A note on three types of quasisymmetric functions

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Abstract

In the context of generating functions for P-partitions, we revisit three flavors of quasisymmetric functions: Gessel's quasisymmetric functions, Chow's type B quasisymmetric functions, and Poirier's signed quasisymmetric functions. In each case we use the inner coproduct to give a combinatorial description (counting pairs of permutations) to the multiplication in: Solomon's type A descent algebra, Solomon's type B descent algebra, and the Mantaci-Reutenauer algebra, respectively. The presentation is brief and elementary, our main results coming as consequences of P-partition theorems already in the literature.

1 Quasisymmetric functions and Solomon's descent algebra

The ring of quasisymmetric functions is well-known (see [12], ch. 7.19). Recall that a quasisymmetric function is a formal series

$$Q(x_1, x_2, \ldots) \in \mathbb{Z}[[x_1, x_2, \ldots]]$$

of bounded degree such that the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ is the same for all $i_1 < i_2 < \cdots < i_k$ and all compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Recall that a composition of n, written $\alpha \models n$, is an ordered tuple of positive integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. In this case we say that α has k parts, or $\#\alpha = k$. We can put a partial order on the set of all compositions of n by reverse refinement. The covering relations are of the form

$$(\alpha_1, \ldots, \alpha_i + \alpha_{i+1}, \ldots, \alpha_k) \prec (\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_k).$$

Let $\mathcal{Q}sym_n$ denote the set of all quasisymmetric functions homogeneous of degree n. The ring of quasisymmetric functions can be defined as $\mathcal{Q}sym := \bigoplus_{n\geq 0} \mathcal{Q}sym_n$, but our focus will stay on the quasisymmetric functions of degree n, rather than the ring as a whole.

The most obvious basis for $\mathcal{Q}sym_n$ is the set of *monomial* quasisymmetric functions, defined for any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$,

$$M_\alpha := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

We can form another natural basis with the *fundamental* quasisymmetric functions, also indexed by compositions,

$$F_\alpha := \sum_{\alpha \preccurlyeq \beta} M_\beta,$$

since, by inclusion-exclusion we can express the M_α in terms of the F_α:

$$M_\alpha = \sum_{\alpha \preccurlyeq \beta} (-1)^{\#\beta - \#\alpha} F_\beta.$$

As an example,

$$F_{(2,1)} = M_{(2,1)} + M_{(1,1,1)} = \sum_{i<j} x_i^2 x_j + \sum_{i<j<k} x_i x_j x_k = \sum_{i \le j < k} x_i x_j x_k.$$

Compositions can be used to encode descent classes of permutations in the following way. Recall that a *descent* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi_i > \pi_{i+1}$, and that an *increasing run* of a permutation π is a maximal subword of consecutive letters $\pi_{i+1}\pi_{i+2}\cdots\pi_{i+r}$ such that $\pi_{i+1} < \pi_{i+2} < \cdots < \pi_{i+r}$. By maximality, we have that if $\pi_{i+1}\pi_{i+2}\cdots\pi_{i+r}$ is an increasing run, then i is a descent of π (if $i \neq 0$), and $i+r$ is a descent of π (if $i + r \neq n$). For any permutation $\pi \in \mathfrak{S}_n$ define the *descent composition*, $C(\pi)$, to be the ordered tuple listing the lengths of the increasing runs of π. If $C(\pi) = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we can recover the descent set of π:

$$\mathrm{Des}(\pi) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}.$$

Since $C(\pi)$ and $\mathrm{Des}(\pi)$ have the same information, we will use them interchangeably. For example the permutation $\pi = (3, 4, 5, 2, 6, 1)$ has $C(\pi) = (3, 2, 1)$ and $\mathrm{Des}(\pi) = \{3, 5\}$.

Recall ([11], ch. 4.5) that a P-partition is an order-preserving map from a poset P to some (countable) totally ordered set. To be precise, let P be any labeled partially ordered set (with partial order $<_P$) and let S be any totally ordered countable set. Then $f : P \to S$ is a P-partition if it satisfies the following conditions:

1. $f(i) \le f(j)$ if $i <_P j$
2. $f(i) < f(j)$ if $i <_P j$ and $i > j$ (as labels)

We let $\mathcal{A}(P)$ (or $\mathcal{A}(P; S)$ if we want to emphasize the image set) denote the set of all P-partitions, and encode this set in the generating function

$$\Gamma(P) := \sum_{f \in \mathcal{A}(P)} x_{f(1)} x_{f(2)} \cdots x_{f(n)},$$

where n is the number of elements in P (we will only consider finite posets). If we take S to be the set of positive integers, then it should be clear that $\Gamma(P)$ is always going to be a quasisymmetric function of degree n. As an easy example, let P be the poset defined by $3 >_P 2 <_P 1$. In this case we have

$$\Gamma(P) = \sum_{f(3)\geq f(2)<f(1)} x_{f(1)}x_{f(2)}x_{f(3)}.$$

We can consider permutations to be labeled posets with total order $\pi_1 <_\pi \pi_2 <_\pi \cdots <_\pi \pi_n$. With this convention, we have

$$\mathcal{A}(\pi) = \{f : [n] \to S \,|f(\pi_1) \leq f(\pi_2) \leq \cdots \leq f(\pi_n) \text{ and } k \in \mathrm{Des}(\pi) \Rightarrow f(\pi_k) < f(\pi_{k+1})\},$$

and

$$\Gamma(\pi) = \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n\\ k\in\mathrm{Des}(\pi)\Rightarrow i_k<i_{k+1}}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

It is not hard to verify that in fact we have

$$\Gamma(\pi) = F_{C(\pi)},$$

so that generating functions for the P-partitions of permutations of $\pi \in \mathfrak{S}_n$ form a basis for $\mathcal{Q}sym_n$.

We have the following theorem related to P-partitions of permutations, due to Gessel [5].

Theorem 1 *As sets, we have the bijection*

$$\mathcal{A}(\pi; ST) \leftrightarrow \coprod_{\sigma\tau=\pi} \mathcal{A}(\tau; S) \oplus \mathcal{A}(\sigma; T),$$

where ST is the cartesian product of the sets S and T with the lexicographic ordering.

Let $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ be two two sets of commuting indeterminates. Then we define the bipartite generating function,

$$\Gamma(\pi)(XY) = \sum_{\substack{(i_1,j_1)\leq(i_2,j_2)\leq\cdots\leq(i_n,j_n)\\ k\in\mathrm{Des}(\pi)\Rightarrow(i_k,j_k)<(i_{k+1},j_{k+1})}} x_{i_1}\cdots x_{i_n}y_{j_1}\cdots y_{j_n}.$$

We will apply Theorem 1 with $S = T = \mathbb{P}$, the positive integers.

Corollary 1 *We have*

$$F_{C(\pi)}(XY) = \sum_{\sigma\tau=\pi} F_{C(\tau)}(X)F_{C(\sigma)}(Y).$$

Following [5], we can define a coalgebra structure on $\mathcal{Q}sym_n$ in the following way. If π is any permutation with $C(\pi) = \gamma$, let $a^{\gamma}_{\alpha,\beta}$ denote the number of pairs of permutations $(\sigma, \tau) \in \mathfrak{S}_n \times \mathfrak{S}_n$ with $C(\sigma) = \alpha$, $C(\tau) = \beta$, and $\sigma\tau = \pi$. Then Corollary 1 defines a coproduct $\mathcal{Q}sym_n \to \mathcal{Q}sym_n \otimes \mathcal{Q}sym_n$:

$$F_\gamma \mapsto \sum_{\alpha,\beta \models n} a^{\gamma}_{\alpha,\beta} F_\beta \otimes F_\alpha.$$

If $\mathcal{Q}sym_n^*$, with basis $\{F_\alpha^*\}$, is the algebra dual to $\mathcal{Q}sym_n$, then by definition it is equipped with multiplication

$$F_\beta^* * F_\alpha^* = \sum_\gamma a^{\gamma}_{\alpha,\beta} F_\gamma^*.$$

Let $\mathbb{Z}\mathfrak{S}_n$ denote the group algebra of the symmetric group. We can define its dual coalgebra $\mathbb{Z}\mathfrak{S}_n^*$ with comultiplication

$$\pi \mapsto \sum_{\sigma\tau=\pi} \tau \otimes \sigma.$$

Then by Corollary 1 we have a surjective homomorphism of coalgebras $\varphi^* : \mathbb{Z}\mathfrak{S}_n^* \to \mathcal{Q}sym_n$ given by

$$\varphi^*(\pi) = F_{C(\pi)}.$$

The dualization of this map is then an injective homomorphism of algebras $\varphi : \mathcal{Q}sym_n^* \to \mathbb{Z}\mathfrak{S}_n$ with

$$\varphi(F_\alpha^*) = \sum_{C(\pi)=\alpha} \pi.$$

The is image of φ is then a subalgebra of the group algebra, with basis

$$u_\alpha := \sum_{C(\pi)=\alpha} \pi.$$

This subalgebra is well-known as *Solomon's descent algebra* [10], denoted $\mathrm{Sol}(A_{n-1})$. Corollary 1 has then given a combinatorial description to multiplication in $\mathrm{Sol}(A_{n-1})$:

$$u_\beta u_\alpha = \sum_{\gamma \models n} a^{\gamma}_{\alpha,\beta} u_\gamma.$$

The above arguments are due to Gessel [5]. We give them here in full detail for comparison with later sections, when we will outline a similar relationship between Chow's type B quasisymmetric functions [4] and $\mathrm{Sol}(B_n)$, and between Poirier's *signed* quasisymmetric functions [9] and the Mantaci-Reutenauer algebra.

2 Type B quasisymmetric functions and Solomon's descent algebra

The type B quasisymmetric functions can be viewed as the natural objects related to type B P-partitions (see [4]). Define the type B posets (with $2n+1$ elements) to be posets labeled distinctly by $\{-n,\ldots,-1,0,1,\ldots,n\}$ with the property that if $i <_P j$, then $-j <_P -i$. For example, $-2 >_P 1 <_P 0 <_P -1 >_P 2$ is a type B poset.

Let P be any type B poset, and let $S = \{s_0, s_1, \ldots\}$ be any countable totally ordered set with a minimal element s_0. Then a type B P-partition is any map $f : P \to \pm S$ such that

1. $f(i) \leq f(j)$ if $i <_P j$
2. $f(i) < f(j)$ if $i <_P j$ and $i > j$ (as labels)
3. $f(-i) = -f(i)$

where $\pm S$ is the totally ordered set

$$\cdots < -s_2 < -s_1 < s_0 < s_1 < s_2 < \cdots$$

If S is the nonnegative integers, then $\pm S$ is the set of all integers.

The third property of type B P-partitions means that $f(0) = 0$ and the set $\{f(i) \mid i = 1, 2, \ldots, n\}$ determines the map f. We let $\mathcal{A}_B(P) = \mathcal{A}_B(P; \pm S)$ denote the set of all type B P-partitions, and define the generating function for type B P-partitions as

$$\Gamma_B(P) := \sum_{f \in \mathcal{A}_B(P)} x_{|f(1)|} x_{|f(2)|} \cdots x_{|f(n)|}.$$

Signed permutations $\pi \in \mathfrak{B}_n$ are type B posets with total order

$$-\pi_n < \cdots < -\pi_1 < 0 < \pi_1 < \cdots < \pi_n.$$

We then have

$$\begin{aligned} \mathcal{A}_B(\pi) = \{f : \pm[n] \to \pm S \mid 0 \leq f(\pi_1) \leq f(\pi_2) \leq \cdots \leq f(\pi_n), \\ f(-i) = -f(i), \\ \text{and } k \in \mathrm{Des}_B(\pi) \Rightarrow f(\pi_k) < f(\pi_{k+1})\}, \end{aligned}$$

and

$$\Gamma_B(\pi) = \sum_{\substack{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ k \in \mathrm{Des}(\pi) \Rightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Here, the type B descent set, $\mathrm{Des}_B(\pi)$, keeps track of the ordinary descents as well as a descent in position 0 if $\pi_1 < 0$. Notice that if $\pi_1 < 0$, then $f(\pi_1) > 0$, and $\Gamma_B(\pi)$ has no x_0 terms, as in

$$\Gamma_B((-3, 2, -1)) = \sum_{0 < i \leq j < k} x_i x_j x_k.$$

The possible presence of a descent in position zero is the crucial difference between type A and type B descent sets. Define a *pseudo-composition* of n to be an ordered tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 \geq 0$, and $\alpha_i > 0$ for $i > 1$, such that $\alpha_1 + \cdots + \alpha_k = n$. We write $\alpha \Vdash n$ to mean α is a pseudo-composition of n. Define the descent pseudo-composition $C_B(\pi)$ of a signed permutation π be the lengths of its increasing runs as before, but now we have $\alpha_1 = 0$ if $\pi_1 < 0$. As with ordinary compositions, the partial order on pseudo-compositions of n is given by reverse refinement. We can move back and forth between descent pseudo-compositions and descent sets in exactly the same way as for type A. If $C_B(\pi) = (\alpha_1, \ldots, \alpha_k)$, then we have

$$\mathrm{Des}_B(\pi) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}.$$

We will use pseudo-compositions of n to index the type B quasisymmetric functions. Define $\mathcal{BQ}sym_n$ as the vector space of functions spanned by the *type B monomial quasisymmetric functions*:

$$M_{B,\alpha} := \sum_{0 < i_2 < \cdots < i_k} x_0^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k},$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is any pseudo-composition, or equivalently by the *type B fundamental quasisymmetric functions*:

$$F_{B,\alpha} := \sum_{\alpha \preccurlyeq \beta} M_{B,\beta}.$$

The space of all type B quasisymmetric functions is defined as the direct sum $\mathcal{BQ}sym := \bigoplus_{n \geq 0} \mathcal{BQ}sym_n$. By design, we have

$$\Gamma_B(\pi) = F_{B,C_B(\pi)}.$$

From Chow [4] we have the following theorem and corollary.

Theorem 2 *As sets, we have the bijection*

$$\mathcal{A}_B(\pi; ST) \leftrightarrow \coprod_{\sigma\tau = \pi} \mathcal{A}_B(\tau; S) \oplus \mathcal{A}_B(\sigma; T),$$

where ST is the cartesian product of the sets S and T with the lexicographic ordering.

We take $S = T = \mathbb{Z}$ and we have the following.

Corollary 2 *We have*

$$F_{B,C_B(\pi)}(XY) = \sum_{\sigma\tau = \pi} F_{B,C_B(\tau)}(X) F_{B,C_B(\sigma)}(Y).$$

The coalgebra structure on $\mathcal{BQ}sym_n$ works just the same as in the type A case. Corollary 2 gives us the coproduct

$$F_{B,\gamma} \mapsto \sum_{\alpha,\beta \Vdash n} b^{\gamma}_{\alpha,\beta} F_{B,\beta} \otimes F_{B,\alpha},$$

where for any π such that $C_B(\pi) = \gamma$, $b^{\gamma}_{\alpha,\beta}$ is the number of pairs of signed permutations (σ, τ) such that $C_B(\sigma) = \alpha$, $C_B(\tau) = \beta$, and $\sigma\tau = \pi$. The dual algebra is isomorphic to $\mathrm{Sol}(B_n)$, where if u_α is the sum of all signed permutations with descent pseudo-composition α, the multiplication given by

$$u_\beta u_\alpha = \sum_{\gamma \Vdash n} b^{\gamma}_{\alpha,\beta} u_\gamma.$$

3 Signed quasisymmetric functions and the Mantaci-Reutenauer algebra

One thing to have noticed about the generating function for type B P-partitions is that we are losing a certain amount of information when we take absolute values on the subscripts. We can think of signed quasisymmetric functions as arising naturally by dropping this restriction.

For a type B poset P, define the *signed generating function* for type B P-partitions to be

$$\overline{\Gamma}(P) := \sum_{f \in \mathcal{A}_B(P)} x_{f(1)} x_{f(2)} \cdots x_{f(n)},$$

where we will write

$$x_i = \begin{cases} u_i & \text{if } i < 0, \\ v_i & \text{if } i \geq 0. \end{cases}$$

In the case where P is a signed permutation, we have

$$\overline{\Gamma}(\pi) = \sum_{\substack{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ s \in \mathrm{Des}_B(\pi) \Rightarrow i_s < i_{s+1} \\ \pi_s < 0 \Rightarrow x_{i_s} = u_{i_s} \\ \pi_s > 0 \Rightarrow x_{i_s} = v_{i_s}}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

so that now we are keeping track of the set of minus signs of our signed permutation along with the descents. For example,

$$\overline{\Gamma}((-3, 2, -1)) = \sum_{0 < i \leq j < k} u_i v_j u_k.$$

To keep track of both the set of signs and the set of descents, we introduce the *signed compositions* as used in [3]. A signed composition α of n, denoted $\alpha \Vvdash n$, is a tuple of nonzero integers $(\alpha_1, \ldots, \alpha_k)$ such that $|\alpha_1| + \cdots + |\alpha_k| = n$. For any signed permutation π we will associate a signed composition $sC(\pi)$ by simply recording the length of increasing runs with constant sign, and then recording that sign. For example, if $\pi = (-3, 4, 5, -6, -2, -7, 1)$, then $sC(\pi) = (-1, 2, -2, -1, 1)$. The signed composition keeps track of both the set of signs and the set of descents of the permutation as we demonstrate with an example. If $sC(\pi) = (-3, 2, 1, -2, 1)$, then we know that π is a permutation in $\mathfrak{S}_9$ such that π_4, π_5, π_6, and π_9 are positive, whereas the rest are all negative. The descents of π are in positions 5 and 6. Note that for any ordinary composition of n with k parts, there are 2^k signed compositions, leading us to conclude that there are

$$\sum_{k=1}^{n} \binom{n-1}{k-1} 2^k = 2 \cdot 3^{n-1}$$

signed compositions of n. The partial order on signed compositions is given by reverse refinement with constant sign, i.e., the cover relations are still of the form:

$$(\alpha_1, \ldots, \alpha_i + \alpha_{i+1}, \ldots, \alpha_k) \prec (\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_k),$$

but now α_i and α_{i+1} have to have the same sign. For example, if $n = 2$, we have the following partial order:

$$\begin{array}{l} (2) \prec (1,1) \\ (-1,1) \\ (1,-1) \\ (-2) \prec (-1,-1) \end{array}$$

We will use signed compositions to index the signed quasisymmetric functions (see [9]). For any signed composition α, define the *monomial signed quasisymmetric function*

$$\overline{M}_\alpha := \sum_{\substack{i_1 < i_2 < \cdots < i_k \\ \alpha_r < 0 \Rightarrow x_{i_r} = u_{i_r} \\ \alpha_r > 0 \Rightarrow x_{i_r} = v_{i_r}}} x_{i_1}^{|\alpha_1|} x_{i_2}^{|\alpha_2|} \cdots x_{i_k}^{|\alpha_k|},$$

and the *fundamental signed quasisymmetric function*

$$\overline{F}_\alpha := \sum_{\alpha \preccurlyeq \beta} \overline{M}_\beta.$$

By construction, we have

$$\overline{\Gamma}(\pi) = \overline{F}_{sC(\pi)}.$$

Notice that if we set $u = v$, then our signed quasisymmetric functions become type B quasisymmetric functions.

Let $\mathcal{SQ}sym_n$ denote the span of the $\overline{M}_\alpha$ (or $\overline{F}_\alpha$), taken over all $\alpha \Vdash n$. The space of all signed quasisymmetric functions, $\mathcal{SQ}sym := \bigoplus_{n\geq 0} \mathcal{SQ}sym_n$, is a graded ring whose n-th graded component has rank $2 \cdot 3^{n-1}$. We will relate this to the Mantaci-Reutenauer algebra.

Theorem 2 is a statement about splitting apart bipartite P-partitions, independent of how we choose to encode the information. So while Corollary 2 is one such way of encoding the information of Theorem 2, the following is another.

Corollary 3 *We have*

$$\overline{F}_{sC(\pi)}(XY) = \sum_{\sigma\tau=\pi} \overline{F}_{sC(\tau)}(X)\overline{F}_{sC(\sigma)}(Y).$$

We define a coalgebra structure on $\mathcal{SQ}sym_n$ as we did in the earlier cases. Let $\pi \in \mathfrak{B}_n$ be any signed permutation with $sC(\pi) = \gamma$, and let $c^{\gamma}_{\alpha,\beta}$ be the number of pairs of permutations $(\sigma, \tau) \in \mathfrak{B}_n \times \mathfrak{B}_n$ with $sC(\sigma) = \alpha$, $sC(\tau) = \beta$, and $\sigma\tau = \pi$. Corollary 3 gives a coproduct $\mathcal{SQ}sym_n \to \mathcal{SQ}sym_n \otimes \mathcal{SQ}sym_n$:

$$\overline{F}_\gamma \mapsto \sum_{\alpha,\beta \Vdash n} c^{\gamma}_{\alpha,\beta} \overline{F}_\beta \otimes \overline{F}_\alpha.$$

Multiplication in the dual algebra $\mathcal{SQ}sym^*_n$ is given by

$$\overline{F}^*_\beta * \overline{F}^*_\alpha = \sum_{\gamma \Vdash n} c^{\gamma}_{\alpha,\beta} \overline{F}^*_\gamma.$$

The group algebra of the hyperoctahedral group, $\mathbb{Z}\mathfrak{B}_n$, has a dual coalgebra $\mathbb{Z}\mathfrak{B}^*_n$ with comultiplication given by the map

$$\pi \mapsto \sum_{\sigma\tau=\pi} \tau \otimes \sigma.$$

By Corollary 3, the following is a surjective homomorphism of coalgebras $\psi^* : \mathbb{Z}\mathfrak{B}^*_n \to \mathcal{SQ}sym_n$ given by

$$\psi^*(\pi) = \overline{F}_{sC(\pi)}.$$

The dualization of this map is an injective homomorphism $\psi : \mathcal{SQ}sym^*_n \to \mathbb{Z}\mathfrak{B}_n$ with

$$\psi(\overline{F}^*_\alpha) = \sum_{sC(\pi)=\alpha} \pi.$$

The image of ψ is then a subalgebra of $\mathbb{Z}\mathfrak{B}_n$ of dimension $2 \cdot 3^{n-1}$, with basis

$$v_\alpha := \sum_{sC(\pi)=\alpha} \pi.$$

This subalgebra is called the *Mantaci-Reutenauer algebra* [6], with multiplication given explicitly by

$$v_\beta v_\alpha = \sum_{\gamma \Vdash n} c^{\gamma}_{\alpha,\beta} v_\gamma.$$

The duality between $\mathcal{SQ}sym_n$ and the Mantaci-Reutenauer algebra was shown in [1], and the bases $\{\overline{F}_\alpha\}$ and $\{v_\alpha\}$ are shown to be dual in [2], but the the P-partition approach to the problem is new. As the Mantaci-Reutenauer algebra is defined for any wreath product $C_m \wr \mathfrak{S}_n$, i.e., any "m-colored" permutation group, it would be nice to develop a theory of colored P-partitions to tell the dual story in general.

In closing, we remark that this same method was put to use in [8], where Stembridge's enriched P-partitions [13] were generalized and put to use to study peak algebras. Variations on the theme can also be found in [7].

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