Bounds for the average L^p-extreme and the L^∞-extreme discrepancy

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Abstract

The extreme or unanchored discrepancy is the geometric discrepancy of point sets in the d-dimensional unit cube with respect to the set system of axis-parallel boxes. For $2 \leq p < \infty$ we provide upper bounds for the average L^p-extreme discrepancy. With these bounds we are able to derive upper bounds for the inverse of the L^∞-extreme discrepancy with optimal dependence on the dimension d and explicitly given constants.

1 Introduction

Let $\mathcal{R}_d$ be the set of all half-open axis-parallel boxes in the d-dimensional unit ball with respect to the maximum norm, i.e.,

$$\mathcal{R}_d = \{[x,y) \,|\, x,y \in [-1,1]^d, x \leq y\},$$

where $[x,y) := [x_1,y_1)\times\ldots\times[x_d,y_d)$ and inequalities between vectors are meant componentwise. It is convenient to identify $\mathcal{R}_d$ with

$$\Omega := \{(\underline{x},\overline{x}) \in \mathbb{R}^{2d} \,|\, -\mathbf{1} \leq \underline{x} \leq \overline{x} \leq \mathbf{1}\},$$

where for any real scalar a we put $\mathbf{a} := (a,\ldots,a) \in \mathbb{R}^d$. The *$L^p$-extreme discrepancy* of a point set $\{t_1,\ldots,t_n\} \subset [-1,1]^d$ is given by

$$D_p(t_1,...,t_n) := \left(\int_\Omega \left| \prod_{l=1}^d (\overline{x}_l - \underline{x}_l) - \frac{1}{n}\sum_{i=1}^n 1_{[\underline{x},\overline{x})}(t_i) \right|^p d\omega(\underline{x},\overline{x}) \right)^{1/p},$$

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where $1_{[\underline{x},\overline{x})}$ denotes the characteristic function of $[\underline{x},\overline{x})$ and $d\omega$ is the normalized Lebesgue measure $2^{-d}d\underline{x}\,d\overline{x}$ on Ω. The L^∞-*extreme discrepancy* is

$$D_\infty(t_1,...,t_n) := \sup_{(\underline{x},\overline{x})\in\Omega} \Big| \prod_{l=1}^{d}(\overline{x}_l - \underline{x}_l) - \frac{1}{n}\sum_{i=1}^{n} 1_{[\underline{x},\overline{x})}(t_i) \Big| ,$$

and the smallest possible L^∞-extreme discrepancy of any n-point set is

$$D_\infty(n,d) = \inf_{t_1,...,t_n\in[-1,1]^d} D_\infty(t_1,...,t_n) .$$

Another quantity of interest is the *inverse of* $D_\infty(n,d)$, namely

$$n_\infty(\varepsilon,d) = \min\{n\in\mathbb{N} \,|\, D_\infty(n,d)\le\varepsilon\} .$$

If we consider in the definitions above the set of all d-dimensional corners

$$\mathcal{C}_d = \{[-\mathbf{1},y) \,|\, y\in[-1,1]^d\}$$

instead of $\mathcal{R}_d$, we get the classical notion of star-discrepancy.

It is well known that the star-discrepancy is related to the error of multivariate integration of certain function classes (see, e.g., [2, 5, 8, 10, 12]). That this is also true for the extreme discrepancy was pointed out by Novak and Woźniakowski in [12]. Therefore it is of interest to derive upper bounds for the extreme discrepancy with a good dependence on the dimension d and explicitly known constants.

Heinrich, Novak, Wasilkowski and Woźniakowski showed in [4] with probabilistic methods that for the inverse $n^*_\infty(\varepsilon,d)$ of the star-discrepancy we have $n^*(\varepsilon,d)\le Cd\varepsilon^{-2}$. The drawback is here that the constant C is not known. In the same paper a lower bound was proved establishing the linear dependence of $n^*_\infty(\varepsilon,d)$ on d. This bound has recently been improved by Aicke Hinrichs to $n^*_\infty(\varepsilon,d)\ge cd\varepsilon^{-1}$ [6]. These results hold also for $n_\infty(\varepsilon,d)$.

In [4], Heinrich et al. presented two additional bounds for $n^*_\infty(\varepsilon,d)$ with slightly worse dependence on d, but explicitly known constants. The first one uses again a probabilistic approach, employs Hoeffding's inequality and leads to

$$n^*_\infty(\varepsilon,d) \le O\big(d\varepsilon^{-2}\big(\ln(d)+\ln(\varepsilon^{-1})\big)\big) .$$

The approach has been modified in more recent papers to improve this bound or to derive similar results in different settings [1, 5, 9]. In particular, it has been implicitly shown in the quite general Theorem 3.1 in [9] that the last bound holds also for the extreme discrepancy (as pointed out in [3], this result can be improved by employing the methods used in [1]).

The second bound was shown in the following way: The authors proved for even p an upper bound for the average L^p-star discrepancy $\mathrm{av}^*_p(n,d)$:

$$\mathrm{av}^*_p(n,d) \le 3^{2/3}2^{5/2+d/p}p(p+2)^{-d/p}n^{-1/2} .$$

(This analysis is quite elaborate, since $\mathrm{av}_p^*(n,d)$ is represented as an alternating sum of weighted products of Stirling numbers of the first and second kind.) The bound was used to derive upper bounds $n_\infty^*(\varepsilon,d) \le C_k d^2 \varepsilon^{-2-1/k}$ for every $k \in \mathbb{N}$. To improve the dependence on d, Hinrichs suggested to use symmetrization. This approach was sketched in [11] and leads to

$$\mathrm{av}_p^*(n,d) \le 2^{1/2+d/p} p^{1/2} (p+2)^{-d/p} n^{-1/2}$$

and $n_\infty^*(\varepsilon,d) \le C_k d \varepsilon^{-2-1/k}$. (Actually there seems to be an error in the calculations in [11], therefore we stated the results of our own calculations—see Remark 4 and 9).

In this paper we use the symmetrization approach to prove an upper bound for the average L^p-extreme discrepancy $\mathrm{av}_p(n,d)$ for $2 \le p < \infty$. Our analysis does not need Stirling numbers and uses rather simple combinatorial arguments. Similar as in [4], we derive from this bound upper bounds for the inverse of the L^∞-extreme discrepancy of the form $n_\infty(\varepsilon,d) \le C_k d \varepsilon^{-2-1/k}$ for all $k \in \mathbb{N}$.

2 Bound for the average L^p-discrepancy

If $\underline{x}$, $\overline{x}$ are vectors in $\mathbb{R}^d$ with $\underline{x} \le \overline{x}$, we use the (non-standard) notation $x := (\overline{x} - \underline{x})/2$. Let $p \in \mathbb{N}$ be even. For $i = 1,...,n$ we define the Banach space valued random variable $X_i : [-1,1]^{nd} \to L^p(\Omega,\, d\omega)$ by $X_i(t)(\underline{x},\overline{x}) = 1_{[\underline{x},\overline{x})}(t_i)$. Then $X_1,...,X_n$ are independent and identically distributed. Note that X_i is Bochner integrable for all $i \in [n]$. If $\mathbb{E}$ denotes the expectation with respect to the normalized measure $2^{-nd}\,dt$, then $\mathbb{E}X_i \in L^p(\Omega,\, d\omega)$ and $\mathbb{E}X_i(\underline{x},\overline{x}) = x_1 ... x_d$ almost everywhere. We obtain

$$\begin{aligned} \mathrm{av}_p(n,d)^p &= \int\limits_{[-1,1]^{nd}} D_p(t_1,...,t_n)^p \, 2^{-nd}\, dt \\ &= \int\limits_{[-1,1]^{nd}} \Big\| \frac{1}{n} \sum_{i=1}^n (X_i(t) - \mathbb{E}X_i) \Big\|_{L^p(\Omega,\, d\omega)}^p 2^{-nd}\, dt \\ &= \mathbb{E}\Big(\Big\| \frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}X_i) \Big\|_{L^p(\Omega,\, d\omega)}^p \Big) . \end{aligned}$$

Let $\varepsilon_1,...,\varepsilon_n : [-1,1]^{nd} \to \{-1,+1\}$ be symmetric Rademacher random variables, i.e., random variables taking the values ± 1 with probability $1/2$. We choose these variables such that $\varepsilon_1,...,\varepsilon_n, X_1,...,X_n$ are independent. Then (see [7, §6.1])

$$\begin{aligned} \mathrm{av}_p(n,d)^p &\le \mathbb{E}\Big(2^p \Big\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i X_i \Big\|_{L^p(\Omega,\, d\omega)}^p \Big) \\ =&\Big(\frac{2}{n}\Big)^p \sum_{i_1,...,i_p=1}^n \int\limits_{[-1,1]^{nd}} \int\limits_{\Omega} \prod_{l=1}^p \Big(\varepsilon_{i_l}(t) 1_{[\underline{x},\overline{x})}(t_{i_l}) \Big)\, d\omega(\underline{x},\overline{x})\, 2^{-nd}\, dt \, . \end{aligned}$$

Let us now consider $(\underline{x},\overline{x}) \in \Omega$, $k \in [p]$, pairwise disjoint indices $i_1, ..., i_k$ and $j_1, ..., j_k \in [p]$ with $\sum_{l=1}^{k} j_l = p$. According to Fubini's Theorem

$$\begin{aligned} J &:= \int\limits_{[-1,1]^{nd}} \Big(\prod_{l=1}^{k} \varepsilon_{i_l}^{j_l}(t)\Big)\Big(\int\limits_{\Omega}\Big(\prod_{l=1}^{k} X_{i_l}^{j_l}(t)(\underline{x},\overline{x})\Big)\,d\omega(\underline{x},\overline{x})\Big)\,2^{-nd}\,dt \\ &= \Big(\prod_{l=1}^{k}\int\limits_{[-1,1]^{nd}} \varepsilon_{i_l}^{j_l}(t)\,2^{-nd}\,dt\Big)\Big(\int\limits_{\Omega}\int\limits_{[-1,1]^{nd}}\Big(\prod_{l=1}^{k} 1_{[\underline{x},\overline{x})}(t_{i_l})\Big)\,2^{-nd}\,dt\,d\omega(\underline{x},\overline{x})\Big). \end{aligned}$$

This yields $J = \int_\Omega (x_1...x_d)^k\,d\omega(\underline{x},\overline{x}) = 2^d(k+1)^{-d}(k+2)^{-d}$ if every exponent j_l is even, and $J = 0$ if there exists at least one odd exponent j_l. Let $T(p,k,n)$ be the number of tuples $(i_1, ..., i_p) \in [n]^p$ with $|\{i_1, ..., i_p\}| = k$ and $|\{l \in [p] \,|\, i_l = i_m\}|$ even for each $m \in [p]$. Our last observation implies

$$\mathrm{av}_p(n,d)^p \le 2^{p+d} n^{-p} \sum_{k=1}^{p/2} \frac{T(p,k,n)}{(k+1)^d(k+2)^d}\,.$$

In the next step we shall estimate the numbers $T(p,k,n)$. For that purpose we introduce further notation. Let

$$M(p/2,k) = \Big\{\nu \in \mathbb{N}^k \,\Big|\, 1 \le \nu_1 \le ... \le \nu_k \le p/2,\ \sum_{i=1}^{k} \nu_k = p/2\Big\}\,,$$

and for $\nu \in M(p/2,k)$ let $e(\nu,i) = |\{j \in [k] \,|\, \nu_j = i\}|$. With the standard notation for multinomial coefficients we get

$$T(p,k,n) = \sum_{\nu \in M(p/2,k)} \binom{p}{2\nu_1, ..., 2\nu_k} \frac{n(n-1)...(n-k+1)}{e(\nu,1)!...e(\nu,p/2)!}\,.$$

If $\sharp(p/2,k,n)$ denotes the number of tuples $(i_1, ..., i_{p/2}) \in [n]^{p/2}$ with $|\{i_1, ..., i_{p/2}\}|$ $= k$, then

$$\sharp(p/2,k,n) = \sum_{\nu \in M(p/2,k)} \binom{p/2}{\nu_1, ..., \nu_k} \frac{n(n-1)...(n-k+1)}{e(\nu,1)!...e(\nu,p/2)!}\,.$$

We want to compare $T(p,k,n)$ with $\sharp(p/2,k,n)$ and are therefore interested in the quantity

$$Q_k^p(\nu) := \binom{p}{2\nu_1, ..., 2\nu_k}\binom{p/2}{\nu_1, ..., \nu_k}^{-1}.$$

To derive an upper bound for $Q_k^p(\nu)$, we prove two auxiliary lemmas.

Lemma 1. *Let $f : \mathbb{N}_0 \to \mathbb{R}$ be defined by $f(r) = [2r(2r-1)...(r+1)](2r)^{-r}$ for $r > 0$ and $f(0) = 1$. Then $f(r+s) \le f(r)f(s)$ for all $r,\ s \in \mathbb{N}_0$.*

Proof. We prove the inequality for an arbitrary s by induction over r. It is evident if $r = 0$. So let the inequality hold for some $r \in \mathbb{N}_0$. The well known relations $\Gamma(x+1) = x\Gamma(x)$ and $\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2)$ for the gamma function lead to

$$f(r) = 2^{2r}\pi^{-1/2}\Gamma(r+1/2)\exp(-r\ln(2r))\,,$$

with the convention $0 \cdot \ln(0) = 0$ when $r = 0$, and

$$\frac{f(r+1+s)}{f(r+1)} = \frac{g(r)}{g(r+s)}\frac{f(r+s)}{f(r)}\,,$$

where $g : [0,\infty) \to [0,\infty)$ is defined by $g(0) = 2$ and

$$g(\lambda) = \Big(1 + \frac{1}{2\lambda+1}\Big)\exp(\lambda\ln(1+1/\lambda))$$

for $\lambda > 0$. The function g is continuous in 0 and its derivative is given by

$$g'(\lambda) = \Big(\ln(1+1/\lambda) - \frac{2}{2\lambda+1}\Big)g(\lambda)$$

for $\lambda > 0$. Since

$$\frac{d}{d\lambda}\Big(\ln(1+1/\lambda) - \frac{2}{2\lambda+1}\Big) = -\frac{1}{\lambda(\lambda+1)(2\lambda+1)^2} < 0$$

and

$$\lim_{\lambda\to\infty}\Big(\ln(1+1/\lambda) - \frac{2}{2\lambda+1}\Big) = 0\,,$$

we obtain $g'(\lambda) \geq 0$. Therefore g is an increasing function. Thus

$$\frac{g(r)}{g(r+s)} \leq 1\,, \text{ which establishes } \frac{f(r+1+s)}{f(r+1)} \leq \frac{f(r+s)}{f(r)} \leq f(s)\,.$$

□

Lemma 2. *Let $k \in \mathbb{N}$, $a_1, ..., a_k \in [0,\infty)$ and $\sigma = \sum_{i=1}^k a_i$. Then*

$$\Big(\frac{\sigma}{k}\Big)^\sigma \leq \prod_{i=1}^k a_i^{a_i}\,.$$

Proof. Let $\sigma > 0$, and consider the functions $s : \mathbb{R}^k \to \mathbb{R}$, $x \mapsto \sum_{i=1}^k x_i$ and

$$f : [0,\infty)^k \to \mathbb{R}\,, x \mapsto \prod_{i=1}^k x_i^{x_i} = \prod_{i=1}^k \exp(x_i\ln(x_i))\,,$$

where we use the convention $0 \cdot \ln(0) = 0$. Let $M = \{x \in (0,\infty)^k \,|\, s(x) = \sigma\}$. Since f is continuous, there exists a point ξ in the closure $\overline{M}$ of M with $f(\xi) = \min\{f(x) \,|\, x \in \overline{M}\}$.

Now let $x \in \overline{M} \setminus M$, which implies $x_\mu = 0$ for an index $\mu \in [k]$. Since $s(x) = \sigma$, there exists a $\nu \in [k]$ with $x_\nu > 0$. Without loss of generality we may assume $\mu = 1$, $\nu = 2$. Then

$$f(x) = \prod_{i=2}^{k} x_i^{x_i} > \left(\frac{x_2}{2}\right)^{x_2} \prod_{i=3}^{k} x_i^{x_i} = f(x'),$$

where $x' = (\frac{x_2}{2}, \frac{x_2}{2}, x_3, ..., x_k)$. Thus ξ lies in M. Since $\operatorname{grad} s \equiv (1, ..., 1) \neq 0$, there exists a Lagrangian multiplier $\lambda \in \mathbb{R}$ with $\operatorname{grad} f(\xi) = \lambda \operatorname{grad} s(\xi)$. From $\operatorname{grad} f(x) = (1 + \ln(x_1), ..., 1 + \ln(x_k)) f(x)$ follows $\xi_1 = ... = \xi_k$, i.e., $\xi_i = \sigma/k$ for $i = 1, ..., k$. $\square$

With the help of Lemma 1 and 2 we conclude

$$Q_k \leq \frac{p^{p/2}}{(2\nu_1)^{\nu_1} ... (2\nu_k)^{\nu_k}} = \Big(\prod_{i=1}^{k} \nu_i^{\nu_i}\Big)^{-1} \Big(\frac{p}{2}\Big)^{p/2} \leq \Big(\frac{1}{k} \sum_{i=1}^{k} \nu_i\Big)^{-p/2} \Big(\frac{p}{2}\Big)^{p/2} = k^{p/2}.$$

Therefore

$$T(p, k, n) \leq k^{p/2} \sharp(p/2, k, n). \tag{1}$$

The last estimate yields

$$\operatorname{av}_p(n, d)^p \leq 2^{p+d} n^{-p} \sum_{k=1}^{p/2} \frac{k^{p/2}}{(k+1)^d (k+2)^d} \sharp(p/2, k, n).$$

If $p \geq 4d$, then

$$\begin{aligned} \operatorname{av}_p(n, d)^p &\leq 2^{p/2+3d} n^{-p} p^{p/2} (p+2)^{-d} (p+4)^{-d} \sum_{k=1}^{p/2} \sharp(p/2, k, n) \\ &\leq 2^{p/2+3d} p^{p/2} (p+2)^{-d} (p+4)^{-d} n^{-p/2}. \end{aligned}$$

If $p < 4d$, then

$$\begin{aligned} \operatorname{av}_p(n, d)^p &\leq 2^{p+d} n^{-p} \sum_{k=1}^{p/2} \big[(k+1)(k+2)\big]^{p/4-d} \sharp(p/2, k, d) \\ &\leq 2^{5p/4} 3^{p/4-d} n^{-p/2}. \end{aligned}$$

Thus we have shown the following theorem:

Theorem 3. *Let p be an even integer. If $p \geq 4d$, then*

$$\operatorname{av}_p(n, d) \leq 2^{1/2+3d/p} p^{1/2} (p+2)^{-d/p} (p+4)^{-d/p} n^{-1/2}.$$

If $p < 4d$, then the estimate $\operatorname{av}_p(n, d) \leq 2^{5/4} 3^{1/4-d} n^{-1/2}$ holds.

For a general $p \in [2, \infty)$ we find a $k \in \mathbb{N}$ with $2k \leq p < 2(k+1)$. Hence there exists a $t \in (0, 1]$ with $1/p = t/2k + (1-t)/2(k+1)$ and from Hölder's inequality we get

$$\operatorname{av}_p(n, d) \leq \operatorname{av}_{2k}(n, d)^t \operatorname{av}_{2(k+1)}(n, d)^{1-t}.$$

Remark 4. The probabilistic argument we used for deriving our upper bound for the average L^p-extreme discrepancy was sketched in [11]. Unfortunately the derivation there contains an error (the number $\sharp(p/2,k,n)$ that appears there has to be substituted by the number $T(p,k,n)$ defined above). For that reason we state here the bounds for the average L^p-star discrepancy $\mathrm{av}_p^*(n,d)$ that we get by mimicking the approach discussed in this section: With the symmetrization argument and (1) we obtain

$$\mathrm{av}_p^*(n,d)^p \leq \left(\frac{2}{n}\right)^p \sum_{k=1}^{p/2} \frac{k^{p/2}}{(k+1)^d} \sharp(p/2,k,n) \,.$$

If $p < 2d$, then $\mathrm{av}_p^*(n,d) \leq 2^{3/2-d/p}\, n^{-1/2}$. If $p \geq 2d$, then

$$\mathrm{av}_p^*(n,d) \leq 2^{1/2+d/p}\, p^{1/2} (p+2)^{-d/p}\, n^{-1/2} \,. \tag{2}$$

3 Application to the L^∞-discrepancy

Now we want to derive an upper bound for the inverse $n_\infty(\varepsilon,d)$ of the L^∞-extreme discrepancy in terms of the average L^p-extreme discrepancy $\mathrm{av}_p(n,d)$. Therefore we define first a "homogeneous version" of the L^∞-extreme discrepancy: For any $h \in (0,1]$ and any $t_1,...,t_n \in \mathbb{R}^d$ let

$$D_\infty^h(t_1,...,t_n) = \inf_{c>0} \sup_{-\mathbf{h} \leq \underline{x} < \overline{x} \leq \mathbf{h}} \Big| \prod_{l=1}^d x_l - c \sum_{i=1}^n 1_{[\underline{x},\overline{x})}(t_i) \Big| \,.$$

Obviously $D_\infty^h(ht_1,...,ht_n) = h^d D_\infty^1(t_1,...,t_n)$. Further quantities of interest are

$$D_\infty^1(n,d) = \inf_{t_1,\dots,t_n \in [-1,1]^d} D_\infty^1(t_1,...,t_n)$$

and

$$n_\infty^1(\varepsilon,d) := \min\{n \in \mathbb{N} \,|\, D_\infty^1(n,d) \leq \varepsilon\} \,.$$

Lemma 5. *For every $\varepsilon > 0$ we have $n_\infty^1(\varepsilon,d) \leq n_\infty(\varepsilon,d) \leq n_\infty^1(\varepsilon/2,d)$.*

The Lemma can be verified by just mimicking the proof of [4, Lemma 2].

Now define for $1 > \varepsilon > 0$, $h = (1+\varepsilon)^{-1/d}$ and all even natural numbers p

$$A_p^d(\varepsilon) := h^{d(p+2)} \int_{[-\mathbf{1},(1-2(1-\varepsilon)^{1/d})\mathbf{1}]} \int_{[(1-\varepsilon)^{1/d}\mathbf{1},\frac{1}{2}(\mathbf{1}-y)]} \left((\varepsilon-1) + \prod_{j=1}^d z_j\right)^p dz\, dy$$

and

$$B_p^d(\varepsilon) := \int_{[-\mathbf{1},-\mathbf{h}]} \int_{[\mathbf{h},\mathbf{1}]} \left(1 - \prod_{l=1}^d x_l\right)^p 2^{-d}\, d\overline{x}\, d\underline{x} \,.$$

Theorem 6. *Let* $\varepsilon \in (0,1)$*. If* $\varepsilon < D^1_\infty(n,d)$*, then we obtain for all even* p *the inequality* $\mathrm{av}_p(n,d) > \min(A^d_p(\varepsilon), B^d_p(\varepsilon))^{1/p}$*. Therefore*

$$n^1_\infty(\varepsilon, d) \leq \min\{n | \, \exists p \in 2\mathbb{N} : \mathrm{av}_p(n,d) \leq \min(A^d_p(\varepsilon), B^d_p(\varepsilon))^{1/p}\}\,.$$

Proof. To verify the theorem, we modify the proof from [4, Thm. 6]: Let $D^1_\infty(n,d) > \varepsilon$. For $h \in (0,1]$ and t_1,...,$t_n \in [-1,1]^d$ we have

$$D^h_\infty(t_1, ..., t_n) = h^d D^1_\infty(t_1/h, ..., t_n/h) > \varepsilon h^d\,.$$

Therefore we find $\underline{x}$, $\overline{x} \in [-h,h]^d$ with $\underline{x} < \overline{x}$ and

$$\Big| \prod_{l=1}^d x_l - \frac{1}{n}\sum_{i=1}^n 1_{[\underline{x},\overline{x})}(t_i) \Big| > \varepsilon h^d\,.$$

Case 1: There holds

$$\prod_{l=1}^d x_l - \frac{1}{n}\sum_{i=1}^n 1_{[\underline{x},\overline{x})}(t_i) > \varepsilon h^d\,.$$

With respect to its volume the box $[\underline{x},\overline{x})$ contains not sufficiently many sample points. This holds also for slightly smaller boxes. If $[\underline{v},\overline{v}) \subseteq [\underline{x},\overline{x})$, then

$$\prod_{j=1}^d v_j - \frac{1}{n}\sum_{i=1}^n 1_{[\underline{v},\overline{v})}(t_i) > \varepsilon h^d - \prod_{j=1}^d x_j + \prod_{j=1}^d v_j\,.$$

This leads to

$$D_p(t_1, ..., t_n)^p > \int_{[\underline{x},\overline{x}]} \int_{[\underline{v},\overline{x}]} \Big(\varepsilon h^d - \prod_{j=1}^d x_j + \prod_{j=1}^d v_j\Big)^p_+ 2^{-d}\, d\overline{v}\, d\underline{v}$$

$$= \int_{[-\mathbf{h},-\mathbf{h}+2x]} \int_{[\underline{z}+2(\mathbf{h}-x),\mathbf{h}]} \Big(\varepsilon h^d - \prod_{j=1}^d x_j + \prod_{j=1}^d (z_j + x_j - h)\Big)^p_+ 2^{-d}\, d\overline{z}\, d\underline{z}\,,$$

where in the last step we made a change of coordinates: $\underline{z} = \underline{v} - \underline{x} - \mathbf{h}$ and $\overline{z} = \overline{v} - \overline{x} + \mathbf{h}$. If we translate edge points v and w, $v \leq w$, of anchored boxes $[0,v)$ and $[0,w)$ by a vector $a \geq 0$, then it is a simple geometrical observation that the volumes of the corresponding anchored boxes satisfy

$$\mathrm{vol}([0,w)) - \mathrm{vol}([0,v)) \leq \mathrm{vol}([0,w+a)) - \mathrm{vol}([0,v+a))\,.$$

In particular, if $w = x$, $v = z + x - \mathbf{h}$ and $a = \mathbf{h} - x$, then

$$\prod_{j=1}^d x_j - \prod_{j=1}^d (z_j + x_j - h) \leq h^d - \prod_{j=1}^d z_j\,.$$

This, and integrating over the variable z instead over $\overline{z}$, leads to

$$D_p(t_1, ..., t_n)^p > \int\limits_{[-\mathbf{h}, -\mathbf{h}+2x]} \int\limits_{[\mathbf{h}-x, \frac{1}{2}(\mathbf{h}-\underline{z})]} \Big((\varepsilon-1)h^d + \prod_{j=1}^{d} z_j\Big)_+^p \, dz \, d\underline{z} \, .$$

We can ignore those vectors z with a component $z_i < (1-\varepsilon)h$, since they satisfy the relation $(\varepsilon-1)h^d + \prod_{j=1}^{d} z_j < 0$. As $x_i > \varepsilon h$ for all $1 \le i \le d$, we get

$$D_p(t_1, ..., t_n)^p > \int\limits_{[-\mathbf{h}, (2\varepsilon-1)\mathbf{h}]} \int\limits_{[(1-\varepsilon)\mathbf{h}, \frac{1}{2}(\mathbf{h}-\underline{z})]} \Big((\varepsilon-1)h^d + \prod_{j=1}^{d} z_j\Big)_+^p \, dz \, d\underline{z}$$

$$\ge \int\limits_{[-\mathbf{h}, (1-2(1-\varepsilon)^{1/d})\mathbf{h}]} \int\limits_{[(1-\varepsilon)^{1/d}\mathbf{h}, \frac{1}{2}(\mathbf{h}-\underline{z})]} \Big((\varepsilon-1)h^d + \prod_{j=1}^{d} z_j\Big)^p \, dz \, d\underline{z} \, .$$

Case 2: There holds

$$\frac{1}{n}\sum_{i=1}^{n} 1_{[\underline{x},\overline{x})}(t_i) - \prod_{l=1}^{d} x_l > \varepsilon h^d \, .$$

The box $[\underline{x}, \overline{x})$ contains too many points, and this is also true for somewhat larger boxes. If $[\underline{x}, \overline{x}) \subseteq [\underline{w}, \overline{w})$, then

$$\frac{1}{n}\sum_{i=1}^{n} 1_{[\underline{w},\overline{w})}(t_i) - \prod_{l=1}^{d} w_l > \varepsilon h^d + \prod_{l=1}^{d} x_l - \prod_{l=1}^{d} w_l \, .$$

This implies

$$D_p(t_1, ..., t_n)^p > \int\limits_{[-\mathbf{1}, \underline{x}]} \int\limits_{[\overline{x}, \mathbf{1}]} \Big(\varepsilon h^d + \prod_{l=1}^{d} x_l - \prod_{l=1}^{d} w_l\Big)_+^p \, 2^{-d} \, d\overline{w} \, d\underline{w}$$

$$\ge \int\limits_{[-\mathbf{1}-\mathbf{h}-\underline{x}, -\mathbf{h}]} \int\limits_{[\mathbf{h}, \mathbf{1}+\mathbf{h}-\overline{x}]} \Big(\varepsilon h^d + \prod_{l=1}^{d} x_l - \prod_{l=1}^{d} (z_l - h + x_l)\Big)_+^p \, 2^{-d} \, d\overline{z} \, d\underline{z} \, ,$$

where we made the substitutions $\overline{z} = \overline{w} - \overline{x} + \mathbf{h}$ and $\underline{z} = \underline{w} - \underline{x} - \mathbf{h}$. If we restrict the domain of integration and use the simple geometric observation mentioned in the discussion of Case 1, we obtain

$$D_p(t_1, ..., t_n)^p > \int\limits_{[-\mathbf{1}, -\mathbf{h}]} \int\limits_{[\mathbf{h}, \mathbf{1}]} \Big((1+\varepsilon)h^d - \prod_{l=1}^{d} z_l\Big)_+^p \, 2^{-d} \, d\overline{z} \, d\underline{z} \, .$$

If we choose $h = (1+\varepsilon)^{-1/d}$, then $D_p(t_1, ..., t_n)^p > B_p^d(\varepsilon)$.

Our analysis results in $D_p(t_1, ..., t_n)^p > \min\{A_p^d(\varepsilon), B_p^d(\varepsilon)\}$ for all $t_1, ..., t_n \in [-1, 1]^d$. Theorem 6 follows now by integration. $\square$

Lemma 7. *Let $\varepsilon \in (0,1/2]$ and $p \geq 4d$ be an even integer. Then*

$$\min(A_p^d(\varepsilon), B_p^d(\varepsilon))^{1/p} \geq \frac{1}{3}\varepsilon\Big(\frac{\varepsilon}{4d}\Big)^{2d/p}.$$

Proof. Let again $h = (1+\varepsilon)^{-1/d}$. From the definition of $B_p^d(\varepsilon)$ follows

$$\begin{aligned} B_p^d(\varepsilon) &\geq \int\limits_{[-(1+\varepsilon/2)^{-1/d}\mathbf{1},-\mathbf{h}]} \int\limits_{[\mathbf{h},(1+\varepsilon/2)^{-1/d}\mathbf{1}]} \big(1-(1+\varepsilon/2)^{-1}\big)^p \, 2^{-d}\, d\overline{x}\, d\underline{x} \\ &= 2^{-d}\big(1-(1+\varepsilon/2)^{-1}\big)^p\big((1+\varepsilon/2)^{-1/d}-(1+\varepsilon)^{-1/d}\big)^{2d}. \end{aligned}$$

As $\varepsilon \leq 1/2$, it is straightforward to verify the inequalities $1-(1+\varepsilon/2)^{-1} \geq 2\varepsilon/5$ and $(1+\varepsilon/2)^{-1/d}-(1+\varepsilon)^{-1/d} \geq \varepsilon/4d$. That implies

$$B_p^d(\varepsilon)^{1/p} \geq 2^{-d/p}\frac{2}{5}\varepsilon\Big(\frac{\varepsilon}{4d}\Big)^{2d/p} \geq 2^{-1/4}\frac{2}{5}\varepsilon\Big(\frac{\varepsilon}{4d}\Big)^{2d/p} \geq \frac{1}{3}\varepsilon\Big(\frac{\varepsilon}{4d}\Big)^{2d/p}.$$

We can estimate $A_p^d(\varepsilon)$ in the following way:

$$\begin{aligned} A_p^d(\varepsilon) &\geq h^{d(p+2)} \int\limits_{[-\mathbf{1},(1-2(1-\varepsilon/2)^{1/d})\mathbf{1}]} \int\limits_{[(1-\varepsilon/2)^{1/d}\mathbf{1},\frac{1}{2}(\mathbf{1}-y)]} (\varepsilon/2)^p \, dz\, dy \\ &= (1+\varepsilon)^{-p-2}(\varepsilon/2)^p\big(1-(1-\varepsilon/2)^{1/d}\big)^{2d}. \end{aligned}$$

Since $1-(1-\varepsilon/2)^{1/d} \geq \varepsilon/2d$, we get

$$\begin{aligned} A_p^d(\varepsilon)^{1/p} &\geq \frac{1}{(1+\varepsilon)^{1+2/p}}\,\frac{\varepsilon}{2}\Big(\frac{\varepsilon}{2d}\Big)^{2d/p} \\ &\geq \frac{1}{1+\varepsilon}\Big(\frac{2^d}{1+\varepsilon}\Big)^{2/p}\frac{\varepsilon}{2}\Big(\frac{\varepsilon}{4d}\Big)^{2d/p} \geq \frac{1}{3}\varepsilon\Big(\frac{\varepsilon}{4d}\Big)^{2d/p}. \end{aligned}$$

□

Let now $k \in \mathbb{N}$, $p = 4kd$ and $\varepsilon \in (0,1/2)$. With Theorem 3 and Lemma 7 it is easily verified that

$$n \geq 9 \cdot 2^{3(1+1/2k)} k^{1-1/k} d\varepsilon^{-2-1/k} \quad \text{ensures} \quad \mathrm{av}_p(n,d) \leq \min(A_p^d(\varepsilon), B_p^d(\varepsilon))^{1/p}.$$

This, Lemma 5 and Theorem 6 lead to the following theorem:

Theorem 8. *Let $\varepsilon \in (0,1/2)$ and $k \in \mathbb{N}$. Then $n_\infty(\varepsilon,d) \leq C_k d\varepsilon^{-2-1/k}$, where the constant C_k is bounded from above by $9 \cdot 2^{5(1+1/2k)}k^{1-1/k}$.*

Remark 9. In a similar way we can use the bound for the average L^p-star discrepancy to calculate an upper bound for the inverse $n_\infty^*(d,\varepsilon)$ of the star discrepancy: With (2), [4, Thm. 6] and [4, Lemma 3] (where we can replace the factor $\sqrt{2/3}$ by 1—cf. with the proof of Lemma 7), we obtain

$$n_\infty^*(d,\varepsilon) \leq 9 \cdot 2^{4+3/k}k^{1-1/k}d\varepsilon^{-2-1/k}. \tag{3}$$

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