ω-Periodic graphs

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Abstract

ω-Periodic graphs are introduced and studied. These are graphs which arise as the limits of periodic extensions of the nearest neighbor graph on the integers. We observe that all bounded degree ω-periodic graphs are amenable. We also provide examples of ω-periodic graphs which have exponential volume growth, non-linear polynomial volume growth and intermediate volume growth.

1 Introduction

In [9] Milnor asked the following question. Does there exist a finitely generated group G such that the volume of the ball of radius n about the identity in the Cayley graph of G grows faster than polynomially but slower than exponentially? This question was answered by Grigorchuk, who constructed a family of groups whose Cayley graphs have intermediate growth [6]. See [8] for a nice description of these groups.

Graphs that have intermediate volume growth also have a connection with long range percolation models in probability. In long range percolation on $\mathbb{Z}$, a random graph is constructed with $\mathbb{Z}$ as its vertex set. The measure is determined by a sequence p_n. For each pair $(u, v) \in \mathbb{Z} \times \mathbb{Z}$ there exists an edge $e_{u,v}$ between u and v with probability $p_{|u-v|}$. The existence of an edge between u and v is determined only by the distance between u and v and is independent of the existence of edges between any other pairs of vertices. Long range percolation on $\mathbb{Z}$ was introduced and studied in [11], [10] and [1] and is commonly used as a model for social networks.

Given a sequence p_n these papers studied the probability that an infinite connected subgraph exists. The more recent papers [4], [3], [7] and [5] considered the case when there is a unique infinite connected subgraph a.s. and studied the volume growth of this graph. In particular they considered the case that $p_1 = 1$ and $p_n = \beta n^{-\alpha}$ for $n > 1$. For these sequences when $\alpha > 2$ the random graph has linear volume growth a.s. When $2 > \alpha > 1$, the random graph has intermediate growth a.s., yet large intervals admit polynomially small boundaries. And when $\alpha \leq 1$, all degrees are infinite. It is conjectured in [3] that when $\alpha = 2$ one gets polynomial volume growth with the degree of the polynomial depending on β.

In this paper we present a simple and natural graph, G, that has intermediate volume growth. Our graph is in some sense a hybrid of the Cayley graphs of the Grigorchuk groups and the graphs from long range percolation. Like the graphs generated by long range percolation our graph is constructed as an extension of the nearest neighbor graph on $\mathbb{Z}$. However our graph has much more regularity than a realization of long range percolation. In particular it occurs as a Schreier graph, the analog of a Cayley graph for a finitely generated group modulo a subgroup, in a natural way.

In addition to the study of one particular graph we also consider a broad family of graphs that contains G.

Definition 1. *A graph G with vertices labelled by $\mathbb{Z}$ is* ω-**periodic** *if it is a union of periodic graphs over $\mathbb{Z}$.*

We show that all ω-periodic graphs are amenable and we illustrate the possible volume growth of ω-periodic graphs. In particular we show that there are ω-periodic graphs of (non-linear) polynomial growth as well as ones with exponential volume growth.

2 The Basic Example

The vertices of G are the integers, $\mathbb{Z}$. Define the sets of edges

$$E_0 = \{(i, i+1) : i \in \mathbb{Z}\}$$

and

$$E_k = \left\{ \left(2^k(j - \frac{1}{2}), 2^k(j + \frac{1}{2}) \right) \right\},$$

for all $j \in \mathbb{Z}$ and $k > 0$.

The graph G has edges

$$E = \bigcup_{k \geq 0} E_k.$$

G is ω-periodic because it is the union of G_k which has edges

$$\bigcup_{i=0}^{k} E_i.$$

We refer to the edges in E_k as the kth layer. The first layer E_1 connects $2i-1$ with $2i+1$ for each i. The kth layer E_k is a copy of E_1 that has been scaled by 2^{k-1}. The degree of every vertex of G (except for 0) is 4. A picture of a portion of G is shown below.

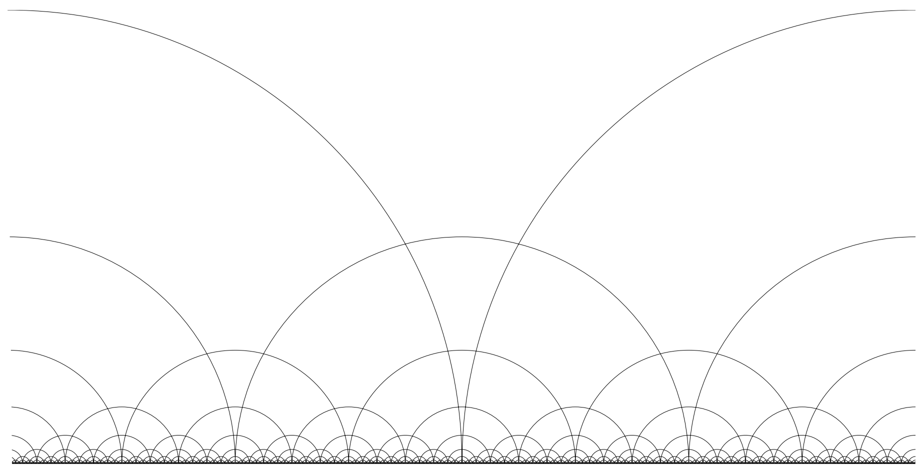

Figure 1: The basic example

We now show how G is realized as a Schreier graph. First we define two maps $u, t : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$. For all but a countable set of $x \in \{0,1\}^{\mathbb{N}}$ there exist $y, z \in \{0,1\}^{\mathbb{N}}$ and $n, p, q \geq 0$ such that

$$x = 1^n \circ 0 \circ y$$

and

$$x = 0^p \circ 1^{q+1} \circ 0 \circ z.$$

For these x we define t and u to be the maps

$$t(x) = t(1^n \circ 0 \circ y) = (0^n \circ 1 \circ y)$$

and

$$u(x) = u(0^p \circ 1^{q+1} \circ 0 \circ z) = (0^p \circ 1 \circ 0^q \circ 1 \circ z).$$

We use the notation $\mathbf{0} = (0,0,0,\dots)$ and $\mathbf{1} = (1,1,1,\dots)$. For all other x we define u and t by

$$t(\mathbf{1}) = \mathbf{0}, \qquad u(\mathbf{0}) = \mathbf{0} \qquad \text{and} \qquad u(0^n \circ \mathbf{1}) = 0^n \circ 1 \circ \mathbf{0}.$$

This group is an example of a self similar action of a group on the set $\{0,1\}^{\mathbb{N}}$. A survey of self similar group actions is given in [2].

Lemma 1. *G is the Schreier graph of the group generated by t and u modulo the stabilizer of $\mathbf{0}$.*

Proof. We start by noting a sequence which is eventually constant is mapped to a sequence which is eventually constant by all of the maps u, u^{-1}, t, and t^{-1}. Also note that for any $m \geq 0$, $t^m(\mathbf{0}) = v$ where

$$\sum_{i=1}^{\infty} v_i 2^i = m$$

and for any $k > 0$, $t^{-k}(\mathbf{0}) = w$ where

$$\sum_{i=1}^{\infty}(1 - w_i)2^i = k - 1.$$

Thus for all g in the group generated by t and u there exists m such that $g(\mathbf{0}) = t^m(\mathbf{0})$. Thus for all g in the group there exists $s = g^{-1}(t^m)$ such that $s(\mathbf{0}) = \mathbf{0}$ and m such that

$$g(s) = t^m.$$

Thus every equivalence class of the group mod the stabilizer of $\mathbf{0}$ contains an element of the form t^m. As $t^m(\mathbf{0}) = t^n(\mathbf{0})$ implies $m = n$ we see that each equivalence class has a unique element of the form t^m. Thus we can associate the vertex m in our graph to the equivalence class containing t^m.

We now check that our graph has the proper edge structure. From the above relations we can check that

$$u\left(t^{2^k(j-\frac{1}{2})}(\mathbf{0})\right) = t^{2^k(j+\frac{1}{2})}(\mathbf{0}) \tag{1}$$

and

$$u^{-1}\left(t^{2^k(j+\frac{1}{2})}(\mathbf{0})\right) = t^{2^k(j-\frac{1}{2})}(\mathbf{0}). \tag{2}$$

If an integer m is not zero then m is of the form $2^k(j + \frac{1}{2})$ for some integers j and $k > 0$. Thus (1) tells us that there is an edge connecting $2^k(j + \frac{1}{2})$ and $2^k(j + \frac{3}{2})$ and (2) tells us that there is an edge connecting $2^k(j + \frac{1}{2})$ and $2^k(j - \frac{1}{2})$. Thus (1) and (2) explain the two edges that emanate from m in layer E_k. The two edges out of m in E_0 connecting m to $m + 1$ and $m - 1$ come from the actions of t and t^{-1}. $\square$

In order to calculate the volume growth of G we would like to calculate the length minimal path between any two integers and then use that to estimate the volume growth. To find a minimal path between two integers u and v we take the following approach. We pick an integer k and we move as quickly as possible from u to a vertex in the kth layer. Then we move in the kth layer and finally we go from the kth layer to v. It is easy to show that a minimal path must take such an approach.

The problem is given u and v how do we identify the optimal k. In general we do not know how to answer that question but we are able to calculate the length minimal paths between 0 and points of the form 2^n. Our main tools are induction and the symmetries of the graphs G_k. By knowing the distance from 0 to 2^k for all $k < n$ we can determine the distance from 0 to 2^n. The inductive relationship is given in Lemma 2 while the formula is determined explicitly in Lemma 3.

Using this information along with the symmetries of G_k we are able to determine the growth rate of $|B_j(0)|$, the number of vertices within distance j of 0. Although $|B_j(0)|$ does not have a simple formula we show in Lemma 5 that $|B_j(0)| \approx j^{.5 \log j}$ and determine $|B_j(0)|$ to within a factor of $16j^2$.

To analyze the growth rate of the ball centered at 0 we make the following definitions. For $i \geq 1$ and $j \in \mathbb{Z}$ let $x_{i,j}$ be the distance from 0 to $(j + 1/2)2^i$ in the graph G_i. The next lemma gives us an inductive relationship for x_i based on x_j, $j < i$. In Lemma 3 we will calculate x_i explicitly.

Lemma 2. *For all i and j*

$$x_{i,j} = x_{i,0} + |j + 1/2| - 1/2.$$

Also $x_1 = 1$, $x_2 = 2$ and for all $i > 2$

$$x_i = x_{i,0} = \min_{0<k<i} 2x_k + 2^{i-k-1} - 1.$$

Proof. The proof is by induction on i. It is easy to check for $i = 1, 2$ that the first formula is true. Fix i and j. Assume the lemma is true for all $k < i$ and for all j.

Fix j. Let P be an oriented path in G_i from 0 to $(j+1/2)2^i$ which has minimal length. Let k be the largest integer such that an edge of P is in E_k. It causes no loss of generality to assume that $k < i$. (This is because there is a first point of the form $(j' + 1/2)2^i$ in P. If the lemma is not true for i and j then it is also not true for i and j'.)

Since P has minimal length and $i > 2$ then it is clear that $k > 0$. Divide P up into three parts, P_1, P_2, and P_3, as follows. Let n_1 be the first point in P of the form $(n_1 + 1/2)2^k$. Let n_2 be the last point in P of the form $(n_2 + 1/2)2^k$. Then P_1 is the portion of P connecting 0 to $(n_1 + 1/2)2^k$, P_2 connects $(n_1 + 1/2)2^k$ to $(n_2 + 1/2)2^k$, and P_3 connects $(n_2 + 1/2)2^k$ to $(j + 1/2)2^i$. Then we have that

$$\begin{aligned}
|P| &= |P_1| + |P_2| + |P_3| \\
&\geq x_{k,n_1} + (n_2 - n_1) + x_{k,2^{i-k-1}-n_2-1+j2^{i-k}} \\
&\geq x_k + |n_1 - 1/2| + 1/2 + (n_2 - n_1) + x_k + |2^{i-k-1} - n_2 + j2^{i-k} - 1/2| - 1/2 \\
&\geq 2x_k + n_1 + (n_2 - n_1) + 2^{i-k-1} - n_2 - 1 + j2^{i-k} \\
&\geq 2x_k + 2^{i-k-1} - 1 + j2^{i-k} \\
&\geq \min_{0<k<i} 2x_k + 2^{i-k-1} - 1 + j2^{i-k}.
\end{aligned}$$

The existence of a path of the minimum distance is easy to construct. □

We now calculate x_i exactly. Let $z_n = n2^n + 1$ and

$$y_n = \frac{n^2 + 3n + 2}{2}.$$

Lemma 3. *$x_{y_n} = z_n$ for all n. Moreover for all i and n, if $y_n < i \leq y_{n+1}$*

$$x_i = z_{n+1} - (y_{n+1} - i)2^n.$$

Proof. The proof is by induction. It is easy to check that the lemma is true for all $i \leq y_1 = 3$. Now assume that the lemma is true for all $j < i$. Note that this implies that the sequence $\{x_j - x_{j-1}\}$ is nondecreasing for all j, $2 \leq j \leq i - 1$.

Let

$$f(k) = f(i, k) = 2x_k + 2^{i-k-1} - 1.$$

By Lemma 2 $x_i = \min_{k<i} f(k)$. Let n be the largest integer such that $y_n < i$. We break the proof up into two cases. The first is when $i < y_{n+1}$ and the second is when $i = y_{n+1}$.
Case 1: We show that the minimum of $f(k)$ occurs at two values, $i-n-1$ and $i-n-2$. More specifically we show that $f(k)$ is decreasing up to $i-n-1$ and increasing afterwards. Let $m = i-n-1$. Since $y_n < i < y_{n+1}$ we have that $y_{n-1} < m \le y_n$. Thus

$$x_m - x_{m-1} = 2^{n-1} \qquad \text{and} \qquad x_{m+1} - x_m \ge 2^{n-1}.$$

We now calculate for $j < m$

$$\begin{aligned} f(j) - f(j-1) &= 2(x_j - x_{j-1}) + (2^{i-j-1} - 2^{i-j}) \\ &\le 2(x_m - x_{m-1}) - 2^{i-j-1} \\ &\le 2 \cdot 2^{n-1} - 2^{i-m} \\ &\le 2^n - 2^{n+1} \\ &< 0. \end{aligned}$$

We also have that

$$f(m) - f(m-1) = 2(x_m - x_{m-1}) + (2^n - 2^{n+1}) = 2 \cdot 2^{n-1} - 2^n = 0.$$

For $l \ge m$ we have

$$\begin{aligned} f(l+1) - f(l) &= 2(x_{l+1} - x_l) + (2^{i-l-2} - 2^{i-l-1}) \\ &\ge 2(x_{m+1} - x_m) - 2^{i-l-2} \\ &\ge 2(2^{n-1}) - 2^{i-m-2} \\ &\ge 2^n - 2^{n-1} \\ &> 0. \end{aligned}$$

From these three calculations it is clear that f obtains its minimum at $i = m-1 = i-n-2$ and $i = m = i-n-1$. By Lemma 2 we have that

$$x_i = 2x_{i-n-1} + 2^n - 1.$$

Note that $y_{n+1} - y_n = n+2$ so if $i = y_{n+1} - l$ then

$$i - n - 1 = y_n - (l-1) \in (y_{n-1}, y_n]$$

so we can apply the induction hypothesis. Doing this we get

$$\begin{aligned} x_i &= 2x_{i-n-1} + 2^n - 1 \\ &= 2x_{y_n-(l-1)} + 2^n - 1 \\ &= 2(z_n - (l-1)2^{n-1}) + 2^n - 1 \\ &= 2z_n - l2^n + 2^n + 2^n - 1 \\ &= 2(n2^n + 1) - l2^n + 2^{n+1} - 1 \\ &= (n+1)2^{n+1} + 1 - l2^n \\ &= z_{n+1} - l2^n. \end{aligned}$$

Case 2: Now we have that $i = y_{n+1}$. We claim that in this case the unique minimum of f occurs at $m = y_n$. By the induction hypothesis we have that

$$x_m - x_{m-1} = 2^{n-1} \qquad \text{and} \qquad x_{m+1} - x_m = 2^n.$$

In exactly the same way as the previous case we calculate for $j \le m$

$$f(j) - f(j-1) \le -2^n$$

and for $l \ge m$

$$f(l+1) - f(l) \ge 2^n.$$

From these calculations it is clear that f obtains its minimum at $m = y_n$. Thus

$$\begin{aligned} x_{y_{n+1}} &= 2z_n + 2^{n+1} - 1 \\ &= 2(n2^n) + 2^{n+1} - 1 \\ &= n2^{n+1} + 2^{n+1} - 1 \\ &= (n+1)2^{n+1} + 1 \\ &= z_{n+1}. \end{aligned}$$

Thus the induction hypothesis is true for i and the lemma is proven. □

Now we use Lemma 3 to estimate the growth rate of the ball around 0 in the graph G.

Lemma 4. *For any $i > 0$ and any m, $0 \le m \le 2^{i-1}$ we have that $d(0, m) \le x_i$.*

Proof. By induction we can see that the distance from any point to the nearest vertex of layer E_k is at most x_k. For any m such that $0 \le m \le 2^{i-1}$ the nearest vertex to m of layer k will lie in the interval $(0, 2^{i-1})$. Thus

$$d(0, m) \le \min_k 2x_k + 2^{i-k-1} - 1 = x_i.$$

□

Lemma 5. *There is a function $G(j)$ (defined below) such that*

$$G(j) \le |B_j(0)| \le 16j^2 G(j).$$

The function $G(j) \approx j^{.5 \log j}$.

Proof. First for $i > 2$ let

$$w_i = \sup_{k>0} 2^{k-1} + 2^k((x_i - 1)/2 - x_k).$$

(We want $i > 2$ because all x_i are odd except $x_2 = 2$.) Thus w_i is the largest integer such that there exists a path from 0 to w_i of length $(x_i - 1)/2$. This makes it is clear that

$$B_{(x_i-1)/2}(0) \subset (-w_i, w_i).$$

Let P be a path of length $(x_i-1)/2$ connecting 0 to w_i and k be such that the longest step in P is of size 2^k. Suppose that $w_i \geq 2^{i-2}$. The graph G_k is symmetric about any point of the form $w_i \pm l2^{k-1}$. Thus by combining P and the reflection of P (about some suitably chosen point) we could construct a path from 0 to 2^{i-1} of length at most $x_i - 1$. This is a contradiction. Thus $w_i < 2^{i-2}$ and

$$B_{(x_i-1)/2}(0) \subset (-2^{i-2}, 2^{i-2}). \tag{3}$$

On the other hand by Lemma 4 gives us that

$$[-2^{i-1}, 2^{i-1}] \subset B_{x_i}(0). \tag{4}$$

Plugging $i = y_{n+1}$ into line (3) and $i = y_n$ into line (4) gives

$$[-2^{y_n-1}, 2^{y_n-1}] \subset B_{x_{y_n}}(0) \subset B_{(x_{y_{n+1}}-1)/2}(0) \subset (-2^{y_{n+1}-2}, 2^{y_{n+1}-2}). \tag{5}$$

If $x_{y_n} \leq j < x_{y_{n+1}}$ then

$$B_{x_{y_n}}(0) \subset B_j(0) \subset B_{y_{n+1}}(0)$$

and

$$[-2^{y_n-1}, 2^{y_n-1}] \subset B_j \subset (-2^{y_{n+2}-2}, 2^{y_{n+2}-2}). \tag{6}$$

We now rewrite this equation using the following definitions. Let

$$f(j) = \sup\{n : \ z_n \leq j\},$$
$$g(j) = 2^{y_{f(j)}-1},$$

and

$$h(j) = 2^{y_{f(j)+2}-2},$$

where was defined earlier to be $y_n = (n^2+3n+2)/2$. Thus line (6) becomes

$$[-g(j), g(j)] \subset B_j \subset (-h(j), h(j)). \tag{7}$$

Then calculating

$$\begin{aligned} \frac{h(j)}{g(j)} &= 2^{y_{f(j)+2}-2-y_{f(j)}+1} \\ &= 2^{.5((f(j)^2+7f(j)+12)-(f(j)^2+3f(j)+2))-1} \\ &= 2^{2f(j)+4}. \end{aligned}$$

By the definitions of z_n and f_j we get the bound $f(j)2^{f(j)} \leq j$ and thus $f(j) < \log(j)$. Putting these two together we get that

$$\frac{h(j)}{g(j)} = 2^{2f(j)+4} < 2^{2\log(j)+4} = 16j^2.$$

Thus

$$[-g(j), g(j)] \subset B_j \subset (-16j^2 g(j) j^2, 16j^2 g(j)).$$

Thus we can pick $G(j) = 2g(j)+1$. Finally we check that

$$G(j) \approx g(j) \approx 2^{.5f(j)^2} \approx j^{.5\log j}.$$

$\square$

3 ω-periodic graphs are amenable

Our general result about ω-periodic graphs is the following.

Proposition 1. *Bounded degree ω-periodic graphs are amenable.*

Proof. Let H be an ω-periodic graph such that every vertex has degree less than D. To show that H is amenable it is enough to show that as n increases the ratio between the size of the boundary of the set $\{0,\ldots,n\}$ and n is approaching 0. Let H_k consist of all $z\in\mathbb{Z}$ such that there exists $z'\in\mathbb{Z}$ with $|z-z'|>k$ and there is an edge in H from z to z'. As H is ω-periodic the sets H_k all have density. Let $d(k)$ be the density of H_k. As H is ω periodic we have that $d(k)\to 0$ as $k\to\infty$. Given ϵ choose k large enough so that $d(k)<\epsilon$. Then choose N large enough so that for all $n>N$

$$|H_k\cap\{0,\ldots,n\}|<2\epsilon n.$$

Hence for all $n>N$

$$|\partial([0,n])|\leq 2k+(2\epsilon n)D.$$

As ϵ was arbitrary we get that

$$\lim_{n\to\infty}\frac{|\partial([0,n])|}{n}=0.$$

□

4 Polynomial Growth

In this section we will use a subgraph $\tilde{G}$ of G in Section 2. We will show that $\tilde{G}$ has nonlinear polynomial growth. Again the vertices of $\tilde{G}$ are the integers, $\mathbb{Z}$, and we define the sets of edges

$$E_0=\{(i,i+1):i\in\mathbb{Z}\}$$

and

$$E_k=\left\{\left(2^k(n-1/2),2^k(n+1/2)\right)\right\},$$

for all $n\in\mathbb{Z}$ and $k>0$. We define the graph $\tilde{G}$ to have edges $E=E_0\bigcup\left(\cup_{k\geq 0}E_{2^k}\right)$.

The proof that the volume of $\tilde{B}_m(0)$ grows polynomially in m is almost exactly like the proof of the volume growth of the full graph in Section 2. First we calculate the distance $\tilde{x}_{2^i}$ from 0 to 2^{2^i-1}. Then we use this information to bound the volume growth. The difference is that we get the formula

$$\tilde{x}_{2^i}=\min_{0<k<i}2\tilde{x}_{2^k}+2^{2^i-2^k-1}-1.$$

We use the notation $\tilde{B}_j(0)$ to be the ball of radius j in $\tilde{G}$ and

$$\tilde{w}_{2^i}=\max_k 2^{2^k-1}+2^{2^k}((\tilde{x}_{2^i}-1)/2-\tilde{x}_{2^k}).$$

This gives us the following lemma.

Lemma 6. *1.* $\tilde{x}_{2^i} = 2\tilde{x}_{2^{i-1}} + 2^{2^{i-1}-1} - 1$

2. $2^{2^{i-1}-1} \leq \tilde{x}_{2^i} \leq 2^{2^{i}-1}$

3. $[-2^{2^i-1}, 2^{2^i-1}] \subset \tilde{B}_{\tilde{x}_{2^i}}(0)$,

4. $B_{2\tilde{x}_{2^i}} \subset (-\tilde{w}_{2^i}, \tilde{w}_{2^i})$ *and*

5. $\tilde{w}_{2^i} \leq 2^{2^i}$.

Proof. The proof of these facts goes exactly as the proof of the corresponding statements in Section 2. $\square$

Lemma 7. *If* $j = \tilde{x}_{2^i}$ *then* $j^2 \leq |\tilde{B}_j(0)| \leq 8j^2$.

Proof. The lower bound follows from condition 3 and the lower bound in condition 2 of Lemma 6. The upper bound follows from conditions 4, 5 and the upper bound in condition 2. $\square$

5 Exponential Growth

In this section we will construct an ω-periodic graph that contains a dyadic tree. Thus the graph has exponential volume growth. Again we let

$$E_0 = \{(i, i+1) : i \in \mathbb{Z}\}$$

be the graph between adjacent integers. Let p_i be the ith prime,

$$l_m = \prod_{i=1}^{i=2^m} (p_i)^{2^i}$$

and

$$t_{m,j} = (p_m)^j, \quad j = 1 \dots 2^{m-1}.$$

Notice that if $t_{m,j} = t_{m',j'}$ then $m = m'$ and $j = j'$. Define the mth layer by

$$E_m = \cup_{k\in\mathbb{Z}} \left(\cup_{j=1}^{2^{m-1}} \left((t_{m,j} + kl_{m+1}, t_{m+1,2j-1} + kl_{m+1}) \cup (t_{m,j} + kl_{m+1}, t_{m+1,2j} + kl_{m+1}) \right) \right).$$

Thus for each j the layer E_m has an edge connecting $t_{m,j}$ to $t_{m+1,2j-1}$ and an edge connecting $t_{m,j}$ to $t_{m+1,2j}$. This is repeated with period l_{m+1}.

Another way to describe E_m is as follows. Let

$$V_m = \{t_{m,j}\}_{j=1}^{2^{m-1}}.$$

Also let $L_m = V_m + \mathbb{Z}l_{m+1}$ and $R_m = V_{m+1} + \mathbb{Z}l_{m+1}$. Then every edge in E_m has its leftmost endpoint in L_m and its rightmost endpoint in R_m. Also every point in L_m is the left hand end point of two edges in E_m. Every point in R_m is the right hand end point of one edge in E_m.

We show that the graph contains a dyadic tree and that it has bounded degree.

Lemma 8. *There is a dyadic tree rooted at* $t_{1,1} = 2$.

Proof. Note that $V_{m+1} \subset L_{m+1} \cap R_m$. The 2^m vertices at distance m from the root are V_{m+1}. $\square$

Lemma 9. $L_m \cap \left(\cup_{j=1}^{m-1} L_j\right) = \emptyset$ *and* $R_m \cap \left(\cup_{j=1}^{m-1} R_j\right) = \emptyset$.

Proof. Fix an m. Every element of $L_m \mod l_{m+1}$ is only divisible by powers of p_m. Every element of $\cup_{j=1}^{m-1} L_j \mod l_{m+1}$ is divisible by at least one prime less than or equal to p_{m-1}. Thus the first two sets are disjoint. Every element of $R_m \mod l_{m+1}$ is only divisible by powers of p_{m+1}. Every element of $\cup_{j=1}^{m-1} R_j \mod l_{m+1}$ is divisible by at least one prime less than or equal to p_m. Thus the last two sets are disjoint. $\square$

Lemma 10. *The degree of any vertex in E is at most five.*

Proof. For any $z \in \mathbb{Z}$ the degree of z is 2 plus twice the number of m such that $z \in L_m$ plus the number of m such that $z \in R_m$. Thus by Lemma 9 the degree of a vertex is at most five. $\square$

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