On an identity for the cycle indices of rooted tree automorphism groups

Stephan G. Wagner*

Institut für Analysis und Computational Number Theory
Technische Universität Graz
Steyrergasse 30, 8010 Graz, Austria
wagner@finanz.math.tugraz.at

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Abstract

This note deals with a formula due to G. Labelle for the summed cycle indices of all rooted trees, which resembles the well-known formula for the cycle index of the symmetric group in some way. An elementary proof is provided as well as some immediate corollaries and applications, in particular a new application to the enumeration of k-decomposable trees. A tree is called k-decomposable in this context if it has a spanning forest whose components are all of size k.

1 Introduction

Pólya's enumeration method is widely used for graph enumeration problems – we refer to [6] and the references therein for instance. For the application of this method, information on the cycle indices of certain groups is needed – mostly, these are comparatively simple examples, such as the cyclic group, the dihedral group or the symmetric group. A very well-known formula gives the cycle index of the symmetric group S_n (we adopt the notation from [6] here):

$$Z(S_n) = \sum_{j_1+2j_2+\ldots+nj_n=n} \prod_{k=1}^{n} \frac{s_k^{j_k}}{k^{j_k} j_k!}. \tag{1}$$

One has

$$\sum_{n=0}^{\infty} Z(S_n) t^n = \exp \sum_{k=1}^{\infty} \frac{s_k}{k} t^k,$$

an identity which is of importance in various tree counting problems (cf. again [6]).

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In the past, several tree counting problems related to the automorphism groups of trees have been investigated. We state, for instance, the enumeration of identity trees (see [7]), and the question of determining the average size of the automorphism group in certain classes of trees (see [9, 10]).

Therefore, it is not surprising that so-called *cycle index series* or *indicatrix series* [2, 8] are of interest in enumeration problems. Given a combinatorial species F, the indicatrix series is given by

$$Z_F(s_1, s_2, \ldots) = \sum_{c_1+2c_2+3c_3+\ldots<\infty} f_{c_1,c_2,c_3,\ldots} \frac{s_1^{c_1} s_2^{c_2} s_3^{c_3} \ldots}{1^{c_1}c_1!2^{c_2}c_2!3^{c_3}c_3!\ldots},$$

where $f_{c_1,c_2,c_3,\ldots}$ denotes the number of F-structures on $n = c_1 + 2c_2 + 3c_3 + \ldots$ points which are invariant under the action of any (given) permutation σ of these n points with cycle type $(c_1, c_2, \ldots)$ (i.e. exactly c_k cycles of length k). See for instance [2, 6, 8] and the references therein for more information on cycle index series. Equivalently, it can be defined via

$$Z_F(s_1, s_2, \ldots) = \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{\sigma\in S_n} \operatorname{fix} F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \ldots \right),$$

where $\operatorname{fix} F[\sigma]$ is the number of F-structures for which the permutation σ is an automorphism and $(\sigma_1, \sigma_2, \ldots)$ is the cycle type of σ [2].

In this note, we deal with the special family $\mathcal{T}$ of rooted trees. Yet another reformulation shows that the cycle index series is also

$$\sum_{T\in\mathcal{T}} Z(\mathrm{Aut}(T)),$$

where $Z(\mathrm{Aut}(T))$ is the cycle index of the automorphism group of T. The following formula for the cycle index series is due to G. Labelle [8, Corollary A2]:

Theorem 1 *The cycle index series for rooted trees is given by*

$$Z_{\mathcal{T}}(s_1, s_2, \ldots) = \sum_{c_1>0} \sum_{c_2,c_3,\ldots\geq 0} \frac{c_1^{c_1-1} s_1^{c_1}}{c_1!} \prod_{i>1} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j \right)^{c_i-1} \left(\sum_{j|i, j\neq i} j c_j \right) s_i^{c_i}.$$

Note that the expression resembles (1), though it is somewhat longer. This result seems to be not too well-known, but it certainly deserves attention. In [8], Labelle proves it in a more general setting, using a multidimensional version of Lagrange's inversion formula due to Good [4]. On the other hand, Constantineau and J. Labelle provide a combinatorial proof in [3].

First of all, we will give a simple proof (though, of course, less general than Labelle's) for this formula, for which only the classical single-variable form of Lagrange inversion will be necessary; then, some immediate corrolaries are stated. Finally, the use of the cycle index series is demonstrated by applying the formula to the enumeration of weighted trees and k-decomposable trees.

2 Proof of the main theorem

By the recursive structure of rooted trees and the multiplicative properties of the cycle index, it is not difficult to see that $Z = Z_{\mathcal{T}}(s_1, s_2, \ldots)$ satisfies the relation

$$Z = s_1 \exp\left(\sum_{m\ge1} \frac{1}{m} Z_m\right),$$

which is given, for instance, in a paper of Robinson [12, p. 344] and the book of Bergeron et al. [2, p. 167]. Here, Z_m is obtained from Z by replacing every s_i with s_{mi}. Now, we prove the following by induction on k:

$$Z = \sum_{\substack{c_1,\ldots,c_k\ge0\\ c_1>0}} \frac{c_1^{c_1-1} s_1^{c_1}}{c_1!} \prod_{i=2}^{k} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j\right)^{c_i-1} \left(\sum_{j|i, j\ne i} j c_j\right) s_i^{c_i}$$
$$\exp\left(\sum_{m>k} \frac{1}{m} \left(\sum_{d|m, d\le k} d c_d\right) Z_m\right)$$

in the ring of formal power series. Then, for finite k, the coefficient of $s_1^{c_1} \ldots s_k^{c_k}$ follows at once, since $\sum_{m>k} \frac{1}{m} \left(\sum_{d|m,d\le k} dc_d\right) Z_m$ doesn't contain the variables $s_1, \ldots, s_k$.

First note that, by Lagrange's inversion formula (cf. [5, 6]), we have

$$w = \sum_{c\ge1} \frac{c^{c-1}}{c!} x^c$$

and

$$\exp(aw) = \sum_{c\ge0} \frac{a(c+a)^{c-1}}{c!} x^c$$

if $w = xe^w$. This yields

$$Z = s_1 \exp\left(Z + \sum_{m\ge2} \frac{1}{m} Z_m\right) = \sum_{c_1\ge1} \frac{c_1^{c_1-1}}{c_1!} s_1^{c_1} \exp\left(\sum_{m\ge2} \frac{c_1}{m} Z_m\right),$$

which is exactly the desired formula for $k = 1$. For the induction step, we note that

$$Z_l = s_l \exp\left(\sum_{m\ge1} \frac{1}{m} Z_{ml}\right)$$

and thus, by the induction hypothesis,

$$\begin{aligned}
Z = & \sum_{\substack{c_1,\ldots,c_{k-1}\geq 0\\ c_1>0}} \frac{c_1^{c_1-1}s_1^{c_1}}{c_1!}\prod_{i=2}^{k-1}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right)s_i^{c_i} \\
& \exp\left(\frac{1}{k}\left(\sum_{d|k,d\neq k} dc_d\right)Z_k + \sum_{m>k}\frac{1}{m}\left(\sum_{d|m,d<k} dc_d\right)Z_m\right) \\
= & \sum_{\substack{c_1,\ldots,c_{k-1}\geq 0\\ c_1>0}} \frac{c_1^{c_1-1}s_1^{c_1}}{c_1!}\prod_{i=2}^{k-1}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right)s_i^{c_i} \\
& \sum_{c_k\geq 0}\frac{1}{c_k!\cdot k}\left(\sum_{j|k,j\neq k} jc_j\right)\left(c_k+\frac{1}{k}\sum_{j|k,j\neq k} jc_j\right)^{c_k-1} s_k^{c_k} \\
& \exp\left(\sum_{l>1}\frac{kc_k}{kl}Z_{kl}\right)\exp\left(\sum_{m>k}\frac{1}{m}\left(\sum_{d|m,d<k} dc_d\right)Z_m\right) \\
= & \sum_{\substack{c_1,\ldots,c_k\geq 0\\ c_1>0}} \frac{c_1^{c_1-1}s_1^{c_1}}{c_1!}\prod_{i=2}^{k}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right)s_i^{c_i} \\
& \exp\left(\sum_{m>k}\frac{1}{m}\left(\sum_{d|m,d\leq k} dc_d\right)Z_m\right).
\end{aligned}$$

This finishes the induction. ■

Corollary 2 *The number $t_n = |\mathcal{T}_n|$ of rooted trees on n vertices is given by*

$$t_n = \sum_{\substack{c_1+2c_2+\ldots=n\\ c_1>0}} \frac{c_1^{c_1-1}}{c_1!}\prod_{i>1}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right).$$

Proof: Simply set $s_1 = s_2 = \ldots = 1$ in the identity

$$\sum_{T\in\mathcal{T}_n} Z(\mathrm{Aut}(T)) = \sum_{\substack{c_1+2c_2+\ldots=n\\ c_1>0}} \frac{c_1^{c_1-1}s_1^{c_1}}{c_1!}\prod_{i>1}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right)s_i^{c_i}.$$

■

As a second corollary, we obtain Cayley's formula for the number of rooted labeled trees.

Corollary 3 *The number of rooted labeled trees on n vertices is given by n^{n-1}.*

Proof: Note that the coefficient of s_1^n in the cycle index of a rooted tree T on n vertices is precisely $|\operatorname{Aut}(T)|^{-1}$. Thus, we have

$$\sum_{T \in \mathcal{T}_n} |\operatorname{Aut}(T)|^{-1} = \frac{n^{n-1}}{n!}.$$

But $\frac{n!}{|\operatorname{Aut} T|}$ is exactly the number of different labelings of T, which finishes the proof. ∎

3 Further applications

Theorem 1 can also be applied to a general class of enumeration problems: let a set $\mathcal{B}$ of combinatorial objects with an additive weight be given, and let $B(z)$ be its counting series. Now, if we want to enumerate trees on n vertices, where an element of $\mathcal{B}$ is assigned to every vertex of the tree, the counting series is given by

$$\sum_{\substack{c_1+2c_2+\ldots=n \\ c_1>0}} \frac{c_1^{c_1-1}}{c_1!} B(z)^{c_1} \prod_{i>1} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j \right)^{c_i-1} \left(\sum_{j|i, j\neq i} j c_j \right) B(z^i)^{c_i}.$$

The coefficient of z equals the total weight. For example, the counting series for rooted weighted trees on n vertices (i.e. each vertex is assigned a positive integer weight, cf. Harary and Prins [7]) is given by

$$W(z) = \sum_{\substack{c_1+2c_2+\ldots=n \\ c_1>0}} \frac{c_1^{c_1-1}}{c_1!} \left(\frac{z}{1-z}\right)^{c_1} \prod_{i>1} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j \right)^{c_i-1} \left(\sum_{j|i, j\neq i} j c_j \right) \left(\frac{z^i}{1-z^i}\right)^{c_i}.$$

The first few instances are

- $n = 1$: $W(z) = \frac{z}{1-z} = z + z^2 + z^3 + \ldots$,
- $n = 2$: $W(z) = \frac{z^2}{(1-z)^2} = z^2 + 2z^3 + 3z^4 + \ldots$,
- $n = 3$: $W(z) = \frac{z^3(2+z)}{(1-z)^2(1-z^2)} = 2z^3 + 5z^4 + 10z^5 + \ldots$.

Finally, we are going to consider a new application of Theorem 1. This example deals with the decomposability of trees: we call a tree k-decomposable (a special case of the general concept of λ-decomposability, see [1, 16]) if it has a spanning forest whose components are all of size k. It has been shown by Zelinka [17] that such a decomposition, if it exists, is always unique. The special case $k = 2$, which has already been investigated by Moon [11] and Simion [13, 14], corresponds to perfect matchings. Now, let $D(x)$ denote the generating function for the number of k-decomposable rooted trees. Since a decomposable rooted tree is made up from a rooted tree on k vertices (the component containing the root) and collections of k-decomposable rooted trees attached to each of these k vertices, we obtain the following functional equation for k-decomposable trees:

$$D(x) = \sum_{\substack{c_1+2c_2+\ldots=k \\ c_1>0}} \frac{c_1^{c_1-1}}{c_1!} E(x)^{c_1} \prod_{i>1} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j \right)^{c_i-1} \left(\sum_{j|i, j\neq i} j c_j \right) E(x^i)^{c_i},$$

where $E(x) = x \exp\left(\sum_{m\geq 1} \frac{1}{m} D(x^m)\right)$. For $k = 2$, we obtain

$$D(x) = x^2 \exp\left(\sum_{m\geq 1} \frac{2}{m} D(x^m) \right),$$

giving the known counting series for trees with a perfect matching (Sloane's A000151 [15], see also [11, 13, 14]):

$$D(x) = x^2 + 2x^4 + 7x^6 + 26x^8 + 107x^{10} + 458x^{12} + \ldots$$

For $k = 3$, to give a new example, we have

$$D(x) = \frac{3x^3}{2} \exp\left(\sum_{m\geq 1} \frac{3}{m} D(x^m) \right) + \frac{x^3}{2} \exp\left(\sum_{m\geq 1} \frac{1}{m} \left(D(x^m) + D(x^{2m}) \right) \right),$$

yielding

$$D(x) = 2x^3 + 10x^6 + 84x^9 + 788x^{12} + \ldots$$

Of course, it is possible to calculate the counting series of k-decomposable rooted trees for arbitrary k in this way. The functional equation can also be used to obtain information about the asymptotic behavior (cf. [6, 16]).

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