the bijection  $f$ , from the first part of the preceding proof, by finding the image of 846572931 under  $f$ :

$$\begin{aligned} f(846572931) &= \bar{u}f(84657)df(1) = \bar{u}udf(4657)dud = \\ &= \bar{u}udu f(465)ddud = \bar{u}udu\bar{u}dddud. \end{aligned}$$

For convenience we have not reduced the permutations in the intermediate steps.

To give an example of how  $g$ , from the second part of the preceding proof, is applied, we first need to describe Krattenthaler's [5, p. 522] bijection  $\Psi$  from  $\mathcal{S}_n(123)$  to  $\mathcal{D}_n$ . Let  $\pi = a_1a_2 \cdots a_n$  be a 123-avoiding permutation. A *right-to-left maximum* is an element  $a_i$  such that  $a_i > a_j$  for all  $j > i$ . Let the right-to-left maxima in  $\pi$  be  $m_1, m_2, \dots, m_s$ , from right to left, so that

$$\pi = \pi_s m_s \cdots \pi_2 m_2 \pi_1 m_1,$$

where  $\pi_i$  is the subword of  $\pi$  between  $m_{i+1}$  and  $m_i$ . If there is an occurrence  $ab$  of 12 in  $\pi_i$  then  $abm_i$  is an occurrence of 123 in  $\pi$ . Therefore, the elements in  $\pi_i$  are in decreasing order. Moreover, we have  $\pi_i < \pi_{i+1}$ .

The Dyck path  $\Psi(\pi)$  is generated from right to left: Read  $\pi$  from right to left. Any right-to-left maximum  $m_i$  is translated into  $m_i - m_{i-1}$  up-steps (with the convention  $m_0 = 0$ ). Any subword  $\pi_i$  is translated into  $|\pi_i| + 1$  down-steps.

We are now ready for an illustration of the bijection  $g$ . The permutation 957841362 is 123-segmented. It has two occurrences of 123, namely 578 and 136. To find the image of 957841362 under  $g$  we proceed as follows:

$$\begin{aligned} g(957841362) &= \Phi\langle (g \circ \text{red})(98), (g \circ \text{red})(541362) \rangle, \\ (g \circ \text{red})(98) &= \Psi(21) = udud, \\ (g \circ \text{red})(541362) &= \Phi\langle (g \circ \text{red})(546), (g \circ \text{red})(12) \rangle, \\ (g \circ \text{red})(546) &= \Psi(213) = uuuddd, \\ (g \circ \text{red})(12) &= \Psi(12) = uudd, \\ \Phi\langle uuuddd, uudd \rangle &= u\bar{u}uddudd, \\ \Phi\langle udud, u\bar{u}uddudd \rangle &= ud\bar{u}\bar{u}uddudd. \end{aligned}$$

Thus  $g(957841362) = ud\bar{u}\bar{u}uddudd$ .

**Corollary 6** For  $k \geq 0$  and  $n \geq 0$  we have

$$|R_n^k(123)| \leq |\mathcal{R}_n^k(132)|.$$

**Proof** The result follows immediately from  $\mathcal{B}_{n,k}^{[2]} \subseteq \mathcal{B}_{n,k}$  and Theorem 5. □

**Corollary 7** *The generating functions  $R(132; q, t)$  and  $R(123; q, t)$  admit the following continued fraction expansions:*

$$R(132; q, t) = \frac{1}{1 - \frac{t + qt^3}{1 - \frac{t + qt^3}{1 - \frac{t + qt^3}{\ddots}}}}, \quad R(123; q, t) = \frac{1}{1 - \frac{t + qt^3}{1 - \frac{t + qt^3}{1 - \frac{t}{1 - \frac{t}{1 - \frac{t}{\ddots}}}}}}.$$

**Proof** The first identity to prove is simply iterating the formula (13). The second identity follows from  $R = B^{[2]}(qt^2, t)$ , (9), and iteration of  $C(t) = 1/(1 - tC(t))$ .  $\square$

**Proposition 8** *The generating function*

$$R(123, 132; p, q, t) = \sum_{\pi \in \mathcal{R}(123) \cap \mathcal{R}(132)} p^{(123)\pi} q^{(132)\pi} t^{|\pi|}$$

*counting  $\{123, 132\}$ -segmented permutations by occurrences of 123 and 132 is the following rational function:*

$$R(123, 132; p, q, t) = \frac{1 - t}{1 - 2t - (p + q)t^3} = \frac{1}{1 - \frac{t + (p + q)t^3}{1 - t}}.$$

**First proof** Let  $n$  be a positive integer, and let  $\pi$  be a  $\{123, 132\}$ -segmented permutation of length  $n$ . We distinguish between three cases:

- (a) If the letter  $n$  is not part of any occurrence of 123 or 132, then we can factor  $\pi$  as  $\pi = \pi_1 n \pi_2$ , where  $\pi_1$  is 12-avoiding,  $\pi_2$  is  $\{123, 132\}$ -segmented, and  $\pi_2 < \pi_1$ .
- (b) If the letter  $n$  is part of an occurrence of 123, then we can factor  $\pi$  as  $\pi = \pi_1 a b n \pi_2$ , where  $\pi_1$  is 12-avoiding,  $\pi_2$  is  $\{123, 132\}$ -segmented, and  $\pi_2 < a < b < \pi_1$ .
- (c) If the letter  $n$  is part of an occurrence of 132, then we can factor  $\pi$  as  $\pi = \pi_1 a n b \pi_2$ , where  $\pi_1$  is 12-avoiding,  $\pi_2$  is  $\{123, 132\}$ -segmented, and  $\pi_2 < a < b < \pi_1$ .

It is clear that an occurrence of 123 can not overlap with an occurrence of 132 without creating a non-segment occurrence of 123 or 132. Therefore, the cases (b) and (c) are mutually exclusive. Thus the generating function  $R = R(123, 132; p, q, t)$  satisfies

$$R = 1 + R(12; 0, t)(t + pt^3 + qt^3)R, \tag{20}$$

where  $R(12; 0, t) = 1/(1 - t)$  is the generating function for 12-avoiding permutations. Solving (20) for  $R$  we obtain the desired result.  $\square$

**Second proof** To give a bijective proof of Proposition 8 we consider lattice paths with three different types of up-steps: let  $\mathcal{T}$  be the language over  $\{u, \bar{u}, \bar{\bar{u}}, d\}$  whose characteristic series is implicitly given by

$$\mathcal{T} = \epsilon + (u + \bar{u} + \bar{\bar{u}})(ud)^*d\mathcal{T},$$

where  $(ud)^* = \epsilon + ud + udud + \dots$ . We may think of a word in the language  $\mathcal{T}$  as a 3-coloured Dyck path whose  $u$ -steps are of height 0 or 1, and whose  $\bar{u}$ - and  $\bar{\bar{u}}$ -steps are of height 0. Applying the substitution  $\mu : \mathbb{C}\langle\langle u, \bar{u}, \bar{\bar{u}}, d \rangle\rangle \rightarrow \mathbb{C}[[p, q, t]]$  defined by

$$\mu = \{ u \mapsto 1, \bar{u} \mapsto p, \bar{\bar{u}} \mapsto q, d \mapsto t \},$$

we see that  $R(123, 132, p, q, t) = \mu(\mathcal{T})(pt^2, qt^2, t)$ . We shall give a bijection

$$f : \mathcal{R}(123) \cap \mathcal{R}(132) \rightarrow \mathcal{T}$$

such that  $|\pi| = |\beta|_u + 3(|\beta|_{\bar{u}} + |\beta|_{\bar{\bar{u}}})$ , where  $\beta = f(\pi)$ . Following the decomposition given in the first proof, we recursively define  $f$  as follows:

$$\begin{aligned} f(\epsilon) &= \epsilon, \\ f(\pi_1 n \pi_2) &= u(f \circ \text{red})(\pi_1) d(f \circ \text{red})(\pi_2), \\ f(\pi_1 a b n \pi_2) &= \bar{u}(f \circ \text{red})(\pi_1) d(f \circ \text{red})(\pi_2), \\ f(\pi_1 a n b \pi_2) &= \bar{\bar{u}}(f \circ \text{red})(\pi_1) d(f \circ \text{red})(\pi_2). \end{aligned}$$

It is straightforward, but tedious, to give the inverse of  $f$ . □

**Example** The permutation 875963124 is  $\{123, 132\}$ -segmented. It has one occurrence of each of the patterns 123 and 132, namely 124 and 596. To illustrate the second proof of Proposition 8 we find the image of 875963124 under  $f$ :

$$f(875963124) = \bar{\bar{u}}f(87)df(3124) = \bar{\bar{u}}udf(7)d\bar{u}f(3)d = \bar{\bar{u}}ududd\bar{u}udd.$$

## 4 Acknowledgments

I thank Petter Brändén for several discussions on the subject of this paper. I also thank Einar Steingrímsson for proofreading a preliminary version of this article.



## 5 Tables

$|\mathcal{B}_{n,k}| :$

$n \setminus k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	4	2					
3	5	15	15	5				
4	14	56	84	56	14			
5	42	210	420	420	210	42		
6	132	792	1980	2640	1980	792	132	
7	429	3003	9009	15015	15015	9009	3003	429

$|\mathcal{B}_{n,k}^{[1]}| :$

$n \setminus k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	3	1					
3	5	9	5	1				
4	14	28	20	7	1			
5	42	90	75	35	9	1		
6	132	297	275	154	54	11	1	
7	429	1001	1001	637	273	77	13	1

$|\mathcal{B}_{n,k}^{[2]}| :$

$n \setminus k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	4	2					
3	5	14	13	4				
4	14	48	62	36	8			
5	42	165	264	217	92	16		
6	132	572	1066	1104	670	224	32	
7	429	2002	4186	5130	3965	1912	528	64

## References

- [1] T. Chow and J. West. Forbidden subsequences and Chebyshev polynomials. *Discrete Math.*, 204(1-3), 119–128, 1999.
- [2] S. Elizalde and M. Noy. Enumeration of subwords in permutations. In *Formal power series and algebraic combinatorics (Tempe, 2001)*, 179–189, Arizona State University, 2001.
- [3] S. Kitaev. Multi-avoidance of generalised patterns. *Discrete Math.*, 260(1-3), 89–100, 2003.
- [4] S. Kitaev and T. Mansour. Simultaneous avoidance of generalized patterns. *Ars Combinatoria*, Vol. LXXV, 2005.
- [5] C. Krattenthaler. Permutations with restricted patterns and Dyck paths. *Advances in Applied Math.*, 27:510, 510–530, 2001.
- [6] G. Kreweras, Sur les éventails de segments, Cahiers du BURO 15, 3-41, 1970.
- [7] T. Mansour and A. Vainshtein. Counting occurrences of 132 in a permutation. *Advances in Applied Math.*, 28:2, 329 – 344, 2002.
- [8] J. Riordan. *Combinatorial identities*. John Wiley & Sons Inc., New York, 1968.
- [9] R. Simion and F. W. Schmidt. Restricted permutations. *European J. Combin.*, 6(4), 383–406, 1985.