An asymptotic Result for the Path Partition Conjecture

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Abstract

The detour order of a graph G, denoted by $\tau(G)$, is the order of a longest path in G. A partition of the vertex set of G into two sets, A and B, such that $\tau(\langle A\rangle) \leq a$ and $\tau(\langle B\rangle) \leq b$ is called an (a,b)-*partition* of G. If G has an (a,b)-partition for every pair (a,b) of positive integers such that $a+b=\tau(G)$, then we say that G is τ-partitionable. The Path Partition Conjecture (PPC), which was discussed by Lovász and Mihók in 1981 in Szeged, is that every graph is τ-*partitionable.* It is known that a graph G of order n and detour order $\tau = n-p$ is τ-partitionable if $p=0,1$. We show that this is also true for $p=2,3$, and for all $p\geq 4$ provided that $n \geq p(10p-3)$.

1 Introduction

The *vertex set* and *edge set* of a graph G is denoted by $V(G)$ and $E(G)$, respectively. The degree of a vertex v in G will be denoted by $d_G(v)$. If H is a subgraph of G, the *open* H-*neighbourhood of* v is the set $N_H(v) = \{u \in V(H) - v \mid uv \in E(G)\}$. If S is a subset of $V(G)$, we write $N_H(S) = \bigcup_{v\in S} N_H(v)$. The subgraph of G induced by S is denoted by $\langle S\rangle$.

A longest path in a graph G is called a *detour* of G. The number of vertices in a detour of G is called the *detour order* of G and denoted by $\tau(G)$. The number of vertices in a longest cycle of G is called the *circumference* of G and denoted by $c(G)$. A graph

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of order n will be called *hamiltonian* or *traceable*, if $c(G) = n$ or $\tau(G) = n$, respectively. The vertex independence number of a graph G is denoted by $\alpha(G)$.

A partition of the vertex set of G into two sets, A and B, such that $\tau(\langle A \rangle) \leq a$ and $\tau(\langle B \rangle) \leq b$ is called an (a, b)-*partition* of G. If G has an (a, b)-partition for every pair (a, b) of positive integers such that $a + b = \tau(G)$, then we say that G is τ-*partitionable.* The following conjecture is known as the Path Partition Conjecture (or the PPC, for short).

Conjecture 1 *Every graph is τ-partitionable.*

The PPC was discussed by Lovász and Mihók in 1981 in Szeged and treated in the theses [10] and [15]. The PPC first appeared in the literature in 1983, in a paper by Laborde, Payan and Xuong [11]. Although that paper dealt mainly with directed graphs, they stated the PPC only for undirected graphs. In 1995 Bondy [2] stated a directed version of the PPC. In [3] the PPC is stated in the language of the theory of hereditary properties of graphs. It is also mentioned in [5]. Results on the PPC and its relationship with other conjectures appear in [4], [6], [7], [8] and [9] . A summary of the conjecture status is given in [7].

A subset S of $V(G)$ is called a P_n-*kernel* of G if $\tau(\langle S \rangle) \leq n - 1$ and every vertex $v \in V(G) - S$ is adjacent to an end-vertex of a path of order $n - 1$ in $\langle S \rangle$ (cf. [5] and [13]). If $\tau(G) = a + b$ and G has a P_{a+1}-kernel S, then $(S, V(G) - S)$ is an (a, b)-partition of G. It is shown in [6] that every graph has a P_n-kernel for every $n \leq 7$, and in [14] it is shown that every graph has a P_8-kernel. These results imply that the PPC holds for $a \leq 7$. However, Aldred and Thomassen [1] have recently constructed a graph that has no P_{364}-kernel.

2 Main Results

In this section we state our two main theorems, together with the main lemmas and the partition strategy used in the proofs. The proofs are presented in Section 4.

The **Partition Strategy**

Let G be a graph of order n and detour order $\tau = n - p$. Our main strategy is to find a subset $A_1 \subset V(G)$ such that $|A_1| = p$ and $|N_{G-A_1}(A_1)| \leq \frac{\tau+1}{2}$.

If $\tau = a + b$; $1 \leq a \leq b$, and we can find such a set A_1, then we choose B to be a subset of $V(G) - A_1$, consisting of exactly b vertices and containing $N_{G-A_1}(A_1)$ (since $b \geq \frac{\tau+1}{2}$, this is possible). Then we set $A_2 = V(G) - A_1 - B$ and put $A = A_1 \cup A_2$. Since $|A_2| = n - p - b = a$, it follows that $\tau(A) \leq \max\{a, p\}$. Thus (A, B) will be an (a, b)-partition if $a \geq p$.

Since we know that the PPC holds for $a \leq 7$, our partition strategy will yield all the necessary partitions if $\tau(A_1) \leq 8$.

The following two lemmas will enable us to find all the necessary partitions when $p = 3$, by applying our partition strategy.

Lemma 2.1

(a) *Let P be a longest path in a nontraceable graph G and let $H = G - V(P)$. If H consists of $k \geq 1$ components, then $|N_P(H)| \leq \frac{\tau(G)-1}{2} + \binom{k}{2}$.*

(b) *Let C be a longest cycle in a nonhamiltonian graph G and $H = G - V(C)$. If H consists of $k \geq 1$ components, then $|N_C(H)| \leq \frac{c(G)}{2} + \binom{k}{2}$.*

Lemma 2.2 *Let G be a graph of order n and detour order $\tau = n - p$ with $p \geq 2$, and let P be a detour of G with vertices labeled $v_1, v_2, \ldots, v_\tau$ such that $d_P(v_1) \leq d_P(v_\tau)$. If $H = G - V(P)$ consists of $k \geq 2$ components $H_1, H_2, \ldots, H_k$, then $|N(v_1) \cup N_P(H_i) \cup N_P(H_j)| \leq \frac{\tau+1}{2}$, for $1 \leq i < j \leq k$.*

Our first theorem follows from Lemma 2.1(a) and Lemma 2.2.

Theorem 2.3 *Let G be a graph of order n and detour order $\tau = n - p$, with $0 \leq p \leq 3$. Then G is τ*-partitionable.

When $n \geq p\,(10p - 3)$ the next lemma allows us to apply our partition strategy when $p \geq 4$, thus yielding (a, b)-partitions when $a \leq p$.

Lemma 2.4 *Let G be a graph of order n and detour order $\tau = n - p$, with $p \geq 4$. Let P be a detour of G and let $H = G - V(P)$. If $|N_P(H)| > \frac{\tau+1}{2}$ then there exists an independent set $Y \subset V(P)$ with $|Y| = p$ such that $|N_{P-Y}(Y)| \leq \frac{\tau-1}{2}$, provided $n \geq p\,(10p-3)$.*

The next lemma enables us to find (a, b)-partitions when $a > p$, provided $n \geq 4p^2 - 6p - 4$.

Lemma 2.5 *Let G be a graph of order n and detour order $\tau = n - p$, with $p \geq 1$. Suppose $\tau = a + b$; $1 \leq a \leq b$. If $a \leq \alpha(G) - p$, then G has an (a, b)-partition.*

Our second theorem uses Lemmas 2.1, 2.4 and 2.5, together with Lemmas 3.3 and 3.4.

Theorem 2.6 *Let G be a graph of order n and detour order $\tau = n - p$, with $p \geq 4$. Suppose $\tau = a + b$; $1 \leq a \leq b$. Then the following hold:*

(a) If $a \geq p$, then G has an (a, b)-partition, provided $n \geq p\,(10p - 3)$.

(b) If $a < p$, then G has an (a, b)-partition, provided $n \geq 4p^2 - 6p - 4$.

Since $p\,(10p - 3) \geq 4p^2 - 6p - 4$ for all $p \geq 4$, we have

Corollary 2.7 *Let G be a graph of order n and detour order $\tau = n - p$, with $p \geq 4$. Then G is τ-partitionable, provided $n \geq p\,(10p - 3)$.*

3 Auxiliary Results

If P is a path in G with a fixed orientation and $u, v \in V(P)$, then v^- and v^+ denote the immediate predecessor and immediate successor of v on P, respectively. We denote the segment of P from u to v by $u\overrightarrow{P}v$ and the reverse segment from v to u by $v\overleftarrow{P}u$. We shall refer to the vertices in the segment $u\overrightarrow{P}v$ as the *interval* $[u, v]$.

Lemma 3.1 *Let G be a connected nontraceable graph with detour order τ and let P be a detour of G, with vertices labelled $v_1 \dots v_\tau$. Let $H = G - V(P)$ and let $H_1, \dots, H_k$ be the components of H. Then the following hold:*

(a) *If $u \in V(P)$, then $N_{H_i}(u) \cap N_{H_i}(u^+) = \emptyset$, for $i = 1, \dots, k$.*

(b) *If $\{u, v\} \subseteq N_P(H_i)$ for some i, then $\{u^+, v^+\} \not\subseteq N_P(H_j)$ for any j.*

(c) *If $u \in N_P(v_1)$, then $u^- \notin N_P(v_\tau)$.*

(d) *If $u \in N_P(H)$, then $u^+ \notin N_P(v_1)$ and $u^- \notin N_P(v_\tau)$.*

Proof.

(a) Suppose u and u^+ both have neighbours in some component H_i of H. Then a path of order greater than τ is obtained from P by replacing the edge uu^+ with a $u - u^+$ path whose internal vertices are in H_i.

(b) Suppose, to the contrary, that $\{u^+, v^+\} \subseteq N_P(H_j)$ for some j. Then it follows from (a) that $i \neq j$. Let Q be a path in H_i from a neighbour of u to a neighbour of v and let R be a path in H_j from a neighbour of u^+ to a neighbour of v^+. Then the path $v_1\overrightarrow{P}u\overrightarrow{Q}\,v\overleftarrow{P}u^+Rv^+\overrightarrow{P}v_\tau$ is longer than P.

(c) If $u^- \in N(v_\tau)$, then $v_1\overrightarrow{P}u^{-1}v_\tau\overleftarrow{P}uv_1$ is a cycle of order τ in G. But then there is a path of order $\tau + 1$ in G consisting of this cycle together with a vertex in $N_H(P)$.

(d) Let h be a neighbour of u in H. If $u^+ \in N(v_1)$, then the path $hu\overleftarrow{P}v_1u^+\overrightarrow{P}v_\tau$ is longer than P. Thus $u^+ \notin N(v_1)$. Similarly, $u^- \notin N(v_\tau)$. ■

Lemma 3.2 *Let G be a nontraceable graph with detour order τ and let P be a detour of G, with vertices labelled $v_1 \dots v_\tau$. Let H_k be a component of $H = G - V(P)$ and denote the neighbours of H_k on P by $u_1, \dots, u_s$, labelled according to the order in which they appear on P. Then:*

(a) *$N_P^+(H_k) = \{u_1^+, \dots, u_s^+\}$ is an independent set.*

(b) *Consider any pair i, j, with $1 \leq i < j \leq s$ and suppose $x \in N_P(u_i^+)$. Then:*

  (i) *If $x \in [v_1, u_i]$ or $x \in [u_j^+, v_\tau]$, then $x^+ \notin N_P(u_j^+)$.*

  (ii) *If $x \in [u_i^{++}, u_j]$, then $x^- \notin N_P(u_j^+)$.*

Proof.

(a) Suppose two vertices, $u_i^+, u_j^+ \in N_P^+(H_k)$ are adjacent to one another. Let Q be a path in H_k from a neighbour of u_i to a neighbour of u_j. Then the path $v_1 \overrightarrow{P} u_i Q\, u_j \overleftarrow{P} u_i^+ u_j^+ \overrightarrow{P} v_\tau$ is longer than P. This contradiction proves that $N_P^+(H_1)$ is an independent set.

(b) (i) Let Q be a path in H_k from a neighbour of u_i to a neighbour of u_j. Suppose $x^+ \in N_P\left(u_j^+\right)$. If $x \in [v_1, u_i]$, then the path $v_1 \overrightarrow{P} x u_i^+ \overrightarrow{P} u_j \overleftarrow{Q} u_i \overleftarrow{P} x^+ u_j^+ \overrightarrow{P} v_\tau$ is longer than P. If $x \in \left[u_j^+, v_\tau\right]$, then the path $v_1 \overrightarrow{P} u_i Q u_j \overleftarrow{P} u_i^+ x \overleftarrow{P} u_j^+ x^+ \overrightarrow{P} v_\tau$ is longer than P.

(ii) If $x \in \left[u_i^{++}, u_j\right]$ and $x^- \in N_P\left(u_j^+\right)$ then $v_1 \overrightarrow{P} u_i Q u_j \overleftarrow{P} x u_i^+ \overrightarrow{P} x^- u_j^+ \overrightarrow{P} v_\tau$ is a path with more vertices than P. ■

The following result is proved in [8]

Lemma 3.3 *Let G be a graph and (a, b) any pair of positive integers such that $\tau(G) = a + b$. If $c(G) \leq b + 2$, then G has an (a, b)-partition.*

The following Lemma was proved in [7].

Lemma 3.4 *Let G be a graph with $\tau(G) = a + b$; $1 \leq a \leq b$. If G has a cycle C of order greater than b such that $|N_C(G - V(C))| \leq b$, then G has an (a, b)-partition.*

Corollary 3.5 *Let C be a longest cycle in a graph G. If $|N_C(G - V(C))| \leq \left\lceil \frac{\tau(G)}{2} \right\rceil$, then G is τ-partitionable.*

Corollary 3.6 *Let C be a longest cycle in a graph G. If $\tau(G) \leq c(G) + 1$, then G is τ-partitionable.*

Proof. Two consecutive vertices of C cannot both have neighbours in $G - V(C)$, otherwise G would have a path of order $c(G) + 2$. Thus $|N_C(G - V(C))| \leq \left\lceil \frac{\tau(G)}{2} \right\rceil$ and hence G is τ-partitionable, by Corollary 3.5. ■

4 Proofs of the Main Results

Proof of Lemma 2.1.

(a) If H is connected, then $|N_P(H)| \leq \frac{\tau(G)-1}{2}$. So let $k \geq 2$. Suppose for two components of H, say H_1, H_2, the neighbourhood of $H_1 \cup H_2$ contains three pairs of consecutive vertices $\{u, u^+\}$, $\{v, v^+\}$ and $\{w, w^+\}$ on P. Assume $u \in N_P(H_1)$. Then, by Lemma 3.1(a) and (b), we must have $\{u, v^+\} \subseteq N_P(H_1)$ and $\{u^+, v\} \subseteq N_P(H_2)$. Now, by Lemma 3.1(a), either we have $w \in N_P(H_1)$ and $w^+ \in N_P(H_2)$, or we have

$w \in N_P(H_2)$,and $w^+ \in N_P(H_1)$. By Lemma 3.1(b) the first case cannot occur, since we cannot have $\{u, w\} \subseteq N_P(H_1)$ and $\{u^+, w^+\} \subseteq N_P(H_2)$. Also, the second case cannot occur, since we cannot have $\{v, w\} \subseteq N_P(H_2)$ and $\{v^+, w^+\} \subseteq N_P(H_1)$. This proves that each of the $\binom{k}{2}$ pairs of components of H has at most two pairs of consecutive vertices on P in their neighbourhood union. Thus, for each neighbour of H on P, the next vertex is a non-neighbour, except in at most $2\binom{k}{2}$ cases.

Since $v_1, v_\tau \notin N_P(H)$ and P has τ vertices, we conclude that

$$1 + 2|N_P(H)| - 2\binom{k}{2} \leq \tau(G),$$

and hence $|N_P(H)| \leq \frac{\tau(G)-1}{2} + \binom{k}{2}$.

(b) By the same arguments as above we conclude that

$$2|N_C(H)| - 2\binom{k}{2} \leq c(G),$$

which gives $|N_C(H)| \leq \frac{c(G)}{2} + \binom{k}{2}$. ■

Proof of Lemma 2.2. If $d_P(v_1) + d_P(v_\tau) \geq \tau$, then there is a cycle containing $v_1, v_2, \ldots, v_\tau$, by Ore's Lemma. Since some vertex of H is adjacent to some vertex on this cycle, we would have a path of order at least $\tau + 1$ in G, a contradiction. Hence we may assume that $d_P(v_1) + d_P(v_\tau) \leq \tau - 1$.

We shall call the case where $d_P(v_1) + d_P(v_\tau) = \tau - 1$ the *saturated case.*

Let H_i and H_j be two components of H. If $N_P(H_i) \cup N_P(H_j) \subseteq N(v_1) \cup N(v_\tau)$, then it follows from Lemma 3.1 that $|N_P(H_i) \cup N_P(H_j) \cup N(v_1)| \leq \frac{\tau-1}{2}$. We may therefore assume that $q \geq 1$ vertices in $N_P(H_i) \cup N_P(H_j)$ are not neighbours of v_1 or v_τ.

Suppose $N_P(H_i) \cup N_P(H_j)$ has d pairs of consecutive vertices. As shown in the proof of Lemma 2.1, $d = 0, 1$, or 2.

We call an interval $I = [v_r, v_s]$ a *t-hole* if $t = s - r + 1$ and no vertex in I is in $N(v_1) \cup N(v_\tau)$ but $v_{r-1}, v_{s+1} \in N(v_1) \cup N(v_\tau)$. We now compare the number of neighbours that H can have in the holes of P with the value that $d_I(v_1) + d_I(v_\tau)$ would have had in the saturated case. We need to consider three types of t-holes:

T1: $v_{r-1} \in N(v_\tau), v_{s+1} \in N(v_1)$:

Since $v_{r+1}, v_{s-1} \notin N_P(H)$, it follows that $|N_I(H_i) \cup N_I(H_j)| \leq \frac{t-1+d}{2}$. In the saturated case, $d_I(v_1) + d_I(v_\tau)$ would have been equal to $t - 1$.

T2: $v_{r-1} \in N(v_1) - N(v_\tau), v_{s+1} \in N(v_\tau) - N(v_1)$:

In this case $|N_I(H_i) \cup N_I(H_j)| \leq \frac{t+1+d}{2}$, and in the saturated case $d_I(v_1) + d_I(v_\tau)$ would have been equal to $t + 1$.

T3:$v_{r-1}, v_{s+1} \in N(v_1), v_{r-1} \notin N(v_\tau)$ $(v_{r-1}, v_{s+1} \in N(v_\tau), v_{s+1} \notin N(v_1))$:

Since $v_{r-1} \notin N(v_\tau)$ $(v_{s+1}) \notin N(v_1)$, it follows that $|N_I(H_i) \cup N_I(H_j)| \leq \frac{t+d}{2}$. In the saturated case $d_I(v_1) + d_I(v_\tau)$ would have been equal to t.

Thus, in each hole I, the value that $d_I(v_1) + d_I(v_\tau)$ would have had in the saturated case is greater than or equal to $2|N_I(H_i) \cup N_I(H_j)| - d$. Since $H_i \cup H_j$ has altogether q neighbours in the holes of P, we have

$$\begin{aligned} d_P(v_1) + d_P(v_\tau) &\leq \tau - 1 - (2q - d) \\ &\leq \tau + 1 - 2q, \text{ since } d \leq 2. \end{aligned}$$

Hence $d(v_1) \leq \frac{\tau+1}{2} - q$ and therefore

$$\begin{aligned} |N(v_1) \cup N_P(H_i) \cup N_P(H_j)| &\leq (\frac{\tau+1}{2} - q) + q \\ &= \frac{\tau+1}{2}. \blacksquare \end{aligned}$$

Proof of Theorem 2.3. If $p = 0, 1$, then G is τ-partitionable (cf. [4]).

Now suppose $p \geq 2$ and P is a detour of G with vertices labelled $v_1, \ldots, v_\tau$, with $d(v_1) \leq d(v_\tau)$. Put $H = G - V(P)$. Then $|V(H)| = p$.

If H has at most two components, put $A_1 = H$. Then it follows from Lemma 2.1 that $|N_{G-A_1}(A_1)| \leq \frac{\tau+1}{2}$, so we get all the necessary partitions by applying our **Partition Strategy**.

If H has three components, H_1, H_2, H_3, put $A_1 = H_1 \cup H_2 \cup \{v_1\}$. Then it follows from Lemma 2.2 that $|N_{G-A_1}(A_1)| \leq \frac{\tau+1}{2}$, so again we get all the necessary partitions by applying our **Partition Strategy**. $\blacksquare$

Proof of Lemma 2.4. Let $u \in V(H)$ be a vertex which has a maximum number of neighbours on P. Then

$$|N_P(u)| > \frac{\tau+1}{2p} = \frac{n-p+1}{2p} \geq \frac{10p^2-4p+1}{2p} > 5p - 2.$$

By Lemma 3.1(a) no vertex in $N_P^+(u)$ is adjacent to u and by Lemma 3.1(b) no two vertices in $N_P^+(u)$ have a common neighbour in H. Hence at most $p-1$ vertices of $N_P^+(u)$ have neighbours in H. Let

$$W = \left\{ w \in N_P^+(u) : N_H(w) = \emptyset \right\}.$$

Then

$$|W| \geq |N_P(u)| - (p-1) \geq 4p.$$

Let the vertices of W be $w_1, \ldots, w_r$, labelled according to the order in which they appear on P. By Lemma 3.2(a), W is an independent set.

Now let I be an interval on P such that all the vertices of I except the first one is in $N(W)$. From Lemma 3.2 we deduce the following:

(1) The set $N_I(w_i)$ consists of consecutive vertices.

(2) $|N_I(w_i) \cap N_I(w_j)| \leq 1$, for $1 \leq i < j \leq r$.

(3) If $I \subseteq [v_1, w_1^-]$, then the I-neighbourhoods of the vertices of W appear in the order $N_I(w_r)$, $N_I(w_{r-1}), \ldots,$ $N_I(w_1)$. Moreover, if $1 \leq i < j \leq r$ and $|N_I(w_i) \cap N(w_k)| = 1$, then $N_I(w_k) \subseteq N_I(w_i) \cap N(w_j)$ for all k such that $i \leq k \leq j$.

(4) If $I \subseteq V[w_s, w_{s+1}^-]$ for some $s \in \{1, \ldots, r-1\}$, then the I-neighbourhoods of the vertices in W appear in the order $N_I(w_s)$, $N_I(w_{s-1}), \ldots,$ $N_I(w_1)$, $N_I(w_r)$, $N_I(w_{r-1}), \ldots,$ $N_I(w_{s+1})$. Now suppose $1 \leq i < j \leq r$ and
$|N_I(w_i) \cap N(w_j)| = 1$. Then the following hold:
If $j \leq s$ or $i \geq s+1$, then $N_I(w_k) \subseteq N_I(w_i) \cap N(w_j)$ for all k such that $i \leq k \leq j$.
If $i \leq s$ and $j \geq s+1$, then $N_I(w_k) \subseteq N_I(w_i) \cap N(w_j)$ for all k such that $k \leq i$ or $k \geq j$.

Let $q = \lfloor \frac{|W|}{p} \rfloor$ and put

$$W_i = \{w_{(i-1)p+1}, \ldots, w_{ip}\} \text{ for } 1 \leq i \leq q.$$

Then $|W_i| = p$, for $i = 1, \ldots, q$.

We now partition $P - v_\tau$ into consecutive intervals $I_1, \ldots I_r$ such that the initial vertex of each of the intervals is not in $N_P(W)$, while all the others are. It now follows from the structure of the I_j-neighbourhoods of the vertices in W (as explained in (1)-(4) above) that

$$\sum_{i=1}^{q} |N_{I_j}(W_i)| \leq |I_j| - 1 + q \text{ for } j = 1, \ldots, r.$$

If $|I_j| \geq 3$, then $|I_j| - 1 + q \leq \frac{|I_j| q}{2}$, since $q \geq 4$. Furthermore, for each $i \in \{1, \ldots, q\}$, we have $\left|N_{I_j}(W_i)\right| = 0$ if $|I_j| = 1$ and $\left|N_{I_j}(W_i)\right| \leq 1$ if $|I_j| = 2$. Thus

$$\sum_{i=1}^{q} |N_{I_j}(W_i)| \leq \frac{|I_j|\, q}{2} \text{ for } j = 1, \ldots, r.$$

Hence

$$\sum_{i=1}^{q} |N_P(W_i) \leq \sum_{j=1}^{r} \frac{|I_j|\, q}{2} = \frac{(\tau - 1)\, q}{2}$$

and hence

$$\min_{1 \leq i \leq q} |N_P(W_i)| \leq \frac{\tau - 1}{2}.$$

Now let Y be a subset W_i achieving this minimum. Then $|N_{G-Y}(Y)| = |N_{P-Y}(Y)| \leq \frac{\tau-1}{2}$. ■

Proof of Lemma 2.5. Let $A \subset V(G)$ be an independent set with $|A| = \alpha(G)$ and set $B = V(G) - A$. Then $\tau(\langle A \rangle) = 1 \leq a$ and $\tau(\langle B \rangle) \leq n - \alpha(G) \leq \tau + p - (a+p) = b$. ■

Proof of Theorem 2.6. Let P be a detour of G and $H = G - V(P)$.

(a) If $|N_P(H)| \leq \frac{\tau+1}{2}$ then, since $\tau(H) \leq p \leq a$, we can apply the **Partition Strategy** with $A_1 = V(H)$.

If $|N_P(H)| > \frac{\tau+1}{2}$, then by Lemma 2.5 there exists an independent set $Y \subset V(P)$ such that $N(Y) = N_P(Y)$ and $|N_P(Y) - Y| \leq \frac{\tau-1}{2}$. Now we can apply the **Partition Strategy** with $A_1 = Y$.

(b) We distinguish two cases.
Case 1: $c(G) \leq n - 2p + 3$:
In this case

$$b = \tau - a = n - p - a \geq n - p - (p-1) = n - 2p + 1.$$

Thus we have $b \geq c(G) - 2$. Hence G has an (a,b)-partition by Lemma 3.3.
Case 2: $n - 2p + 3 < c(G)$:
If $\alpha(G) \geq 2p - 1$, then G has an (a,b)-partition by Lemma 2.5, so we may assume that $\alpha(G) \leq 2p - 2$. Let C be a longest cycle of G. Let $H = G - V(C)$ and suppose H has k components. Then $k \leq \alpha \leq 2p - 2$. Let $|N_C(H)| = t$. By Lemma 3.4 we may assume that $b \leq t - 1$ and by Lemma 3.3, $t \leq \frac{c(G)}{2} + \binom{k}{2}$. Thus

$$b \leq \frac{c(G)}{2} + \binom{2p-2}{2} - 1.$$

By Corollary 3.6 we may assume that $\tau \geq c(G) + 2$. Now

$$b = \tau - a \geq c(G) + 2 - (p-1).$$

It follows that

$$\begin{aligned} c(G) - p + 3 &\leq \frac{c(G)}{2} + \binom{2p-2}{2} - 1; \\ \text{i.e.} \quad c(G) &\leq 4p^2 - 8p - 2. \end{aligned}$$

But by our assumption, $c(G) \geq n - 2p + 3$; hence

$$\begin{aligned} n - 2p + 3 &\leq 4p^2 - 8p - 2, \\ \text{i.e.} \quad n &\leq 4p^2 - 6p - 5, \end{aligned}$$

contradicting our assumption. ■

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